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GERRYMANDERING AND THE
LIMITS OF REPRESENTATIVE DEMOCRACY

By

Kai Hao Yang and Alexander K. Zentefis

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Gerrymandering and the Limits of Representative Democracy*

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March 23, 2022

Abstract

We assess the capacity of gerrymandering to undermine the will of the people in a representative democracy. Citizens have political positions represented on a spectrum, and electoral maps separate people into districts. We show that unrestrained gerrymandering can severely distort the composition of a legislature, potentially leading half the population to lose all representation of their views. This means that, under majority rule in the congress, gerrymandering enables politicians to enact any legislation of their choice as long as it falls within the interquartile range of the political spectrum. Just as worrisome, gerrymandering can rig any legislation to pass instead of the median policy, which would otherwise prevail in a referendum against any other choice.

JEL classification: D72, D78, D82, K16

Keywords: Gerrymandering, representatives, legislature, Bayesian persuasion

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1 Introduction

Drawing election districts in America for political advantage dates as far back as the colonies, but the Massachusetts incident in 1812 stamped a permanent name to the practice. Having recently won full control of the Commonwealth’s Legislature, Jefferson’s Democratic-Republicans passed a redistricting bill that cut up, joined, and divided counties in an extraordinary manner. The governor at the time, Elbridge Gerry, signatory of the Declaration of Independence and early advocate for the Bill of Rights, apparently regarded the redistricting bill personally repugnant, but nevertheless complied with his party and signed the policy into law. Outraged, the opposing Federalists denounced the law, blamed Gerry for the policy’s invention, and satirized a particular district’s serpentine shape by calling it a “Gerry-mander,” forever associating partisan redistricting with poor Elbridge. In a blistering critique of the law, they wrote that the organization of districts was manipulated to “secure a majority, in defiance of the will of the people.”¹

Concerning the perils of gerrymandering, were the Federalists right? Notably, can unrestrained redistricting defeat the declared voice of a citizenry, and in so doing, threaten the integrity of representative institutions? Social scientists who have carefully examined gerrymandering have largely directed their energies towards ways to measure it, how to administer it optimally, or identifying its downstream consequences on areas of interest, like policy, polarization, and voting.² But we do not yet understand the full lengths to which gerrymandering can pervert the composition of a legislative body and upset the self-determination of an electorate. If a resolute party eyes a desired outcome in an election or in policy, can it shrewdly assign the populace into organized districts on a map so as to expressly achieve that purpose?

In this paper, we show that the answer, by and large, is “yes.” We study an environment with a continuum of citizens. Each person has an ideal position on political issues, and we represent the spectrum of people’s positions as an interval. An observer could say that

¹For a history of U.S. partisan redistricting through the mid-19th century, see [Griffith \(1907\)](#). For biographies of Elbridge Gerry see [Austin \(1828-1829\)](#) and [Billias \(1976\)](#). The quote from the Federalists is from the February 13th, 1812 edition of the Boston Gazette. See also the February 6th edition for another Federalist editorial about the 1812 Massachusetts law, and the March 26th edition for the political cartoon of the Gerry-mander, “a new species of Monster” fabricated to mock the grotesque figure of the newly drawn election district of Essex County, Massachusetts.

²For measurement, see [Grofman and King \(2007\)](#); [Chambers and Miller \(2010\)](#); [Fryer Jr and Holden \(2011\)](#); [McGhee \(2014\)](#); [Stephanopoulos and McGhee \(2015\)](#); [Duchin \(2018\)](#); [Tapp \(2019\)](#). For optimal gerrymandering under either partisan or social welfare objectives, see [Owen and Grofman \(1988\)](#); [Sherstyuk \(1998\)](#); [Shotts \(2001\)](#); [Coate and Knight \(2007\)](#); [Friedman and Holden \(2008\)](#); [Puppe and Tasnádi \(2009\)](#); [Gul and Pesendorfer \(2010\)](#); [Bracco \(2013\)](#); [Ely \(2019\)](#); [Friedman and Holden \(2020\)](#); [Kolotilin and Wolitzky \(2020\)](#). For studies on consequences, see [Shotts \(2002\)](#); [Besley and Preston \(2007\)](#); [McCarty, Poole, and Rosenthal \(2009\)](#); [Hayes and McKee \(2009\)](#); [Caughey, Tausanovitch, and Warshaw \(2017\)](#); [Stephanopoulos and Warshaw \(2020\)](#).

those on the left side of the interval are more liberal, whereas those on the right are more conservative, but this interpretation is not the only one, as the divide might split the politics of the left-wing from the right-wing.

Under this representation, we define a *map* as a way to split the distribution of citizens into electoral districts, where each district itself is a distribution of political positions. Given a map, citizens then elect their districts' representatives. Election results abide by the median voter property á la [Downs \(1957\)](#) and [Black \(1958\)](#). That is, elected representatives must have ideal positions at the medians of their districts. The representatives then form a legislature.

As our main result, we characterize all possible compositions of the legislature that a map can induce. Two extreme legislatures are important in this regard: an “all-left” body and an “all-right” body. In the former, every representative occupying the legislature has an ideal position that is left of the median voter's ideal, whereas in the latter, every representative is to the right. Hence, in the all-left legislature, only the views of citizens with ideal positions left-of-median are represented, whereas in the all-right legislature, only the views of citizens with positions right-of-median are represented.

[Theorem 1](#) shows that a map can induce *any* legislature within the bounds of these two extreme bodies. Mathematically speaking, there exists a map such that the distribution of the elected representatives' ideal positions coincides with a distribution H if and only if $H_R \leq H \leq H_L$, where H_L (H_R , respectively) is the distribution of citizens' ideal positions conditional on being left (right, respectively) of the median. In other words, a map can induce every distribution bounded by the two extremes—the all-left legislature (H_L) and the all-right legislature (H_R)—in the sense of first-order stochastic dominance.

Consequently, as members of the most extreme legislatures, H_L and H_R , only hold views consistent with half the population, [Theorem 1](#) implies that gerrymandering can inflict serious damage on the citizenry by depriving many of their just claim to representation. Furthermore, because a map can obtain any distribution of representatives between the all-left and the all-right, if political conspirators crave a particular composition of the legislature in this range, they have the recipe to get it, and by so doing, block the aspirations of the people.

To better understand the intuition behind [Theorem 1](#), consider the following map that induces, say, the all-right legislature, where every position right-of-the-median is represented and none of the positions left-of-the-median are. In this map, every position on the right is placed into a separate district from one other, whereas all positions on the left are uniformly pooled into each district, which spreads them out to narrowly lose each district's election. In other words, each district contains one and only one position on the right and just enough positions on the left so that the (only) position that is right of the population median becomes a median of that district. As a result, every district eventually elects a representative whose

ideal position equals the separated position on the right, leading to an all-right legislature.

Knowing all possible distributions of representatives, we then pick a canonical voting procedure for the legislature, simple majority rule, and ask: Under this procedure, what scope of legislative outcomes can gerrymandering achieve? The broader the scope, the greater the power relinquished to a partisan group to construct a map that carries out a desired intent, thereby suspending the dominion of the people. [Corollary 1](#) shows that the subversive power of gerrymandering is alarmingly broad. Given the initial distribution of citizens' political views, gerrymandering can procure *any* legislation within the 25th and 75th percentiles of ideal positions. Thus, the scope of gerrymandering's subversion of legislative policy is the interquartile range of the population's political views, and it reaches that far out no matter the properties of the distribution of ideals. But more or less extreme legislative policies can be secured if enough of the citizenry subscribes to them.

Tracing the range of legislative outcomes that gerrymandering can obtain offers but one aspect from which to describe the practice's subversive scope. Another is to ask: What legislation can a map engender that would defeat the median voter's ideal in a head-to-head vote in the legislature? If all citizens voted genuinely in a referendum within a political system governed by any Condorcet voting method, the median position would prevail. No majority of voters would agree to an alternative. To what extent can gerrymandering undermine this Condorcet winner? The answer is more striking: In some circumstances, such as when the distribution of voters' ideal positions is uniform, [Corollary 2](#) implies that maps can be designed so that *any* bill can defeat the median legislative policy in a congressional vote, wholly bankrupting the promise of representative democracy.

Recognizing the subversive power of gerrymandering, we then explore remedies that can weaken or undue its effects. A major benefit of our framework is distilling the complex problem of designing maps, which are high-dimensional objects, into the far simpler problem of choosing a value within an interval of legislative outcomes, which is a one-dimensional object. A common proposal to impede gerrymandering is to strip the majority party of the power to draw districts and grant that authority to a bipartisan committee. In our setting, a negotiation between committee members over maps can be modeled as a two-person bargaining game over surplus, which corresponds to the interquartile range of possible legislation under majority rule. If both sides were infinitely patient negotiators and bargained per the model in [Rubinstein \(1982\)](#), the legislative policy chosen would be the position held by the median voter, consistent with majority rule and striking out all injurious effects of gerrymandering.

Even after the mapmaking stage, remedies are possible. We explore changes to legislative voting rules. We show that no voting procedure can narrow the interquartile range of possible

legislative outcomes whenever such a procedure would pass legislation that a majority of the representatives would most prefer to pass. However, if one’s objective, instead, is to shrink the range of legislative policies that can defeat the median in a head-to-head vote, we show that a supermajority voting threshold can do exactly that.

In the final part of the paper, we study several extensions of our baseline setting, including when maps must be drawn on two-dimensional geographical planes; when they must respect state boundaries; when the number of districts are fixed; when districts within a state must be of equal population; and when residual uncertainty affects elections after a map is drawn. Overall, the extensions maintain the same general spirit of our main results.

Related Literature. This paper relates to the literature on optimal gerrymandering. In a notable contribution, [Owen and Grofman \(1988\)](#) characterize optimal district maps that maximize either the expected number of seats or the probability of winning a majority of a certain party, within a setting featuring individual-level uncertainty, but not aggregate uncertainty. [Friedman and Holden \(2008\)](#) solve for optimal district maps that maximize the expected number of seats for a designer who has imperfect (but nearly perfect) information about voters’ preferences, as part of a setting with aggregate uncertainty.³ Meanwhile, [Gul and Pesendorfer \(2010\)](#) consider a strategic setting where two parties who control fixed shares of regions can draw district maps simultaneously. In their model, both aggregate and individual uncertainty are present, and both parties observe a noisy signal of voters’ party registration. In a similar spirit to ours, [Kolotilin and Wolitzky \(2020\)](#) cast optimal gerrymandering as a Bayesian persuasion problem. They characterize the optimal maps in a setting with both aggregate and (linear) individual uncertainty, and a designer has perfect information about voters’ types when structuring a map to maximize the expected number of seats.

Generally, these papers focus on either finding optimal maps for a single designer or modeling strategic interactions between two parties. With different assumptions about the information available to the designer(s), as well as the form of uncertainty at the voting stage, the authors characterize the solutions and discuss the structures of their optimal or equilibrium maps. By contrast, this paper characterizes every possible composition of a legislative body that a map can generate. In so doing, we allow a map to allocate citizens according to their preferences in arbitrary ways, and we abstract from both individual-level and aggregate-level uncertainty in our baseline model. From this perspective, our results complement existing findings, as we identify an upper bound to the set of feasible outcomes

³See [Friedman and Holden \(2020\)](#) for an extension where two parties can draw election maps at the same time.

in this literature.

Our paper also connects to the Bayesian persuasion literature. Since a map in our model is, in fact, equivalent to a Blackwell experiment for a one-dimensional state, our framework corresponds to that of [Kamenica and Gentzkow \(2011\)](#). In this regard, this paper shares the same spirit as several recent works that describe the set of feasible outcomes across all possible information structures in various environments (see, for instance, [Bergemann, Brooks, and Morris 2015](#); [Bergemann, Brooks, and Morris 2017](#); [Roesler and Szentes 2017](#); [Condorelli and Szentes 2020](#); [Bergemann, Brooks, and Morris 2021](#); [Yang 2021](#); [Armstrong and Zhou 2022](#); [Elliot, Galeotti, Koh, and Li 2022](#); [Haghpanah and Siegel 2022](#); [Haghpanah and Siegel forthcoming](#); and [Condorelli and Szentes forthcoming](#)).⁴

Furthermore, since our main result can be regarded as a complete characterization of the distribution of medians of posteriors induced by all possible signals, our paper contrasts with Bayesian persuasion problems in which the sender’s objective depends only on the mean (see, for instance, [Gentzkow and Kamenica 2016](#); [Kolotilin, Li, Mylovanov, and Zapechelnyuk 2017](#); [Kolotilin 2018](#); and [Dworczak and Martini 2019](#)). Since the characterization of [Rothschild and Stiglitz \(1970\)](#) does not apply to medians of posteriors, persuasion problems where the sender’s objective depends on medians is relatively less understood than those where the sender’s objective depends on means. Our main result reduces the dimensionality of this class of persuasion problems so that it is without loss for the sender to choose from distributions characterized by our [Theorem 1](#).

Outline. The paper proceeds as follows: [Section 2](#) provides an informal, illustrative example of gerrymandering in our setting that spotlights its subversive effects. [Section 3](#) formally establishes the setting. [Section 4](#) presents the central result on gerrymandering’s power to the limit representative democracy. [Section 5](#) studies remedies that constrict the subversive scope of gerrymandering, and [Section 6](#) provides extensions of the baseline model. [Section 7](#) concludes.

2 Gerrymandering Illustrated

In this section, we provide a simple illustration of gerrymandering in our framework to shed light on its power to undermine representative democracy. [Figure I](#) provides a circular representation of a big city and the surrounding suburban and rural areas. The urban core

⁴It is noteworthy that proposition 1 of [Gomberg, Pance, and Sharma \(2021\)](#) shares the same flavor as our [Theorem 1](#). However, the authors assume that each district elects a *mean* candidate as opposed to the median, and, hence, their characterization follows from Blackwell’s theorem and properties of the majorization order, which do not apply to our setting.

is labeled Region 1; the suburbs, Region 2; and the rural zones are split into Regions 3 and 4. All regions share equal population.

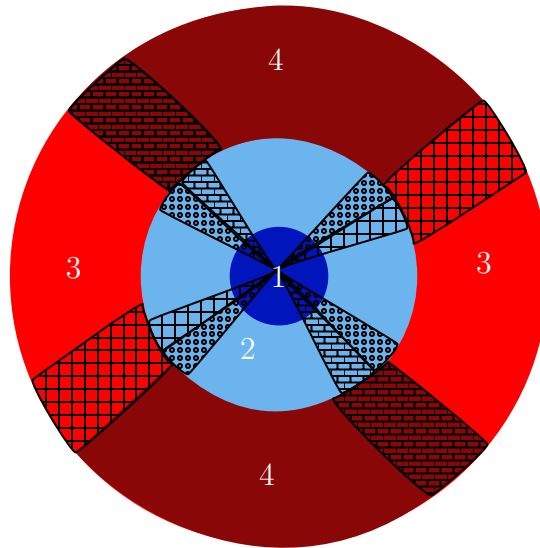


Figure I
GERRYMANDERED MAP

Citizens identify with one of four ideal positions over political issues. All citizens within a region hold the same ideal position and are uniformly distributed over each region. The number assigned to each region matches the ideal position of the area’s inhabitants. Residents of the big city lean heavily left, which we illustrate by coloring Region 1 dark blue. Suburban residents of Region 2 are closer to the median, but still left of it, which we represent by coloring the area light blue. Rural residents of Region 3 lean right, so that the section is colored light red; and finally, those living in rural Region 4 lean heavily right, making that region colored dark red.

A map is a partition of the entire circle of regions, and each element of the partition represents a district. Given a map, citizens of each district then elect a representative with an ideal position matching their median political view. As different maps can induce different groups of representatives, a natural question arises: What compositions of the legislature can a map create?

We propose an example map in the figure, which we represent as a collection of different patterned sections. This map demonstrates that very extreme compositions of the legislature are possible. In the map, a small fraction of citizens with positions 1 and 2—those who belong to the section patterned with dots in [Figure I](#)—are drawn into one, “X”-looking, district.

This district would certainly elect a representative whose position is either 1 or 2.

In the meantime, the remaining population is uniformly partitioned into many equally-sized districts. These districts can be categorized into two types. The first type is illustrated by the two rectangular and two triangular sections patterned with bricks that join at the center of the city. (Only one type of this district is shown as an example.) Collectively, districts of this first type form a partition of the population consisting of half the remaining left-leaning population, as well as all citizens with position 4. As a result, citizens with position 4 would constitute a majority in each of these districts, and thus, each of these districts would elect a representative whose ideal position is also 4. The second type of district is illustrated by the two rectangular and two triangular sections patterned with squares that too join at the center. (Only one example of this second type of district is shown.) For the same reason as before, each of these districts would elect a representative whose ideal position is 3. Notice that, by construction, all districts share the same population under this map. Nonetheless, all but one elected representative would have a position of either 3 or 4, whereas a lone representative would subscribe to position 1 or 2.

This example map demonstrates the formidable power of gerrymandering: All right-leaning citizens have representatives who match their political views, whereas virtually all left-leaning citizens do not. In essence, this map achieves a nearly “all-right” legislature, consisting of almost all members with ideal positions right-of-median. The map accomplishes this feat by following this strategy: (i) separate the two right-leaning positions (i.e., 3 and 4) so that citizens with those positions are never drawn into the same district, and (ii) pool the right-leaning citizens with the left-leaning ones in a way that the right-leaning positions have just enough support to win in every district they belong to.

Figure I illustrates, in a simple way, the power of unrestrained gerrymandering to manipulate the makeup of a legislative body. In what follows, we consider a richer environment and characterize every possible composition of a legislature that a map can produce.

3 Model

A continuum of citizens vote, and each citizen has single-peaked preferences over positions on political issues. The variety of these positions is represented by the unit interval $X := [0, 1]$, and each citizen is identified by an ideal position $x \in X$. The distribution of citizens’ ideal positions is given by a continuous and strictly increasing CDF F .⁵

A map segments citizens into electoral *districts*, where each district elects a *representative*.

⁵Continuity and strict monotonicity of F are, in fact, not necessary; the setting can naturally be extended to any CDF F , even discrete ones over a finite set of political positions. The set of citizens may also be finite rather than a continuum. We impose these assumptions here for the ease of exposition.

Formally, let \mathcal{D} be the collection of CDFs with supports on X .⁶ A map is defined as a probability distribution m over \mathcal{D} such that

$$\int_{\mathcal{D}} D(x)m(dD) = F(x), \forall x \in X. \quad (1)$$

That is, a map, m , splits the distribution of citizens' ideal positions F into districts in $\text{supp}(m)$ so that the collection of districts average back to F .⁷ Let \mathcal{M} denote the collection of all maps. As a consequence, for any map $m \in \mathcal{M}$, each $D \in \text{supp}(m)$ represents a district of citizens.

Election results at the district-level follow the median voter property. That is, given any map $m \in \mathcal{M}$, the elected representative of each district $D \in \text{supp}(m)$ must have an ideal position that is a median of D .⁸ Specifically, let $\text{med}(D) := [D^{-1}(1/2), D^{-1}(1/2^+)]$ denote the set of medians of D . Any elected representative in district D must be an element of $\text{med}(D)$.⁹ As the median of a distribution may not be unique in general, the exact distribution of representatives of a district D depends on tie-breaking rules in each district. Henceforth, we describe district-level tie-breaking rules by a transition probability $r : \mathcal{D} \rightarrow \Delta(X)$ with $\text{supp}(r(D)) \subseteq \text{med}(D)$ for all D . The set of all district-level tie-breaking rules is denoted by \mathcal{R} .

Given any map $m \in \mathcal{M}$ and any tie-breaking rule $r \in \mathcal{R}$, let $H(\cdot|m, r)$ be the distribution of the ideal positions of the elected representatives induced by map m . That is,

$$H(x|m, r) := \int_{\mathcal{D}} r([0, x]|D)m(dD).$$

Our goal is to characterize the set of distributions of representatives' ideal positions that can be induced by some map $m \in \mathcal{M}$ and some tie-breaking rule $r \in \mathcal{R}$.

Because we allow for arbitrary ways of splitting the distribution of citizens' ideal positions,

⁶ \mathcal{D} is endowed with the weak-* topology and the associated Borel σ -algebra.

⁷In other words, each district is defined as the conditional distribution of the ideal positions of citizens who belong to that district. In other words, we implicitly assume that a map can allocate citizens according to their ideal positions in arbitrary ways. In [Section 6](#), we further discuss sufficient conditions under which any map of this form can be obtained by partitioning an underlying geographical space of citizens, as well as several extensions that relax this assumption.

⁸Any district-level election system that meets the Condorcet criterion satisfies the median voter property ([Downs 1957](#); [Black 1958](#)). An example is majority voting with two office-seeking candidates. Alternatively, any citizen-candidateship voting system with sufficiently high entry costs would elect a median representative in each district. An extensive empirical literature suggests that median voter preferences can more or less be mapped onto a single-issue space satisfying an ideal point ([Congleton 2004](#)). See, for example, [Poole and Daniels \(1985\)](#), [Congleton and Bennett \(1995\)](#), and [Gerber and Lewis \(2004\)](#).

⁹Recall that x is a median of D if and only if $D(x) \geq 1/2$ and $1 - D(x^-) \geq 1/2$. Thus, the set of medians of D is an interval $[D^{-1}(1/2), D^{-1}(1/2^+)]$, where D^{-1} is the (left-continuous) quantile of D .

every possible map in the literature on gerrymandering can be represented by some $m \in \mathcal{M}$.¹⁰ From this perspective, our result can be regarded as identifying the largest possible set of distributions of representatives that a map can engineer.

4 The Limits of Representative Democracy

4.1 Distributions of Representatives

In that follows, we characterize the set of all possible distributions of representatives' ideal positions. To this end, we first introduce two crucial distributions. For any $x \in X$, let

$$H_L(x) := \begin{cases} 2F(x), & \text{if } x \in [0, F^{-1}(1/2)] \\ 1, & \text{if } x \in (F^{-1}(1/2), 1] \end{cases},$$

and

$$H_R(x) := \begin{cases} 0, & \text{if } x \in [0, F^{-1}(1/2)] \\ 2F(x) - 1, & \text{if } x \in (F^{-1}(1/2), 1] \end{cases}.$$

In other words, H_L and H_R are distributions of representatives that only reflect one side of voters' political positions relative to the median of the population. The distribution H_L describes an “all-left” legislature, in which each representative elected has an ideal position that is left of the median voter's ideal. Conversely, H_R represents an “all-right” legislature, in which all representatives are positioned to the right of the median voter.

The distributions H_L and H_R are arguably extreme legislative bodies, as the views of fifty percent of the citizens are completely unrepresented. Nonetheless, our main characterization below shows that these distributions can, in fact, be induced by some maps. In fact, as shown by [Theorem 1](#), a map can procure *any* distribution bounded by H_L and H_R in the sense of first-order stochastic dominance. This leads to our main result.

Theorem 1 (Limits of Representative Democracy). *For any distribution $H : X \rightarrow [0, 1]$, the following are equivalent:*

1. *There exists a map $m \in \mathcal{M}$ and tie-breaking rule $r \in \mathcal{R}$ such that $H(x) = H(x|m, r)$ for all $x \in X$.*
2. *$H_R(x) \leq H(x) \leq H_L(x)$ for all $x \in X$.*

¹⁰For instance, for any aggregate state and for any realized taste shock per voter, a strategy profile of the two parties in [Gul and Pesendorfer \(2010\)](#) induces a map that has four elements in its support. Each element is assigned a probability equal to the fraction of the population that the left/right party controls multiplied by the probability that voters are on the left/right according to that party's strategy.

The set of possible distributions of representatives' ideal positions is depicted in [Figure II](#). The all-left distribution, H_L , is colored blue, whereas the all-right distribution, H_R , is colored red. The green dotted curve represents the population distribution of citizens' ideal positions, F . According to [Theorem 1](#), any distribution H bounded by H_L and H_R (for instance, the black curve in the figure) can be induced by a map and a tie-breaking rule.

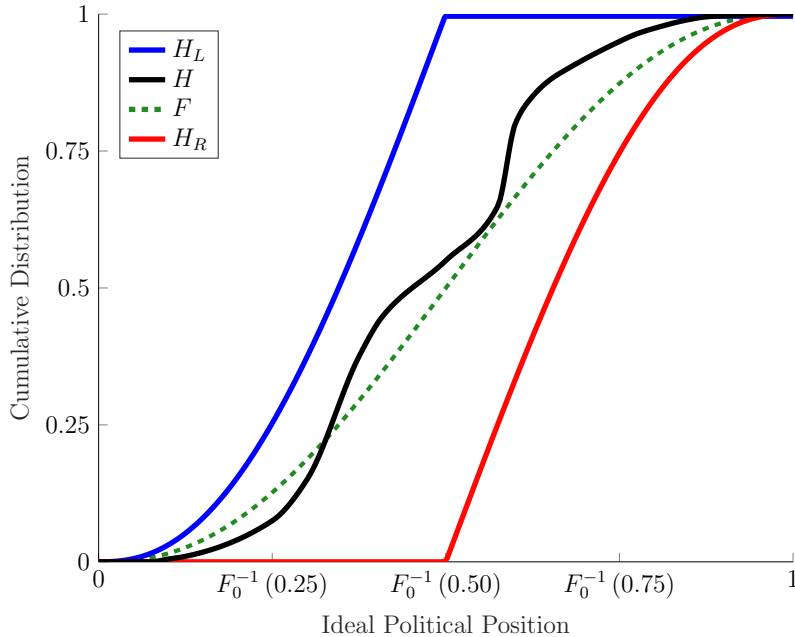


Figure II
LEGISLATURE DISTRIBUTIONS

The proof of [Theorem 1](#) can be found in the Appendix. To better understand the intuition behind the result, consider any map $m \in \mathcal{M}$ under which (almost) all districts $D \in \text{supp}(m)$ have a unique median. In this case, for any tie-breaking rule $r \in \mathcal{R}$, the induced distribution of representatives' ideal positions must coincide with the distribution of district medians. Furthermore, since each district has a unique median, for any $x \in X$, the share of representatives with ideal positions left of x must equal the share of districts in which at least $1/2$ of the citizens in that district have ideal positions left of x . Namely,

$$H(x|m, r) = m(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\}) = m(\{D \in \mathcal{D} | D(x) \geq 1/2\}).$$

From (1), for $\{D(x) | D \in \text{supp}(m)\}$ to have an average of $F(x)$, the probability that $D(x)$ is at least $1/2$ must be at most $2F(x)$ whenever $1/2 \geq F(x)$. Similarly, the probability that $D(x)$ is at least $1/2$ must be at least $2F(x) - 1$ whenever $1/2 < F(x)$.¹¹ As a result, it must

¹¹More specifically, for any fixed $x \in X$, we may regard $D(x)$ as a random variable whose distribution is

be that $H_R(x) \leq H(x|m, r) \leq H_L(x)$ for all $x \in X$. The proof in the Appendix extends this observation to cases when some districts $D \in \text{supp}(m)$ may have multiple medians.

As for the converse part of [Theorem 1](#), the proof is completed by (i) approximating any $H_R \leq H \leq H_L$ with a distribution that has finite support; (ii) finding a map and a tie-breaking rule that induces the approximating distribution; and (iii) demonstrating that there exists some map and some tie-breaking rule in the limit that induce H (see details in the Appendix). Below, we reveal the intuition by constructing maps and tie-breaking rules (without approximation) that induce the extreme distributions H_L and H_R . To this end, for any $z \in [0, F^{-1}(1/2)]$, define district D_L^z as

$$D_L^z(x) := \begin{cases} 0, & \text{if } x \in [0, z) \\ \frac{1}{2}, & \text{if } x \in [z, F^{-1}(1/2)) \\ F(x), & \text{if } x \in [F^{-1}(1/2), 1] \end{cases},$$

for all $x \in X$. Similarly, for any $z \in [F^{-1}(1/2), 1]$, define district D_R^z as

$$D_R^z(x) := \begin{cases} F(x), & \text{if } x \in [0, F^{-1}(1/2)) \\ \frac{1}{2}, & \text{if } x \in [F^{-1}(1/2), z) \\ 1, & \text{if } x \in [z, 1] \end{cases},$$

for all $x \in X$. Now define maps $m_L, m_R \in \mathcal{M}$ as

$$m_L(\{D_L^z | z \leq x\}) := H_L(x); \text{ and } m_R(\{D_R^z | z \leq x\}) := H_R(x),$$

for all $x \in X$. By construction, both m_L and m_R satisfy [\(1\)](#). Furthermore, for any $D_L^z \in \text{supp}(m_L)$, $\text{med}(D_L^z) = [z, F^{-1}(1/2)]$. Likewise, for any $D_R^z \in \text{supp}(m_R)$, $\text{med}(D_R^z) = [F^{-1}(1/2), z]$. Together with the tie-breaking rule that selects the smallest (largest, resp.) median in each district, the elected representative in district D_L^z (D_R^z , resp.) must have an ideal position of z . Therefore, under map m_L (m_R , resp.), every political candidate with an ideal position to the left (right, resp.) of the population median $F^{-1}(1/2)$ is elected per district, while no candidate with a position on the other side of the median is elected. This, in turn, implies that the distribution of representatives' ideal positions must coincide with H_L (H_R , resp.). [Figure III](#) illustrates these maps by plotting districts $D_L^z, D_L^{z'}, D_R^y, D_R^{y'}$ for some $z < z' < F^{-1}(1/2) < y < y'$.

In essence, m_L and m_R induce the most extreme compositions of the legislature by making each district as competitive as possible. For example, to induce a legislature to only represent

implied by m . As a result, [\(1\)](#) implies that $D(x)$ dominates $F(x)$ in the convex order. With this observation, inequalities here then follow directly from the characterization of [Rothschild and Stiglitz \(1970\)](#).

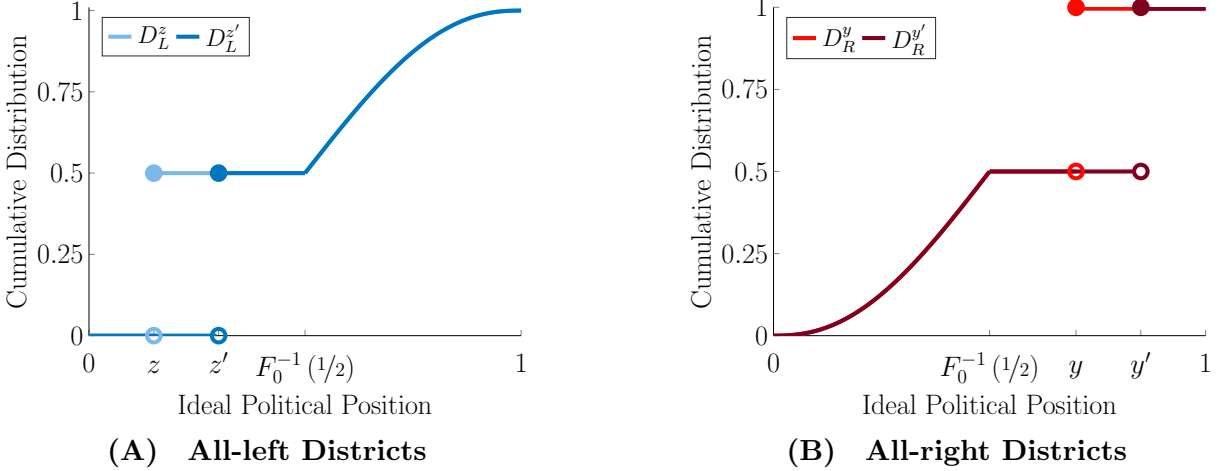


Figure III
ALL-LEFT AND ALL-RIGHT DISTRICTS

citizens on the left of the median voter, the map m_L separates citizens holding each ideal position on the left into different districts and then uniformly pools citizens holding all the ideal positions on the right with each one of these left positions. This pattern creates districts where each contain one and only one left position that is held by exactly $1/2$ the share of the citizens in the district.¹² With the tie-breaking rule that selects the smallest median, each district would elect the political candidate holding the (one and only one) position to the left of the population median. The map m_L generates an all-left legislature by ensuring that, in every district, a candidate with a position to the left of the median has just enough of votes to win.

While the tie-breaking rule seems to play a crucial role under these extreme maps, this is the case only because the extreme distributions are non-generic. For a generic distribution bounded by H_L and H_R , there exists a map where almost all districts have a unique median, leaving no room for tie-breaking.

¹²Note that this structure is reminiscent of the F^p districting of Gul and Pesendorfer (2010), as well as the *segregate-pool* districting described by Kolotilin and Wolitzky (2020). The difference is that under either of their maps, a citizen (type) is either maximally separated from all other citizens (types), or pooled with the rest of them, whereas under our map m_R (m_L , resp.), a citizen that is right (left, resp.) of the median is separated from all other citizens on the right (left, resp.) and pooled with all other citizens on the left (right, resp.). The maps m_L , m_R are neither “packing” nor “cracking,” but are a combination of segregation and pooling, in the sense that segregated citizens are at the same time pooled with those on the other side of the median. This, in turn, is of a similar fashion as the “matching extremes” structure à la Friedman and Holden (2008), except that matched citizens in each district only contain the “extremes” on one side.

4.2 Implications for Enacted Legislation

As [Theorem 1](#) provides a complete characterization of all possible compositions of the legislature, we may next explore the implications for legislative policy. Doing so requires specifying the voting rules deciding how representatives enact legislation. But we may again exploit the median voter property by supposing that the representatives pass the legislation that is a median of the distribution of representatives' ideal positions.¹³ Hence, given a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$, we assume that the enacted *legislative outcomes* must be an element of $\text{med}(H(\cdot|m, r))$.

Let $\mathcal{Z} \subseteq X$ denote the set of legislative outcomes that a map can attain. That is,

$$\mathcal{Z} := \{z \in X | z \in \text{med}(H(\cdot|m, r), \text{ for some } m \in \mathcal{M}, r \in \mathcal{R}\}.$$

Relying on [Theorem 1](#), the next corollary describes the set \mathcal{Z} of achievable legislative outcomes under any voting procedure of the congress satisfying the median voter property.

Corollary 1. *Under a legislative voting procedure satisfying the median voter property, a map can achieve the set of legislative outcomes $\mathcal{Z} = [F^{-1}(1/4), F^{-1}(3/4)]$.*

Proof. By [Theorem 1](#), for any map $m \in \mathcal{M}$ and for any tie-breaking rule $r \in \mathcal{R}$, $H_R(x) \leq H(x|m, r) \leq H_L(x)$ and, hence, $H_R^{-1}(x) \leq H^{-1}(x|m, r) \leq H^{-1}(x^+|m, r) \leq H_L^{-1}(x^+)$, for all $x \in X$. This, in turn, implies that

$$F^{-1}(1/4) \leq H^{-1}(1/2|m, r) \leq H^{-1}(1/2^+|m, r) \leq F^{-1}(3/4).$$

Conversely, consider any $z \in [F^{-1}(1/4), F^{-1}(3/4)]$, since $H_L(z) \geq 1/2$ and $H_R(z) \leq 1/2$, there exists $\lambda \in [0, 1]$ such that $\lambda H_L(z) + (1 - \lambda)H_R(z) = 1/2$, and that $\lambda H_L + (1 - \lambda)H_R$ is strictly increasing in a neighborhood of z . Since $\lambda H_L + (1 - \lambda)H_R$ must satisfy condition 2 of [Theorem 1](#), there exists a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$ such that $H(x|m, r) = \lambda H_L(x) + (1 - \lambda)H_R(x)$ for all $x \in X$ and is strictly increasing in a neighborhood of z . Therefore, $\text{med}(H(\cdot|m, r)) = \{z\}$, as desired. ■

According to [Corollary 1](#), a congress whose voting procedure satisfies the median voter property can enact *any* legislation within the interquartile range of citizens' ideal positions.¹⁴

¹³Whether the median voter property is true in legislative practice is debated among scholars, but empirical evidence suggests that it is, at least, a good approximation. For data on the median legislator being decisive, see [McCarty, Poole, and Rosenthal \(2001\)](#), [Bradbury and Crain \(2005\)](#), and [Krehbiel \(2010\)](#). In a model of sequential bargaining between representatives in which each period a policy is proposed and voted on, [Cho and Duggan \(2009\)](#) show that as representatives become increasingly patient, the set of policies that can pass in any subgame perfect equilibrium converges to the median representative's ideal position.

¹⁴It is noteworthy that the full-information and infinite districts limit of [Friedman and Holden \(2008\)](#) also

The expansive range of achievable legislation is possible even when the median voter property holding at both the district-level and the legislative-level would imply that only the median legislative policy *should* be enacted if gerrymandered maps did not manipulate the composition of the legislature. The subversive scope of gerrymandering is thus established. A skillfully designed map can enact legislation well beyond the median citizen’s ideal position, indeed, as far as the ends of the interquartile range of the population’s distribution of political views.

In fact, not only can a map trigger non-median outcomes in the legislature, but many policies can also defeat the median in a head-to-head vote. Specifically, if we instead consider a majority voting rule among the representatives, and we compare the median citizen’s ideal position to other positions, [Corollary 2](#) below characterizes the set of legislative outcomes $z \in X$ that can have majority congressional support compared to the median citizen’s ideal position. To state this result, let $\underline{z} := \max\{2F^{-1}(1/4) - F^{-1}(1/2), 0\}$ and $\bar{z} := \min\{2F^{-1}(3/4) - F^{-1}(1/2), 1\}$.

Corollary 2. *For any $z \in X$, the following are equivalent:*

1. *There exists a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$ such that under $H(\cdot|m, r)$ the share of representatives with ideal positions closer to z than to $F^{-1}(1/2)$ is at least $1/2$.*
2. *$z \in [\underline{z}, \bar{z}]$.*

Proof. We first prove that 2 implies 1. Consider any $z \in [\underline{z}, \bar{z}]$, if $z = F^{-1}(1/2)$, then 1 is trivially satisfied. Suppose that $z < F^{-1}(1/2)$. Note that if the distribution of representatives’ ideal positions is H_L , then the share of representatives whose ideal positions are closer to z than to $F^{-1}(1/2)$ would be $2F((F^{-1}(1/2) + z)/2)$, which, in turn, is at least $1/2$, as $z \geq \underline{z}$. Similarly, suppose that $z > F^{-1}(1/2)$. If the distribution of representatives’ ideal positions is H_R , then the share of representatives whose ideal position is closer to z than to $F^{-1}(1/2)$ would be $2(1 - F((F^{-1}(1/2) + z)/2))$, which, in turn, is at least $1/2$, as $z \leq \bar{z}$. Therefore, by [Theorem 1](#), 1 is satisfied for all $z \in [\underline{z}, \bar{z}]$.

Conversely, to prove that 2 implies 1, fix any $z \in X$ and suppose that there exists a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$ such that under $H(\cdot|m, r)$, the share of representatives with ideal positions closer to z than to $F^{-1}(1/2)$ is at least $1/2$. That is, $H((F^{-1}(1/2) + z)/2|m, r) \geq 1/2$ if $z \leq F^{-1}(1/2)$ and $H((F^{-1}(1/2) + z)/2|m, r) \leq 1/2$ if $z >$

leads to the conclusion that the induced outcome under the left-leaning party’s optimal gerrymander equals the 25th percentile. [Corollary 1](#) here, using [Theorem 1](#), provides a simpler proof and characterizes the entire set of possible outcomes.

$F^{-1}(1/2)$. By [Theorem 1](#), it then follows that

$$2F\left(\frac{F^{-1}(1/2) + z}{2}\right) \geq H\left(\frac{F^{-1}(1/2) + z}{2} \middle| m, r\right) \geq \frac{1}{2}$$

if $z \leq F^{-1}(1/2)$ and

$$2F\left(\frac{F^{-1}(1/2) + z}{2}\right) - 1 \leq H\left(\frac{F^{-1}(1/2) + z}{2} \middle| m, r\right) \leq \frac{1}{2}$$

if $z > F^{-1}(1/2)$. Together with the monotonicity of F , this, in turn, implies $\underline{z} \leq z \leq \bar{z}$, as desired. ■

Given a collection of preferences over legislative outcomes, recall that a *Condorcet winner* is defined as an outcome that has majority support when compared to any other alternative. In our setting, where every citizen has single-peaked preferences over positions in X , a Condorcet winner always exists and must be the median citizen's ideal position, $F^{-1}(1/2)$. Despite this fact, [Corollary 2](#) demonstrates that for any legislative outcome $z \in [\underline{z}, \bar{z}]$, there exists a map (and a tie-breaking rule) such that z has majority support of the representatives compared to the Condorcet winner $F^{-1}(1/2)$. In fact, under some distributions, such as uniform, we would have $\underline{z} = 0$ and $\bar{z} = 1$, which is a complete reversal of the desirable property of the Condorcet winner and a total bankruptcy of the promise of political representatives to serve on behalf of the people.

5 Remedies

In this section, we use the characterizations in [Section 4](#) to discuss possible remedies to gerrymandering, from the mapmaking process to legislative voting rules.

5.1 Mapmaking

In most U.S. states, one of two institutions draw state and Congressional election districts: (1) the state legislatures themselves or (2) a commission ([Brennan Center for Justice 2019](#)). Advocates for redistricting reform often regard the second body, notably bipartisan or independent commissions, as superior to the first in producing fair maps (see, for instance, [Kubin 1996](#); [Cox 2006](#); [Cain 2011](#); and [Gartner 2019](#)). Any commission in practice inevitably would involve some negotiation between members. Without the characterizations in [Section 4](#), modeling such negotiations is challenging, as maps are complex objects that involve distributions over distributions. A benefit of [Theorem 1](#) and [Corollary 1](#) is that they

greatly reduce the complexity of this negotiation process. After all, according to [Theorem 1](#), the set of all possible maps is, in fact, equivalent to the set of all distributions bounded by the two extremes, H_L and H_R . Moreover, with the assumption that the legislative voting procedure satisfies the median voter property, the relevant objects of negotiation are further reduced to a one-dimensional interval, $[F^{-1}(1/4), F^{-1}(3/4)]$.

Regarding negotiation over the mapmaking process, consider the simple theoretical case of a bipartisan commission involving just two members, one from the left side of the median and one from the right. Instead of arguing over maps, they instead would argue over the division of the interquartile range of the citizenry’s distribution of ideal positions. Whatever division they might agree upon, there would exist a map that accompanies that agreement. Their interaction could be represented as a bargaining game, and, depending on the chosen model, the equilibrium could appeal to several canonical results. For example, if the game followed the setting of [Rubinstein \(1982\)](#), as both sides became increasingly patient negotiators, the bipartisan commission would agree to a map that coincided with the median citizen’s ideal point, thus satisfying the will of the majority. The only legislative outcome possible would be the median, and any damage done by gerrymandering would be entirely erased.

This example is just one way to study reforms to redistricting at the stage when maps are developed. Endless other models of mapmaking are possible. Whatever the model, we have shown that the choices of participants in the environment need not be over high-dimensional maps, but potentially over the much simpler set of one-dimensional legislative outcomes that such maps induce.

5.2 Legislative Voting Rules

In addition to bipartisan negotiations during the mapmaking process, another way to mitigate the effects of gerrymandering is through the voting procedure in the legislature. Although [Corollary 1](#) concludes from [Theorem 1](#) that a rather wide range of legislation is possible, that conclusion relies on the assumption that the legislative voting system satisfies the median voter property. Consequently, a natural question arises: Might an alternative voting procedure restrain the effect of gerrymandering and reduce the range of possible legislation?

To better formulate this question, we may regard a legislative voting procedure as a mapping from the distribution of representatives’ ideal positions to a legislative outcome. Of course, if any arbitrary voting procedure—regardless of complexity and practicality—is under consideration, then the answer to the question must be “yes.” After all, always enacting the median citizen’s ideal position regardless of how representatives vote (i.e., a constant mapping that maps every distribution to $F^{-1}(1/2)$) is feasible.

A more reasonable thought experiment would be to impose some minimal and reasonable

requirements on the legislative voting procedure. One natural requirement would be that it must reflect the will of a majority whenever it is unambiguous. In other words, for any distribution of representatives' ideal positions where more than $1/2$ of the representatives have an ideal position $x \in X$, then a reasonable legislative voting system must yield outcome x .

From this perspective, the aforementioned question can be reframed as the following: Does a mapping from the set of distributions of representatives' ideal positions to legislative outcomes exist such that its image is narrower than the interquartile range *and* reflects any unambiguous majority will at the same time? From [Corollary 3](#) below, unfortunately, the answer is “no.” To state this observation, let \mathcal{H} be the collection of distributions of representatives' ideal positions that a map can induce.

Corollary 3. *Consider any $\mathbb{C} : \mathcal{H} \rightarrow X$. Suppose that $\mathbb{C}(H) = x$ for all H that assigns probability greater than $1/2$ to x . Then, $\mathbb{C}(\mathcal{H}) \supseteq (F^{-1}(1/4), F^{-1}(3/4))$.*

Proof. Consider any $x \in (F^{-1}(1/4), F^{-1}(3/4))$. Let H^x be a distribution that assigns probability $2 \cdot \min\{F(x), 1 - F(x)\}$ to x , and probability $1 - 2 \cdot \min\{F(x), 1 - F(x)\}$ to $F^{-1}(1/2)$. Then $H_R \leq H^x \leq H_L$. Therefore, by [Theorem 1](#), $H^x \in \mathcal{H}$, which in turn implies that $\mathbb{C}(H^x) = x$, as desired. ■

While [Corollary 3](#) might suggest a pessimistic answer to whether legislative voting procedures can remedy gerrymandering from the perspective of [Corollary 1](#), changes to voting rules may still be useful in providing remedies from the point of view of [Corollary 2](#). Specifically, instead of using majority rule when comparing alternatives to the median citizen's ideal position, the legislature can potentially adopt other criteria. We say that a legislature adopts an α -absolute majority rule against $F^{-1}(1/2)$ when legislation $z \in X$ can defeat the median citizen's ideal position, $F^{-1}(1/2)$, in a head-to-head vote, if and only if at least a share $\alpha \in [1/2, 1]$ of the representatives prefer z over $F^{-1}(1/2)$.

Clearly, α -absolute majority rule raises the threshold to defeat the median, and, hence, it should be expected to limit the set of alternatives that can pass the legislature over the median under a map and tie-breaking rule. [Corollary 4](#) describes the exact degree to which the effect of gerrymandering can be alleviated in this regard. To state this result, let $\underline{z}(\alpha) := \max\{2F^{-1}(\alpha/2) - F^{-1}(1/2), 0\}$ and $\bar{z}(\alpha) := \min\{2F^{-1}(1 - \alpha/2) - F^{-1}(1/2), 1\}$.

Corollary 4. *For any $z \in X$ and for any $\alpha \in [1/2, 1]$, the following are equivalent:*

1. *There exists a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$ such that under $H(\cdot|m, r)$, z defeats the median under the α -absolute majority rule.*
2. *$z \in [\underline{z}(\alpha), \bar{z}(\alpha)]$.*

The proof of [Corollary 4](#) is analogous to that of [Corollary 2](#) and is therefore omitted. To better understand [Corollary 4](#), it is noteworthy that $\underline{z}(\alpha)$ is increasing in α and $\bar{z}(\alpha)$ is decreasing in α . Moreover, $\underline{z}(1) = \bar{z}(1) = F^{-1}(1/2)$. In other words, according to [Corollary 4](#), the set of alternatives that can defeat the median, $F^{-1}(1/2)$, shrinks as the threshold α increases. When unanimity is required, no alternative legislation can defeat the median.

The supermajority voting procedure of the type described in [Corollary 4](#) can protect the median from defeat and undue the power of gerrymandering to decide legislative outcomes. But the median legislative policy might not be the most progressive one, as half the citizenry must subscribe to its merit. Thus, one could think of the median as a stand-in for the status quo. Elevating the threshold for changing the status quo in the legislature may partially remedy gerrymandering, but it may also stall progressive change. This deterrence to change resembles the consequences of the filibuster in the United States Senate. There, a supermajority of members must agree to invoke cloture to close debate on a bill ([United States Senate Committee on Rules and Administration 2022](#)). Nowadays, legislation that is filibustered effectively requires a supermajority to pass. [Corollary 4](#) reveals that a filibuster-type rule in a legislative body (or, at least, a supermajority requirement to end debate) can partly cure the subversive effects of gerrymandering, but at the cost of potentially hindering progress.

6 Extensions

Here, we discuss several extensions of the baseline model to illustrate how the framework accommodates several salient features of political redistricting in practice.

Geographic Maps. Thus far, we model a map as splitting the distribution of citizens' ideal positions into several district distributions that average back to the population distribution. In practice, districts can only be drawn on a geographic map. To better connect our model to gerrymandering in reality, we further explain whether and when it is possible to generate a map $m \in \mathcal{M}$ in our setting by actually drawing election districts on a physical map.

Drawing districts on this kind of map can be regarded as partitioning a two-dimensional space that is spanned by latitude and longitude. Thus, the question of whether and when someone can generate a map $m \in \mathcal{M}$ in our model by drawing election districts on a geographic map can be recast as whether and when it is possible to generate a distribution m over distributions by partitioning an underlying characteristic space.

More specifically, let $\Theta := [0, 1]^2$ denote a geographic map. Suppose that every citizen who resides at the same location $\theta \in \Theta$ shares the same ideal position $\mathbf{x}(\theta)$, where $\mathbf{x} : \Theta \rightarrow X$

is a measurable function. Furthermore, suppose that citizens are distributed on Θ according to a density function $\phi > 0$. Under this setting, theorem 1 of [Yang \(2020\)](#) ensures that for any $m \in \mathcal{M}$ with countable support, there exists a countable partition of Θ , such that the distributions of citizens' ideal positions within each element coincide with the distributions in the support of m . Indeed, together with the fact that any line segment must have measure zero under ϕ , these partitions can even be chosen such that each element is connected and compact, which are requirements for drawing districts in many states ([Brennan Center for Justice 2019](#)).¹⁵

If we further assume that \mathbf{x} is non-degenerate, in the sense that each of its indifference curves $\{\theta \in \Theta | \mathbf{x}(\theta) = x\}_{x \in X}$ is isomorphic to the unit interval, then theorem 2 of [Yang \(2020\)](#) ensures that for any $m \in \mathcal{M}$, there exists a partition on Θ that generates the same distributions in each district.

State Boundaries. The baseline model in [Section 3](#) allows for any arbitrary way to split the population distribution. In the U.S., however, election maps can only be drawn within the boundary of a state, which implicitly imposes constraints on which maps are feasible. To extend our model so that maps can only be drawn within state boundaries, suppose that there are $N \in \mathbb{N}$ states. Ideal positions of citizens in state i are distributed according to F_i . Each state i has λ_i share of the total population. A map in this setting can be regarded as a collection $\{m_i\}_{i=1}^N$ of distributions on \mathcal{D} such that

$$\int_{\mathcal{D}} D(x) m_i(dD) = F_i(x),$$

for all $x \in X$ and for all $i \in \{1, \dots, N\}$. Given a map $\{m_i\}_{i=1}^N$ and a tie-breaking rule $\{r_i\}_{i=1}^N$, the distribution of the representatives' ideal positions is

$$H(x | \{m_i\}_{i=1}^N, \{r_i\}_{i=1}^N) := \sum_{i=1}^N \lambda_i H(x | m_i, r_i),$$

where $H(\cdot | m_i, r_i)$ is the conditional distribution of the ideal positions of representatives elected in state i . As in the baseline model, let $H_L^i(x) := \min\{2F_i(x), 1\}$ and let $H_R^i(x) := \max\{2F_i(x) - 1, 0\}$ for all $x \in X$. Under this extension, our main result can be readily generalized, as described in [Proposition 1](#) below.

Proposition 1. *For any distribution $H : X \rightarrow [0, 1]$, the following are equivalent:*

¹⁵If any element of the partition is not connected, then one can “connect” the disjoint parts by drawing a line segment that connects them. Since line segments have measure zero, this would not affect the conditional distribution of citizens' ideal positions.

1. There exists a map $\{m_i\}_{i=1}^N$ and a tie-breaking rule $\{r_i\}_{i=1}^N$ such that

$$H(x) = H(x|\{m_i\}_{i=1}^N, \{r_i\}_{i=1}^N)$$

for all $x \in X$.

2. $H(x) = \sum_{i=1}^N \lambda_i H_i(x)$ for all $x \in X$ and for some $\{H_i\} \subseteq \mathcal{D}$ such that $H_R^i \leq H_i \leq H_L^i$ for all i .

Finite Number of Districts. In practice, electoral maps contain only a finite number of districts. Each U.S. state is apportioned a fixed number of districts every 10 years, and the total number of districts nationwide is 435. Our model can readily satisfy this reality. Notice that whenever $x \in X$ is a median of a collection of distributions $D \in \mathcal{D}$, x must also be a median of any mixture of those distributions. Therefore, a natural extension of [Theorem 1](#) follows.

Proposition 2. *For any distribution $H : X \rightarrow [0, 1]$ with $|\text{supp}(H)| = K \in \mathbb{N}$, the following are equivalent:*

1. There exists a map $m \in \mathcal{M}$ with $|\text{supp}(m)| = K$ and a tie-breaking rule $r \in \mathcal{R}$ such that $H(x) = H(x|m, r)$ for all $x \in X$.
2. $H_R(x) \leq H(x) \leq H_L(x)$ for all $x \in X$.

Equal-Population Requirement. In the baseline model, we regard a distribution over distributions m as a map and each of its realizations $D \in \text{supp}(m)$ as a district. The distribution of representatives is then implied by each district D and the overall distribution m . In particular, m dictates relative population shares of any two districts. In practice, one might argue that each district should have the same population. This criterion is equivalent to the “one person, one vote” requirement imposed by US case law on Congressional and state legislative districts. The requirement stipulates that districts within states must have equal population as is practicable ([Smith 2014](#)).

An equal population requirement can immediately be accommodated, and the reason why is simple. Suppose that under a map m , there is a district D that is larger than another district D' in terms of its size. We may further split D uniformly so that it becomes several smaller districts whose conditional distributions all remain D . This way, we construct districts that are of the same size.

Formally, consider any map $m \in \mathcal{M}$. Since \mathcal{D} is a standard Borel space, there exists a measurable function $M : [0, 1] \rightarrow \mathcal{D}$ such that

$$L(\{u \in [0, 1] | M(u) \in \mathcal{A}\}) = m(\{D \in \mathcal{A}\}),$$

for any measurable subset \mathcal{A} of \mathcal{D} , where L is the Lebesgue measure on \mathbb{R} . Therefore, we may regard each district implied by a map m as $M(u)$, where u is uniformly distributed on $[0, 1]$.

Residual Uncertainty. Another extension is to further restrict the set of feasible maps by introducing residual uncertainty even after a map is drawn (e.g., aggregate uncertainty of citizens' preferences or limitations to map-drawing technology so that only some characteristics that are correlated with citizens' ideal positions are observed). Specifically, a general model that allows for residual uncertainty is as follows: Citizens carry an observable characteristic $\theta \in \Theta$. Across citizens, characteristics are distributed according to a probability measure ν_0 . For any characteristic $\theta \in \Theta$, ideal positions of citizens with characteristic θ are distributed according to $F(\cdot|\theta)$. A map is defined as a probability measure m over $\Delta(\Theta)$ so that

$$\int_{\Delta(\Theta)} \nu(A) m(d\nu) = \nu_0(A),$$

for all measurable sets $A \subseteq \Theta$.

Characterizing all possible compositions of a legislative body under residual uncertainty is beyond the scope of this paper; the set of possible distributions of representatives' ideal positions depends largely on the correlation structure implied by $F(\cdot|\theta)$. Here, we provide a specific example. Namely, we assume that $|\Theta| < \infty$ and that $\{\text{supp}(F(\cdot|\theta)) | \theta \in \Theta\}$ form a partition of Θ .¹⁶ In this case, the proof of [Theorem 1](#) can be readily applied, which leads to the following extension.

Proposition 3. *Let N be the largest $n \in \mathbb{N}$ under which the partition induced by $X_n := \{F^{-1}(1/2n), \dots, F^{-1}(2n-1/2n)\}$ is coarser than the partition $\{\text{supp}(F(\cdot|\theta)) | \theta \in \Theta\}$. Then, for any $n < N$ and for any $H \in \mathcal{F}$ with $\text{supp}(H) = X_n$, there exists a map m and a tie-breaking rule r such that $H(x) = H(x|m, r)$ if and only if $H_R(x) \leq H(x) \leq H_L(x)$ for all $x \in X_n$.*

¹⁶In fact, the same argument can be applied to a similar generalization, where $\Theta = X$ and $F(\cdot|\theta)$ is unimodal with its (unique) median being equal to the mode. Note that this generalization corresponds to the setting of [Friedman and Holden \(2008\)](#). In this generalized setting, the same conclusion would hold.

7 Conclusion

Gerrymandering’s dreadful talent to stifle the voice of an electorate has been understood for two centuries. But in this paper, we rigorously measure the maximum possible extent of that suppression. We represent the variety of a citizenry’s political positions as a spectrum on the unit interval. We characterize all possible distributions of elected representatives’ ideal positions in a legislature that a map can engender. Unrestrained gerrymandering can pervert the composition of a legislature so severely that half the population loses representation of their political positions. No matter the distribution of positions, a partisan body may cleverly design a district map that can procure any but the 25 percent most extreme-left or extreme-right legislation under majority rule. When the criterion changes from majority rule to legislation that can defeat voters’ median preference, the subversive scope of gerrymandering can inch even further, reaching the maximum possible extent when political views are uniformly distributed. In that case, gerrymandering can engender maps that empower any bill to defeat the median in a head-to-head vote in the legislature.

We explore reforms that can undue or confine the damage of gerrymandering. A significant advantage of our framework is that it simplifies any discussion over district maps, which are high-dimensional objects, into eligible legislative policies, which are one-dimensional. The simplification permits a far easier analysis of remedies in the mapmaking process. For instance, a bipartisan commission with members negotiating a redistricting plan can be modeled as a two-person bargaining game over splitting a surplus. Canonical results from bargaining theory then imply that gerrymandering’s entire subversive scope can be erased if both parties are sufficiently patient negotiators. In that case, all agreed maps would comply with the will of the majority.

Beyond mapmaking, reforms to the voting rules of the elected legislative body can help undue the effects of gerrymandering. Supermajority voting rules limit the set of policies that can defeat the median. As the fraction of representatives necessary for approving a bill increases, the set of legislation that can win against the median position in a head-to-head vote shrinks. But the median might represent the status quo, and voting procedures that greater insulate the median from defeat grant more power to the minority, which can entrench the status quo and arrest progress.

We provide several extensions of our baseline model to demonstrate how it makes room for notable features of gerrymandering in practice. Maps can display as two-dimensional geographical diagrams, and redistricting can respect state boundaries. The number of districts can be finite and satisfy the equal-population requirement. And residual uncertainty can disturb voters’ preferences. Even so, our framework leaves room for continuing work.

Preferences might depart from being single-peaked, citizens may engage in strategic voting or be subject to behavioral biases in their decisions, and politicians may manipulate information so as to alter voter perceptions or turnout. Incorporating these aspects and others is a promising research area left for the future.

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Appendix

Proof of Theorem 1

To show that 1 implies 2, consider any $m \in \mathcal{M}$ and any $r \in \mathcal{R}$. By definition of $H(\cdot|m, r)$, it must be that for all $x \in X$,

$$H(x|m, r) \leq m(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\}) = m(\{D \in \mathcal{D} | D(x) \geq 1/2\}).$$

Now consider any $x \in X$. Clearly, $m(\{D \in \mathcal{D} | D(x) \geq 1/2\}) \leq 1$ as m is a probability measure. Moreover, let $M_x^+(z) := m(\{D \in \mathcal{D} | D(x) \geq z\})$ for all $z \in X$. From (1), it follows that the left-limit of $1 - M_x^+$ is a CDF and a mean-preserving spread of a Dirac measure at $F(x)$. Therefore, whenever $x \leq F^{-1}(1/2)$, it must be that $1/2 \geq F(x)$, and, hence, $M_x^+(1/2)$ can at most be $2F(x)$ to have a mean of $F(x)$.¹⁷ Together, this implies that $m(\{D \in \mathcal{D} | D(x) \geq 1/2\}) \leq H_L(x)$ for all $x \in X$.

In the meantime, by the definition of $H(\cdot|m, r)$, again, it must be that for all $x \in X$,

$$H(x^-|m, r) \geq m(\{D \in \mathcal{D} | D^{-1}(1/2^+) < x\}) = m(\{D \in \mathcal{D} | D(x) > 1/2\}).$$

Consider any $x \in X$. Since m is a probability measure, it must be that $m(\{D \in \mathcal{D} | D(x) > 1/2\}) \geq 0$ for all $x \in X$. Furthermore, let $M_x^-(z) := m(\{D \in \mathcal{D} | D(x) > z\})$ for all $z \in X$. From (1), it follows that $1 - M_x^-$ is a CDF and a mean-preserving spread of a Dirac measure at $F(x)$. Therefore, whenever $x \geq F^{-1}(1/2)$, it must be that $1/2 \leq F(x)$, and, hence, $M_x^-(1/2)$ must be at least $2F(x) - 1$ to have a mean of $F(x)$.¹⁸ Together, this implies that $m(\{D \in \mathcal{D} | D(x) > 1/2\}) \geq H_R(x)$ for all $x \in X$, which, in turn, implies that $H_R(x) \leq H(x^-|m, r) \leq H(x|m, r) \leq H_L(x)$ for all $x \in X$.

To prove that 2 implies 1, we first consider the case where H is a step function whose jumps are a subset of $X_n := \{F^{-1}(1/2n), \dots, F^{-1}(2n-1/2n)\}$, for some $n \in \mathbb{N}$. In this case, we may represent H as a probability distribution $\eta = \{\eta_j\}_{j=1}^{2n-1}$ on the set X_n . We first claim that for any step function H with jumps in a subset of X_n and satisfies

$$2F(x) - 1 + \frac{1}{n} \leq H(x) \leq 2F(x), \forall x \in X,$$

there exists a map $m \in \mathcal{M}$ and a tie-breaking rule $r \in \mathcal{R}$, such that $H(x|m, r) = H(x)$ for all $x \in X$. To see this, first notice that the collection of CDFs that satisfy these conditions can be characterized by the following conditions:

$$\eta_j \geq 0, \forall j \in \{1, \dots, 2n-1\}, \tag{A.2}$$

and

$$\sum_{j=1}^{2n-1} \eta_j = 1, \tag{A.3}$$

¹⁷More specifically, to maximize the probability at $1/2$, a mean-preserving spread of $F(x)$ must assign probability $2F(x)$ at $1/2$, and probability $1 - 2F(x)$ at 0.

¹⁸More specifically, to minimize the probability at $1/2$, a mean-preserving spread of $F(x)$ must assign probability $2F(x) - 1$ at 1, and probability $2(1 - F(x))$ at 0.

as well as

$$\begin{aligned} \sum_{j=1}^k \eta_j &\leq 2F(F^{-1}(k/2n)) = \frac{k}{n}, \forall k \in \{1, \dots, n\}, \\ \sum_{j=1}^k \eta_{2n-j} &\leq 2F(F^{-1}(k/2n)) = \frac{k}{n}, \forall k \in \{1, \dots, n\}. \end{aligned} \quad (\text{A.4})$$

Denote the convex set of vectors η that satisfy (A.2), (A.3), and (A.4) by \mathcal{H}_n , and notice that the extreme points of \mathcal{H}_n must have at least $2n - 1$ inequalities among (A.2), (A.3), and (A.4) that bind (see Proposition 15.2 of [Simon 2011](#)).

Now, consider the following map m^* that assigns probability $1/n$ to each district D^k for all $k \in \{1, \dots, n\}$, where D^k is the conditional CDF on the set

$$[F^{-1}(k-1/2n), F^{-1}(k/2n)] \cup [F^{-1}(n+k-1/2n), F^{-1}(n+k/2n)].$$

Note that, for any $j \in \{1, \dots, n\}$,

$$F^{-1}(j/2n) \in \text{med}(D^k) \iff k \leq j, \quad (\text{A.5})$$

and

$$F^{-1}(2n-j/2n) \in \text{med}(D^k) \iff j \leq k. \quad (\text{A.6})$$

As a result, for any tie-breaking rule $r \in \mathcal{R}$ that always selects medians in X_n in each district, the probability distribution associated with $H(\cdot|m^*, r)$ must satisfy (A.3). We now argue that for any probability distribution $\eta \in \mathcal{H}_n$ where at least $2n - 2$ of the inequalities among (A.2) and (A.4) bind, there exists a tie-breaking rule $r \in \mathcal{R}$, such that η has CDF $H(\cdot|m^*, r)$.

Indeed, for any $\eta \in \mathcal{H}$, such that at least $2n - 2$ inequalities among (A.2) and (A.4) bind, η must be in a one dimensional affine space. Moreover, by the nature of these constraints, (A.3) implies that at most n out of $2n - 2$ binding constraints are from (A.4). Meanwhile, for any $j \in \{1, \dots, n\}$, by (A.5) and (A.6), both $F^{-1}(j/2n) \in X_n$ and $F^{-1}(2n-j/2n) \in X_n$ can be assigned at most j/n probability under any tie-breaking rule. Conversely, (A.5) and (A.6) also imply that for any subset $\mathcal{J} \subseteq \{1, \dots, 2n - 1\}$ with $|\mathcal{J}| \leq n$, and, for any $k, l \notin \mathcal{J}$, there exists a tie-breaking rule such that the total probability assigned to $\{F^{-1}(j/2n)\}_{j \in \mathcal{J}}$ equals to $|\mathcal{J}|/n$ and every other elements of X_n , except for at most $F^{-1}(k/2n)$ and $F^{-1}(l/2n)$, is assigned with probability zero. Together, there exists a tie-breaking rule $r \in \mathcal{R}$ such that η has CDF $H(\cdot|m^*, r)$. Since every extreme point of \mathcal{H} must have at least $2n - 2$ inequalities among (A.2) and (A.4) binding, and since, for any such η , there exists a tie-breaking rule $r \in \mathcal{R}$ such that η has CDF $H(\cdot|m^*, r)$, any element of \mathcal{H}_n must be in the convex hull of finitely many probability distributions associated with CDFs of form $H(\cdot|m^*, r)$, for some $r \in \mathcal{R}$, as desired. Furthermore, since $r \mapsto H(\cdot|m^*, r)$ is affine, it then follows that, for any CDF H whose associated probability distribution is in \mathcal{H}_n , there exists a tie-breaking rule $r \in \mathcal{R}$ such that $H(x) = H(x|m^*, r)$ for all $x \in X$.

Finally, consider any CDF H that satisfies condition 2. Since F is continuous and strictly increasing, there must exist a sequence $\{H_n\}$ such that the probability distribution η_n associated with H_n is in \mathcal{H}_n and

that $\{H_n\} \rightarrow H$ under the weak-* topology. Moreover, for any $n \in \mathbb{N}$, there exists a map $m_n \in \mathcal{M}$ and a tie-breaking rule $r_n \in \mathcal{R}$ such that $H_n(x) = H(x|m_n, r_n)$ for all $x \in X$. Since \mathcal{M} is a compact set under the weak-* topology, after possibly taking a subsequence, $\{m_n\}$ converges to some $m \in \mathcal{M}$ under the weak-* topology on \mathcal{M} . In the meantime, by the definition of $H(\cdot|m, r)$, for any $n \in \mathbb{N}$, we must have

$$H_n(x) = H(x|m_n, r_n) \leq m_n(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\}) = m_n(\{D \in \mathcal{D} | D(x) \geq 1/2\}),$$

for all $x \in X$, and

$$H_n(x^-) = H(x^-|m_n, r_n) \geq m_n(\{D \in \mathcal{D} | D^{-1}(1/2) < x\}) = m_n(\{D \in \mathcal{D} | D(x) > 1/2\}),$$

for all $x \in X$.

Notice that, for any $\{D_n\} \subset \mathcal{D}$ such that $D_n \rightarrow D$ under the weak-* topology and that $D_n(x) \geq 1/2$ for all $x \in X$ and for all $n \in \mathbb{N}$,

$$\frac{1}{2} \leq \lim_{n \rightarrow \infty} D_n(x) \leq \limsup_{n \rightarrow \infty} D_n(x) \leq F(x)$$

for all $x \in X$. Hence, the set $\{D \in \mathcal{D} | D(x) \geq 1/2\}$ is closed in \mathcal{D} . By similar arguments, the set $\{D \in \mathcal{D} | D(x) > 1/2\}$ is open in \mathcal{D} . Together, since $\{m_n\} \rightarrow m$ under the weak-* topology, for any $x \in X$ at which H is continuous,

$$\begin{aligned} H(x) &= \lim_{n \rightarrow \infty} H_n(x) = \lim_{n \rightarrow \infty} H(x|m_n, r_n) \\ &\leq \limsup_{n \rightarrow \infty} m_n(\{D \in \mathcal{D} | D(x) \geq 1/2\}) \\ &= m(\{D \in \mathcal{D} | D(x) \geq 1/2\}) \\ &= m_n(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\}), \end{aligned}$$

and

$$\begin{aligned} H(x) &= \lim_{n \rightarrow \infty} H_n(x^-) = \lim_{n \rightarrow \infty} H_n(x^-|m_n, r_n) \\ &\geq \liminf_{n \rightarrow \infty} m_n(\{D \in \mathcal{D} | D(x) > 1/2\}) \\ &\geq m(\{D \in \mathcal{D} | D(x) > 1/2\}) \\ &= m_n(\{D \in \mathcal{D} | D^{-1}(1/2^+) < x\}). \end{aligned}$$

Moreover, since $x \mapsto m(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\})$ is right-continuous and $x \mapsto m(\{D \in \mathcal{D} | D^{-1}(1/2^+) < x\})$ is left-continuous, it must be that

$$m(\{D \in \mathcal{D} | D^{-1}(1/2^+) < x\}) \leq H(x^-) \leq H(x) \leq m(\{D \in \mathcal{D} | D^{-1}(1/2) \leq x\}), \forall x \in X.$$

As a result, there exists a tie-breaking rule $r \in \mathcal{R}$ such that $H(x|m, r) = H(x)$ for all $x \in X$. This completes the proof. ■