Supplemental Material to Distributions of Posterior Quantiles and Economic Applications

By

Kai Hao Yang and Alexander K. Zentefis

March 2022

Revised July 2022

COWLES FOUNDATION DISCUSSION PAPER NO. 2328RS

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Online Appendix of Distributions of Posterior Quantiles and Economic Applications

Kai Hao Yang* Alexander K. Zentefis†

July 6, 2022

OA.1 Properties of Distributions of Posterior Quantiles

OA.1.1 Proof of Corollary 2

For any $H \in \mathcal{H}_\tau$, Theorem 1 implies that

$$\overline{F}_0^\tau(\underline{\omega}) \leq \overline{F}_0^\tau(\overline{\omega}) \leq H(\underline{\omega}) \leq H(\overline{\omega}) \leq \underline{F}_0^\tau(\underline{\omega}) \leq \underline{F}_0^\tau(\overline{\omega}).$$

Therefore,

$$(\overline{F}_0^\tau(\underline{\omega})) - \underline{F}_0^\tau(\overline{\omega}))^+ \leq H(\overline{\omega}) - H(\underline{\omega}) \leq \underline{F}_0^\tau(\overline{\omega}) - \overline{F}_0^\tau(\underline{\omega}).$$

Conversely, for any $\eta \in [(\overline{F}_0^\tau(\underline{\omega})) - \underline{F}_0^\tau(\overline{\omega}))^+, (\underline{F}_0^\tau(\overline{\omega}) - \overline{F}_0^\tau(\underline{\omega}))]$, if $\underline{\omega} \leq F_0^{-1}(\tau)$, then let H be defined as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < (\underline{F}_0^\tau)^{-1}(\eta + \underline{F}_0^\tau(\underline{\omega})) \\ \eta + \underline{F}_0^\tau(\underline{\omega}), & \text{if } \omega \in [(\underline{F}_0^\tau)^{-1}(\eta + \underline{F}_0^\tau(\underline{\omega})), (\overline{F}_0^\tau)^{-1}((\eta + \underline{F}_0^\tau(\underline{\omega}))^+) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq (\overline{F}_0^\tau)^{-1}((\eta + \underline{F}_0^\tau(\underline{\omega}))^+). \end{cases},$$

for all $\omega \in \mathbb{R}$; if $\underline{\omega} > F_0^{-1}(\tau)$, then define H as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < (\underline{F}_0^\tau)^{-1}(\overline{F}_0^\tau(\overline{\omega}) - \eta) \\ \overline{F}_0^\tau(\overline{\omega}) - \eta, & \text{if } \omega \in [(\underline{F}_0^\tau)^{-1}(\overline{F}_0^\tau(\overline{\omega}) - \eta), (\overline{F}_0^\tau)^{-1}((\overline{F}_0^\tau(\overline{\omega}) - \eta)^+) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq (\overline{F}_0^\tau)^{-1}((\overline{F}_0^\tau(\overline{\omega}) - \eta)^+). \end{cases}.$$

In both cases, $H(\overline{\omega}) - H(\underline{\omega}) = \eta$ and $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$. By Theorem 1, $H \in \mathcal{H}_\tau$ as well. This completes the proof. ■

OA.1.2 Proof of Theorem 2

For any $F_0 \in \mathcal{F}$ and for any $\tau \in (0, 1)$, let $\mathcal{M}_\tau^0(F_0)$ denote the collection of signals $\mu \in \mathcal{M}(F_0)$ such that

$$\mu(\{F \in \mathcal{F} | F^{-1}(\tau) = F^{-1}(\tau^+)\}) = 1.$$

*Yale School of Management, Email: kaihao.yang@yale.edu

†Yale School of Management, Email: alexander.zentefis@yale.edu

When there is no confusion, we write $\mathcal{M}^0_\tau(F_0)$ as $\mathcal{M}^0_\tau$. Note that for any $\mu \in \mathcal{M}^0_\tau$, $H^\tau(\omega|\mu, r) = H^\tau(\omega|\mu, r')$ for all $\omega \in \mathbb{R}$ and for all $r, r' \in \mathcal{R}$. Henceforth, we denote the distribution of posterior quantiles induced by $\mu \in \mathcal{M}^0_\tau$ as $H^\tau(\omega|\mu)$.

To prove Theorem 2, we first establish two lemmas that generalize Lemma 1 and Lemma 2 in the main text.

Lemma OA.1. *For any $\tau \in (0,1)$, and for any nondecreasing and right-continuous function $g : [0,1] \to [0,1]$, the following are equivalent:*

1. *For any $F_0 \in \mathcal{F}$ and for any $H \in \mathcal{F}$ such that $H \succeq g \circ F_0$ ($H \preceq g \circ F_0$, resp.), there exists $\mu \in \mathcal{M}^0_\tau(F_0)$ such that $H^\tau(\omega|\mu) = H(\omega)$.*

2. *For any $H \in \mathcal{F}$ such that $H \succeq g$ ($H \preceq g$, resp.), there exists $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ such that $H^\tau(\omega|\tilde{\mu}) = H(\omega)$.*

Proof. Since the CDF of the uniform distribution on $[0,1]$ is in $\mathcal{F}$ and since U is the identity function, 1 implying 2 follows immediately from the definition of $\tilde{r}_0$.

Conversely, suppose that 2 holds. Consider any $F_0 \in \mathcal{F}$ and any $H \in \mathcal{F}$ such that $H \succeq g \circ F_0$. We now show that there exists $\mu \in \mathcal{M}^0_\tau$ such that $H^\tau(\omega|\mu) = H(\omega)$. To this end, let $\widetilde{H}(q) := H(F_0^{-1}(q))$ for all $q \in \mathbb{R}$. Since g is nondecreasing, it must be that $\widetilde{H} \succeq g$. Therefore, there exists $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ such that

$$\widetilde{H}(q) = H^\tau(q|\tilde{\mu}) = \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq q\}).$$

for all $q \in [0,1]$. Next, for any $F_0 \in \mathcal{F}$, define μ as

$$\mu(A) := \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A\}),$$

for all measurable $A \subseteq \mathcal{F}$. We claim that $\mu \in \mathcal{M}^0_\tau(F_0)$. Indeed, for any measurable $A \subseteq \mathcal{F}$, $\mu(A) = \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A\}) \geq 0$. Meanwhile, $\mu(\mathcal{F}) = \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in \mathcal{F}\}) = \tilde{\mu}(\mathcal{F}) = 1$. Furthermore, for any measurable $A \subseteq \mathcal{F}$, let

$$F_0^{-1} \circ A := \{F_0^{-1} \circ F | F \in A\},$$

and note that $F \circ F_0 \in A$ if and only if $F \in F_0^{-1} \circ A$ for all $F \in \mathcal{F}$. Thus, for any disjoint collection of measurable sets $\{A_n\} \subseteq \mathcal{F}$,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) =& \tilde{\mu}\left(\left\{F \in \mathcal{F} \middle| F \circ F_0 \in \bigcup_{n=1}^{\infty} A_n\right\}\right) \\
=& \tilde{\mu}\left(\left\{F \in \mathcal{F} \middle| F \in F_0^{-1} \circ \bigcup_{n=1}^{\infty} A_n\right\}\right) \\
=& \sum_{n=1}^{\infty} \tilde{\mu}(F_0^{-1} \circ A_n) \\
=& \sum_{n=1}^{\infty} \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A_n\}) \\
=& \sum_{n=1}^{\infty} \mu(A_n).
\end{aligned}$$

Consequently, μ is indeed a probability measure on $\mathcal{F}$.

In addition, note that, for any $\omega \in \mathbb{R}$,

$$\int_{\mathcal{F}} F(\omega)\mu(\mathrm{d}F) = \int_{\mathcal{F}} F(F_0(\omega))\tilde{\mu}(\mathrm{d}F) = F_0(\omega),$$

which in turn implies that $\mu \in \mathcal{M}(F_0)$. Lastly, since F_0 has a support on an interval of $\mathbb{R}$, F_0 is strictly increasing on its support. As a result,

$$\mu(\{F \in \mathcal{F} | F^{-1}(\tau) = F^{-1}(\tau)\}) = \tilde{\mu}(\{F \in \mathcal{F} | F_0^{-1} \circ F^{-1}(\tau) = F_0^{-1} \circ F^{-1}(\tau^+)\}) = 1,$$

and hence $\mu \in \mathcal{M}_\tau^0(F_0)$.

As a result, for any $\omega \in \mathbb{R}$,

$$\begin{aligned} H(\omega) = \widetilde{H}(F_0(\omega)) =& \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq F_0(\omega)\}) \\ =& \tilde{\mu}(\{F \in \mathcal{F} | F_0^{-1} \circ F^{-1}(\tau) \leq \omega\}) \\ =& \mu(\{F \in \mathcal{F} | F^{-1}(\tau) \leq \omega\}) \\ =& H^\tau(\omega | \mu), \end{aligned}$$

as desired. The case for $H \preceq g \circ F_0$ is analogous. This completes the proof. ■

To set the stage for the next lemma, we note that for any $\tau \in (0,1)$ and for any $\underline{\tau}, \overline{\tau}$ such that $0 < \underline{\tau} < \tau < \overline{\tau} < 1$, $\underline{U}^{\overline{\tau}} \preceq \overline{U}^{\underline{\tau}}$ whenever $\tau - \underline{\tau}$ and $\overline{\tau} - \tau$ are small enough. For any such $\underline{\tau}, \overline{\tau}$, let $\mathcal{I}^*_{\underline{\tau},\overline{\tau}} := \mathcal{I}(\underline{U}^{\overline{\tau}}, \overline{U}^{\underline{\tau}})$.

Lemma OA.2. *H is an extreme point of $\mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ if and only if there exists $0 \leq \underline{x} \leq \overline{x} \leq \underline{\tau} \leq \tau \leq \overline{\tau} \leq \underline{y} \leq \overline{y}$; countable sets I, J; and sequences $\{\underline{x}_i, \overline{x}_i\}_{i \in I}, \{\underline{y}_j, \overline{y}_j\}_{j \in J} \subseteq \mathbb{R}$ such that $\underline{U}^{\overline{\tau}}(\overline{x}) = \overline{U}^{\underline{\tau}}(\underline{y})$; that $\underline{x} \leq \underline{x}_i \leq \overline{x}_i \leq \underline{x}_{i+1} \leq \overline{x} < \underline{y} \leq \underline{y}_j \leq \overline{y}_j \leq \underline{y}_{j+1} \leq \overline{y}$ for all $i \in I, j \in J$; and that*

$$H(\omega) = \begin{cases} 0, & \text{if } \omega < \underline{x} \\ \underline{U}^{\overline{\tau}}(\underline{x}_i), & \text{if } \omega \in [\underline{x}_i, \overline{x}_i) \\ \underline{U}^{\overline{\tau}}(\omega), & \text{if } \omega \in [\underline{x}, \overline{x}) \backslash \cup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ \underline{U}^{\overline{\tau}}(\overline{x}), & \text{if } \omega \in [\overline{x}, \underline{y}) \\ \overline{U}^{\underline{\tau}}(\underline{y}_j), & \text{if } \omega \in [\underline{y}_j, \overline{y}_j) \\ \overline{U}^{\underline{\tau}}(\omega), & \text{if } \omega \in [\underline{y}, \overline{y}) \backslash \cup_{j \in J} [\underline{y}_j, \overline{y}_j) \\ 1, & \text{if } \omega \geq \overline{y} \end{cases}, \tag{OA.1}$$

for all $\omega \in \mathbb{R}$.

Proof. Embed $\mathcal{I}^*_{\underline{\tau},\overline{\tau}} \subseteq \mathcal{F}$ into the collection $L^1([0,1])$ of integrable functions on $[0,1]$. Note that $\mathcal{I}^*_\tau$ is a convex subset of a normed linear space $L^1([0,1])$. Consider any $H \in \mathcal{I}^*_\tau$ that takes the form of (OA.1), and any $\widehat{H} \in L^1([0,1])$ such that $\widehat{H}(\tilde{\omega}) \neq 0$ for some $\tilde{\omega} \in [0,1]$. Suppose that $H(\tilde{\omega}) \in \{\underline{U}^{\overline{\tau}}(\tilde{\omega}), \overline{U}^{\underline{\tau}}(\tilde{\omega})\}$. Then clearly either $H(\tilde{\omega}) + \widehat{H}(\tilde{\omega}) > \underline{U}^{\overline{\tau}}(\tilde{\omega})$ or $H(\tilde{\omega}) - \widehat{H}(\tilde{\omega}) < \overline{U}^{\underline{\tau}}(\tilde{\omega})$ and hence, either $H + \widehat{H} \notin \mathcal{I}^*_\tau$ or $H - \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. Meanwhile, suppose that $\tilde{\omega} \in [\underline{x}_i, \overline{x}_i)$ for some $i \in I$ or $\tilde{\omega} \in [\underline{y}_j, \overline{y}_j)$ for some $j \in J$. If either $H + \widehat{H} \notin \mathcal{F}$ or $H - \widehat{H} \notin \mathcal{F}$, then clearly either $H + \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ or $H - \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. If, on the other hand, both $H + \widehat{H}$ and $H - \widehat{H}$ are in $\mathcal{F}$, then it must be that either $H(\omega) + \widehat{H}(\omega) = \underline{U}^{\overline{\tau}}(\underline{x}_i) + \widehat{H}(\tilde{\omega}) > \underline{U}^{\overline{\tau}}(\underline{x}_i)$ for all

$\omega \in [\underline{x}_i, \overline{x}_i)$, or $H(\omega) - \widehat{H}(\omega) = \overline{U}^{\underline{\tau}}(\overline{y}_j) - \widehat{H}(\tilde{\omega}) < \overline{U}^{\underline{\tau}}(\overline{y}_j)$, for all $\omega \in [\underline{y}_j, \overline{y}_j)$. Therefore, there must exist $\hat{\omega} \in \mathbb{R}$ such that either $H(\hat{\omega}) + \widehat{H}(\hat{\omega}) \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ or $H(\hat{\omega}) - \widehat{H}(\hat{\omega}) \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$.

Conversely, suppose that $H \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ does not take form of (OA.1). Then there exists $\underline{\omega} < \overline{\omega}$ and $\underline{\eta} < \overline{\eta}$ such that $H(\underline{\omega}^-) \leq \underline{\eta} \leq H(\underline{\omega})$, $H(\overline{\omega}^-) \leq \overline{\eta} \leq H(\overline{\omega})$; that $\overline{U}^{\underline{\tau}}(\overline{\omega}) \leq \underline{\eta} < \overline{\eta} \leq \underline{U}^{\overline{\tau}}(\underline{\omega})$; and that $\underline{\eta} < H(\omega) < \overline{\eta}$ for some $\omega \in (\underline{\omega}, \overline{\omega})$. Then, since the set of extreme points of nondecreasing functions that map from $[\underline{\omega}, \overline{\omega}]$ to $[\underline{\eta}, \overline{\eta}]$ must only take values in $\{\underline{\eta}, \overline{\eta}\}$ (see, for instance, lemma 2.7 of Börgers, 2015), there exists a non-zero, integrable function $\widetilde{H} : [\underline{\omega}, \overline{\omega}] \to [\underline{\eta}, \overline{\eta}]$ such that both $H + \widetilde{H}$ and $H - \widetilde{H}$ are nondecreasing, right-continuous functions from $[\underline{\omega}, \overline{\omega}]$ to $[\underline{\eta}, \overline{\eta}]$. As a result, for any $\omega \in [\underline{\omega}, \overline{\omega}]$, it must be that

$$\max\{H(\omega) + \widetilde{H}(\omega), H(\omega) - \widetilde{H}(\omega)\} \leq \overline{\eta} \leq \underline{U}^{\overline{\tau}}(\underline{\omega}) \leq \underline{U}^{\overline{\tau}}(\omega) \tag{OA.2}$$

and that

$$\min\{H(\omega) + \widetilde{H}(\omega), H(\omega) - \widetilde{H}(\omega)\} \geq \underline{\eta} \geq \overline{U}^{\underline{\tau}}(\overline{\omega}) \geq \overline{U}^{\underline{\tau}}(\omega), \tag{OA.3}$$

for all $\omega \in [\underline{\omega}, \overline{\omega}]$. Now let $\widehat{H} : [0,1] \to \mathbb{R}$ be defined as

$$\widehat{H}(\omega) := \begin{cases} \widetilde{H}(\omega), & \text{if } \omega \in [\underline{\omega}, \overline{\omega}] \\ 0, & \text{otherwise} \end{cases},$$

for all $\omega \in [0,1]$. Clearly, $\widehat{H} \in L^1([0,1])$. Moreover, for any $\omega \in [0,1]$, from (OA.2) and (OA.3), together with $H \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$, it follows that

$$\overline{U}^{\underline{\tau}}(\omega) \leq \min\{H(\omega) + \widehat{H}(\omega), H(\omega) - \widehat{H}(\omega)\} \leq \max\{H(\omega) + \widehat{H}(\omega), H(\omega) - \widehat{H}(\omega)\} \leq \underline{U}^{\overline{\tau}}(\omega),$$

for all $\omega \in [0,1]$. Meanwhile, since $\underline{\eta} \in [H(\underline{\omega}^-), H(\underline{\omega})]$ and $\overline{\eta} \in [H(\overline{\omega}^-), H(\overline{\omega})]$, it must be that

$$H(\omega) + \widehat{H}(\omega) = H(\omega) - \widehat{H}(\omega) = H(\omega) \leq H(\underline{\omega}^-) \leq \underline{\eta},$$

for all $\omega \leq \underline{\omega}$; while

$$H(\omega) + \widehat{H}(\omega) = H(\omega) - \widehat{H}(\omega) = H(\omega) \geq H(\overline{\omega}) \geq \overline{\eta},$$

for all $\omega \geq \overline{\omega}$. As a result, both $H + \widehat{H}$ and $H - \widehat{H}$ are nondecreasing and right-continuous. It then follows that $H + \widehat{H} \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ and $H - \widehat{H} \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$, and hence H is not an extreme point of $\mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. This completes the proof. ■

Proof of Theorem 2. By Theorem 1,

$$H^0_\tau \subseteq \mathcal{H}_\tau = \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0).$$

We now show that $\mathcal{I}(\underline{F}^{\tau+\varepsilon}_0, \overline{F}^{\tau-\varepsilon}_0) \subseteq \mathcal{H}^0_\tau$ for all $\varepsilon > 0$ small enough. To this end, first notice that since F_0 has full support on an interval in $\mathbb{R}$, $\mathcal{I}(\underline{F}^{\tau+\varepsilon}_0, \overline{F}^{\tau-\varepsilon}_0)$ is well-defined when ε is small enough. Then, notice that since the functions $q \mapsto \min\{q/\tau+\varepsilon, 1\}$ and $q \mapsto \max\{q-(\tau-\varepsilon)/1-(\tau-\varepsilon), 0\}$ are nondecreasing for $\varepsilon > 0$ small enough, Lemma OA.1 implies that it suffices to prove $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon} \subseteq \mathcal{H}^0_\tau$. To show this, consider any $\varepsilon > 0$ small enough and consider any extreme point H of $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$. By Lemma OA.2, H must take the form of (OA.1) for some $\underline{x}, \overline{x}, \underline{y}, \overline{y} \in [0,1]$ and countable sequences $\{\underline{x}_i, \overline{x}_i\}_{i \in I}$ and $\{\underline{y}_j, \overline{y}_j\}_{j \in J}$, such that

$\underline{x} \leq \underline{x}_i \leq \overline{x}_i \leq \underline{x}_{i+1} \leq \overline{x} < \underline{y} \leq \underline{y}_j \leq \overline{y}_j \leq \underline{y}_{j+1} \leq \overline{y}$ for all $i \in I, j \in J$. Now define two classes of distributions, $\{\underline{U}^\omega\}_{\omega\in[0,\overline{x}]}$ and $\{\overline{U}^\omega\}_{\omega\in[\underline{y},1]}$, as follows:

$$\underline{U}^\omega(x) := \begin{cases} 0, & \text{if } x < \omega \\ \frac{\overline{x}}{\underline{y}-(\tau+\varepsilon)+\overline{x}}, & \text{if } x \in [\omega, \tau+\varepsilon) \\ \frac{x-(\tau+\varepsilon)+\overline{x}}{1-\underline{y}+\overline{x}}, & \text{if } x \in [\tau+\varepsilon, \underline{y}) \\ 1, & \text{if } x \geq \underline{y} \end{cases}; \text{ and } \overline{U}^\omega(x) := \begin{cases} 0, & \text{if } x < \overline{x} \\ \frac{x-\overline{x}}{1-\underline{y}+\tau-\varepsilon-\overline{x}}, & \text{if } x \in [\overline{x}, \tau-\varepsilon) \\ \frac{\tau-\varepsilon-\overline{x}}{1-\underline{y}+\tau-\varepsilon-\overline{x}}, & \text{if } x \in [\overline{x}, \omega) \\ 1, & \text{if } x \geq \omega \end{cases}.$$

Note that since $\underline{U}^{\tau+\varepsilon}(\overline{x}) = \overline{U}^{\tau-\varepsilon}(\underline{y})$, it follows that $(1-\tau+\varepsilon)\overline{x} = (\tau+\varepsilon)(\underline{y}-\tau+\varepsilon)$, and hence, $\underline{U}^\omega(x) = \tau+\varepsilon$ for all $x \in [\omega, \underline{y}]$ and $\overline{U}^\omega(x) = \tau-\varepsilon$ for all $x \in [\overline{x}, \omega)$. As a result, $\mathbb{Q}^\tau(\underline{U}^\omega) = \{\omega\}$ for all $\omega \in [0, \overline{x}]$ and $\mathbb{Q}^\tau(\overline{U}^\omega) = \{\omega\}$ for all $\omega \in [\underline{y}, 1]$. Moreover, for any $i \in I$ and for any $j \in J$, let $\underline{U}^i$ and $\overline{U}^j$ be defined as

$$\underline{U}^i(x) := \frac{1}{\overline{x}_i - \underline{x}_i}\int_{\underline{x}_i}^{\overline{x}_i} \underline{U}^\omega(x)\,\mathrm{d}\omega; \text{ and } \overline{U}^j(x) := \frac{1}{\overline{y}_j - \underline{y}_j}\int_{\underline{y}_j}^{\overline{y}_j} \overline{U}^\omega(x)\,\mathrm{d}\omega,$$

for all $x \in \mathbb{R}$. Notice that, by construction, $\underline{U}^i, \overline{U}^j \in \mathcal{F}$ and $\mathbb{Q}^\tau(\underline{U}^i) = \{\overline{x}_i\}$, $\mathbb{Q}^\tau(\overline{U}^j) = \{\underline{y}_j\}$, for all $i \in I$ and $j \in J$. For any $\omega \in \text{supp}(H)$, let $F_\omega \in \mathcal{F}$ be defined as[1]

$$F_\omega(x) := \begin{cases} \underline{U}^\omega(x), & \text{if } \omega \in [0,\overline{x}] \setminus \cup_{i\in I} [\underline{x}_i, \overline{x}_i] \\ \underline{U}^i(x), & \text{if } \omega \in [\underline{x}_i, \overline{x}_i] \\ \overline{U}^\omega(x), & \text{if } \omega \in [\underline{y}, 1] \setminus \cup_{j\in J} [\underline{y}_j, \overline{y}_j] \\ \overline{U}^j(x), & \text{if } \omega \in [\underline{y}_j, \overline{y}_j] \end{cases},$$

for all $x \in \mathbb{R}$.

Now define $\tilde{\mu}$ as

$$\tilde{\mu}(\{F_\omega \in \mathcal{F} | \omega \leq x\}) := H(x),$$

for all $x \in \mathbb{R}$. By construction, $\text{supp}(\tilde{\mu}) = \{\underline{U}^\omega\}_{\omega\in[0,\overline{x}]\setminus\cup_{i\in I}[\underline{x}_i,\overline{x}_i]} \cup \{\underline{U}^i\}_{i\in I} \cup \{\overline{U}^\omega\}_{\omega\in[\underline{y},1]\setminus\cup_{j\in J}[\underline{y}_j,\overline{y}_j]} \cup \{\overline{U}^j\}_{j\in J}$. Furthermore, for any $x \in [0,1]$,

$$\int_{\mathcal{F}} F(x)\tilde{\mu}(\mathrm{d}F) = \int_0^1 F_\omega(x) H(\mathrm{d}\omega) = x,$$

and hence $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$. Then, for all $\omega \in \mathbb{R}$, $H^\tau(\omega|\tilde{\mu}) = H(\omega)$ for all $\omega \in \mathbb{R}$, as desired.

Now, consider any $H \in \mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$. Since $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$ is a convex and compact set in a metrizable space, Choquet's theorem implies that there exists a probability measure $\Lambda_H \in \Delta(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$ that assigns probability 1 to $\text{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$ such that

$$H(\omega) = \int_{\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}} \widetilde{H}(\omega)\Lambda_H(\mathrm{d}\widetilde{H}).$$

In the meantime, define the linear functional $\Xi : \mathcal{M}^0_\tau(U) \to \mathcal{F}$ as

$$\Xi(\tilde{\mu})[\omega] := \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq \omega\}),$$

[1] Similar to the proof of Theorem 1, as a convention, define $\underline{U}^i(x) := \underline{U}^\omega(x)$ for all x if $\underline{x}_i = \overline{x}_i = \omega$. Similarly, define $\overline{U}^j(x) := \overline{U}^\omega(x)$ for all x if $\underline{y}_j = \overline{y}_j = \omega$.

for all $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ and for all $\omega \in \mathbb{R}$. Now define a probability measure $\widetilde{\Lambda}$ on $\mathcal{M}^0_\tau(U)$ by

$$\widetilde{\Lambda}_H(A) := \Lambda_H(\{\Xi(\tilde{\mu})|\tilde{\mu} \in A\}),$$

for all $A \subseteq \mathcal{M}^0_\tau(U)$. Then, since $\Lambda(\text{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})) = 1$ and since, for any $\widetilde{H} \in \text{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$, there exists $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ such that $H(\omega) = H^\tau(\omega|\tilde{\mu})$, it must be that $\widetilde{\Lambda}_H(\mathcal{M}^0_\tau(U)) = 1$, and hence $\widetilde{\Lambda}_H$ is a probability measure on $\mathcal{M}(U)$.

As a result, since Ξ is linear, let $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ be defined as

$$\tilde{\mu}(A) := \int_{\mathcal{M}^0_\tau(U)} \mu(A)\widetilde{\Lambda}_H(\mathrm{d}\mu),$$

for all measurable $A \subseteq \mathcal{F}$. Then, since Ξ is linear, it follows that

$$\begin{aligned} H(\omega) = \int_{\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}} \widetilde{H}(\omega)\Lambda_H(\mathrm{d}\widetilde{H}) = &\int_{\mathcal{M}^0_\tau(U)} \Xi(\mu)[\omega]\widetilde{\Lambda}_H(\mathrm{d}\mu) \\ =&\Xi(\tilde{\mu})[\omega] \\ =&H^\tau(\omega|\tilde{\mu}), \end{aligned}$$

and therefore, $H \in \mathcal{H}^0_\tau$. By Lemma OA.1, it then follows that $H \in \mathcal{I}(\underline{F}_0^{\tau+\varepsilon}, \overline{F}_0^{\tau-\varepsilon})$ for all $\varepsilon > 0$ small enough. Thus, there exists $\bar{\varepsilon} > 0$ such that

$$\bigcup_{0<\varepsilon<\bar{\varepsilon}} \mathcal{I}(\underline{F}_0^{\tau+\varepsilon}, \overline{F}_0^{\tau-\varepsilon}) \subseteq \mathcal{H}^\tau_0 \subseteq \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau).$$

This completes the proof. ∎

OA.1.3 Exposed Points of $\mathcal{I}^*_\tau$

Lemma 2 in the main text characterizes the extreme points of $\mathcal{I}^*_\tau$, which are then used to construct a signal μ and a selection $r \in \mathcal{R}$ that generates an arbitrary $H \in \mathcal{I}^*_\tau$. In addition, the characterization of extreme points of $\mathcal{I}^*_\tau$ facilitates solving problems of the form

$$\sup_{H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)} \int_{\mathbb{R}} V(\omega)H(\mathrm{d}\omega), \tag{OA.4}$$

for some measurable $V : \mathbb{R} \to \mathbb{R}$. A natural question is whether there are any irrelevant extreme points that never correspond to solutions of (OA.4). As we show in Theorem OA.1 below, the answer is "no."

Theorem OA.1. *Every extreme point of $\mathcal{I}^*_\tau$ is exposed.*

Proof. Consider any extreme point $\widehat{H}$ of $\mathcal{I}^*_\tau$. By Lemma 2, $\widehat{H}$ must take the form of (3). To show that H is an exposed point of $\mathcal{I}^*_\tau$, we construct a measurable function $V : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} V(\omega)\widehat{H}(\mathrm{d}\omega) \geq \int_{\mathbb{R}} V(\omega)H(\mathrm{d}\omega)$$

for all $H \in \mathcal{I}^*_\tau$, with the equality holding only at $H = \widehat{H}$. To this end, define $\underline{v}$ and $\overline{v}$ as

$$\underline{v}(\omega) := \begin{cases} -1, & \text{if } \omega \in [\underline{x}, \overline{x})\backslash \cup_{i\in I} [\underline{x}_i, \overline{x}_i) \\ 0, & \text{otherwise} \end{cases}, \quad \text{and } \overline{v}(\omega) := \begin{cases} 1, & \text{if } \omega \in [\underline{y}, \overline{y})\backslash \cup_{j\in J} [\underline{y}_j, \overline{y}_j) \\ 0, & \text{otherwise} \end{cases},$$

for all $\omega \in \mathbb{R}$, and let $V : \mathbb{R} \to \mathbb{R}$ be defined as

$$V(\omega) := \begin{cases} \int_{(-\infty,\omega)} [\underline{v}(\omega) + \overline{v}(\omega)]\widehat{H}(\mathrm{d}\omega), & \text{if } \omega < \overline{x} \\ \int_{(-\infty,\omega]} [\underline{v}(\omega) + \overline{v}(\omega)]\widehat{H}(\mathrm{d}\omega), & \text{if } \omega \geq \overline{x} \end{cases}.$$

Notice that V is upper-semicontinuous, strictly decreasing on $\mathrm{supp}(\widehat{H}) \cap [\underline{x}, \overline{x}]$, strictly increasing on $\mathrm{supp}(\widehat{H}) \cap [\underline{y}, \overline{y}]$, and is constant otherwise. Moreover, for any $i \in I$ and for any $\omega \leq \underline{x}_i$, $V(\omega) > V(\overline{x}_i)$; while for any $j \in J$ and for any $\omega \geq \overline{y}_j$, $V(\omega) > V(\underline{y}_j)$. Thus, for any $H \in \mathcal{I}^*_\tau$, it must be that

$$\int_{\mathbb{R}} V(\omega)H(\mathrm{d}\omega) \leq \int_{\mathbb{R}} V(\omega)\widehat{H}(\mathrm{d}\omega). \tag{OA.5}$$

Furthermore, by the definition of V, (OA.5) is binding only if $H = \widehat{H}$. Therefore, $\widehat{H}$ is exposed. ■

OA.2 Omitted Details

OA.2.1 Optimality of H^*

Here, we show that H^* defined in (5) is a solution of (4) when W is quasi-concave with a peak at $1/2$. Notice that, for any realized $x \in [0, 1]$, and for any $H \in \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0)$, if $x \leq F_0^{-1}(1/4)$, then $W(1 - H(x^-)) = W(H(x^-)) \leq W(\underline{F}_0^{1/2}(x^-))) = W(1 - \underline{F}^\tau_0(x))$, since $\underline{F}_0^{1/2}(x^-) \leq 1/2$ and W is increasing on $[0, 1/2]$; if $x \geq F_0^{-1}(3/4^+)$, then $W((1-H(x^-))) = W(H(x^-)) \leq W(\overline{F}^\tau_0(x^-)) = W((1-\overline{F}_0^{1/2}(x^-)))$, since $\overline{F}^{1/2}(x^-) \geq 1/2$ and W is decreasing on $[1/2, 1]$; while if $x \in (F_0^{-1}(1/2)(1/4), F_0^{-1}(3/4^+))$, clearly $W(1 - H(x^-)) \leq W(1/2)$. Therefore, for any $x \in [0, 1]$, $W((1 - H(x^-))) \leq W(1 - H^*(x^-))$, which in turn implies that

$$\int_0^1 W((1 - H(x^-)))G(\mathrm{d}x) \leq \int_0^1 W((1 - H^*(x^-)))G(\mathrm{d}x).$$

Moreover, when W is strictly quasi-concave, and when both F_0 and G have full support on the same interval, the inequality must be strict for any $H \neq H^*$ with positive G-measure.

OA.2.2 Optimality of H^{**}

To see that H^{**} is optimal, note that for any $H \in \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0)$,

$$\begin{aligned}\int_0^1 v_S(\omega)H(\mathrm{d}\omega) &= \int_0^{\omega_0} v_S(\omega)H(\mathrm{d}\omega) + \int_{\omega_0}^1 v_S(\omega)H(\mathrm{d}\omega) \\ &\leq v_S(\omega_0)H(\omega_0) + \int_{\omega_0}^1 v_S(\omega)H(\mathrm{d}\omega) \\ &\leq v_S(\omega_0)\underline{F}^\tau_0(\omega_0) + \int_{\omega_0}^1 v_S(\omega)\underline{F}^\tau_0(\mathrm{d}\omega) \\ &= \int_0^1 v_S(\omega)H^{**}(\omega),\end{aligned}$$

where the first inequality follows from $v_S(\omega) \leq v_S(\omega_0)$ for all $\omega \leq \omega_0$, and the second inequality follows from $H \preceq \underline{F}^\tau_0$ and v_S being nonincreasing on $[\omega_0, 1]$.

OA.2.3 Optimal Signal for a Quasi-Convex v_S

Consider the same problem in (8) with $\Omega = [0,1]$, and consider the case where v_S is quasi-convex with a minimum at $\omega_0 \in [0, F_0^{-1}(\tau)]$. For any $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$, let $\overline{H}$ be defined as

$$\overline{H}(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < \underline{\omega}_H \\ \underline{F}_0^\tau(\underline{\omega}_H), & \text{if } \omega \in [\underline{\omega}_H, \overline{\omega}_H) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq \overline{\omega}_H \end{cases},$$

for all $\omega \in [0,1]$, where $\underline{\omega}_H$ and $\overline{\omega}_H$ are defined so that $\underline{F}_0^\tau(\underline{\omega}_H) = H(\omega_0) = \overline{F}_0^\tau(\overline{\omega}_H)$. Then, since v_S is nonincreasing on $[0, \omega_0]$ and is nondecreasing on $[\omega_0, 1]$,

$$\int_0^1 v_S(\omega) H(\mathrm{d}\omega) \leq \int_0^1 v_S(\omega) \overline{H}(\mathrm{d}\omega). \tag{OA.6}$$

Thus, it is without loss to search across $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ that take the form of

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < \underline{\omega}^\alpha \\ \underline{F}_0^\tau(\underline{\omega}^\alpha), & \text{if } \omega \in [\underline{\omega}^\alpha, \overline{\omega}^\alpha) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq \overline{\omega}^\alpha \end{cases}, \tag{OA.7}$$

for all $\omega \in [0,1]$, where $\underline{\omega}^\alpha$ and $\overline{\omega}^\alpha$ are defined so that $\underline{F}_0^\tau(\underline{\omega}^\alpha) = \alpha = \overline{F}_0^\tau(\overline{\omega}^\alpha)$. Hence, the solution of (8) must take the form of (OA.7) with $\alpha \in [0, \underline{F}_0^\tau(\omega_0)]$ that maximizes

$$\frac{1}{\tau} \int_0^{\underline{\omega}^\alpha} v_S(\omega) F_0(\mathrm{d}\omega) + \frac{1}{1-\tau} \int_{\overline{\omega}^\alpha}^1 v_S(\omega) F_0(\mathrm{d}\omega),$$

whose first-order condition, when F_0 has a density $f_0 > 0$, is simply $v_S(\underline{\omega}^\alpha) = v_S(\overline{\omega}^\alpha)$. In this case, the solution can be characterized by the system of equations

$$\underline{F}_0^\tau(\underline{\omega}) = \overline{F}_0^\tau(\overline{\omega}) \text{ and } v_S(\underline{\omega}) = v_S(\overline{\omega}). \tag{OA.8}$$

Moreover, if v_S is strictly quasi-convex and if F_0 has full support on $[0,1]$, then the inequality in (OA.6) must be strict, unless H takes the form of (OA.7). In this case, the solution is unique. Note that this solution cannot be attained by the "single-dipped" signal defined by Kolotilin, Corrao, and Wolitzky (2022), as noted in footnote 17.

As an example, suppose that $F_0 = U$. If $v_S(\omega) = (\omega - 1/2)^2$, then (OA.8) has a unique solution where $\underline{\omega} = 1/4$ and $\overline{\omega} = 3/4$, and hence H^* is optimal.

OA.2.4 Theorem 1 and the Securability Theorem

In section 5.1, we demonstrate how Theorem 1 can be applied to characterize the sender's equilibrium payoffs in a cheap talk game with transparent motives (as in Lipnowski and Ravid, 2020), where the receiver's optimal actions are τ-quantiles for each posterior. We prove our claims in the main text formally in this section. Recall that for any upper-semicontinuous function v_S and for any $v^* \in \mathbb{R}$, the set $\{\omega \in [0,1] | v_S(\omega) < v^*\}$, if non-empty, can be written as the union of countably many disjoint open intervals $\{(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)\}_{i=1}^{I_{v^*}}$ for some $I_{v^*} \in \mathbb{N} \cup \{\infty\}$. Below we formally state the characterization of the sender's equilibrium payoffs.

Proposition OA.1. *For any $v^* \geq v_0 := \max_{\omega \in \mathbb{Q}^\tau(F_0)} v_S(\omega)$, there exists a perfect Bayesian equilibrium in which the sender's payoff is v^* if and only if $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$.*

Proof. By theorem 1 of Lipnowski and Ravid (2020), it suffices to show that for any $v^* \geq v_0$, $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$ if and only if there exists a signal $\mu \in \mathcal{M}$ and a receiver's best response function such that the sender's payoff under μ is at least v^* with probability 1. By Theorem 1, the latter is equivalent to the existence of some $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H assigns probability zero to the set $\{\omega \in [0,1] | v_S(\omega) < v^*\} = \cup_{i=1}^{I_{v^*}} (\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$.

Consider any $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H is a constant on $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. Then Corollary 2 implies that $\underline{F}_0^\tau(\underline{\omega}_{v^*}^i) \geq \overline{F}_0^\tau(\overline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. Conversely, suppose that $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. First consider the case where $I_{v^*} < \infty$. In this case, it is without loss to reorder the intervals $\{(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)\}_{i=1}^{I_{v^*}}$ so that $\overline{\omega}_{v^*}^i < \underline{\omega}_{v^*}^{i+1}$. Now let H be defined as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega \notin \cup_{i=1}^{I_{v^*}} [\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \\ \underline{F}_0^\tau(\underline{\omega}_{v^*}^i), & \text{if } \omega \in [\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \end{cases},$$

for all $\omega \in \mathbb{R}$. Then, since $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$, it must be that $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$. Now consider the case where $I_{v^*} = \infty$. Since $\cup_{i=1}^\infty (\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \subseteq [0,1]$, it must be that $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) \leq 1 < \infty$. Thus, for any $\varepsilon > 0$, there exists $\bar{I}_{v^*}^\varepsilon \in \mathbb{N}$ such that $\sum_{i=\bar{I}_{v^*}^\varepsilon}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) < \varepsilon$. If $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) = 1$, then $H(\omega) := \sum_{i=1}^\infty \mathbf{1}_{[\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)}(\omega) \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ is well-defined on $[0,1]$ and is in $\mathcal{F}$. Moreover, $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ by definition. In contrast, if $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) < 1$, then for $\varepsilon = (1 - \sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i))/2$, take $\bar{I}_{v^*}^\varepsilon$. It then follows that there exists a finite collection of disjoint intervals $\{\underline{x}_{v^*}^j, \overline{x}_{v^*}^j\}_{i=j}^{\bar{I}_{v^*}^\varepsilon}$ such that, for all $i \in \mathbb{N}$, $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \subseteq (\underline{x}_{v^*}^j, \overline{x}_{v^*}^j)$ for some $j \in \{1, \ldots, \bar{I}_{v^*}^\varepsilon\}$ and $\underline{F}_0^\tau(\underline{x}_{v^*}^i) \geq \overline{F}_0^\tau(\overline{x}_{v^*}^i)$. Then, as shown above, there exists $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H is constant on $(\underline{x}_{v^*}^j, \overline{x}_{v^*}^j)$ for all $j \in \{1, \ldots, \bar{I}_{v^*}^\varepsilon\}$ and hence constant on $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$ for all $i \in \mathbb{N}$, as desired. ■

To take an example, we consider the case in the main text where $\tau = 1/2$, $F_0 = U$, and $v_S(\omega) = (\omega - 1/2)^2$. Notice that for any $v^* \geq v_0 = 0$, $I_{v^*} = 1$, and $\underline{\omega}_{v^*}^1 = 1/2 - \sqrt{v^*}$, and $\overline{\omega}_{v^*}^1 = 1/2 + \sqrt{v^*}$. Therefore, $\underline{F}_0^{1/2}(\underline{\omega}_{v^*}^1) \geq \overline{F}_0^{1/2}(\overline{\omega}_{v^*}^1)$ if and only if $2(1/2 - \sqrt{v^*}) \geq 2(1/2 + \sqrt{v^*}) - 1$, which simplifies to $v^* \leq 1/16$. As a result, v^* is the sender's equilibrium payoff under some perfect Bayesian equilibrium if and only if $v^* \in [0, 1/16]$.

Comparing the two models with and without sender commitment in this special case, we make note that the sender's payoff that can be induced by the optimal signal H^* with commitment is disjoint with the set of payoffs that are securable without commitment.

References

Börgers, T. (2015): *An Introduction to the Theory of Mechanism Design*, Oxford University Press.

Kolotilin, A., R. Corrao, and A. Wolitzky (2022): "Persuasion as matching," Working Paper. Massachusetts Institute of Technology, Cambridge, MA.

Lipnowski, E. and D. Ravid (2020): "Cheap talk with transparent motives," *Econometrica*, 88, 1631–60.