Yale University

EliScholar – A Digital Platform for Scholarly Publishing at Yale

Cowles Foundation Discussion Papers | Cowles Foundation

5-19-2021

Calibrated Click-Through Auctions: An Information Design Approach

Dirk Bergemann
Yale University

Paul Duetting
Google Research

Renato Paes Leme
Google Research

Song Zuo
Google Research

Follow this and additional works at: https://elischolar.library.yale.edu/cowles-discussion-paper-series

 Part of the Economics Commons

Recommended Citation

Bergemann, Dirk; Duetting, Paul; Paes Leme, Renato; and Zuo, Song, "Calibrated Click-Through Auctions: An Information Design Approach" (2021). *Cowles Foundation Discussion Papers*. 2618.
https://elischolar.library.yale.edu/cowles-discussion-paper-series/2618

This Discussion Paper is brought to you for free and open access by the Cowles Foundation at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Cowles Foundation Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.

CALIBRATED CLICK-THROUGH AUCTIONS: AN INFORMATION DESIGN APPROACH

By

Dirk Bergemann, Paul Duetting, Renato Paes Leme, and Song Zuo

May 2021

COWLES FOUNDATION DISCUSSION PAPER NO. 2285



COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Calibrated Click-Through Auctions: An Information Design Approach

Dirk Bergemann* Paul Duetting† Renato Paes Leme‡ Song Zuo§

May 19, 2021

Abstract

We analyze the optimal information design in a click-through auction with fixed valuations per click, but stochastic click-through rates. While the auctioneer takes as given the auction rule of the click-through auction, namely the generalized second-price auction, the auctioneer can design the information flow regarding the click-through rates among the bidders. A natural requirement in this context is to ask for the information structure to be calibrated in the learning sense. With this constraint, the auction needs to rank the ads by a product of the bid and an *unbiased* estimator of the click-through rates, and the task of designing an optimal information structure is thus reduced to the task of designing an optimal unbiased estimator.

We show that in a symmetric setting with uncertainty about the click-through rates, the optimal information structure attains both social efficiency and surplus extraction. The optimal information structure requires private (rather than public) signals to the bidders. It also requires correlated (rather than independent) signals, even when the underlying uncertainty regarding the click-through rates is independent. Beyond symmetric settings, we show that the optimal information structure requires partial information disclosure.

Keywords: Click-Through Rates, Information Design, Second-Price Auction, Calibration, Private Signals, Public Signals, Conflation.

JEL Classification: D44, D47, D82

*Yale University, dirk.bergemann@yale.edu

†Google Research, duetting@google.com

‡Google Research, renatoppl@google.com

§Google Research, szuo@google.com

1 Introduction

In the world of digital advertising, whether in display or in search advertising, the allocation mechanism is commonly an auction. Independent of the details of the auction format, the mechanism typically elicits from each bidder the willingness-to-pay for the item. Importantly, the auction frequently uses additional information to determine the ranking in a search auction or the assignment in the display advertisement. The additional information frequently concerns the quality of the match between the bidder (the advertiser) and the item (attention of the consumer). Importantly, this additional information, such as the expected click-through rate or expected transaction rate, is often held by the platform that manages the auction, or the publisher on whose site the consumer is present. Consequently, the relevant information for the auction comes from many information sources, some of it provided by the advertiser (the bidder), some of it provided by the publisher (the seller), or the auction platform.

As the relevant information arises from many sources, the allocation mechanism must determine how much information to elicit and how to integrate this information into the allocation mechanism. We pursue this question in a simple, yet prominent setting—namely, the click-through auction as analyzed in Edelman et al. (2007) and Varian (2007), who refer to this type of auction as the *generalized second-price auction*, or *position auction*, respectively. In this environment, the utility of a bidder is given by the product of the willingness-to-pay of the bidder per click of the consumer, and the click-through rate of the consumer. In the following, we shall simply refer to this auction mechanism as the *click-through auction*.

Edelman et al. (2007) and Varian (2007) consider a complete information setting where the willingness-to-pay of each bidder and the click-through rate for each position are assumed to be known by all auction participants. They showed that there is an equilibrium in the bidding game of the click-through auction in which the bidder with the highest product of the willingness-to-pay and the click-through rate wins. Thus, the click-through auction supports the socially efficient outcome, and if the auction offers several positions or rankings, then the socially efficient outcome extends to all offered positions. Significantly, in equilibrium the price for each position reflects the marginal contribution of each bidder to the social surplus. Thus, the resulting payoffs of the bidders and the auctioneer are as they would be in the direct Vickrey-Groves-Clarke (VCG) mechanism. The payoff equivalence of the click-through auction with the corresponding VCG mechanism is critical as it suggests that the mechanism is robust to the introduction of private information about either the willingness-to-pay or the click-through rate.

In the subsequent analysis, we take as given the click-through auction format *and* the profile of

willingness-to-pay across bidders. In contrast to the above contributions, we allow for randomness and uncertainty in the click-through rates. The publisher or auction platform must then decide how much of the information to share with the auction participants. For the subsequent analysis, we do not distinguish between the publisher and the auction platform, and simply refer to either (or both of them) as the seller. The central instrument, and thus the focus of our analysis, is the *information policy* of the seller with respect to the true click-through rates generated by any given search event.

We approach the problem of determining the optimal information policy in a number of steps in increasing generality. We focus on randomness in the click-through rates and maintain complete information about the willingness-to-pay of each bidder throughout the analysis. Each search event generates a random click-through rate for each of the participating bidders. The search environment is therefore described by a joint distribution of the click-through rates across the bidders. The joint distribution of the click-through rates is common information to all the participants in the auction, bidders and seller, and is thus given by a common prior distribution. The initial information of the bidders is given by the common prior distribution. In contrast, the seller is assumed to know the realized click-through rates of each search event. The information policy of the seller then has to determine how much information to disclose about the realized click-through rates before bidding begins.

With the randomness of the click-through rates it is natural to constrain the information design of the platform to be consistent with the prior distribution. We therefore refer to an information policy that maintains the law of iterated expectation as *calibrated* (in the sense of Foster and Vohra (1997)). Among all possible information policies, the complete information policy and the zero information policy are both leading examples, as well as extremal information policies. Under a complete information policy, the seller completely discloses all information to the bidders. It is thus as if the bidders were in a complete information environment. By contrast, in a zero information policy the seller does not disclose any information about the realized click-through rates. In consequence, each bidder acts as if the realized click-through rate is always equal to the ex ante expected click-through rate. These two extremal information policies have dramatically different payoff implications.

With stochastic click-through rates, social efficiency and revenue will depend on which information about the click-through rates is disclosed. With complete information, the resulting allocation is always efficient. But as the competitive position of each bidder can vary across the realized click-through rates, the resulting revenue of the seller can be low due to weak competition. By contrast, with zero information about the click-through rates, the resulting allocation will typically fail to be socially

efficient. As the bidding behavior cannot reflect any information about the click-through rates, the socially relevant information fails to be reflected in the auction outcome. Yet, as the bidding reflects only the expected click-through rate, and hence only the mean of the click-through rate (and not the higher moments of the click-through distribution), the resulting bids will have zero variance, and thus be more competitive.

Our first result, Proposition 3.1, shows that a statistically independent information structure can never improve the revenue over the no-disclosure information policy. Our second result, Corollary 3.2, shows that with any level of uncertainty in the click-through rates, the seller strictly prefers no-disclosure to full-disclosure. Thus, the seller always favors competition over information disclosure. We then ask whether there exists an improved information policy that can realign social efficiency with the revenue of the seller.

Our main result, Theorem 4.1, establishes that a calibrated and correlated information policy can completely align social efficiency and revenue maximization. In particular, when the common prior distribution over the click-through rates is symmetric across the bidders, we can then explicitly construct an information policy such that the socially efficient allocation is always realized and the competition between the bidder levels the information rent of the bidders to the ex-ante level. Given the symmetry of the common prior distribution, this implies that the bidders compete their residual surplus down to zero.

We also provide an explicit construction of the correlated information structure (or signalling scheme). Interestingly, the optimal information structure is an interior information structure; that is, it is neither zero nor complete information disclosure. The information structure balances two conditions that are necessary to attain the socially efficient allocation while maintaining competition: (i) it provides sufficient information to rank the alternative allocations according to social efficiency, and (ii) it limits the variance in the posterior beliefs of the competing bidders so that their equilibrium bids remain arbitrarily close to support competitive bids.

The optimality of a noisy information structure has some significant implications in the world of digital advertising. Since the optimal information structure remains noisy, better click-through-rate predictions, achieved through improved learning, may not necessarily lead to better auction results. Thus, there might be limits to the returns of more elaborate machine learning algorithms to inform the prediction problem. Importantly, the calibration constraint makes sure that the ranking score always remains an unbiased click-through-rate predictor.

The optimality of an interior information structure remains even when we move away from the symmetric stochastic environment. In Proposition 4.5, we show that the socially efficient allocation and revenue maximization remain perfectly aligned as long as the expected click-through rate is equalized across bidders, even when the support of the ex-post realized click-through rates can vary across bidders. Finally, in Theorem 5.1, we show that an interior information structure remains part of the optimal information design with stochastic click-through rates, even when the expected click-through rates across the bidders differ, and therefore one bidder is stronger from an ex-ante perspective. In particular, the optimal information structure releases less information about the winning bidder than about the losing bidder. This suggests that the optimal information structure in an asymmetric auction seeks to strengthen the weak bidder with additional information, relative to the strong bidder.

1.1 Related Literature

Bergemann and Pesendorfer (2007) and Eső and Szentes (2007) are among the first to investigate the design of optimal information structure in an auction setting. Bergemann and Pesendorfer (2007) consider an auction environment with n bidders and independent private values. In their setting, the bidders initially have no private information, and the platform can distribute any information. Given the independent value assumption, Bergemann and Pesendorfer (2007) restrict the auctioneer to transmit information regarding bidder i's valuation only to bidder i. Thus, there is a restriction in the signalling scheme. By contrast, the seller can adapt the selling mechanism to the information structure, and is thus not restricted to the second-price auction. Consequently, the seller jointly optimizes auction and information policies. Eső and Szentes (2007) extend the analysis of Bergemann and Pesendorfer (2007) to permit the bidder to initially have private information, and then allow the seller to propose the optimal mechanism.

More recently, a number of contributions have analyzed the optimal information structure for a given mechanism or auction format. Bergemann et al. (2015) consider the optimal information structure for a seller who uses third-degree price discrimination, and Bergemann et al. (2017) pursue this question in the context of the first-price auction. Badanidiyuru et al. (2019) consider a second-price auction where the information about the valuation of bidder i is partially shared between the bidder and the platform. The limiting cases are that: (i) the bidder has all the private information (the standard setting of auction theory), and (ii) the platform has all the private information.

Considering the randomness of the click-through rates, earlier literature notices that a modified

version of the click-through rates may positively impact the revenue of the click-through auction. In this context, the adjusted click-through rates in Lahaie and Pennock (2007) and Mahdian and Sundararajan (2015) are not calibrated, as the true click-through rate is "squashed" or "boosted" for the purpose of ranking. This suggests a natural distinction between the analysis of optimal calibrated and non-calibrated information structures.

The design of optimal information structures in a strategic setting remains an area with many open questions. In the current setting, we explicitly allow for private information disclosure, rather than public information disclosure. By contrast, in most of the preceding literature the information disclosure was either public, as in Arieli and Babichenko (2019), or independent across bidders, as in Bergemann and Pesendorfer (2007). Here, we are allowing for, and importantly showing, the optimality of private and correlated information structures.

In the current setting, the auctioneer can substantially improve the revenue by using correlated instead of independent signals. The significance of correlation among the signals for the revenue maximizing mechanism has been observed earlier in Cremer and McLean (1985, 1988). They establish that correlation in the private values among bidders can be used in an optimal mechanism to extract the full surplus. While our results also highlight the increased power of correlated signals relative to independent signals, the setting and arguments differ substantially. In Cremer and McLean (1985, 1988), the auctioneer is free to choose the optimal mechanism, while we take the generalized second-price auction as given. In Cremer and McLean (1985, 1988), the signal of each bidder is the private value of the bidder, and the values need to be correlated for the full surplus extraction result to be obtained. In our setting, the values of the bidders are known, and the click-through rates themselves can either be independent or correlated. We choose the information structure so that the signals are sufficiently informative, yet yield competitive interim expectations. By contrast, in Cremer and McLean (1985, 1988), the payments of each bidder depend on the reports of the other bidders, so that the individual reporting strategy seeks to maximize a scoring function. Thus, the correlation of the signals in these two settings seeks to achieve very different objectives, and accordingly the construction differs significantly. In particular, Cremer and McLean (1985, 1988) take as given the signals and then design the optimal transfer function. We take as given the transfer function, namely the payment rules, and then design

the signals to maintain competition. [1]

2 Model

We will analyze the setting of click-through auctions where bidders are ranked by the product of their value (expressed as a maximum willingness-to-pay per click) and an unbiased estimator of the click-through rates. Our main goal will be to study how to engineer such unbiased estimators to achieve better revenue-efficiency trade-offs. In the language of information design, we will keep the auction format fixed and vary the information structure.

To allow us to focus on the information structure, we will work in the full information model where each bidder $i = 1, \ldots, n$ has a fixed and known value $v_i \geq 0$ representing their willingness-to-pay for a click. Our central object of study will be the click-through rates (CTRs): before the auction, a vector $r = (r_1, \ldots, r_n) \in [0,1]^n$ will be drawn from a joint prior distribution G, which is a multi-dimensional distribution that may display correlation across the bidders' CTRs.

The click-through rates are known by the auctioneer: typically, the platform is the one building a machine learning model to estimate them. The auctioneer must now decide on a score/signal $s = (s_1, \ldots, s_n) \in [0,1]^n$ to rank the bidders. The design space will be to design a joint probability distribution ρ on pairs (r, s) such that the marginal on r is G:

$$\int_{r \in R} dG(r) = \int_{r \in R} \int_s d\rho(s, r), \quad \forall \text{ measurable } R \subseteq [0,1]^n.$$

This joint probability distribution will be referred as the *information structure*. For notational convenience, it will be useful to assume that both G and ρ are discrete distributions with finite support, and hence we can write $g(r)$ for the probability of a given vector r under distribution G and $x(r, s)$ as the probability of a pair (r, s) under distribution ρ. With that notation, the information structure is a function $x : [0,1]^n \times [0,1]^n \to [0,1]$ satisfying:

$$\sum_s x(r, s) = g(r), \quad \forall r \in [0,1]^n.$$

Auction Mechanics Again for simplicity, we focus on the single slot setting where the goal of the auction is to select a single winner $i^* \in [n]$. The winner will be selected as the bidder having the largest

[1] We should also note that correlation is not always part of an optimal information structure. In the case of weak competition, that is, when the efficient allocation always assigns the object to the same bidder, then the optimal information structure will not use correlated signals.

$s_i v_i$, with a symmetric tie-breaking rule. The winner's cost per click is then:

$$p_{i^*} = \max_{j \neq i^*} \frac{v_j s_j}{s_{i^*}}.$$

The winner only pays when there is a click, which happens with probability r_{i^*}. Hence, the expected revenue from this auction is $r_{i^*} p_{i^*}$.

Calibration The auctioneer is restricted to ranking with the unbiased estimator of the click-through rates. Hence, the information structure will be valid only if it is calibrated in the Foster-Vohra sense (Foster and Vohra, 1997). An information structure is called *calibrated* if the posterior, given any signal realization s_i' for bidder i, matches with the signal itself, i.e.,

$$\mathbb{E}[r_i | s_i = s_i'] = s_i'. \tag{1}$$

If the CTR and signal space is discrete, then we can write calibration as:

$$\sum_{(r,s);s_i = s_i'} x(r,s) \cdot (r_i - s_i') = 0, \quad \forall i, s_i'.$$

There are two important examples of calibrated information structures:

- Full-disclosure: where $s_i = r_i$ almost surely.
- No-disclosure: where $s_i = \mathbb{E}[r_i]$ almost surely.

Since the calibration constraint is imposed on every bidder separately, it is possible to create information structures that combine disclosure and no-disclosure. For example, given two bidders, we can consider an information structure where bidder 1 receives only one signal, and the signal is equal to the ex-ante expectation of the click-through rate, thus $s_1 = \mathbb{E}[r_1]$; and bidder 2 receives as many signals as click-through rates, thus $s_2 = r_2$. This forms a calibrated information structure.

Whenever $s_i = \mathbb{E}[r_i]$ we will say that we *fully bundle* bidder i. Whenever $s_i = r_i$ we will say that we *unbundle* bidder i. If neither is the case we will say that we *partially bundle* the bidder.

Independence and Correlation A information structure is called *independent* if signals s_j for $j \neq i$ do not offer additional information on the expectation of r_i beyond s_i. Formally:

$$\mathbb{E}[r_i | s = s'] = \mathbb{E}[r_i | s_i = s_i'], \quad \forall i, \forall s. \tag{2}$$

Both full-disclosure and no-disclosure information structures are independent.

Whenever we do not assume independence, we will say that an information structure is *correlated*. Below, we give an example of a correlated and calibrated information structure. This will also serve as an example of how information structures will be illustrated throughout the paper. Consider the two-bidder setting where CTRs are $r \in \{(1/2, 1/2), (1/2, 1), (1, 1/2), (1, 1)\}$, each with probability $1/4$ (hence r_1 and r_2 are independent). We represent an information structure where rows correspond to pairs of CTRs (r_1, r_2) and columns correspond to pairs of signals (s_1, s_2). Each entry of the matrix will correspond to $x(r, s)$ which is the probability of the event that the CTRs are r and the signals are s. The following table shows the "*flipping the square*" structure, which we will discuss in detail in Section 4.1:

r \ s	$(3/4-\epsilon, 3/4-\epsilon)$	$(3/4-\epsilon, 3/4+\epsilon)$	$(3/4+\epsilon, 3/4-\epsilon)$	$(3/4+\epsilon, 3/4+\epsilon)$
$(1/2, 1/2)$	ϵ	0	0	$1/4-\epsilon$
$(1/2, 1)$	0	$1/4$	0	0
$(1, 1/2)$	0	0	$1/4$	0
$(1, 1)$	$1/4-\epsilon$	0	0	ϵ

One can check that while the calibration constraints (equation (1)) hold, the independence condition (equation (2)) does not. So, this is a calibrated, correlated information structure.

Symmetric vs Asymmetric Environments Finally, we will say that the environment is *symmetric* when random variables $v_1 r_1, \ldots, v_n r_n$ are exchangeable. Whenever symmetry does not hold, we will say that the environment is *asymmetric*.

3 Independent Information Structures

A first step in the analysis of optimal information design is a focus on independent signals. With independent signals, the signal s_i of each bidder i contains all the information about the CTR r_i that the auctioneer releases before the auction. Thus, bidder i could not learn anything more about the true CTR from any other bidder. In turn, the information that bidder i receives from the auctioneer is the maximal information that is available before the auction to place an informed bid.

3.1 Independent Signals and Two Bidders

We begin the analysis with two bidders and then generalize the insight to many bidders.

Proposition 3.1 (Independent and Calibrated Signalling)**.** *In a two-bidder environment, the expected revenue of an independent and calibrated information structure cannot exceed the one from no-disclosure.*

Proof. We start by computing the expected revenue given signals s_1 and s_2. If $v_1 s_1 \geq v_2 s_2$, then the revenue can be written as:

$$\mathbb{E}\left[r_1 \cdot \frac{v_2 s_2}{s_1} \middle| s_1, s_2\right] = \mathbb{E}\left[r_1 | s_1, s_2\right] \cdot \frac{v_2 s_2}{s_1} = \mathbb{E}\left[r_1 | s_1\right] \cdot \frac{v_2 s_2}{s_1} = s_1 \cdot \frac{v_2 s_2}{s_1} = v_2 s_2,$$

where the second equality follows from the independence of the signaling scheme and the third equality follows from calibration.

Therefore, we can write the expected revenue as:

$$\textsc{Rev} = \mathbb{E}[\min(v_1 s_1, v_2 s_2)] \leq \min(\mathbb{E}[v_1 s_1], \mathbb{E}[v_2 s_2]) = \min(v_1 \mathbb{E}[s_1], v_2 \mathbb{E}[s_2]) = \min(v_1 \mathbb{E}[r_1], v_2 \mathbb{E}[r_2]),$$

where the first inequality follows from Jensen's inequality and the concavity of the minimum. The last equality follows from calibration. Finally, note that $\min(v_1 \mathbb{E}[r_1], v_2 \mathbb{E}[r_2])$ is the revenue from no-disclosure. □

We have thus shown that full-disclosure can never revenue-dominate no-disclosure. We next show that generically full-disclosure is, in fact, strictly revenue-dominated by zero no-disclosure.

Corollary 3.2 (Zero vs. Complete Information Disclosure)**.** *In the two-bidder environment, if both $v_1 r_1 > v_2 r_2$ and $v_1 r_1 < v_2 r_2$ occur with positive probability, then the revenue from no-disclosure strictly dominates the revenue from full-disclosure.*

Proof. If both events $v_1 r_1 > v_2 r_2$ and $v_1 r_1 < v_2 r_2$ occur with positive probability, then Jensen's inequality holds with a strict inequality $\mathbb{E}[\min(v_1 r_1, v_2 r_2)] < \min(\mathbb{E}[v_1 r_1], \mathbb{E}[v_2 r_2])$ in the previous proof. □

The argument suggests that the revenue dominance result does not extend to more than two bidders. With more than two bidders, the smaller of the two highest realizations determines the price, and the expectation of the smaller of the two highest is now larger than the unconditional expectation of the second highest click-through rate.

3.2 Independent Signals with Many Bidders

Indeed, the power of independent signalling is much improved in the presence of competition, and we now consider the case of more than two bidders, $n > 2$. We show by example that an independent symmetric-calibration signal can improve the revenue, and thus partial revelation is better than no- or full-disclosure.

Consider a symmetric three-bidder environment with $v_1 = v_2 = v_3 = 1$. For each bidder i let $r_i = 0$ with probability $2/3$, and $r_i = 1$ with probability $1/3$. Moreover, assume that r_1, r_2 and r_3 are independent. It is simple to check that full revelation has revenue $7/27$ and no-revelation has revenue $9/27$. This can be improved by the following signaling scheme with partial bundling:

r_i \ s_i	$s_i = 0$	$s_i = 4/9$
$r_i = 0$	$1/4$	$5/12$
$r_i = 1$	0	$1/3$

The revenue of partial disclosure is $4/9$ whenever at least two of the bidders have the high signal, which happens with probability $1 - (1/4)^3 - 3(1/4)^2(3/4) = 27/32$. Hence, the overall revenue of this partial disclose scheme is $3/8$, which dominates both full-disclosure and no-disclosure.

Instead of trying to optimize for the optimal independent information structure, we will move to the more powerful model of correlated information structures in the next section.

The results here mirror earlier results by Board (2009), who considers an ascending auction in a private value setting without click-through rates. In his analysis, he restricts attention to independent signals and establishes that with two bidders, the seller's revenue is smaller *with* than *without* information disclosure. He further shows that in a symmetric model, as the number of bidders become arbitrarily large, complete information disclosure eventually revenue-dominates zero information disclosure.

4 Correlated Structures in Symmetric Environments

We obtain a much stronger result if we allow the information structure to be correlated across bidders. The information flow allows influence over the level of competition to some extent. This allows us to conflate auction items and restore some market thickness (see Levin and Milgrom (2010)).

4.1 A First Example: Flipping the Square

To showcase the power of correlated information structures, we start with an example where the optimal structure is rather counterintuitive. Consider two bidders with values $v_1 = v_2 = 1$ and independent click-through rates distributed uniformly in $\{1/2, 1\}$. Or rather: the vector r is uniformly distributed in $\{(1/2, 1/2), (1/2, 1), (1, 1/2), (1, 1)\}$. With an independent signaling scheme, the optimal information structure is no-disclosure, which yields revenue equal to $3/4$. With a correlated signaling scheme, one can obtain arbitrarily close to $7/8$ revenue, which is optimal since it corresponds to the welfare $\mathbb{E}[\max_i v_i r_i]$ of the optimal allocation.

The information structure is described in the following table:

r \ s	$(3/4-\epsilon, 3/4-\epsilon)$	$(3/4-\epsilon, 3/4+\epsilon)$	$(3/4+\epsilon, 3/4-\epsilon)$	$(3/4+\epsilon, 3/4+\epsilon)$
$(1/2, 1/2)$	ϵ	0	0	$1/4-\epsilon$
$(1/2, 1)$	0	$1/4$	0	0
$(1, 1/2)$	0	0	$1/4$	0
$(1, 1)$	$1/4-\epsilon$	0	0	ϵ

To see that this information structure is calibrated, observe that $s_1 = 3/4 - \epsilon$ with probability $1/2$. This probability event can be decomposed in two: with probability $1/4 + \epsilon$ we output this signal with $r_1 = 1/2$, and with the remaining $1/4 - \epsilon$ probability we have $r_1 = 1$. Hence:

$$\mathbb{E}[r_1 | s_1 = 3/4 - \epsilon] = \frac{(1/4+\epsilon)\cdot 1/2 + (1/4-\epsilon)\cdot 1}{1/2} = 3/4 - \epsilon.$$

The counterintuitive nature of this mapping can best be seen when depicted as in Figure 1. We map the CTRs in $\{1/2, 1\}$ to two values $\{3/4-\epsilon, 3/4+\epsilon\}$ around the mean. The symmetric pairs, $(1/2, 1/2)$ and $(1, 1)$, are mapped with high probability into symmetric, but order-reversed pairs, $(3/4+\epsilon, 3/4+\epsilon)$ and $(3/4-\epsilon, 3/4-\epsilon)$, respectively. The calibration is nonetheless achieved by the off-diagonal pairs $(1/2, 1)$ and $(1, 1/2)$ that are mapped into order-preserving signals with probability 1, $(3/4-\epsilon, 3/4+\epsilon)$ and $(3/4+\epsilon, 3/4-\epsilon)$, respectively. The ϵ perturbation in the mapping of the diagonal pairs then achieves the ordering of the signals.

We notice the click-through signals s_i maintain the efficient ranking of the alternatives, and thus guarantee an efficient outcome in the auction. The revenue in the auction is given by:

$$\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{\frac{3}{4}-\epsilon}{\frac{3}{4}+\epsilon} + \frac{1}{4}\cdot\frac{\frac{3}{4}-\epsilon}{\frac{3}{4}+\epsilon} + \frac{1}{4} = \frac{21-4\epsilon}{24+32\epsilon} = \frac{7}{8} - O(\epsilon),$$

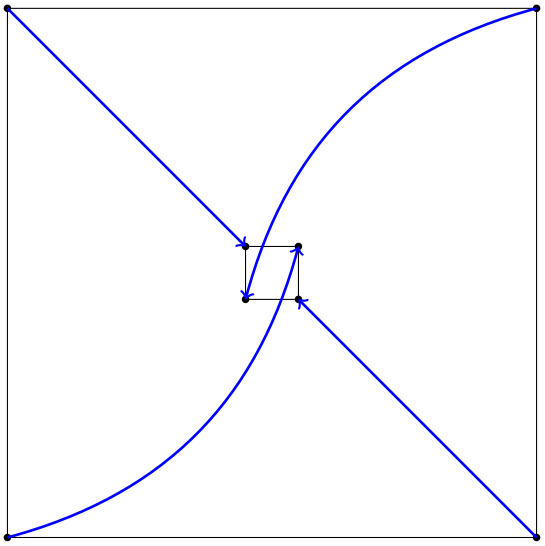


Figure 1: Depiction of the "flipping the square" structure (with ϵ-flows omitted).

which means that almost the entire surplus is extracted. The auction uses a *uniform* tie-breaking rule, thus allocating the object with equal probability if the signals are equal across the bidders.

4.2 A Second Example: Dispersion Along the Diagonal

The second example maintains symmetry across the bidders but has correlated click-through rates. The resulting information structure is more subtle, reflecting the need to balance information necessary to support an efficient allocation with information to support competition.

We present the construction, which we refer to as "dispersion along the diagonal," for a small number of signals. Our main result in this section (Theorem 4.2) builds on a generalization of this construction to more signals (see Lemma 4.1 and Figure 2).

Consider again two bidders with values $v_1 = v_2 = 1$ and click-through rates either $(1/2, 1)$ or $(1, 1/2)$, with probability $1/2$ each. The CTRs are thus perfectly negatively correlated and the social surplus is 1. The revenue under full-disclosure would be $1/2$, and under no-disclosure it would be $3/4$. With no-disclosure, the price-per-click is always competitive, as $\mathbb{E}[r_1]/\mathbb{E}[r_2] = 1$, but the auction fails to lead to the efficient allocation with probability $1/2$. With the following information flow, we attain a revenue of $0.79 > 3/4$:

r \ s	$(0.6, 0.6)$	$(0.6, 0.75)$	$(0.75, 0.6)$	$(0.75, 0.9375)$	$(0.9375, 0.75)$	$(0.9375, 0.9375)$
$(1/2, 1)$	$2/15$	$2/5$	0	$2/5$	0	$1/15$
$(1, 1/2)$	$2/15$	0	$2/5$	0	$2/5$	$1/15$

This information flow lowers the probability of an inefficient allocation from $1/2$ to $1/5$ and attains an equilibrium price closer to 1. In particular,

$$\min \left| \frac{r_i}{r_j} \right| = \frac{1}{2} < \frac{4}{5} = \min \left| \frac{s_i}{s_j} \right|.$$

The information flow in this example generates some symmetric click-through *signals* in the absence of symmetric click-through *rates*. The symmetric signals in the presence of asymmetric rates create some inefficiency in the allocation. But the symmetric click-through rates create the basis for signals that are adjacent, in the sense that they are nearby, yet signal the correct ranking of the underlying click-through rates. If we increase the numbers of signals in the construction of the information flow, we can then reduce the revenue loss and bring it arbitrarily close to zero. This is the following content of Lemma 4.2.

4.3 Optimal Information Structure

We can now state and establish the first main result, showing that for any n-bidder symmetric environment it is possible to construct an information structure extracting revenue that is arbitrarily close to the optimal surplus.

Theorem 4.1 (Full Surplus Extraction in Symmetric Environments)**.** *For every symmetric n-bidder environment, there exists a randomized and calibrated correlated information structure whose revenue is arbitrarily close to full surplus extraction.*

As a building block, we will consider the special case of a symmetric environment of two bidders where $v_1 = v_2 = v$ and

$$\Pr[r = (l, h)] = \Pr[r = (h, l)] = 1/2,$$

for two values $0 \leq l < h \leq 1$. We will then reduce the general symmetric case to a composition of information structures for pairs (h, l). The optimal information structure will be to disperse the signals along the diagonal as depicted in Figure 2.

Lemma 4.2 (Dispersion Along the Diagonal)**.** *Consider the symmetric setting with two bidders with values $v_1 = v_2 = v$ where the click-through rate vector is either (l, h) or (h, l), each with probability $1/2$. Then, for every $\epsilon > 0$, there is a calibrated, correlated information structure with revenue $vh - \epsilon$.*

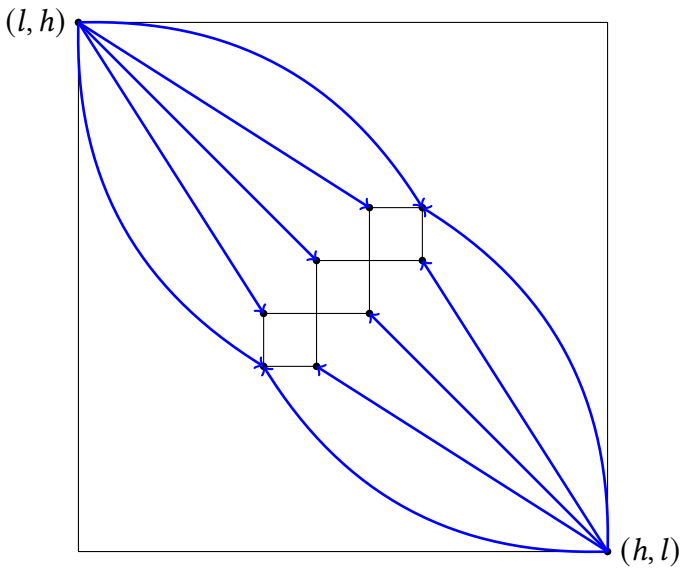


Figure 2: Depiction of the "dispersion along the diagonal" structure for Lemma 4.2.

Proof. We assume without loss of generality that $v = 1$. To prove the lemma, we construct an information structure with a finite set of signals, $S \subset [0, 1]$. The key to this construction is to (i) properly select the signal set S, and (ii) come up with a discretized and calibrated information structure $x(r, s)$ for $r \in \{(l, h), (h, l)\}$ and $s \in S$ that achieves almost optimal revenue.

We consider the following construction with parameters $\delta > 0$ and $x_0 > 0$ to be determined later (see Figure 2 for an illustration).

1. Signal set $S = \{s_{-K}, \ldots, s_0, s_1, \ldots, s_K\}$, where $s_0 = (l + h)/2$, $s_k = s_0 \cdot (1 + \delta)^k$ for $-K \le k \le K$,
$$K = \left\lfloor \log_{(1+\delta)} \frac{2h}{l + h} \right\rfloor - 1;$$
2. $x((l, h), (s_k, s_{k+1})) = x((h, l), (s_{k+1}, s_k)) = x_k$ for $-K \le k \le K - 1$, where
$$x_k = \frac{h - s_k}{s_k - l} \cdot x_{k-1} = x_0 \prod_{\kappa=1}^{k} \frac{h - s_\kappa}{s_\kappa - l}, \text{ when } 1 \le k \le K - 1,$$
$$x_k = \frac{s_{k+1} - l}{h - s_{k+1}} \cdot x_{k+1} = x_0 \prod_{\kappa=k}^{-1} \frac{s_{\kappa+1} - l}{h - s_{\kappa+1}}, \text{ when } -K \le k \le -1;$$

3. $x((l,h),(s_{-K},s_{-K})) = x((h,l),(s_{-K},s_{-K})) = y$, where

$$y = \frac{s_{-K} - l}{l + h - 2s_{-K}} \cdot x_{-K};$$

4. $x((l,h),(s_K,s_K)) = x((h,l),(s_K,s_K)) = z$, where

$$z = \frac{h - s_K}{2s_K - l - h} \cdot x_{K-1}.$$

In the rest of the proof, we first verify that the construction is a valid calibrated and correlated information structure, then show that by choosing a sufficiently small δ, the revenue is at least $h - \epsilon$.

Step 1: We verify that the signals in S are valid probabilities, i.e., $S \subset [0,1]$.

For sufficiently small $\frac{h-l}{3(h+l)} > \delta > 0$, $K - 1 = \left\lfloor \log_{(1+\delta)} \frac{2h}{l+h} \right\rfloor - 1 \geq 1$. For all $-K \leq k \leq K$,

$$s_k \leq s_K = \frac{l+h}{2} \cdot (1+\delta)^K \leq \frac{l+h}{2} \cdot \frac{2h}{l+h} \cdot \frac{1}{1+\delta} = \frac{h}{1+\delta} < h \leq 1;$$
$$s_k \geq s_{-K} = \frac{l+h}{2} \cdot (1+\delta)^{-K} \geq \frac{l+h}{2} \cdot \frac{l+h}{2h} \cdot (1+\delta) \geq \frac{4hl}{4h} \cdot (1+\delta) > l \geq 0.$$

Therefore, $S \subset [0,1]$ is a valid finite signal space.

Step 2: We verify that the parameters x_k, y, z are non-negative.

Since $s_k \in (l,h)$, by the construction of x_k for $k \neq 0$, $x_k/x_0 > 0$. For y and z, since $s_{-K} < s_0 = (l+h)/2 < s_K$,

$$y/x_{-K} = \frac{s_{-K} - l}{l + h - 2s_{-K}} = \frac{(s_{-K} - l)/2}{(l+h)/2 - s_{-K}} > 0, \quad z/x_{K-1} = \frac{h - s_K}{2s_K - l - h} = \frac{(h - s_K)/2}{s_K - (l+h)/2} > 0.$$

Therefore, as long as $x_0 > 0$, all probability terms are positive.

Step 3: We verify that we can choose x_0 such that:

$$\sum_s x((l,h),s) = \sum_s x((h,l),s) = 1/2$$

As we showed, the coefficients, x_k/x_0, y/x_0, and z/x_0 are all positive and fixed. Then with x_0 defined below,

$$x_0 = \frac{1}{2} \cdot \frac{1}{y/x_0 + z/x_0 + \sum_k x_k/x_0} > 0,$$

we have $\sum_s x((l,h),s) = \sum_s x((h,l),s) = y + z + \sum_k x_k = 1/2$.

Step 4: We check that the calibration constraints are satisfied.

For $-K+1 \le k \le K-1$, we verify the calibration constraint for sending signal s_k to bidder 1 as follows:

$$\sum_{s\in S} x((l,h),(s_k,s)) \cdot (l - s_k) + x((h,l),(s_k,s)) \cdot (h - s_k) = x_k \cdot (l - s_k) + x_{k-1} \cdot (h - s_k)$$
$$= x_{k-1} \cdot \frac{h - s_k}{s_k - l} \cdot (l - s_k) + x_{k-1} \cdot (h - s_k) = 0.$$

When sending signal s_K to bidder 1:

$$\sum_{s\in S} x((l,h),(s_K,s)) \cdot (l - s_K) + x((h,l),(s_K,s)) \cdot (h - s_K) = z \cdot (l - s_K) + x_{K-1} \cdot (h - s_K) + z \cdot (h - s_K)$$
$$= \frac{h - s_K}{2s_K - l - h} \cdot x_{K-1} \cdot (l - s_K) + x_{K-1} \cdot (h - s_K) + \frac{h - s_K}{2s_K - l - h} \cdot x_{K-1} \cdot (h - s_K) = 0.$$

When sending signal s_{-K} to bidder 1:

$$\sum_{s\in S} x((l,h),(s_{-K},s)) \cdot (l - s_{-K}) + x((h,l),(s_{-K},s)) \cdot (h - s_{-K})$$
$$= x_{-K} \cdot (l - s_{-K}) + y \cdot (h - s_{-K}) + y \cdot (h - s_{-K})$$
$$= x_{-K} \cdot (l - s_{-K}) + \frac{s_{-K} - l}{l + h - 2s_{-K}} \cdot x_{-K} \cdot (l - s_{-K}) + \frac{s_{-K} - l}{l + h - 2s_{-K}} \cdot x_{-K} \cdot (h - s_{-K}) = 0.$$

As the construction is symmetric for bidder 1 and 2, we omit the verification of the calibration constraints for bidder 2.

Step 5: Now that we verified this is a calibrated signaling scheme, we bound its revenue.

Note that when $s \in \{(s_k, s_{k+1}), (s_{k+1}, s_k)\}_{k=-K}^{K-1}$, the auction allocates the item efficiently and the auctioneer extracts almost all the surplus. More specifically, when $s = (s_k, s_{k+1})$ for CTR profile (l,h), or $s = (s_{k+1}, s_k)$ for CTR profile (h,l), the conditional expected revenue is

$$h \cdot \frac{s_k}{s_{k+1}} = h/(1+\delta) > h - \epsilon/2, \text{ when } \delta < \frac{\epsilon}{2h - \epsilon}.$$

Therefore, we remain to prove that the probability of not extracting revenue $h/(1+\delta)$ is sufficiently small, i.e., $y, z < \epsilon/8$.

Recall that $h \geq s_K \cdot (1+\delta)$, with sufficiently small $\delta < (h-l)/2(h+l)$,

$$\begin{aligned} z = x_{K-1} \cdot \frac{h - s_K}{2s_K - l - h} &\leq x_{K-1} \cdot \frac{h - h/(1+\delta)}{2h/(1+\delta) - l - h} \\ &= x_{K-1} \cdot \frac{\delta h}{h - l - \delta \cdot (h+l)} < \delta \cdot \frac{2h}{h-l} \cdot x_{K-1} < \delta \cdot \frac{2h}{h-l}, \end{aligned}$$

which is less than $\epsilon/8$ when $\delta < \frac{h-l}{16h} \cdot \epsilon$.

Similarly, $s_{-K} = (1+\delta)^{-K} \cdot (l+h)/2 \leq \frac{l+h}{2h} \cdot (1+\delta) \cdot (l+h)/2 < (1+\delta)(h+3l)/4$, with sufficiently small $\delta < (h-l)/2(h+3l)$,

$$\begin{aligned} y = x_{-K} \cdot \frac{s_{-K} - l}{l + h - 2s_{-K}} &\leq x_{-K} \cdot \frac{(1+\delta)(h+3l)/4 - l}{l + h - (1+\delta)(h+3l)/2} \\ &= x_{-K} \cdot \frac{h - l + \delta \cdot (h+3l)}{2h - l - 2\delta \cdot (h+3l)} < \frac{3}{2} \cdot x_{-K}. \end{aligned}$$

It suffices to show $x_{-K} < \epsilon/12$ with a sufficiently small δ. Note that $s_{-K} < \cdots < s_{-1} < s_0 = (l+h)/2 < s_1 < \cdots < s_K$, we then have $x_0 > x_1 > \cdots > x_{K-1}$ and $x_0 > x_{-1} > \cdots > x_{-K}$. Then

$$1/2 = y + z + \sum_{k=-K}^{K-1} x_k > (K+1) \cdot x_{-K}.$$

Therefore, when $\delta < \frac{\epsilon}{6} \cdot \log \frac{2h}{l+h}$, x_{-K} can be bounded by $\epsilon/12$:

$$x_{-K} < 1/2(K+1) < \frac{1}{2\log_{(1+\delta)}(2h/(l+h))} = \frac{\log(1+\delta)}{2\log(2h/(l+h))} < \frac{\delta}{2\log(2h/(l+h))}.$$

In summary, for any given $\epsilon > 0$, we can conclude the proof with a sufficiently small δ:

$$\begin{aligned} \delta < \frac{h-l}{3(h+l)} &\Longrightarrow K - 1 \geq 1, \\ \delta < \frac{\epsilon}{2h - \epsilon} &\Longrightarrow h/(1+\delta) \geq h - \epsilon/2, \\ \delta < \min\left(\frac{h-l}{2(h+l)}, \frac{h-l}{16h} \cdot \epsilon\right) &\Longrightarrow z < \epsilon/8, \\ \delta < \min\left(\frac{h-l}{2(h+3l)}, \frac{\epsilon}{6} \cdot \log\frac{2h}{l+h}\right) &\Longrightarrow y < \epsilon/8. \end{aligned}$$

□

Lemma 4.2 is stated for a very special case within the class of symmetric environments. Both bidders have value 1, each bidder has only one of *two possible* click-through rates, and the click-through rates are *perfectly negatively* correlated. However, the result can now be extended immediately to a general symmetric environment. The extension is based on two simple observations:

Corollary 4.3 (High-Low Pairing)**.** *Consider n bidders with values $v_i \equiv v$ and click-through rates uniformly distributed between two profiles where r and r' are such for two bidders $i, j \in [n]$ that we have: $r_i = r'_j > r_j = r'_i \geq r_k, r'_k$ for any $k \neq i, j$. Then, for any $\epsilon > 0$ there is a calibrated, correlated information structure with revenue $vr_i - \epsilon$.*

Proof. Treat bidders i and j as the high/low pair in Lemma 4.2 and do full-disclosure for any other bidder. The signal is still calibrated and only i and j win the item since their signals will be above the signals of any other bidder. Hence, the revenue bound in Lemma 4.2 still holds. □

The next lemma shows that information structures can be composed, in the sense that if we decompose a distribution of click-through rates and design an information structure for each of them, we can later compose them without loss in calibration.

Lemma 4.4 (Signal Composition)**.** *Let $\mathcal{G}'$ and $\mathcal{G}''$ be distributions over click-through rate profiles r of n bidders and let $\mathcal{F}'$ and $\mathcal{F}''$ be corresponding calibrated information structures given by joint distributions over vector pairs (r, s) such that the r-marginals are $\mathcal{G}'$ and $\mathcal{G}''$ respectively.*

Let $\mathcal{G}$ be the distribution obtained by sampling from $\mathcal{G}'$ with probability λ, and $\mathcal{G}''$ with probability $1-\lambda$. Define a distribution $\mathcal{F}$ similarly. Then, $\mathcal{F}$ is a calibrated information structure for $\mathcal{G}$ and

$$\textsc{Rev}(\mathcal{F}) = \lambda \textsc{Rev}(\mathcal{F}') + (1-\lambda)\textsc{Rev}(\mathcal{F}'').$$

Proof. The r-marginal of $\mathcal{F}$ is clearly $\mathcal{G}$ and $\textsc{Rev}(\mathcal{F}) = \lambda\textsc{Rev}(\mathcal{F}') + (1-\lambda)\textsc{Rev}(\mathcal{F}'')$. The only non-trivial part is to check that $\mathcal{F}$ is calibrated, which we do below:

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathcal{F}}[r_i | s_i = s_i'] &= \frac{\mathbb{E}_{\mathcal{F}}[r_i \mathbf{1}\{s_i - s_i'\}]}{\Pr_{\mathcal{F}}(s_i = s_i')} = \frac{\lambda \mathbb{E}_{\mathcal{F}'}[r_i \mathbf{1}\{s_i - s_i'\}] + (1-\lambda)\mathbb{E}_{\mathcal{F}''}[r_i \mathbf{1}\{s_i - s_i'\}]}{\lambda \Pr_{\mathcal{F}'}(s_i = s_i') + (1-\lambda)\Pr_{\mathcal{F}''}(s_i = s_i')} \\ &= \frac{\lambda \mathbb{E}_{\mathcal{F}'}[s_i' \mathbf{1}\{s_i - s_i'\}] + (1-\lambda)\mathbb{E}_{\mathcal{F}''}[s_i' \mathbf{1}\{s_i - s_i'\}]}{\lambda \Pr_{\mathcal{F}'}(s_i = s_i') + (1-\lambda)\Pr_{\mathcal{F}''}(s_i = s_i')} = s_i'.\end{aligned}$$

□

Proof of Theorem 4.1. Consider an n-bidder symmetric environment and assume for simplicity that the distribution over CTRs is discrete. For every profile of CTRs where bidder i has the highest CTR r_i and bidder j has the second highest CTR r_j (breaking ties lexicographically), we can pair with a profile where the CTRs of i and j are reversed. This leads to a decomposition of the original distribution of CTRs into distributions with support two of the form studied in Corollary 4.3. The results follow from applying Corollary 4.3 together with the composition technique in Lemma 4.4. □

It turns out that the idea behind the construction in this section can be generalized to work under the weaker requirement of *equal means*. The main building block for this is the following generalization of Lemma 4.2, which we prove in Appendix A.

Proposition 4.5 (Generalized Dispersion)**.** *Consider a setting with two bidders with $v_1 = v_2 = v$ and a support size two distribution over click-through rate vectors (r_1, r_2) such that $\mathbb{E}[r_1] = \mathbb{E}[r_2]$. Then for every ϵ, there is a calibrated and correlated information structure with revenue $v \cdot \mathbb{E}[\max(r_1, r_2)] - \epsilon$.*

5 Correlated Structures Beyond Symmetric Environments

Our analysis so far has maintained a weak notion of ex-ante symmetry among the bidders. In Proposition 4.5, the bidders were assumed to have the same ex-ante mean in click-through rates, but may have different supports and different distributions in terms of the ex-post click-through rates. In the presence of ex-ante symmetry we established that an optimal information design supported a revenue level equal to the full social surplus. In this section, we pursue a more limited objective for general asymmetric environments. The central result of this section, Theorem 5.1, establishes that optimal information design remains a powerful instrument to increase revenue, irrespective of the joint distribution of the click-through rates. In particular, we can prove that there always exist information structures that revenue-dominate either no-disclosure or full-disclosure information structures.

For this general environment, we do not provide a complete identification of the optimal information design across all possible configurations of the joint distributions of click-through rates. However, the arguments developed so far will be sufficient to offer significant qualitative characterizations of optimal information design.

Different information structures offer different levels of informativeness, and in fact form a *lattice* (see Figure 3). The no-disclosure information structure is the minimal information policy. The full-disclosure information structure is the maximal information policy. Together, they form the set of *extremal* informational policies. We refer to every information policy that is not extremal as *moderate*, and to any information policy that does not consist of a combination of full- or no-disclosure as *interior*.

Theorem 5.1 establishes that in all environments, symmetric or asymmetric, there exists a moderate information structure that presents a strict improvement. We then show that in some sense this is the strongest result that we can obtain in a general asymmetric environment. Theorem 5.2 establishes that in an environment in which one bidder always has the highest click-through rate, the optimal information structure is moderate, but not interior. Finally, Proposition 5.4 shows that in other environments the optimal information structure must indeed be not only moderate, but in fact interior.

5.1 Extremal Structures are Dominated

For the remainder of this section we consider an environment in which there are two bidders with values $v_1 = v_2 = 1$, and two possible click-through rate configurations, namely (r_1, r_2) and (r_1', r_2'). Without loss

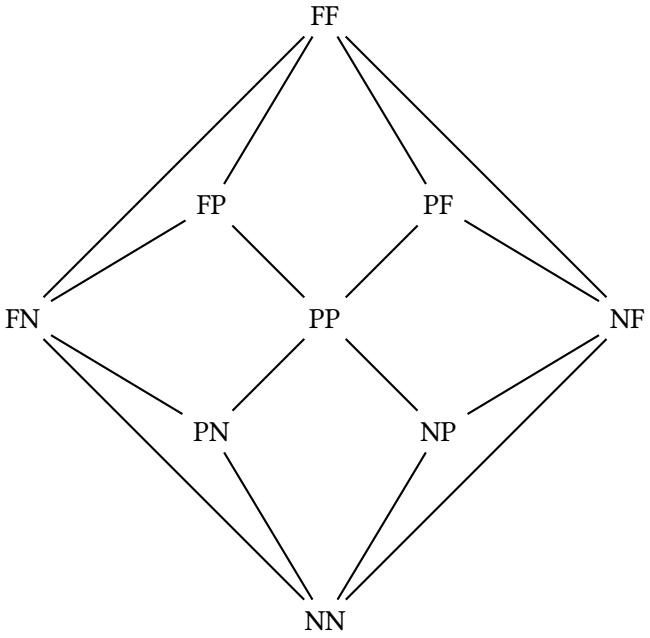


Figure 3: Depiction of the disclosure lattice for two bidders. The label XY of a vertex denotes the pair of policies X and Y used by bidder 1 and bidder 2. N stands for no-disclosure, P for partial disclosure, and F for full-disclosure. All information structures inside the square are interior.

of generality, we can always label the identities of the bidders and the click-through rates so that:

$$r_1 \geq r_1', r_2, r_2'.$$

The prior probability of the pair (r_1, r_2) is p, the other pair (r_1', r_2') has the complementary probability $1 - p$. Let $\mu_1 = p \cdot r_1 + (1 - p) \cdot r_1'$ and $\mu_2 = p \cdot r_2 + (1 - p) \cdot r_2'$ be the expected click-through rates of bidder 1 and bidder 2, respectively.

The main result of this section is the following theorem.

Theorem 5.1 (Moderate Information Structures)**.** *There always exists a moderate information structure that strictly dominates any extremal information structure.*

The theorem will be established in several steps. We split our analysis to match the nature of the competition between the bidders. In the ex-ante symmetric case, the bidder with the highest click-through rate was variable across the realizations, and we thus had a variable winner. In the asymmetric setting, the bidder with the highest click-through rate may be uniform across click-through rates. Given

the ranking, we will split our analysis into the following cases:

Uniform Winner: $r_1' \geq r_2'$	Variable Winner: $r_1' < r_2'$
Congruent Loser: $r_2 \geq r_2'$	Incongruent Loser: $r_2 < r_2'$
Weak Competition: $r_2 \leq \mu_1$	Strong Competition: $r_2 > \mu_1$

5.2 Moderate vs. Interior Structures

We next show by construction that Theorem 5.1 cannot be strengthened from moderate to interior structures in the general asymmetric setting. For this, we consider the case of the uniform winner with a congruent loser under weak competition. In this setting, bidder 1 has the higher click-through rate in all realizations of click-through rates, and is thus the uniform winner. Moreover, the ranking across realizations is the same for all bidders, and hence congruent. Finally, the expected click-through rate of the winner is larger than the maximum of the realized click-through rates of the loser.

For this setting, we can identify the uniquely optimal information structure. Namely, it is optimal to bundle the click-through rates of the winner and to unbundle the click-through rates of the loser. The bundling of the click-through rates of the winner generates more competitive prices, and hence higher revenue for the auctioneer. Thus, the optimal information structure is moderate, but not interior.

Theorem 5.2 (Uniform Winner, Congruent Loser, Weak Competition)**.** *The optimal information structure for the uniform winner, congruent loser, and weak competition case leaves the loser unbundled and fully bundles the winner.*

The main tool in our proof of Theorem 5.2 is the following lemma.

Lemma 5.3 (Chebyshev's sum inequality (Hardy et al., 1988))**.** *Given two sequences $a_1 \geq a_2 \geq \ldots a_n \geq 0$ and $b_1 \geq b_2 \geq \ldots b_n \geq 0$ that are monotone in the same direction, and a set of non-negative weights $w_i \geq 0$ (not necessarily monotone), then:*

$$\left(\sum_i w_i a_i b_i\right) \cdot \left(\sum_i w_i\right) \geq \left(\sum_i w_i a_i\right) \cdot \left(\sum_i w_i b_i\right).$$

If $\{a_i\}$ and $\{b_i\}$ sequences are monotone in different directions (one increasing and one decreasing), the inequality holds in the opposite direction.

(Aside: The probabilistic interpretation of Lemma 5.3 is that if A and B are two positively-correlated random variables, then $\mathbb{E}[AB] \geq \mathbb{E}[A] \cdot \mathbb{E}[B]$.)

Proof of Theorem 5.2. We will start with a generic solution $x(r,s)$ and show that using two applications of Chebyshev's sum inequality (Lemma 5.3) we can bound it with respect to

$$p \cdot r_1 \cdot \frac{r_2}{\mu_1} + (1-p) \cdot r_1' \cdot \frac{r_2'}{\mu_1}, \tag{3}$$

which is the revenue obtained from leaving the loser unbundled and bundling the winner. We will proceed in three steps.

Step 1: Bounding the revenue. Consider any information structure defined by $x(r,s)$. For each signal $s = (s_1, s_2)$, the revenue in the event that bidder 1 wins is $(x(r,s)r_1 + x(r',s)r_1') \cdot s_2/s_1$. If bidder 2 wins, the revenue is $(x(r,s)r_2 + x(r',s)r_2') \cdot s_1/s_2$. In the second case, observe that since $s_2 \geq s_1$ we have $\frac{s_1}{s_2} < 1 < \frac{s_2}{s_1}$, and since we are in the uniform winner case, we know that $x(r,s)r_1 + x(r',s)r_1' \geq x(r,s)r_2 + x(r',s)r_2'$. Hence, we can bound:

$$(x(r,s)r_2 + x(r',s)r_2') \cdot \frac{s_1}{s_2} \leq (x(r,s)r_1 + x(r',s)r_1') \cdot \frac{s_2}{s_1},$$

and write:

$$\textsc{Rev} \leq \sum_s (x(r,s)r_1 + x(r',s)r_1') \cdot \frac{s_2}{s_1}.$$

Step 2: Unbundling the loser. Substituting s_2 by the calibration constraint we obtain:

$$\begin{aligned}
\textsc{Rev} &\leq \sum_{s_2} \sum_{s_1} \left(x(r,(s_1,s_2)) \frac{r_1}{s_1} + x(r',(s_1,s_2)) \frac{r_1'}{s_1} \right) \cdot \frac{\sum_{s_1} x(r,(s_1,s_2))r_2 + x(r',(s_1,s_2))r_2'}{\sum_{s_1} x(r,(s_1,s_2) + x(r',(s_1,s_2))} \\
&\leq \sum_{s_2} \sum_{s_1} \left(x(r,(s_1,s_2)) \frac{r_1 r_2}{s_1} + x(r',(s_1,s_2)) \frac{r_1' r_2'}{s_1} \right),
\end{aligned}$$

where the second inequality follows from Chebyshev's sum inequality with

$$\{a_i\}_i = \left(\frac{r_1}{s_1^1}, \ldots, \frac{r_1}{s_1^n}, \frac{r_1'}{s_1^1}, \ldots, \frac{r_1'}{s_1^n} \right) \quad \text{and} \quad \{b_i\}_i = (r_2, \ldots, r_2, r_2', \ldots, r_2').$$

Congruence implies that the sequence $\{b_i\}$ is sorted. The sequence $\{a_i\}$ is sorted because $r_1' \leq s_1^1 \leq \ldots \leq s_1 \leq r_1$, and hence $\frac{r_1}{s_1} \geq 1 \geq \frac{r_1'}{s_1}$. Now that the loser is unbundled, there is no longer any need to keep track of s_2. To simplify notation we will define:

$$\tilde{x}(r,s_1) = \sum_{s_2} x(r,(s_1,s_2))$$

and re-write our current bound on the objective as:

$$\textsc{Rev} \leq \sum_{s_1} \frac{\tilde{x}(r,s_1)r_1 r_2 + \tilde{x}(r',s_1)r_1' r_2'}{s_1}.$$

Step 3: Bundling the winner. We will replace s_1 according to the calibration constraint in the expression above and replace:

$$\lambda(s_1) = \frac{\tilde{x}(r, s_1)}{\tilde{x}(r, s_1) + \tilde{x}(r', s_1)} \quad \text{and} \quad \lambda'(s_1) = \frac{\tilde{x}(r', s_1)}{\tilde{x}(r, s_1) + \tilde{x}(r', s_1)}.$$

We obtain:

$$\begin{aligned}\textsc{Rev} &\leq \sum_j (\tilde{x}(r, s_1) r_1 r_2 + \tilde{x}(r', s_1) r_1' r_2') \cdot \frac{\tilde{x}(r, s_1) + \tilde{x}(r', s_1)}{r_1 \tilde{x}(r, s_1) + r_1' \tilde{x}(r', s_1)} \\ &= \sum_j \frac{\lambda(s_1) r_1 r_2 + \lambda'(s_1) r_1' r_2'}{\lambda(s_1) r_1 + \lambda'(s_1) r_1'} \cdot (\tilde{x}(r, s_1) + \tilde{x}(r', s_1)).\end{aligned}$$

Now we can apply Chebyshev's sum inequality one more time with:

$$a(s_1) = \frac{\lambda(s_1) r_1 r_2 + \lambda'(s_1) r_1' r_2'}{\lambda(s_1) r_1 + \lambda'(s_1) r_1'}, \quad b(s_1) = \lambda(s_1) r_1 + \lambda'(s_1) r_1', \quad w(s_1) = \tilde{x}(r, s_1) + \tilde{x}(r', s_1).$$

Note that we can reorder signals s_1 in any order we wish. Let us reorder the signals so that $\lambda(s_1)$ is increasing. This immediately implies that $b(s_1)$ is increasing. Namely, $b(s_1) = \lambda(s_1) r_1 + (1 - \lambda(s_1)) r_1' = \lambda(s_1)(r_1 - r_1') + r_1'$. For $a(s_1)$, we can take the derivative in λ,

$$\frac{d}{d\lambda} a(s_1) = \frac{r_1 \cdot r_1' \cdot (r_2 - r_2')}{(r_1 \lambda - r_1'(1 - \lambda))^2} > 0.$$

Thus, $a(s_1)$ is also increasing in λ, allowing us to apply the inequality to obtain:

$$\sum_{s_1} w(s_1) a(s_1) \leq \left(\sum_{s_1} w(s_1) a(s_1) b(s_1) \right) \frac{\sum_{s_1} w(s_1)}{\sum_{s_1} w(s_1) b(s_1)},$$

which translates to (since $\sum_{s_1} w(s_1) = 1$):

$$\textsc{Rev} \leq \frac{\sum_{s_1} \tilde{x}(r, s_1) r_1 r_2 + \tilde{x}(r', s_1) r_1' r_2'}{\sum_{s_1} \tilde{x}(r, s_1) r_1 + \tilde{x}(r', s_1) r_1'} = \sum_{s_1} \tilde{x}(r, s_1) r_1 \frac{r_2}{\mu_1} + \tilde{x}(r', s_1) r_1' \frac{r_2'}{\mu_1} = p r_1 \frac{r_2}{\mu_1} + (1 - p) r_1' \frac{r_2'}{\mu_1}.$$

This is the revenue obtained by bundling the winner and unbundling the loser as desired. □

The impact of competitive prices via competitive click-through rates can be so strong that even a weaker position of the winning bidder can actually increase the revenue of the auctioneer. In fact, as long as we maintain a uniform winner and a congruent loser, it is revenue-increasing for the winner to have a lower click-through rate in the low click-through rate.

For the next result we stay with the uniform winner setting, but now flip the ranking of click-through rates across click-through realizations. Thus, we consider the case of the incongruent loser. We can verify that the competition is guaranteed to be weak in the above sense.

Now, we can show that an interior information structure will always be the optimal information structure. In contrast, the moderate information structure that was optimal in the congruent setting can be shown to perform worse than either of the extremal information structures.

Proposition 5.4 (Uniform Winner and Incongruent Loser)**.** *With a uniform winner and incongruent loser, there is always an interior information structure that is revenue-improving over all exterior information structures. In particular, partially bundling the winner* and *the loser is revenue-improving, relative to all exterior information structures:*

r \ s	(r_1, x)	(z, y)	(z, x)	(r_1', y)
(r_1, r_2)	$p - q$	q	0	0
(r_1', r_2')	0	0	q'	$p' - q'$

for suitably chosen $x < y \leq z$.

These two cases demonstrate how rich the optimal structures in the asymmetric setting can be, but they also emphasize that the same guiding principles that governed the optimal policy in symmetric settings are in play. Namely, as in the symmetric setting, the optimal schemes that we identify are not extremal, maintain efficiency, and seek to strengthen competition through calibrated signals.

6 Conclusion

A large share of online advertising, whether in search advertising, display advertising, or social networks, is allocated by various auction mechanisms. The auction seeks to form a match between the viewer and competing advertisers. The viewers are typically heterogeneous in many dimensions: their characteristics, their preferences, their (past) shopping behavior, their browsing history, and many other aspects, observable and unobservable. A critical dimension of the heterogeneity is the click-through rate of the user for a specific advertiser.

The disclosure policy of the auctioneer regarding the click-through rates therefore influences the distribution of bids, holding fixed the distribution of preferences among the bidders. By disclosing less, the auctioneer in effect bundles certain features, or as Levin and Milgrom (2010) suggest, conflates features of the viewer. The process of conflation influences the thickness or thinness of the market—in other words, the strength of the competition.

In our analysis, conflation was achieved by bundling the information regarding click-through rates in an optimal manner. The optimal information structure conveys just enough information so that the resulting bidding process ranks the bidders according to the true social value of each bidder. At the same time, the information is released only partially to maintain bids as close as possible to a perfectly competitive level. We show that this requires that the information is provided to each bidder as private information at an individual level, rather than as public information on a market level.

An alternative approach might have been to directly apply a version of the revelation principle to the information design problem (see Bergemann and Morris (2019)). The revelation principle for information design would suggest that the auctioneer only requires as many signals as allocations to be implemented. This would suggest that a public signal that informs the market about the identity of the winner and loser would be sufficient. In a two-bidder environment, this would suggest that a binary public information structure would be sufficient. Indeed, the binary signal would lead to an efficient allocation. But as our analysis and Example 2 have shown in detail, this would lead to a lower revenue for the seller than what is attainable by optimal information design. The apparent failure of the revelation principle here is related to the click-through auction. We deliberately fixed the auction mechanism to trace the implications of information design in a given auction mechanism. But by adopting the rules of the click-through auction, we effectively limit the message space of each type, which is the fundamental source of failure of the revelation principle, as famously observed by Green and Laffont (1986).

A significant advance in our information design problem is to allow for multi-dimensional and private information in a strategic setting. By contrast, the most recent result in the design of optimal information structure requires one-dimensional, or equivalently symmetric, solutions to optimal design (see Kleiner et al. (2020)). A general approach to optimal multi-dimensional information design in strategic settings is a wide-open question. In the current context, we could make progress by insights specific to the auction setting.

Throughout this work we maintained the assumption of complete information, regarding the value of each bidder. While this setting is commonly adopted in the analysis of sponsored search auctions, it would clearly be an important issue for how the optimal information policy is influenced by private information, regarding the valuation of each bidder. We would then be in a setting where both the auctioneer and bidders have private information. This also remains a wide-open issue, even in a setting with a single receiver, and progress is currently being made only in specific settings, either binary actions and states or multiplicative separable settings (see Kolotilin et al. (2017) and Candogan and

Strack (2021), respectively). We would hope that the current auction setting would give us additional tools to address these issues eventually.

References

I. Arieli and Y. Babichenko. 2019. Private Bayesian Persuasion. *Journal of Economic Theory* 182 (2019), 185–217.

A. Badanidiyuru, K. Bhawalkar, and H. Xu. 2019. Targeting and Signaling in Ad Auctions. In *Proceedings of the 29th ACM-SIAM Symposium on Discrete Algorithms*. 2545—2563.

D. Bergemann, B. Brooks, and S. Morris. 2015. The Limits of Price Discrimination. *American Economic Review* 105 (2015), 921–957.

D. Bergemann, B. Brooks, and S. Morris. 2017. First Price Auctions with General Information Structures: Implications for Bidding and Revenue. *Econometrica* 85 (2017), 107–143.

D. Bergemann and S. Morris. 2019. Information Design: A Unified Perspective. *Journal of Economic Literature* 57 (2019), 44–95.

D. Bergemann and M. Pesendorfer. 2007. Information Structures in Optimal Auctions. *Journal of Economic Theory* 137 (2007), 580–609.

S. Board. 2009. Revealing Information in Auctions: The Allocation Effect. *Economic Theory* 38 (2009), 125–135.

O. Candogan and P. Strack. 2021. Optimal Disclosure of Information to a Privately Informed Receiver. (2021).

J. Cremer and R. McLean. 1985. Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist When Demands are Interdependent. *Econometrica* 53 (1985), 345–361.

J. Cremer and R. McLean. 1988. Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions. *Econometrica* 56 (1988), 1247–1258.

B. Edelman, M. Ostrovsky, and M. Schwartz. 2007. Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. *American Economic Review* 97 (2007), 242–259.

P. Eső and B. Szentes. 2007. Optimal Information Disclosure in Auctions. *Review of Economic Studies* 74 (2007), 705–731.

D. Foster and R. Vohra. 1997. Calibrated Learning and Correlated Equilibrium. *Games and Economic Behavior* 21 (1997), 40–55.

J. Green and J. Laffont. 1986. Partially Verifiable Information and Mechanism Design. *Review of Economic Studies* 53 (1986), 447–456.

G. H. Hardy, J. E. Littlewood, and G. Pólya. 1988. *Inequalities.* Cambridge University Press, Cambridge, UK.

Andreas Kleiner, Benny Moldovanu, and Philipp Strack. 2020. *Extreme Points and Majorization: Economic Applications.* Technical Report.

A. Kolotilin, M. Li, T. Mylovanov, and A. Zapechelnyuk. 2017. Persuasion of Privately Informed Receiver. *Econometrica* forthcoming (2017).

S. Lahaie and D. Pennock. 2007. Revenue Analysis of a Family of Ranking Rules for Keyword Auctions. In *ACM-EC 2007*. 50–56.

J. Levin and P. Milgrom. 2010. Online Advertising: Heterogeneity and Conflation in Market Design. *American Economic Review Papers and Proceedings* 100 (2010), 603–607.

M. Mahdian and M. Sundararajan. 2015. *Robust Mechanisms for Sponsored Search.* Technical Report. Google Research.

H.R. Varian. 2007. Position Auctions. *International Journal of Industrial Organization* 25 (2007), 1163–1178.

A Full Surplus Extraction with Equal Means

Lemma (Lemma 4.5 Restate)**.** *Consider a setting with two bidders with click-through rates (l, a) and (h, b), where $0 \le l < h$ and $a \ne b \ge 0$, with probability*

$$\Pr[(l,a)] = \frac{h-b}{h+a-l-b}, \qquad \Pr[(h,b)] = \frac{a-l}{h+a-l-b},$$

and the same mean

$$\mu = \frac{ha - lb}{h+a-l-b}.$$

Then, for every $\epsilon > 0$, there is a calibrated, correlated information structure that extracts at least $1 - O(\sqrt{\epsilon})$ fraction of the welfare as revenue.

Note that we can, without loss of generality, assume that $a \ne l$ and $b \ne h$. Otherwise, because of the symmetric mean, when $a = l$ we must have $b = h$, and vice versa. In this special case, the problem is trivial (full-disclosure is optimal).

Consider the following information structure:

- Mapping (l, a) to (s_{2k}, s_{2k+1}) with probability x_k for $-K \le k \le K$;
- Mapping (h, b) to (s_{2k}, s_{2k-1}) with probability x'_k for $-K \le k \le K$.

In particular,

$$s_i = \begin{cases} h^* = \min\{h, \max\{a,b\}\}, & i = 2K+1; \\ s \cdot (1+\epsilon)^i, & -2K \le i \le 2K; \\ l^* = \max\{l, \min\{a,b\}\}, & i = -2K-1. \end{cases}$$

In general, we require that $s_{-2K-1} < s_{-2K} < s_{2K} < s_{2K+1}$, that is

$$l^* < s \cdot (1+\epsilon)^{-2K} < s \cdot (1+\epsilon)^{2K} < h^*.$$

Note that either $a > s_{2K}, \dots, s_{-2K} > b$ or $a < s_{2K}, \dots, s_{-2K} < b$.

We claim that as $\epsilon \to 0$, there exists a valid information structure of the above form satisfying the calibration constraint and extracting at least $1 - O(\sqrt{\epsilon})$ of the welfare as revenue.

Consider the calibration constraints: For $-K \le k \le K$, the first buyer receives signal s_{2k} when the CTR profiles are (l, a) or (h, b), with probability x_k and x'_k, respectively, i.e.,

$$h \cdot x'_k + l \cdot x_k = s_{2k} \cdot (x'_k + x_k) \iff (h - s_{2k}) x'_k = (s_{2k} - l) x_k.$$

Similarly, for the second buyer receiving signal s_{2k+1},

$$a \cdot x_k + b \cdot x'_{k+1} = s_{2k+1} \cdot (x_k + x'_{k+1}) \iff (a - s_{2k+1})x_k = (s_{2k+1} - b)x'_{k+1}.$$

Therefore, we have

$$x_{k+1} = \frac{h - s_{2k+2}}{s_{sk+2} - l} x'_{k+1} = \frac{h - s_{2k+2}}{s_{2k+2} - l} \cdot \frac{a - s_{2k+1}}{s_{2k+1} - b} x_k,$$

and similarly,

$$x'_{k+1} = \frac{a - s_{2k+1}}{s_{2k+1} - b} x_k = \frac{a - s_{2k+1}}{s_{2k+1} - b} \cdot \frac{h - s_{2k}}{s_{2k} - l} x'_k.$$

Note that the coefficients here are all positive,

$$\frac{a - s_{2k+1}}{s_{2k+1} - b} > 0,$$

because either $a > s_{2k+1} > b$ or $a < s_{2k+1} < b$.

Thus,

$$x_k = x_0 \prod_{i=1}^{k} \frac{h - s_{2i}}{s_{2i} - l} \cdot \frac{a - s_{2i-1}}{s_{2i-1} - b} \qquad x_{-k} = x_0 \prod_{i=1}^{k} \frac{s_{2-2i} - l}{h - s_{2-2i}} \cdot \frac{s_{1-2i} - b}{a - s_{1-2i}}$$

$$x'_k = x'_0 \prod_{i=1}^{k} \frac{h - s_{2i-2}}{s_{2i-2} - l} \cdot \frac{a - s_{2i-1}}{s_{2i-1} - b} \qquad x'_{-k} = x'_0 \prod_{i=1}^{k} \frac{s_{-2i} - l}{h - s_{-2i}} \cdot \frac{s_{1-2i} - b}{a - s_{1-2i}}$$

Therefore, the construction is valid, if and only if the following probability constraint is satisfied:

$$\sum x_k = \Pr[(l, a)] = \frac{h - b}{h + a - l - b}, \qquad \sum x'_k = \Pr[(h, b)] = \frac{a - l}{h + a - l - b},$$

and

$$x_k, x'_k \geq 0.$$

It remains to argue the existence of such an information structure (s, ϵ, K) with $\epsilon \to 0^+$, and for the purpose of the revenue bound $x_K, x'_{-K} = o(1)$. This is established in Lemma A.1 and Lemma A.2 below.

Lemma A.1. *For sufficiently small* $0 < \epsilon < (\min\{h^*/\mu, \mu/l^*\})^{1/4} - 1$ *and* $K < \frac{1}{4} \log_{1+\epsilon} \min\{h^*/\mu, \mu/l^*\}$, *there exist* $s \in (s^-, s^+)$ *such that the above construction is a valid signaling scheme, where*

$$s^- = \mu \cdot (1 + \epsilon)^{-2K}, \qquad s^+ = \mu \cdot (1 + \epsilon)^{2K}.$$

Lemma A.2. *As* $\epsilon \to 0^+$ *and* $K = \frac{\sqrt{\epsilon}}{4} \log_{1+\epsilon} \min\{h^*/\mu, \mu/l^*\} = O(1/\sqrt{\epsilon})$, *the valid signaling scheme defined in Lemma A.1 has* $x_K, x'_{-K} = O(\sqrt{\epsilon})$.

Proof of Lemma A.1. Note that from the previous analysis, we have

$$x_k = A_k(s,\epsilon)\cdot x_0, \quad x'_k = A'_k(s,\epsilon)\cdot x'_0,$$

where $A_k(s,\epsilon)$ and $A'_k(s,\epsilon)$ are coefficients depending on s and ϵ.

Therefore, the construction is valid if and only if the following two conditions are satisfied:

- The probability constraints are satisfied, i.e.,

$$x_0\sum_{k=-K}^{K} A_k(s,\epsilon) = \Pr[(l,a)], \qquad x'_0\sum_{k=-K}^{K} A'_k(s,\epsilon) = \Pr[(h,b)]; \tag{4}$$

- All probabilities are non-negative, or equivalently,

$$\forall k\in[-K,K],\ A_k(s,\epsilon), A'_k(s,\epsilon)\geq 0. \tag{5}$$

For any sufficiently small $0<\epsilon<(\min\{h^*/\mu,\mu/l^*\})^{1/4}-1$, there exists $1\leq K<\frac{1}{4}\log_{1+\epsilon}\min\{h^*/\mu,\mu/l^*\}$. Then, for any $s\in(s^-,s^+)$ we have

$$l^* < \mu\cdot(1+\epsilon)^{-4K} = s^-\cdot(1+\epsilon)^{-2K} < s_{-2K},\ldots,s_{2K} < s^+\cdot(1+\epsilon)^{2K} = \mu\cdot(1+\epsilon)^{4K} < h^*.$$

Hence, for all $-K\leq k\leq K$, we know that (5) is satisfied. To conclude the proof, we next argue that there exists $s\in(s^-,s^+)$ such that (4) is also satisfied.

Note that x_k and x'_k are inter-dependent, i.e.,

$$x'_k = x_k\cdot\frac{s_{2k}-l}{h-s_{2k}}.$$

In particular, if $s_{2k}<\mu$, then

$$\frac{s_{2k}-l}{h-s_{2k}} < \frac{\mu-l}{h-\mu} = \frac{\frac{ha-lb}{h+a-l-b}-l}{h-\frac{ha-lb}{h+a-l-b}} = \frac{(h-l)(a-l)}{(h-l)(h-b)} = \frac{a-l}{h-b} = \frac{\Pr[(h,b)]}{\Pr[(l,a)]},$$

and vice versa.

Now,

$$\sum_{k=-K}^{K} x'_k = \sum_{k=-K}^{K} x_k\cdot\frac{s_{2k}-l}{h-s_{2k}} = x_0\sum_{k=-K}^{K} A_k(s,\epsilon)\cdot\frac{s_{2k}-l}{h-s_{2k}},$$

and so (4) can be satisfied if and only if

$$\frac{\sum_{k=-K}^{K} A_k(s,\epsilon)\cdot\frac{s_{2k}-l}{h-s_{2k}}}{\sum_{k=-K}^{K} A_k(s,\epsilon)} = \frac{\Pr[(h,b)]}{\Pr[(l,a)]}.$$

With ϵ fixed, the left-hand-side is a continuous function of s, denoted as $\gamma(s)$. When $s = s^-$, $s_i < \mu$ for all $-2K \le i \le 2K$. So,

$$\gamma(s^-) < \frac{\sum_{k=-K}^{K} A_k(s,\epsilon) \cdot \frac{\Pr[(h,b)]}{\Pr[(l,a)]}}{\sum_{k=-K}^{K} A_k(s,\epsilon)} = \frac{\Pr[(h,b)]}{\Pr[(l,a)]}.$$

Similarly, we have

$$\gamma(s^+) > \frac{\Pr[(h,b)]}{\Pr[(l,a)]}.$$

Therefore, there must exist $s \in (s^-, s^+)$ such that

$$\gamma(s) = \frac{\Pr[(h,b)]}{\Pr[(l,a)]},$$

which implies that (4) is satisfied, hence completing the proof. □

Proof of Lemma A.2. We now argue that for sufficiently small ϵ and properly selected K, the information structure defined by Lemma A.1 has $x_K, x'_{-K} = o(1)$.

Consider the following Taylor expansion near μ,

$$\frac{h-z}{z-l} = \frac{h-\mu}{\mu-l} + O(z-\mu) = \frac{h-b}{a-l} + O(z-\mu).$$

Similarly,

$$\frac{a-z}{z-b} = \frac{a-\mu}{\mu-b} + O(z-\mu) = \frac{a-l}{h-b} + O(z-\mu).$$

Therefore,

$$A_k(s,\epsilon) = 1 + O(\Delta s), \quad A'_k(s,\epsilon) = 1 + O(\Delta s),$$

where $\Delta s = \max\{|s_{-2K} - \mu|, |s_{2K} - \mu|\}$.

Then, we have

$$x_K = x_0 \cdot A_K(s,\epsilon) = \Pr[(l,a)] \cdot \frac{A_K(s,\epsilon)}{\sum_{-K}^{K} A_k(s,\epsilon)} = \Pr[(l,a)] \cdot \frac{1}{2K+1} + O(\Delta s),$$

and similarly,

$$x'_{-K} = x'_0 \cdot A'_{-K}(s,\epsilon) = \Pr[(h,b)] \cdot \frac{A'_{-K}(s,\epsilon)}{\sum_{-K}^{K} A'_k(s,\epsilon)} = \Pr[(h,b)] \cdot \frac{1}{2K+1} + O(\Delta s).$$

In other words, as long as $K \to \infty$ and $\Delta s \to 0$, $x_K, x'_{-K} \to 0$.

In particular, as $\epsilon \to 0$, the following K satisfy the desired property:

$$K = \frac{\sqrt{\epsilon}}{4} \log_{1+\epsilon} \min\{h^*/\mu, \mu/l^*\} = O(\sqrt{\epsilon}/\ln(1+\epsilon)) = O(1/\sqrt{\epsilon}) \to \infty,$$

in the meanwhile,

$$\Delta s \le \mu \cdot (1+\epsilon)^{4K} - \mu \le \mu(\min\{h^*/\mu, \mu/l^*\}^{\sqrt{\epsilon}} - 1) = O(\sqrt{\epsilon}) \to 0.$$ □

B Additional Results and Proofs for Asymmetric Setting

In Section B.1 we provide a proof of Proposition 5.4 for the uniform winner and incongruent loser case, which shows that the optimal information structure must be an interior information structure. In Section B.2 and Section B.3 we state additional propositions for (i) the uniform winner and congruent loser case and (ii) the variable winner case, which establish that the optimal information structure must be moderate. Together, these results show Theorem 5.1, whose proof appears in Section B.4.

B.1 Uniform Winner, Incongruent Loser

Proof of Proposition 5.4. Note that in this case $r_1 \geq r_1' \geq r_2' \geq r_2$ and hence also $\mu_1 \geq \mu_2$. Thus, the revenue from the extremal information structures is given by $\mu_2 = pr_2 + (1-p)r_2'$. Consequently, no-disclosure and full-disclosure yield the same expected revenue. Moreover, unbundling the winner and bundling the loser yield the same revenue as no-disclosure or full-disclosure:

$$r_1 p \left(\frac{r_2 p + r_2' (1-p)}{r_1} \right) + r_1' (1-p) \left(\frac{r_2 p + r_2' (1-p)}{r_1'} \right) = p r_2 + (1-p) r_2'$$

We then consider bundling the winner and unbundling the loser:

$$r_1 p \left(\frac{r_2}{r_1 p + r_1' (1-p)} \right) + r_1' (1-p) \left(\frac{r_2'}{r_1 p + r_1' (1-p)} \right)$$

This yields a strictly lower revenue than no-disclosure with an incongruent loser:

$$\begin{aligned} & r_1 p \left(\frac{r_2}{r_1 p + r_1' (1-p)} \right) + r_1' (1-p) \left(\frac{r_2'}{r_1 p + r_1' (1-p)} \right) - \left(p r_2 + r_2' (1-p) \right) \\ = \; & \frac{p (1-p) \left(r_1 - r_1' \right) \left(r_2 - r_2' \right)}{p r_1 + r_1' (1-p)} < 0, \end{aligned}$$

as the incongruent loser is defined by the property

$$\left(r_1 - r_1' \right) \left(r_2 - r_2' \right) < 0.$$

Thus, the exact opposite result to the congruent loser. Finally, unbundling the winner and bundling the loser yields the same revenue as no-disclosure or full-disclosure.

It therefore suffices to show that we can construct an interior information structure such that revenue improves in comparison to no-disclosure. We begin by partially bundling the loser only. This is

a single-agent optimization problem:

r \ s	(r_1, x)	(r_1, y)	(r_1', x)	(r_1', y)
(r_1, r_2)	$p - q$	q	0	0
(r_1', r_2')	0	0	q'	$p' - q'$

, (6)

where $p' = 1 - p$. This information structure leads the following revenue:

$$r_1 \left((p - q) \frac{x}{r_1} + q \frac{y}{r_1} \right) + r_1' \left(q' \frac{x}{r_1'} + (p' - q') \frac{y}{r_1'} \right) \tag{7}$$

where r_1 and r_1' cancel:

$$((p - q)\, x + qy) + (q'x + (p' - q')\, y)$$

and the calibration constraints are given by:

$$x = \frac{r_2 (p - q) + r_2' q'}{(p - q) + q'}$$

and

$$y = \frac{r_2 q + r_2' (p' - q')}{q + p' - q'}.$$

Thus, we have revenue indifference (as expected):

$$((p - q)\, x + qy) + (q'x + (p' - q')\, y) = r_2 p + p' r_2'.$$

It follows that the introduction of (q, q'), subject to the probability constraints, leaves revenue unchanged. Clearly, we can always choose (q, q') so that

$$y > x. \tag{8}$$

However, we now notice that if we consider the submatrix:

(r_1, y)	(r_1', x)
q	0
0	q'

and $y > x$, then this submatrix represents an instance of the congruent winner with weak competition case, and hence by Theorem 5.2 it is optimal to bundle the winner. Thus,

(z, y)	(z, x)
q	0
0	q'

with

$$z = \frac{r_1 q + r_1' q'}{q + q'}.$$

Moreover, given that the benefit of bundling is increasing in the difference $y - x$, it follows that we have an interior solution, as a higher y and a lower x mean that q and q' have to become smaller. □

B.2 Uniform Winner, Congruent Loser

Proposition B.1 (Uniform Winner, Congruent Loser)**.** *With a uniform winner and a congruent loser, partially bundling the winner* and *the loser is revenue-improving relative to the extremal information structures. Both partial unbundling of the loser:*

r \ s	(μ_1, μ_1)	(μ_1, μ_1)	(μ_1, r_2')
(r_1, r_2)	p	0	0
(r_1', r_2')	0	$1 - p - q$	q

(9)

or partial bundling of the winner:

r \ s	(r_2, r_2)	(r_2, r_2')	(r_1', r_2')
(r_1, r_2)	p	0	0
(r_1', r_2')	0	$1 - p - q$	q

(10)

for suitably chosen q generates a strictly higher revenue than the extremal information structures.

Proof of Proposition B.1. We begin by observing that in the uniform winner and congruent loser case the revenue from no- and full-disclosure is equal to

$$\mu_2 = p r_2 + (1 - p) r_2'.$$

Next, we argue that the information structure in (9) yields a strict revenue improvement. Its revenue is given by:

$$r_1 p + r_1'(1 - p - q) + r_1' q \frac{r_2'}{\mu_1},$$

where the calibration constraint is given by:

$$\frac{r_2 p + r_2'(1 - p - q)}{1 - q} = r_1 p + r_1'(1 - p) = \mu_1,$$

where we remind the reader that

$$\mu_1 = r_1 p + r_1'(1-p)$$

and thus:

$$q = -\frac{\left(pr_1 - pr_2 - r_1'\left(p-1\right) + r_2'\left(p-1\right)\right)}{r_2' - pr_1 + r_1'\left(p-1\right)}$$

Hence, the revenue difference is

$$\begin{aligned} & r_1 p + r_1'(1-p-q) + q r_1' \frac{r_2'}{\mu_1} - \left(r_2 p + (1-p)\, r_2'\right) \\ = \;& p \frac{r_1 - r_1'}{(1-p)\, r_1' + p r_1} \left((1-p)\left(r_1' - r_2'\right) + p\left(r_1 - r_2\right)\right) > 0, \end{aligned}$$

where the inequality holds due to the uniform winner property.

It remains to establish that the revenue of (10) is also greater than the revenue from the extremal information structures. The revenue of (10) is:

$$r_1 p + r_1'(1-p-q)\frac{r_2'}{r_2} + r_1' q \frac{r_2'}{r_1'},$$

where the calibration constraint is given by:

$$\frac{r_1 p + r_1'(1-p-q)}{1-q} = r_2,$$

and thus:

$$q = -\frac{1}{r_1' - r_2}\left(r_2 - p r_1 + r_1'\left(p-1\right)\right).$$

The revenue difference is therefore

$$\begin{aligned} & r_1 p + r_1'(1-p-q)\frac{r_2'}{r_2} + r_1' q \frac{r_2'}{r_1'} - \left(r_2 p + (1-p)\, r_2'\right) \\ = \;& \frac{p}{r_2}\left(r_1 - r_2\right)\left(r_2 - r_2'\right) > 0, \end{aligned}$$

where the inequality holds by the congruent loser property. □

B.3 Variable Winner

Proposition B.2 (Variable Winner)**.** *With a variable winner, a moderate information structure improves on the extremal information structures.*

Proof. Note that the revenue from full-disclosure is $pr_1\frac{r_1'}{r_1} + (1-p)r_2'\frac{r_2}{r_2'} = pr_1' + (1-p)r_2$, which is no more than μ_2 because $r_1' \leq r_2'$, but also no more than μ_1 because $r_2 \leq r_1$. So, the revenue from full-disclosure

is at most $\min\{\mu_1, \mu_2\}$. The revenue from no-disclosure, on the other hand, is μ_2 if $\mu_1 \geq \mu_2$, and μ_1 otherwise. It is therefore equal to $\min\{\mu_1, \mu_2\}$.

We construct a moderate information structure that strictly improves upon this. Without loss of generality, we assume that $\mu_1 > \mu_2$. We consider the following candidate information structure:

r \ s	(r_1, r_2)	(r_1', r_2')	(r_1'', r_2'')
(r_1, r_2)	$\epsilon\lambda$	0	$1-\epsilon$
(r_1', r_2')	0	$\epsilon(1-\lambda)$	$1-\epsilon$

(11)

the construction of which we now detail. By the hypothesis of a variable winner, namely

$$r_1 > r_2, \quad r_1' < r_2', \tag{12}$$

we can find a unique convex combination λ and $1-\lambda$ of the click-through rate configurations (r_1, r_2) and (r_1', r_2'), such that the expected click-through rate of both bidders is the same. Thus, in expectation they have the same expected click-through rate, which we call $\hat{r}$:

$$\lambda r_1 + (1-\lambda) r_1' = \lambda r_2 + (1-\lambda) r_2' = \hat{r}. \tag{13}$$

The remaining probability is collected in a single calibrated signal that yields the posterior expectation of the click-through rates:

$$r_i'' = \frac{(p-\epsilon\lambda) r_i + (1-p-\epsilon(1-\lambda)) r_i'}{(p-\epsilon\lambda) + (1-p-\epsilon(1-\lambda))}.$$

This yields the above information structure (11). Now, we can always chose $\epsilon > 0$ sufficiently small so that $r_1'' \geq r_2''$.

If we restrict attention to the subset of signals (r_1, r_2), (r_1', r_2'), then it is as if we have a bidding game with equal means of the CTRs by construction of (13). By Lemma 4.5, we know that on this subset of signals we can extract the full surplus generated by the click-through rates. In the remaining profile, we run the bidding game as if we have a zero-disclosure policy. With this, we know that the resulting revenue is given by

$$\epsilon\lambda r_1 + (1-\epsilon)\lambda r_2' + (1-\epsilon)\frac{(p-\epsilon\lambda) r_2 + (1-p-\epsilon(1-\lambda)) r_2'}{(p-\epsilon\lambda) + (1-p-\epsilon(1-\lambda))}. \tag{14}$$

By contrast, in the no-disclosure policy the revenue would be μ_2, which can be written as

$$\epsilon\lambda r_2 + (1-\epsilon)\lambda r_2' + (1-\epsilon)\frac{(p-\epsilon\lambda) r_2 + (1-p-\epsilon(1-\lambda)) r_2'}{(p-\epsilon\lambda) + (1-p-\epsilon(1-\lambda))}, \tag{15}$$

and by the hypothesis of the variable winner (see (12)), $r_1 > r_2$, and the above information structure (11) strictly improves upon the no-disclosure policy. □

B.4 Proof of Theorem 5.1

Proof of Theorem 5.1. The claim follows from combining Proposition 5.4 with Proposition B.1 and Proposition B.2. □