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By

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Equilibrium Uniqueness in Entry Games with Private Information*

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Abstract

We study equilibrium uniqueness in entry games with private information. Our framework embeds models commonly used in applied work, allowing rich forms of firm heterogeneity and selective entry. We introduce the notion of strength, which summarizes a firm's ability to endure competition. In environments of applied interest, an equilibrium in which entry strategies are ranked according to strength, called herculean equilibrium, always exists. Thus, when the entry game has a unique equilibrium, it must be herculean. We derive simple sufficient conditions guaranteeing equilibrium uniqueness and, consequently, robust counterfactual analyses.

JEL: D21, D43, L11, L13

Keywords: Entry, Oligopolistic markets, Private information

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1 Introduction

Understanding firms' market entry decisions is a key element of economic policy and regulation. Predicting whether there will be timely entry after a merger or regulatory change requires a framework that determines the number and types of competitors. More broadly, a model with endogenous entry, prices, product characteristics, and welfare outcomes can be used to evaluate policies prospectively. When performing such analysis, researchers use the counterfactual equilibrium of an estimated model to assess the impact of the policy under consideration. A common challenge in this setting is the existence of multiple equilibria. Under multiplicity, the model may not yield a unique prediction to the applied question, difficulting policy analysis (Berry and Tamer, 2006; Borkovsky *et al.*, 2015). Computing these multiple equilibria may also prove challenging when using numerical methods, which may limit the researcher's ability to gain a complete understanding of the impacts of a policy of interest (Iskhakov *et al.*, 2016).

We study equilibrium uniqueness in entry games with private information. Our framework allows for rich forms of firm *heterogeneity* and *selective* entry. Our main contribution is to provide a sufficient condition that guarantees equilibrium uniqueness. The condition is solely based on the model's fundamentals and verifying it does not require equilibrium computation. In many common applications, the sufficient condition can be checked by performing a simple calculation. For example, in the influential articles of Roberts and Sweeting (2013) and Grieco (2014), the authors use numerical methods to show that their fitted models have a unique equilibrium. Using their estimates and our sufficient condition, we can confirm equilibrium uniqueness in their fitted models, highlighting the usefulness of our results. Thus, our findings provide new tools for applied researchers studying entry.

We characterize firms' equilibrium behavior using a simple index that we call *strength*, which summarizes a firm's ability to endure competition. The strength of a firm is the unique *symmetric* threshold-strategy that makes the firm indifferent to enter the market. A stronger firm is simultaneously more willing to enter the market than a weaker competitor, despite facing more competition. The use of strength in entry games is similar to the use of a Gittins index in multi-armed bandit games (Gittins, 1979), as it aggregates the game's relevant information, facilitating the search for equilibria. For the class of models studied, we show that there always exists an equilibrium in which the threshold strategies for entry are ordered according to strength. We call this a *herculean* equilibrium. Thus, when

an entry game has a unique equilibrium, it must be herculean. Identifying the herculean equilibrium, via the firms' strength, is the starting point to develop our sufficient condition for uniqueness. Among the advantages of using strength to find equilibria is that it reduces the computing power necessary to estimate entry models, as strength provides bounds for the herculean equilibrium strategies.

Our proposed framework embeds entry models commonly used in applied work. It accommodates a large variety of post-entry models, including auctions and competitions in price or quantity. The framework also allows for rich forms of firm heterogeneity. Firms are allowed to differ in their payoff functions or in their distribution of types, capturing that firms might be heterogeneous in their public characteristics (e.g., firms might differ in their product characteristics, geographic locations, or in their levels of vertical integration). Payoffs might depend on both actions and the realized types of competitors, allowing a level of strategic interaction often ignored by the entry literature (auctions being an exception). For example, if firms are privately informed about their marginal costs of production, facing a potential competitor with a lower marginal cost decreases a firm's expected profit. The magnitude of this decrease depends on the firms' realized marginal costs, their degree of product substitutability, and the number of entrants. We enrich the set of models available to applied researchers by including these environments.

In the theoretical literature on market entry, Mankiw and Whinston (1986) study welfare in a symmetric model under complete information. Closer to our approach, Brock and Durlauf (2001) examine a symmetric environment in which privately-informed agents choose a binary action. Our modeling shares the idea that both the action and type of an agent affect the payoffs of other agents, but differs in that entry decisions are strategic *substitutes* and in that we allow for asymmetric agents. There is a large literature of costly entry into auctions. Levin and Smith (1994) examine a symmetric scenario in which bidders learn their valuations after entry. Samuelson (1985) studies ex-ante symmetric bidders, which are perfectly and privately informed about their valuations before entry. In Ye (2007), bidders are partially informed at the moment of entry and fully learn their valuations after entry occurs. Our framework embeds these informational environments, as a firm's private information might correspond to its type or to a signal about its type. We provide a general equilibrium characterization and an equilibrium uniqueness result to a wider class of entry models.

In the empirical literature, Bresnahan and Reiss (1990, 1991) and Berry (1992) develop the first empirical models of market entry that explicitly accounted for the strategic interaction between post-entry market competition and firms' entry decisions. Under complete information, the entry game often contains multiple equilibria. Tamer (2003) show that, without further assumptions, multiple equilibria can lead to set, rather than point, identification.[1] Using numerical methods, Seim (2006) show that private information may solve the problem of equilibrium multiplicity. Berry and Tamer (2006), however, construct examples of multiple equilibria under private information, raising the question of when uniqueness can be achieved. We contribute to this discussion by identifying a sufficient condition guaranteeing equilibrium uniqueness in entry games with private information.

The importance of allowing for private information in entry models lies beyond the possibility of solving the multiple equilibria problem. Using complementary methodologies, Grieco (2014) and Magnolfi and Roncoroni (2021) test and reject the hypothesis that firms possess complete information at the moment of entry. Furthermore, compared to models that allow for private information, they show that assuming complete information delivers model estimates that can lead to qualitatively different predictions. Roberts and Sweeting (2013, 2016) provide evidence of selection at the moment of entry, which cannot be accounted for by complete information models. Finally, when firms receive signals about their true type, we can observe behavior consistent with ex-post regret; i.e., entry into an ex-ante profitable market but with negative observed outcomes. This type of outcomes is incompatible with complete information models.

The article is organized as follows. Section 2 introduces the model, discusses its properties, and provides examples illustrating its scope. Section 3 introduces and discusses the notions of firm strength and herculean equilibrium. Our main results are presented in Section 4, which shows that the existence of a herculean equilibrium is guaranteed and provides a sufficient condition for when the herculean equilibrium is the unique equilibrium of the game. Finally, Section 5 concludes. All the proofs are relegated to Appendix A.

[1] Sweeting (2009) shows that multiplicity can help with the model's identification when the values of staying 'in' or 'out' of the market are both endogenously determined. De Paula and Tang (2012) show that multiplicity can be used to infer the signs of strategic interactions.

2 A Model of Market Entry

2.1 The Baseline Model

Set up. Consider n firms simultaneously deciding on whether to enter a market. Firms are privately informed about their *type* v_i (a scalar), summarizing the firm's information about its profitability upon entering the market.[2] Firm i's *post-entry* profit depends on the entry decision of every firm, firm i's type, and the types of other entrants. The value v_i distributes according to F_i; a continuously differentiable atomless distribution, with full support on $[a, b]$ where $a, b \in \overline{\mathbb{R}}$ (the extended reals). The draws of types are independent across firms but not (necessarily) identically distributed.

Let $e_i \in \{0, 1\}$ be an indicator function taking the value 1 when firm i enters the market, and 0 otherwise. Denote by $e = (e_1, e_2, \ldots, e_n)$ the vector of *ex-post* entry decisions, we also refer to e as the (realized) market structure. Let $E_i = \{e : e_i = 1\}$ be the set of market structures in which firm i enters. For a given market structure e, define $I(e) = \{i : e_i = 1\}$ to be the set of firms participating in market e. Similarly, define $I_i(e) = \{j \neq i : e_j = 1\}$ and $O_i(e) = \{j \neq i : e_j = 0\}$ to be the set of i's competitors that are *in* and *out* of the market under structure e, respectively. Denote by $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ the vector with the realized types of every firm. Similarly, v_{-i} represents the realized types of every firm except firm i and $v_e \equiv (v_k)_{k \in I(e)}$ the vector of realized types for every firm participating in market structure e. Finally, for any market structure in which firm i enters ($e \in E_i$), $e \setminus i$ denotes the market structure e without firm i in it. Similarly, for a market structure in which firm j stays out ($e \notin E_j$), $e \cup j$ denotes the market structure e but with firm j participating in it.

With a slight abuse of notation, let $\pi_i(v_e)$ be a real valued function representing firm i's *post-entry* profit when the realized market structure is e and the realized types are v_e. By adopting this notation, we implicitly assume that the types of non-entrants are payoff irrelevant. To illustrate the workings of the notation observe that $\pi_i(v_i)$ represents firm i's post-entry profit when i is the sole entrant and draws v_i. Similarly, $\pi_i(\mathbf{v}) = \pi_i(v_i, v_{-i})$ represents i's profit when every firm enters the market and the vector of realized types is given by $\mathbf{v}$. We normalize the payoff of a non-entrant to zero. Finally, we assume that $\pi_i(v_e)$ is continuous, integrable (with

[2] Although v_i could also represent an informative *signal* about firm i's true type, to ease exposition we simply refer to it as type.

finite expectation) in each dimension of v_e, and differentiable in v_i. We denote such derivative by $\pi_i'(v_e)$.

The timing of the game is as follows. Before making any entry decision, each firm privately observes v_i. After observing v_i and without observing v_{-i}, each firm independently and simultaneously decides whether to enter the market. After entry decisions are made, market structure e is realized and each firm entering the market gets a payoff $\pi_i(v_e)$. The tuple $(\pi_i, F_i)_{i=1}^n$—which includes the number of potential entrants n—is commonly known by every potential entrant.

Main assumptions. For a given market structure e in which firm i enters the market ($e \in E_i$), firm i's profit function satisfies the following four properties.

A1 (**Monotonicity**)**:** The profit function $\pi_i(v_e)$ is weakly increasing in v_i and strictly increasing if firm i is the sole entrant.

A1 gives economic meaning to the firms' type. Upon entering the market, and regardless of the realized market structure e, firm i' profit increases in v_i. In terms of traditional competition models, a higher v_i can represent a lower marginal cost of production, a lower entry cost, a higher product quality, a better managerial ability, or a higher valuation for a good in an auction.[3]

A2 (**Competition**)**:** For each $j \in I_i(e)$, $\pi_i(v_e)$ is weakly decreasing in v_j. For each $j \in O_i(e)$, $\pi_i(v_e) \geq \pi_i(v_{e \cup j})$.

A2 concerns the impact of competition on profits. It states that competition weakly decreases profits. In particular, $\pi_i(v_e)$ decreases with entry or when faced with more productive (higher type) competitors.

A3 (**Substitutes**)**:** For every firm j, there exists a market structure e in which j does not participate ($j \in O_i(e)$) such that $\pi_i(v_e) > \pi_i(v_{e \cup j})$ for $v_j \geq \underline{v}_j$.

A3 says that for every potential entrant, there exists a market structure for which that entrant is a substitute. This is a minimal assumption about the degree of substitutability among firms, as it does not require that every pair of firms to be direct competitors. The impact that firm j has on i, for instance, can be through affecting the equilibrium behavior of other firms participating under market structure e.[4] Most models of competition, however, satisfy a stronger version of A3 in

[3]The weakly increasing assumption accommodates environments with intense rivalry, such as price competitions with homogeneous goods or second-price auctions, in which competition might preclude higher types to increase ex-post payoffs.

[4]Consider a Hotelling model in which firms 1 and 2 are located at each end of the street. If transport costs are high, entry by 1 does not affect 2's profit if they are the only entrants. Entry by 1 can harm 2, however, if there is a third firm located in between 1 and 2.

which entry by firm j has the potential to decrease firm i's profit under any market structure e.[5]

Before stating the next assumption, define $\phi(v_e) = \prod_{j \in I(e)} f_j(v_j)$ to be the joint density of types of every firm participating in market structure e.

A4 (Costly and interior entry): There exist values $\underline{v}_i < \overline{v}_i$ in the interior of the support of $F_i(v_i)$—i.e., $\underline{v}_i, \overline{v}_i \in (a, b)$—such that:

(i) $\pi_i(\underline{v}_i) = 0$ and,

(ii)
$$\int_{[a,b]^{n-1}} \pi_i(\overline{v}_i, v_{-i}) \phi(v_{-i}) d^{n-1} v_{-i} = 0,$$

where the multiple integral is over each of the $n-1$ dimensions of v_{-i}.

A4 concerns the nature of the entry problem. Condition (i) simply states that entry is costly. Firms need a sufficiently good type, $\underline{v}_i > a$, to be willing to enter the market as the sole entrant. Jointly with assumption A2, A4 implies that, when $v_i < \underline{v}_i$, firm i would never choose to enter the market under any market structure. That is, the value $\underline{v}_i$ represents the minimal type required to enter the market.[6] Condition (ii) states that any firm will enter the market if its type is sufficiently high. In particular, there exists a value $\overline{v}_i < b$ such that drawing $v_i > \overline{v}_i$ ensures entry, even if every potential competitor always enters the market. The assumption that $[\underline{v}_i, \overline{v}_i] \subset (a, b)$ guarantees that every equilibrium is interior; i.e., no firm optimally chooses either to never or to always enter the market.

Strategies and equilibrium. A *cutoff* strategy for firm i is a threshold x_i such that firm i enters the market whenever $v_i \geq x_i$ and stays out otherwise. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector of cutoff strategies. Firm i's expected profit of entering the market with type v_i and facing opponents playing the cutoff strategies $\mathbf{x}_{-i}$ is

$$\Pi_i(v_i, \mathbf{x}_{-i}) \equiv \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \int_{(x_j)_{j \in I_i(e)}}^{b} \pi_i \left(v_i, v_{e \setminus i} \right) \phi(v_{e \setminus i}) d^{n_e - 1} v_{e \setminus i} \right\} \quad (1)$$

[5] In our analysis, we can dispense of A3. If firm j is never a substitute of i, however, it does not affect i's equilibrium behavior and becomes irrelevant to determine whether i's behavior is consistent with a unique equilibrium. We adopt A3 for brevity in the proofs.

[6] Notice that A4 is compatible with firms making non-negative variable profit upon entry. It simply states that the entry cost is sufficiently high, so that the non-negative variable profit cannot overcome the entry cost for low types.

where $n_e = \sum_j e_j$ is the number of entrants in market structure e.[7] Firm i's expected profit from entry consists of the probability-weighted sum of i's expected payoffs in each market structure in which firm i participates, i.e., $e \in E_i$. The payoffs in market structure e correspond to the expectation of $\pi_i(v_i, v_{e\setminus i})$ over the realizations of types of the competitors in e, $v_{e\setminus i}$. This expectation takes into account that competitor $j \in I_i(e)$ only enters the market if its valuation is above its cutoff x_j. Appendix B shows that (1) is strictly increasing in firm i's type v_i. Firm i's expected profit also increases in an opponent's cutoff, x_j; a higher entry cutoff lowers the competitor's probability of entry, inducing firm i to face less competition.

A Bayesian *equilibrium* is a vector of cutoff strategies $\mathbf{x}$ such that, for every firm i, $\Pi_i(\mathbf{x}) = 0$; i.e., in equilibrium, when opponents play their equilibrium cutoff strategy $\mathbf{x}_{-i}$, firm i is indifferent to enter the market when draws a type equal to its equilibrium cutoff, x_i. Online Appendix C shows that an equilibrium always exists and that *every* equilibrium is in cutoff strategies; i.e., our focus on cutoff strategies is without loss of generality. We denote the partial derivative of $\Pi_i(\mathbf{x})$ with respect to x_i by $\Pi_i'(\mathbf{x})$.

2.2 Selective Entry

Recent empirical work on market entry has shown the need to account for selection in the entry process (c.f. Roberts and Sweeting, 2013). Selective entry occurs when firms are partially informed about their type before making their entry decisions and only become fully informed after costly entry has occurred.[8] Our framework accommodates selective entry models by adding a weak affiliation assumption between firms' private information and their true type.

Let $F_i(v_i, \theta_i)$ be firm i's joint cumulative distribution of *signals* v_i and *types* θ_i with support on $[a, b] \times [c, d]$ with $c < d$. The distributions F_i are independent across firms and not necessarily identically distributed. Before making their costly entry decisions, a firm privately observes its signal v_i, which allows it to make inferences about its true type, θ_i. Firms learn their type after entering the market. Let $F_i(v_i) = \int_c^d F_i(v_i, s)ds$ and let $F_i(\theta_i|v_i) = F_i(v_i, \theta_i)/F_i(v_i)$ be the CDF of θ_i

[7] The following (standard) notation is used throughout: $\sum_\emptyset k = 0$, $\prod_\emptyset k = 1$, and $\int_\emptyset k dx = k$.

[8] Selective entry allows firms to have *ex-post* regret. For instance, complete and private information models cannot account for a market with a sole entrant (the most profitable outcome under our assumptions) having negative post-entry profit. This outcome, however, is feasible under selective entry.

conditional on v_i.

A5 (**Affiliated Signals**)**:** For $v_i' > v_i$, $F_i(\theta_i|v_i') < F_i(\theta_i|v_i)$ for all θ_i.

A5 states that higher signals lead to a higher expected type in terms of first order stochastic dominance (FOSD) (c.f. Marmer *et al.*, 2013; Gentry and Li, 2014). Let $\hat{\pi}_i(\theta_e)$ be firm i's profit under market structure e and the vector of types for every participating firm is $\theta_e = (\theta_j)_{j \in I(e)}$. Then, we re-interpret $\pi_i(v_e)$ as

$$\pi_i(v_e) = \int_c^d \hat{\pi}_i(\theta_e) \prod_{k \in I(e)} f_k(\theta_k|v_k) d^{n_e}\theta_e$$

where the integral is across the n_e dimensions of θ_e. Given the properties of FOSD, it is straightforward to see that if the profit function $\hat{\pi}_i(\theta_e)$ satisfies analogous conditions to A1-A4, then $\pi_i(v_e)$ would also satisfy A1-A4, and the results presented below go through.

2.3 Model Discussion

An important feature of the model is that it allows for general forms of publicly observed *ex-ante* firm heterogeneity. Firms can differ in their distribution of types F_i. The model also allows for firm heterogeneity in the profit function $\pi_i(v_e)$; i.e., even if firms face the same draws of types, profits might be different. Heterogeneity in profits may come from firms having different entry costs, production costs, production capacities, product characteristics, contracts with suppliers, or different geographic locations. The heterogeneity in profitability may also be due to the way firms compete *after* entry has occurred. Firm heterogeneity can accommodate the existence of dominant firms or a predetermined order of play in the *post-entry* market, such as, competition à la Stackelberg. The proposed framework can also accommodate firms receiving aggregate or idiosyncratic random shocks after entry. In such cases, $\pi_i(v_e)$ would correspond to the *expected* post-entry profit. Finally, we highlight that the model can also accommodate entry into occupied markets. That is, even though $\pi_i(v_i)$ denotes the profit of a single entrant, the market may already have firms competing in it.

The proposed formulation of $\pi_i(v_e)$, however, does impose some restrictions on the nature of *post-entry* competition. First, $\pi_i(v_e)$ is a function rather than a correspondence, imposing that either the post-entry game has a unique equilibrium or, under multiplicity of post-entry equilibria, there is market consensus about

which equilibrium will be played. Second, $\pi_i(v_e)$ does not depend on the profile of cutoff strategies $\mathbf{x}$. A natural interpretation for the model is that entering firms' private information becomes public *after* entry occurs but *before* firms compete in the product market. Consequently, firms carry no beliefs about their competitors' private information to the post-entry game. Finally, our setting is consistent with cases where the type is irrelevant for post-entry strategies (e.g., firms are privately informed about their entry costs) or that no information is revealed after entry; $\pi_i(v_e)$ is only observed at the end of the game.

To see why the omission of $\mathbf{x}$ in $\pi_i(v_e)$ is restrictive, consider the case in which v_e remains private in the post-entry game but the market structure e is observed. In such scenario, firms may base their strategies in the post-entry game on their beliefs about the private information of their competitors. Through Bayesian updating, these *beliefs* would depend on the strategy profile $\mathbf{x}$ and the observed market structure e, making it part of the post-entry profit function. Although important, the analysis of such models lies outside of the scope of this article.

2.4 Examples

We present several examples used in applied work that satisfy our assumptions to illustrate the relevance of our results.

Example 1 (Linear model)**.** We say that the profit function is linear when $\pi_i(v_e) = \pi_i(e) + v_i$, where $\pi_i(e)$ is firm i's profit under market structure e, which does not depend on the realization of v_e. In this scenario, the type of a competitor j does not directly affect firm i's payoff. Firm j's type does, however, affect firm i indirectly through j's entry decision. The most common interpretation of the linear model is that $-v_i$ represents firm i's entry cost. Below we discuss three variations used in the empirical literature:[9]

(a) **Heterogeneous competition:** In studying entry into the video retail industry, Seim (2006) used a linear model of the form

$$\pi_i(v_e) = \eta_i + \sum_{j \in I_i(e)} \delta_{ij} + v_i$$

[9]In addition to the articles mentioned in the examples, other work involving linear entry models with private information include: Aguirregabiria and Mira (2007); Bajari *et al.* (2007); Pakes *et al.* (2007); Pesendorfer and Schmidt-Dengler (2008); Sweeting (2009); Aradillas-Lopez (2010); Bajari *et al.* (2010); De Paula and Tang (2012); Vitorino (2012); Mazzeo *et al.* (2016).

where η_i is a scalar summarizing both market and firm characteristics.[10] The term $\delta_{ij} > 0$ captures how entry by firm j affects i's profit. This model captures different degrees of substitution among firms, as entry by different competitors may have a different impact on firm i's profit. This differential effect, however, is independent of the number of competitors entering the market.

(b) **Decreasing marginal impact of competition:** In studying entry into airlines routes, Berry (1992) studies a complete information version of the following entry model[11]

$$\pi_i(v_e) = \eta_i - \delta_i \sum_{k=1}^{n_e-1} r_i^{k-1} + v_i$$

where n_e is the number of entrants in market structure e. In this model, the impact that entry has on i's profit is independent of the entrant's identity. The model, however, captures that the marginal impact of entry is decreasing in the number of competitors. Entry by a new competitor decreases profits at a fraction $r_i \in [0, 1]$ of the previous entrant. In Example (a), as the number of entrants increases, profits diverge to $-\infty$. In this model, on the other hand, the effect of competition is bounded by $\eta_i - \delta_i/(1 - r_i) + v_i$.

(c) **Auctions with private entry cost:** Consider an auction environment in which each bidder is privately informed about its entry cost and only learns its valuation (and potentially the identity of participating competitors) after paying the entry cost (c.f. Krasnokutskaya and Seim, 2011, for the case of a first-price auction). This environment is captured by $\pi_i(v_e) = \pi_i(e) + v_i$ where $\pi_i(e)$ represents firm i's expected profit of participating in an auction under realized market structure e. Because entry costs are independent of valuations, entry strategies do not affect the bidding behavior in market structure e.

Observe that, in a scenario with $n = 2$ potential competitors, as in Grieco (2014), the three models above can be represented by $\pi_i(v_e) = \eta_i - \mathbb{I}_{j \in I(e)}\delta_i + v_i$, where $\mathbb{I}_{j \in I(e)}$ is an indicator reflecting entry by the competitor.

Example 2 (Auctions with selective entry)**.** Consider a second-price auction where bidders are partially (and privately) informed about their own valuation before making entry decisions. The valuation of bidder i is given by $\theta_i = v_i\varepsilon_i$, where

[10] Observe that, although the term η_i is commonly known by the firms', an econometrician may not observe some elements in η_i. Typically, $\eta_i = X_i\beta_i + \zeta_i$ where X_i is a vector of observed firm and market characteristics and ζ_i is heterogeneity unobserved by the econometrician.

[11] Berry (1992) instead uses a logarithmic version of the model: $\pi_i(v_e) = \eta_i - \delta_i \ln(n_e - 1) + v_i$.

the *signal* v_i is observed before the participation decision is made and the *noise* $\varepsilon_i \sim G_i$, which is independent from v_i, is observed after paying the participation cost $K_i > 0$ but before submitting a bid.[12]

For a given realization of signals and market structure v_e, define $\Phi(s, v_e) = \prod_{j \in I(e)} G_j(s/v_j)$ to be the probability that every firm participating in market structure e obtains a valuation less than s. Then, if $r \geq 0$ is the reserve price of the auction, the (expected) payoff of a firm that participates under v_e is:

$$\pi_i(v_e) = \int_{r/v_i}^{b} \left(\int_{-\infty}^{v_i \varepsilon_i} (v_i \varepsilon_i - \max\{r, s\}) d\Phi\left(s, v_{e \setminus i}\right) \right) dG_i(\varepsilon_i) - K_i.$$

A participating firm i pays the entry cost K_i and, given the signal v_i, bidder i values the good by $\theta_i = v_i \varepsilon_i$, which distributes according to $G_i(\varepsilon)$. Participating firms submit a bid equal to their valuation only if they value the good more than the reserve price r. Bidder i obtains the good when it is the highest valuation firm. The distribution of the highest valuation among i's opponents is $\Phi(s, v_e)$. It can be readily checked that this model satisfies assumptions A1-A5. Variations of this model have been studied by Roberts and Sweeting (2013, 2016), Gentry and Li (2014), and Sweeting and Bhattacharya (2015).

Example 3 (Oligopolistic competition)**.** Our framework also accommodates traditional forms of oligopolistic competition. For instance, entry into a market in which firms compete in prices under differentiated products can be modeled with a logit demand, such as

$$\pi_i(v_e) = (p_i - c_i) S_i(v_e) \mathcal{M} - K_i, \quad \text{where} \quad S_i(v_e) = \frac{D_i}{D} \frac{D^\lambda}{(1 + D^\lambda)}$$

is firm i's market share, which is determined by $D_i = \exp((\eta_i + v_i - \alpha p_i)/\lambda)$ and $D = \sum_{j \in I(e)} D_j$. The model is described by the market size $\mathcal{M}$ as well as firm i's entry costs K_i, marginal cost c_i and product/market characteristics, η_i. The parameter α captures consumers' tastes and $\lambda \in [0, 1]$ captures the strength of the consumers' outside option. Every potential entrant commonly knows all these parameters. The vector of equilibrium prices, $p_e = (p_j)_{j \in I(e)}$, and market shares are a function of the realized market structure e and the draws of types of the entrants,

[12] If we further assume that $\mathbb{E}(\varepsilon_i) = 1$, the signal v_i becomes an unbiased predictor of the valuation θ_i. That is, for a given realization of v_i, the expected valuation is given by $\mathbb{E}(\theta_i | v_i) = v_i \int_{-\infty}^{\infty} \varepsilon_i dG_i(\varepsilon_i) = v_i$.

v_e. In this scenario, v_i represents a product characteristic (such as quality) that is privately known before entry decisions are made, but becomes publicly known after entry occurs. This model satisfies assumptions A1-A4. Complete information versions of this model (i.e., not incorporating the v_i term) have been studied by Ciliberto *et al.* (2020) in the context of entry and by Bresnahan (1987), Berry (1994), and Berry *et al.* (1995) when the number of competitors is exogenous.

3 Strength and Herculean Equilibrium

This section introduces two key concepts: firm *strength* and *herculean* equilibrium. Strength uses the game fundamentals—$(F_i, \pi_i)_{j=1}^n$—to rank firms according to their ability to endure competition. We use strength to identify the equilibrium that remains when the game has a unique equilibrium: the herculean equilibrium. Identifying the herculean equilibrium is the starting point to develop our sufficient condition for equilibrium uniqueness.

Definition (Strength)**.** Let $\sigma_i(v) \equiv \Pi_i(v, \ldots, v)$, where $\Pi_i(\mathbf{x})$ is given by (1). The *strength* of firm i is the unique number $s_i \in \mathbb{R}$ that solves $\sigma_i(s_i) = 0$; i.e.,

$$\sigma_i(s_i) = \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi_i \left(s_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} = 0. \tag{2}$$

We say that firm i is *stronger* than firm j if $s_i < s_j$.

Lemma 1. *$\sigma_i(s)$ is strictly increasing and crosses zero once.*

The strength of firm i is the unique cutoff s_i that is a best response to every other competitor playing the same cutoff strategy s_i. Intuitively, strength ranks firms according to their ability to endure competition. A lower cutoff strategy for firm i implies that the firm is more likely to enter the market, as it enters for lower types. Similarly, a lower cutoff strategy by competitors implies that firm i is more likely to face competition. Firm i being stronger than firm j ($s_i < s_j$) indicates that firm i, despite facing more competition than j, is more likely than j to enter the market. Lemma 1 shows that strength is well defined, as it assigns a unique scalar s_i to each firm i and, therefore, delivers a complete ranking of the firms. Strength motivates our next definition.

Definition (Herculean Equilibrium)**.** An equilibrium is called *herculean* if equilibrium cutoffs are ordered by strength, with stronger firms playing *lower* cutoffs.

Intuitively, because stronger firms are more able to endure competition, they should be more inclined to enter the market than weaker firms. Therefore, an equilibrium in which cutoffs are ordered by strength should naturally emerge in entry games. Figure 1 illustrates this intuition graphically in an environment with two asymmetric entrants. The functions $\sigma_1(s)$ and $\sigma_2(s)$ define the strength of each firm. By Lemma 1, the functions $\sigma_i(s)$ are strictly increasing, crossing the horizontal axis once, at s_1 and s_2, respectively. In this scenario, firm 1 is stronger than firm 2, as $s_1 < s_2$.

[Figure 1 around here]

We use strength to construct a herculean equilibrium in this asymmetric scenario. Let $b_i(x)$ be firm's i best response to a cutoff x. Because expected profits are increasing in the firm's type, firm i's best response decreases in x; i.e., when faced with less competition (higher x), firm i is willing to enter at a lower type. Assume that firm 2 plays the cutoff s_2. Because best responses are decreasing, we have $b_1(s_2) < b_1(s_1) = s_1$ (see Figure 1), where the equality follows from the definition of strength. In turn, if firm 1 plays $b_1(s_2) < s_2$, we have $b_2(b_1(s_2)) > b_2(s_2) = s_2$, where strength was again used in the equality. Consequently, we have shown that $b_1(s_2) < s_1 < s_2 < b_2(b_1(s_2))$. Continuing with these iterated best responses, we can construct monotonic sequences. By assumption A4, these sequences are bounded, converging to an equilibrium $x_1 < x_2$ in which cutoffs are ordered by strength—an herculean equilibrium. The previous argument does not preclude the existence of multiple herculean or non-herculean equilibria. The next section shows the existence of herculean equilibria more broadly and provides a sufficient condition guaranteeing equilibrium uniqueness.

4 Existence and Uniqueness

In this section, we show the existence of a herculean equilibrium in three settings that are commonly used in applied work. As a consequence of this result, if the entry game has a unique equilibrium, it must be herculean. We establish a sufficient condition guaranteeing equilibrium uniqueness and provide examples illustrating how the proposed condition can be used in practice. The next definition is instrumental for the sufficient condition.

Definition (Expected profit loss)**.** For any vector of cutoff strategies $\mathbf{x}$ define the *expected profit loss* that entry by firm j inflicts on firm i to be

$$\Delta_{i,j}(\mathbf{x}) = \sum_{e \in E_i \setminus E_j} \left\{ \left(\prod_{k \in O_i(e)} F_k(x_k) \right) \int_{(x_k)_{k \in I_i(e)}}^{\infty} \delta_{i,j}(x_i, x_j, v_{e\setminus i}) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} > 0 \quad (3)$$

where $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) = \pi_i(x_i, v_{e\setminus i}) - \pi_i(x_i, x_j, v_{e\setminus i})$ is firm i's profit loss inflicted by firm j entry in market structure e when the realized types of the other entrants is $v_{e\setminus i}$. By assumption A2, we know that $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) \geq 0$.

The expected profit loss captures the decrease in profit that firm i experiences when firm j marginally decreases its entry cutoff x_j and firm i draws type x_i.[13] A small change in x_j only affects firm i's expected profit at firm j's pivotal draw, $v_j = x_j$. At that draw, firm j's entry occurs, inducing firm i to lose $\Delta_{i,j}(\mathbf{x})$. The expected profit loss will help us characterize how firm j's best response to firm i's behavior affects firm i's profitability. As we shall see below, if firm j's best response has a bounded effect in firm i's profitability (and reciprocally), the entry game has a unique equilibrium. Although assumption A2 only implies that $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) \geq 0$, together with assumption A3 we have that $\Delta_{i,j}(\mathbf{x}) > 0$.

4.1 Ex-ante Symmetric Games

We now study existence and uniqueness of herculean equilibrium in the context of ex-ante symmetric firms; i.e., firms that have the same ex-ante characteristics but different ex-post outcomes due to particular realizations of the firms' type. Symmetric entry games have been studied, for example, by Bresnahan and Reiss (1990, 1991), in the context of complete information, and by Brock and Durlauf (2001), Sweeting (2009), and Grieco (2014) in the context of private information.

We say that firm i's profit function is *anonymous* if, for every market structure $e \in E_i$, firm i's profit function does not depend on the identities of the entrants; i.e., $\pi_i(v_e) = \pi_i(v_i, \mathbf{v}_{n_e - 1})$ where $\mathbf{v}_r$ is an r-dimensional vector of realized types and n_e is the number of entrants in e. An entry game is called *symmetric* when every firm has the same distribution of types, $F_i(v_i) = F(v_i)$, and profit functions are anonymous and symmetric, $\pi_i(v_e) = \pi(v_i, \mathbf{v}_{n_e - 1})$.

Proposition 1. *In symmetric entry games, there exists a unique herculean equilibrium, where a firm's cutoff is given by its strength. That is,* $x_i = s$ *for every*

[13] Observe that $\Delta_{i,j}(\mathbf{x})$ does not integrate over the i and j dimensions.

firm i, where s is the unique number that solves

$$\sigma(s) = \sum_{r=0}^{n-1} \left\{ \binom{n-1}{r} F(s)^{n-1-r} \int_s^b \pi(s, \mathbf{v}_r) \phi(\mathbf{v}_r) d^r \mathbf{v}_r \right\} = 0.$$

Moreover, the herculean (i.e., symmetric) equilibrium is the only equilibrium of the game if the condition

$$\frac{f(x_i)}{F(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1, \tag{4}$$

holds for any pair of firms i and j, and for every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}, \overline{v}]$.

In symmetric entry games, there always exists a unique herculean equilibrium, as, under symmetry, the herculean cutoffs coincide with the firms' strength and, by Lemma 1, strength is uniquely defined. Proposition 1's main contribution is to provide a sufficient condition under which no asymmetric equilibria exists. The sufficient condition (4) is a stability condition. It guarantees that the gain in firm i's expected profit induced by an increase in its own cutoff x_i cannot be overcome by any best response by its competitors'.

To illustrate the main steps of the proof, consider the case with two symmetric firms. Let $b_2(x_1)$ be firm 2's unique best response to x_1, $b_2(x_1)$ is decreasing in x_1. Observe that every equilibrium of the game satisfies $\Pi_1(x_1, b_2(x_1)) = 0$. Consequently, if $\Pi_1(x_1, b_2(x_1))$ is strictly increasing in x_1, the function crosses zero once and a unique equilibrium exists. Let $\mathbf{x} = (x_1, b_2(x_1))$, differentiating $\Pi_1(x_1, b_2(x_1))$ with respect to x_1 we obtain

$$\frac{d\Pi_1(\mathbf{x})}{dx_1} = \Pi_1'(\mathbf{x}) + b_2'(x_1) \frac{\partial \Pi_1(\mathbf{x})}{\partial x_2} > \Pi_1'(\mathbf{x}) - \frac{f(x_1)}{F(x_1)} \Delta_{1,2}(\mathbf{x}) > 0.$$

The first step of the uniqueness proof consists of providing a bound to the interaction between the slope of firm 2's best response and its impact on firm 1's profit. Sufficient condition (4) plays an important role in establishing this bound, as is used to establish the first inequality. The second inequality follows from condition (4) as well.

Computing the equilibrium is not necessary to check whether condition (4) holds, as it only makes use of the information given in the fundamentals of the game. As shown below, depending on the application, condition (4) might require only a simple calculation. Because of symmetry, the condition only needs to hold

for any pair of potential firms. It is sufficient to check it where entry is feasible—i.e., $x_k \in [\underline{v}, \overline{v}]$—as deviations outside this range are always outside the equilibrium path.

The next set of examples illustrate how sufficient condition (4) works in practice. We discuss some of its properties and how it changes with competition. Moreover, we use it to show that the empirical model in Grieco (2014) has a unique equilibrium.

Example 4 (Symmetric linear model)**.** Consider the linear model of Example 1(b) in a symmetric environment. In this scenario, the post-entry profit of a given firm i becomes

$$\pi_i(v_e) = \eta - \delta \sum_{k=1}^{n_e-1} r^{k-1} + v_i.$$

Under symmetry, the model embeds Example 1(a) when $r = 1$. Sufficient condition (4) holds if, for $x_i \in [\underline{v}, \overline{v}]$, the following inequality is satisfied (see Online Appendix D for a step-by-step derivation)

$$\frac{f(x_i)}{F(x_i)} < \frac{1}{\delta F(\overline{v}) \left(r + F(\overline{v})(1-r)\right)^{n-2}}, \tag{5}$$

where $\underline{v} = -\eta$ and $\overline{v} = \delta(1-r^{n-1})/(1-r) - \eta$. That is, the reversed hazard rate of F needs be bounded above by the inverse of $\delta F(\overline{v}) \left(r + F(\overline{v})(1-r)\right)^{n-2}$. We use condition (5) to illustrate relevant properties:

(a) **Log-concave distributions.** When F is log-concave, its reversed hazard rate $f(x_i)/F(x_i)$ is decreasing in x_i.[14] Consequently, the sufficient condition (5) reduces to $f(\underline{v})/F(\underline{v}) < \left(\delta F(\overline{v}) \left(r + F(\overline{v})(1-r)\right)^{n-2}\right)^{-1}$. For instance, if types distribute type-I extreme value (as in Seim, 2006)[15] and $r = 1$, it can be readily checked that the sufficient condition for uniqueness becomes $\eta + \ln(\delta) < \exp(\eta - (n-1)\delta)$; a restriction to the parameters of the model. Figure 2a illustrates this restriction for different number of potential entrants. The area outside the curves represents the combination of parameters delivering a unique equilibrium. Inside the curves, the game might have a unique or multiple equilibria.

[14] If $G(x) = \ln(F(x))$ is concave, then $G''(x) = d(f(x)/F(x))/dx < 0$. Examples of log-concave distributions include normal, exponential, extreme value, logistic, and gamma.

[15] A type-I extreme value distribution with location parameter 0 and scale parameter λ is given by $F(v) = \exp(-\exp(-v/\lambda))$. Then, its inverted hazard rate is given by $f(v)/F(v) = \exp(-v/\lambda)/\lambda$, which is decreasing in v. This distribution is called standard when $\lambda = 1$.

[Figure 2 around here]

(b) **Uniqueness and competition.** Continuing with the previous example, we now explore the effect of competition on sufficient condition (5). In the linear model, competition manifests through two channels: the number of potential competitors, n, and the decreasing marginal impact of entry, r. For the latter, observe that an increase in r decreases the right hand side of (5)—making the condition harder to satisfy—directly through r (entry has more impact) and indirectly by increasing $\overline{v}$ (enlarging the set of possible deviations). Consistent with the traditional logic of entry games, the prospects of facing multiple equilibria increase with the gains from coordinating entry; i.e., when the profit losses from competition become large. As a consequence, when comparing both panels of Figure 2, we can see that an increase in the marginal impact of entry, r, shrinks the set of parameters that deliver a unique equilibrium.

Increasing the number of competitors, n, also increases the set of possible deviations $\overline{v}$. In contrast, a larger n has the countervailing effect of increasing the expected number of entrants, captured by $(r+F(\overline{v})(1-r))^{n-2}$. When the marginal effect of entry is decreasing (i.e., when $r < 1$), a larger number of entrants decreases the expected profit loss (3) that firm j inflicts on firm i. This makes firm i less susceptible to entry, increasing the set of parameters for which uniqueness occurs. The interaction between these effects makes the set of parameters satisfying (5) to change non-monotonically with n (see Figure 2b). This stands in contrast to the scenario with a constant marginal effect of entry ($r = 1$). There, only the effect of increasing $\overline{v}$ remains, making the set of parameters satisfying (5) shrink with the number of potential entrants n (see Figure 2a). In the limit, as n becomes unboundedly large, the restriction for $r = 1$ becomes the tightest and equal to $\eta + \ln(\delta) < 0$. In contrasts, when $r < 1$, the model always has a unique equilibrium, as the right hand side of (5) goes to infinity.

(c) **Equlibrium multiplicity and uniqueness under Normality.** Suppose types distribute $N(0, \sigma)$, which is log-concave. Berry and Tamer (2006) observe that, in a game with two entrants ($n = 2$) and under the assumption $\delta > \eta$, the entry game has multiple equilibria when it converges to a complete information game ($\sigma \to 0$) and has a unique equilibrium when the private information dominates ($\sigma \to \infty$). We can use sufficient condition (5), $\delta F(\overline{v}) f(\underline{v})/F(\underline{v}) < 1$, to provide a tighter characterization. Suppose, for instance, that $\eta = 0$ and $\delta = 1$

(so that, $\underline{v} = 0$ and $\overline{v} = 1$). Then, because both $f(0)/F(0) = 2/\sqrt{2\pi\sigma^2}$ and $F(\overline{v})$ are decreasing in σ, there is a threshold $\hat{\sigma} = 0.7298$ such that $\sigma > \hat{\sigma}$ guarantees equilibrium uniqueness.

For example, if $\sigma = 1/4$ the game has three equilibria. The herculean equilibrium, which is symmetric and given by the cutoff strategy $x_i = 0.2055$, and two asymmetric equilibria, given by $x_i = 0.041$ and $x_{3-i} = 0.435$, for $i \in \{1, 2\}$. Similarly, if $\sigma = 1$, the game has a unique equilibrium given by $x_i = 0.3596$.

(d) **A concrete application.** When studying entry of supercenters into rural grocery markets, Grieco (2014) estimates a symmetric incomplete information model under the assumption that $v_i \sim N(0, 1)$ and two potential entrants ($n = 2$). In the smallest market, where coordination among entrants is more relevant, the model estimates are given by $\eta = -3.838$ and $\delta = 0.851$.[16] Using the log-concavity property of the normal distribution, in conjunction with the model estimates, sufficient condition (5) becomes $\delta F(\overline{v}) f(\underline{v})/F(\underline{v}) = 10^{-4} < 1$. We can conclude that the equilibrium is unique.

4.2 Two Groups of Firms

We now extend our results to games in which entrants can be divided into two groups according to their public characteristics. Within each group, firms are ex-ante symmetric. Across groups, however, firms can differ in their distribution of types and profit functions. In applied work, models of two groups of entrants have been used, for example, to study the timberwood industry (mills and loggers) by Athey *et al.* (2011) and Roberts and Sweeting (2013, 2016) as well as to study in highway procurement auctions (favored and non-favored bidders) by Krasnokutskaya and Seim (2011). The two-group structure may arise naturally in applications where firms can be divided in incumbents and entrants, high and low quality firms, local and international producers, discount and traditional retailers, or legacy and low-cost airlines, among other examples.

Formally, let G_g be the set of firms belonging to group $g \in \{1, 2\}$. Group g consists of $n_g \in \mathbb{N}$ potential entrants (so that, $n_1 + n_2 = n$) described by the pair (π_g, F_g). Let $g(i)$ be the group of firm i. We assume that profits are *symmetric* and *anonymous* within a group. That is, firm i's profit under market structure e is now

[16] See Table 7, page 329: $\eta = \mu_0 - \mu_4 = -1.222 - 2.158 = -3.838$. Condition (4) also holds for every other specification in the paper.

equal to $\pi_i(v_e) = \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k)$ where r and k are the number of entrants, other than i, from group $g(i)$ and $-g(i)$, respectively. The vectors $\mathbf{v}_r$ and $\mathbf{v}_k$ represent the draws of valuations of such entrants.

Because firms are within-group symmetric, firms in the same group have equal strength. A herculean equilibrium, thus, consists of group-symmetric strategies in which the strongest group plays the lowest cutoff. To formally characterize a group-symmetric equilibrium, define $\varphi_g(\mathbf{v}_r) = \prod_{j=1}^{r} f_g(v_j)$ to be the probability density that r firms belonging to group g draw the vector $\mathbf{v}_r$. For a pair of cutoffs $\hat{\mathbf{x}} = (x^1, x^2)$ describing group-symmetric strategies by the opponents, firm i's expected profit of entering the market with a draw of v_i, when there are r and k entrants, other than firm i, from group $g(i)$ and $-g(i)$, is given by

$$\mathbb{E}[\pi_i(v_i, r, k)|\hat{\mathbf{x}}] = \int_{x^1}^{b} \left(\int_{x^2}^{b} \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k) \varphi_{-g(i)}(\mathbf{v}_k) d^k \mathbf{v}_k \right) \varphi_{g(i)}(\mathbf{v}_r) d^r \mathbf{v}_r$$

where the integrals are over the r and k dimensions of $\mathbf{v}_r$ and $\mathbf{v}_k$. Then, when faced with group-symmetric strategies $\hat{\mathbf{x}}$ (i.e., $x_j = x^{g(j)}$ for every firm $j \neq i$), firm i's expected profit of entering the market under valuation v_i is

$$\Pi_i(v_i, \mathbf{x}_{-i}) = \sum_{k=0}^{n_j} \left\{ \binom{n_j}{k} F_j(x_j)^{n_j - k} \left[\sum_{r=0}^{n_i - 1} \binom{n_i - 1}{r} F_i(x_i)^{n_i - 1 - r} \mathbb{E}[\pi_i(v_i, r, k)|\hat{\mathbf{x}}] \right] \right\},$$

where for ease in notation, we use i and j instead of $g(i)$ and $-g(i)$ as it leads to no confusion. The previous expression corresponds to equation (1) in the context of two groups of firms playing group-symmetric strategies. To understand the previous expression, fix a market structure in which r and k firms of group i and j participate in the market. Because there are n_j firms in group j, there are 'n_j choose k' possibilities to obtain a market structure with k competitors from group j. Each of these possibilities occur with probability $F_j(x_j)^{n_j - k}$; i.e., the probability that $n_j - k$ firms obtain a low draw and stay out of the market. Similarly, there are '$n_i - 1$ choose r' possibilities to observe r competitors from i's group, each occurring with probability $F_i(x_i)^{n_i - 1 - k}$. The expression above is, thus, obtained by summing across every possible market structure.

A pair of strategies $\hat{\mathbf{x}}$ constitutes a group-symmetric equilibrium if and only if, for each firm i, $x_i = x^{g(i)}$ and the vector of strategies $\mathbf{x}$ satisfies $\Pi_i(\mathbf{x}) = 0$. Without loss of generality, let group 1 be the strongest group. The following theorem is the main result of this subsection.

Theorem 1. A herculean equilibrium always exists. The herculean equilibrium $\mathbf{x}$ satisfies $x_1 < s_1 < s_2 < x_2$, where s_g and x_g are the strength and the equilibrium cutoff of group g. Moreover, the herculean equilibrium is the unique equilibrium of the game if these four conditions

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } j \in G_{g(i)}, \tag{6}$$

$$n_{g(j)} \frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } j \in G_{-g(i)}, \tag{7}$$

hold for any pair of firms i and j, and for every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}_{g(k)}, \overline{v}_{g(k)}]$.

Theorem 1 has two main results. First, as a herculean equilibrium always exists, the theorem shows that strength is the right notion to characterize the firms' relative competitiveness. In many empirical applications, where multiplicity of equilibria is a concern, the model estimation is based on assuming that firms play an equilibrium that is ex-ante 'intuitive' given the fundamentals of the model (c.f. Roberts and Sweeting, 2013). As we show in Example 5 below, although intuitive orders are consistent with the proposed notion of strength, they are restrictive in that many applications might not have an ex-ante 'intuitive' order. Strength, in turn, provides an order in any entry game. Theorem 1 also provides bounds on the herculean equilibrium cutoffs—$x_1 \in (\underline{v}_1, s_1)$ and $x_2 \in (s_2, \overline{v}_2)$—which speeds up numerical computation of herculean equilibria.

Second, Theorem 1 provides four conditions that need to be satisfied for equilibrium uniqueness—two conditions per group. The within-group condition (6) is analogous to the sufficient condition for uniqueness in symmetric entry games (4).[17] This condition bounds the change in profit due to deviations from firms within the same group; i.e., it guarantees that no profitable deviation from a group-symmetric strategy exists. Empirical applications usually restrict their analysis to group-symmetric strategies. Condition (6), thus, guarantees that this restriction is without loss. If an applied researcher determines that within-group *asymmetric* strategies are not relevant for the application at hand, this condition can be dispensed.

The cross-group condition (7), on the other hand, guarantees that the herculean

[17] Proposition 1 is a particular case of Theorem 1 when one of the groups has no members. We chose to split the results for clarity in the exposition and because the symmetric model has value on its own.

equilibrium is the unique group-symmetric equilibrium of the game. This condition bounds the change in profit due to a group-symmetric deviation from the opposing group. Observe that the left hand side of condition (7) is multiplied by the number of firms in group j. In group-symmetric strategies, there are $n_{g(j)}$ firms deviating simultaneously; thus, the condition needs to bound $n_{g(j)}$ deviations at the same time. Comparing conditions (6) and (7), we can see that the former condition does not directly depend on $n_{g(j)}$ (because $j \in G_{g(i)}$, $n_{g(j)}$ is the number of competitors in the same group as firm i). This is so, because we can exploit the within-group symmetry among firms to obtain a 'tighter' bound. Below we show that condition (7) might not necessarily be more restrictive than condition (6).

Example 5 (Linear model)**.** Consider the linear model of Example 4(b). In particular, assume that the marginal impact of competition is constant, $r = 1$, and that firm i's type distributes type-I extreme value with scale parameter λ_i. Firm i's profit is given by

$$\pi_i(v_e) = \eta_i - (n_e - 1)\delta_i + v_i$$

where $\delta_i > 0$. In this context, the group g's strength is obtained by picking any firm $i \in G_g$ and solving (2)

$$s_i = \delta_i \sum_{k \neq i} (1 - F_k(s_i)) - \eta_i.$$

Group g's strength negatively depends on the expected number of entrants when every firm plays the cutoff strategy s_i, $\sum_{k \neq i}(1 - F_k(s_i))$, weighted by the impact that each entrant has on profits, δ_i. Strength positively depends on the public characteristics of the firm. Because the relation between η_i and δ_i was studied in Example 4, we simplify the analysis below by assuming $\eta_i = 0$ for both groups. Consequently, the relevant range for i's cutoffs is given by $\underline{v}_i = 0$ and $\overline{v}_i = (n-1)\delta_i$.

Let $\hat{\delta}_i = \delta_i/\lambda_i$. Using the log-concave property of extreme value distributions, conditions for uniqueness (6) and (7) become (see Online Appendix D for details)

$$C_{i,j} = \begin{cases} \exp\left(-(n-1)\hat{\delta}_i\right) > \ln\left(\hat{\delta}_i\right) & \text{if } j \in G_{g(i)} \\ \exp\left(-(n-1)\hat{\delta}_j\right) > \ln\left(n_j\hat{\delta}_i\right) & \text{if } j \in G_{-g(i)} \end{cases}. \tag{8}$$

As discussed in Example 4(b), an increase in the total number of potential entrants (from either group) make both constrains more strict when $r = 1$. Finally, observe that the $C_{i,i}$ constraint simplifies to a threshold value for $\hat{\delta}_i$.

We use this example to illustrate how the notion of strength helps discerning the (herculean) cutoff order before computing equilibrium. Then, we show how the sufficient conditions for uniqueness restrict the set of parameters under various number of potential entrants. Finally, we use our sufficient conditions to show that Roberts and Sweeting (2013, 2016) have a unique equilibrium.

(a) **Strength.** Suppose that $\delta_1 < \delta_2$ and $\lambda_2 > \lambda_1$. In this scenario, a firm in group 1 is less affected by competition than a firm in group 2. In addition, the types of a firm in group 1 stochastically dominate those of a firm of group 2 (in the relevant range for entry, $v_i \geq \underline{v} = 0$). In this scenario, it is 'intuitive' to think that group 1 is more competitive than group 2, thus we expect group 1 to enter more often; i.e., to play a lower entry cutoff.

If, in turn, we have that $\delta_1 < \delta_2$ and $\lambda_2 < \lambda_1$, we cannot use an intuitive criterion because each parameter drives competitiveness in a different direction. Although group 1 is less affected by competition compared to group 2 ($\delta_1 < \delta_2$), group 2 is more likely to draw higher types ($\lambda_1 > \lambda_2$). We can use our notion of strength to discern ex-ante the cutoff order in a herculean equilibrium. This is useful because a herculean equilibrium is guaranteed to exist, regardless of whether the game has a unique or multiple equilibria.

[Figure 3 around here]

To illustrate the previous point, consider the case with two asymmetric firms (i.e., $n_1 = n_2 = 1$) characterized by $\delta_1 = 1$ and $\delta_2 = \lambda_2 = 5/4$. The intuitive criterion allows to rank firms, and discern a suitable equilibrium, only when $\lambda_1 \geq \lambda_2$. Figure 3 depicts the firms' strength as a function of λ_1. The strength of firm 1 is a constant ($s_1 = 0.4909$), as it does not depend on λ_1. Firm 2 becomes weaker (s_2 increases) when firm 1 becomes more competitive by drawing higher types. Consistent with the 'intuitive' criterion, firm 1 is stronger when $\lambda_1 > \lambda_2$; firm 1 is simultaneously less sensitive to entry and draws higher types. As λ_1 decreases, firm 1 remains stronger until it reaches $\lambda^s \equiv 0.7058$, the value of λ_1 which makes both firms equally strong. When $\lambda_1 < \lambda^s$, firm 2 becomes the stronger firm in the game. Figure 3 also shows the herculean equilibrium for each value of λ_1. Consistent with the previous analysis, firm 1 plays the lowest cutoff whenever it is the stronger firm in the game. In summary, for values of $\lambda_1 > \lambda_s$, firm 1 plays a lower cutoff, despite $\lambda_1 < \lambda_2$

(b) **Uniqueness with two asymmetric firms.** Continuing with the previous example of an entry game with two asymmetric firms ($n_1 = n_2 = 1$), Figure 4a depicts the set of parameters that satisfy restrictions $C_{1,2}$ and $C_{2,1}$. Aligned with the intuition that equilibrium multiplicity tends to occur when coordination among firms is important—that is, when the market is likely to support only one firm in equilibrium—multiplicity arises when the profit loss from entry by a competitor, δ_i, is high; or when a firm is unlikely to obtain a high type (low λ_i).

[Figure 4 around here]

(c) **Uniqueness with three asymmetric firms.** To see how the sufficient condition changes when we increase the number of potential entrants, suppose instead that we have three firms: two belonging to group 1 and one belonging to group 2. In this scenario, conditions for uniqueness (6) and (7) become three restrictions on the model parameters. These restrictions are shown in Figure 4b. The restriction $C_{1,1}$, that entry by a firm in group 1 imposes in the other group 1 firm, becomes $\hat{\delta}_1 < 1.1138$. As can be observed, in this example, the restriction $C_{1,2}$ is actually redundant. To have a unique equilibrium, a firm in group 1 is more constrained by the behavior of firms in its own group, than the behavior of the firm in the other group. The restriction that group 1 imposes in group 2, $C_{2,1}$, also tightens, as the curve shifts downwards. As mentioned above, because this example assumes a constant marginal impact of competition ($r = 1$), the set of parameters delivering uniqueness shrinks with the number of competitors. If $r < 1$, on the other hand, the set may either shrink or expand.

Example 6 (Uniqueness in a second-price auction with selective entry.)**.** Using a second-price auction model with selective-entry, Roberts and Sweeting (2013, 2016) study the USFS timber auctions. The auction consists of two groups of potential entrants, millers and loggers (groups 1 and 2, respectively). Before entry, each firm observes a signal $v_i = \theta_i \varepsilon_i$, where θ_i is firm i's valuation for a tract and ε_i represents the signal's noise. For the representative (mean) auction they estimate $\ln \theta_i \sim N(\mu_{g(i)}, 0.3321)$ (with $\mu_1 = 3.9607$ and $\mu_2 = 3.5824$) and $\ln \varepsilon_i \sim N(0, 0.8579)$. The estimated entry costs is \$2.0543/mfb (dollars per thousand board foot) and the auction's reserve price is \$27.77/mfb.[18] Searching numerically, they found a single

[18] See Tables 3 or 4 in Roberts and Sweeting (2013, 2016), respectively.

equilibrium. We prove, for the representative auction, that the game has indeed a unique equilibrium. In the scenario, with two asymmetric entrants $n_1 = n_2 = 1$, we find that the left hand side of condition (7) for millers and loggers are 0.2104 and 0.0017 (both less than one); as a consequence, the game has a unique equilibrium. Online Appendix E offers details on the computations, as well of a discussion of strength and herculean equilibrium for this auction.

4.3 Quasi-symmetric games

In an entry game, there are two elements that determine payoffs: the distribution of types $F_i(v_i)$ and the profit function $\pi_i(v_e)$. A game is called quasi-symmetric when firms differ only in one of these two elements. In this section we extend our results to quasi-symmetric environments in which the n potential entrants might be ex-ante asymmetric. Formally, an entry game is called *quasi-symmetric in distributions* when firms have symmetric and anonymous profit functions, and their distributions of types, $F_i(v_i)$, are ordered in terms of first order stochastic dominance (FOSD). Without loss of generality, we order firms so they satisfy $F_i(v) \leq F_{i+1}(v)$ for all v. Similarly, a game is *quasi-symmetric in profit* when firms have symmetric distributions of types and anonymous profit functions that for any realization v_e, satisfy $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$, where $\mathbf{v}_r$ is an r-dimensional vector of realized types and n_e is the number of entrants in e.[19]

Quasi-symmetry is a common assumption in empirical applications. In the context of *complete* information entry games, quasi-symmetry has been used as an equilibrium selection criteria when multiplicity of equilibria exists. For example, Berry (1992) uses a quasi-symmetric in profit model, in which firms with lower entry costs are assumed to enter first (see also Jia, 2008, which uses profitability as a selection criterion). In the context of private information, Roberts and Sweeting (2013, 2016) use a model in which firms are quasi-symmetric in distributions; and Vitorino (2012) uses a linear model in which firms are quasi-symmetric in payoffs.[20]

Lemma 2. *Suppose an entry game in which firms are quasi-symmetric (either in profit or distribution). Then, firms are ordered by strength, with $s_i < s_{i+1}$; that is, firm 1 is the strongest and firm n the weakest.*

[19] Our results below also extend to environments in which firms are ranked consistently in both dimensions; i.e., $F_i(v) \leq F_{i+1}(v)$ for all v and $\pi_i(v, \mathbf{v}_{n_e-1}) \geq \pi_{i+1}(v, \mathbf{v}_{n_e-1})$ for all v_e.

[20] In a context of entry into a second-price auction without selection, Tan and Yilankaya (2006) and Cao and Tian (2013) study equilibrium uniqueness in quasi-symmetric environments under the restriction that bidders belong to one of two possible groups.

The previous lemma shows that the firms' ranking provided by strength coincides with the order given by quasi-symmetry. Reinforcing the idea that strength is the right notion to measure the firms' relative competitiveness. The following theorem is the main result of this section.

Theorem 2. In quasi-symmetric entry games, there always exists a *herculean* equilibrium. Moreover, a herculean equilibrium is the unique equilibrium of the game if the following condition holds

$$(n-1)\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \tag{9}$$

for every pair of firms i, j and every vector $\mathbf{x}$ such that each dimension satisfies $x_k \in [\underline{v}_k, \overline{v}_k]$, and the game is: i) quasi-symmetric in profit or, ii) quasi-symmetric in distribution and the profit loss does not depend on the type of competitors, i.e., $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) = \delta_i(x_i, n_e)$.

As in the previous two environments, a herculean equilibrium always exists. Furthermore, the herculean equilibrium: i) coincides with the symmetric equilibrium when the game is symmetric; ii) coincides with intuitive cutoff orders in games with more than two groups of players when such order exists, and; iii) in a two group scenario, is defined when no intuitive order exists. Therefore, the herculean equilibrium would be useful for empirical analysis.

Observe that Theorem 2 is not a particular case nor a generalization of Theorem 1. While the former can handle more than two groups of asymmetric firms, the latter allows for more degree of firm heterogeneity between the two groups. There are also differences in the sufficient condition for uniqueness. The induction method used in the proof of Theorem 2 needs to handle simultaneous deviations by each of the $n-1$ competitors in the game, independently of whether a subset of firms are symmetric or not. Theorem 1, on the other hand, exploits the within-group symmetry to provide a weaker sufficient condition.

Although Theorem 2 says that condition (9) needs to hold for every pair of potential entrants, the quasi-symmetric structure usually translates in that the condition needs to be checked only for a specific pair of firms. As illustrated in the examples below, this is always true for the linear model in example 1; if the condition holds for a particular pair of firms, it holds for every other pair. Which these two firms are, depends on the type of quasi-symmetry.

Example 7 (Linear model)**.** Consider the following linear model (Example 1)

$$\pi_i(v_e) = \eta_i - (n_e - 1)\delta_i + v_i.$$

where v_i distributes $N(\mu_i, 1)$. We explore sufficient condition (9) under different forms of quasi-symmetry. Start by observing that the profit loss is independent of the type and number of competitors, as $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) = \delta_i$. This holds regardless of the game being quasi-symmetric in profit or in distribution; thus, for uniqueness, we only need to verify condition (9). Using the log-concavity property of the normal distribution, condition (9) holds if, for every pair of firms i and j, the following inequality is satisfied

$$(n-1)\delta_i F_j\left(\overline{v}_j\right) \frac{f_i(\underline{v}_i)}{F_i(\underline{v}_i)} < 1, \tag{10}$$

where $\underline{v}_i = -\eta_i$ and $\overline{v}_i = (n-1)\delta_i - \eta_i$. We show that, in linear-quasi-symmetric environments, if condition (9) holds for one (specific) pair of firms, it holds for every pair of firms.

(a) **Quasi-symmetric in distribution.** Suppose $\delta_i = \delta$ and $\eta_i = \eta$ for every firm i. That is, firms are quasi-symmetric in distribution, where firms are ordered by the mean of their type distribution μ_i with the strongest firm having the highest μ_i. In this scenario, sufficient condition (10) simplifies to $(n-1)\delta F_j(\overline{v}) f_i(\underline{v})/F_i(\underline{v}) < 1$. Using that, for a given $\underline{v}$, the inverted hazard rate increases in μ_i and stochastic dominance ($F_n(\overline{v}) \geq F_i(\overline{v})$ for all i), the condition holds for *every* pair of firms, if it holds for $i = 1$ and $j = n$.

(b) **Quasi-symmetric in profit I.** Suppose instead that $\delta_i = \delta$ and $\mu_i = \mu$ for every firm i. Firms are quasi-symmetric in profit, where the strongest firms has the highest value of η_i. In this scenario, sufficient condition (10) becomes $(n-1)\delta F(\overline{v}_j) f(\underline{v}_i)/F(\underline{v}_i) < 1$. Because the inverted hazard rate is decreasing in v_i, the condition holds for every pair of firms, if it holds for $i = 1$ and $j = n$ (as $\underline{v}_1 \leq \underline{v}_i$ and $\overline{v}_n \geq \overline{v}_i$ for all i).

(c) **Quasi-symmetric in profit II.** Finally, suppose that $\eta_i = \eta$ and $\mu_i = \mu$ for every firm i. Firms, then, are quasi-symmetric in profit, where the strongest firm is the less sensitive to entry (has a lower δ_i). In this scenario $\underline{v}_i = -\eta$ for every i, and condition (10) becomes $(n-1)\delta_i F(\overline{v}_j) f(-\eta)/F(-\eta) < 1$. Because the two weakest firms are the ones with the highest δ_i and $\overline{v}_j$, pick $\kappa = \max\{\delta_n F(\overline{v}_{n-1}), \delta_{n-1} F(\overline{v}_n)\}$ and the condition holds for every pair of firms, if

$(n-1)\kappa f(-\eta)/F(-\eta) < 1$.

5 Concluding Remarks

In this article, we developed a sufficient condition guaranteeing equilibrium uniqueness in the context of entry games under private information. The proposed framework embeds most of the existing entry models studied in the literature, accommodating various forms of firm heterogeneity and selection. With the aid of strength we are able to identify the herculean equilibrium; the type of equilibrium that remains when the games has a unique equilibrium. Strength can reduce the computational burden of calculating equilibria with heterogeneous firms, as it provides bounds for the herculean equilibrium.

When further exploring the set of sufficient conditions provided, we put special emphasis to models in which private information enters the payoffs linearly. The linear model is the most common model used in the applied literature. There, the proposed conditions reduce to a set of simple calculations. The conditions provide clear intuitions on how competition among firms affects the possibility of having a unique equilibrium. We used our sufficient conditions jointly with the estimates in empirical articles to illustrate that their empirical model have a unique equilibrium, demonstrating the usefulness of the results.

The focus of this article is on static entry games with private information. We put special emphasis in developing a framework that embeds most of the applied work on endogenous market formation. Beyond the results presented, we see these new developments as the starting point to study equilibrium uniqueness in dynamics games with entry. We hope the tools developed here enable further research in dynamic environments.

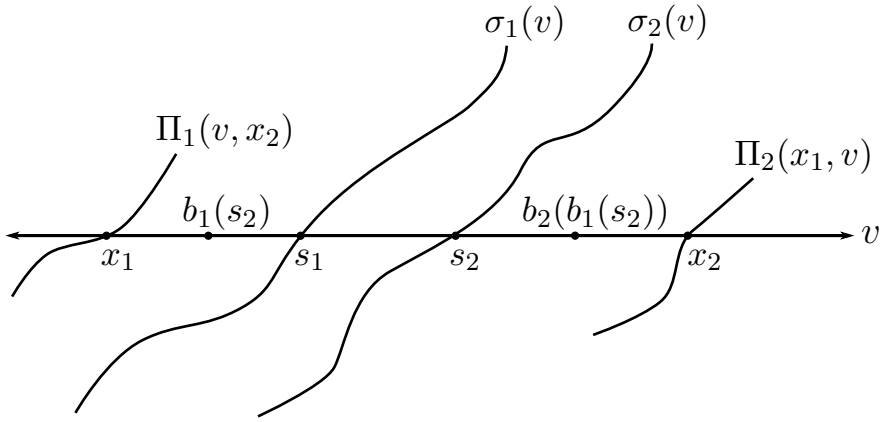
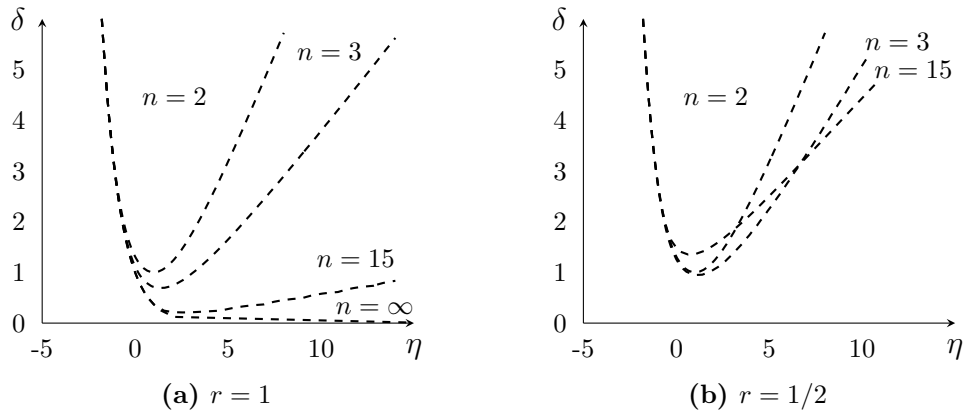


Figure 1: Construction of a herculean equilibrium from iterated best responses. Starting from firm 2's strength, s_2, firm 1's best response is lower than its own strength, s_1. Similarly, firm 2's best response to firm 1's best response is higher than s_2. These iterated best responses are monotonic and bounded, converging to a herculean equilibrium.



(a) $r = 1$ **(b)** $r = 1/2$

Figure 2: Equilibrium uniqueness in a symmetric linear model with a standard type-I extreme value distribution. Area outside curves represents the set of parameters δ and η that deliver a unique equilibrium. In Panel (a), the set of parameters satisfying uniqueness shrinks with the number of potential entrants, n. In Panel (b), the set reacts non-monotonically. Restriction fades away when n becomes unboundedly large.

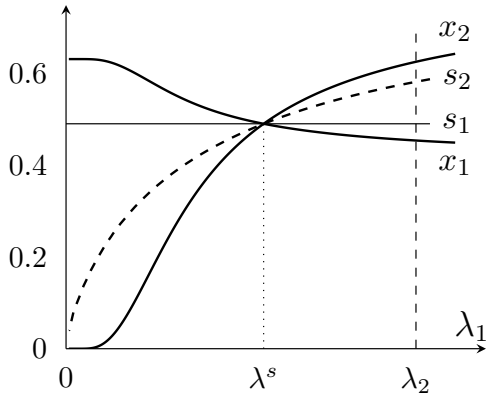
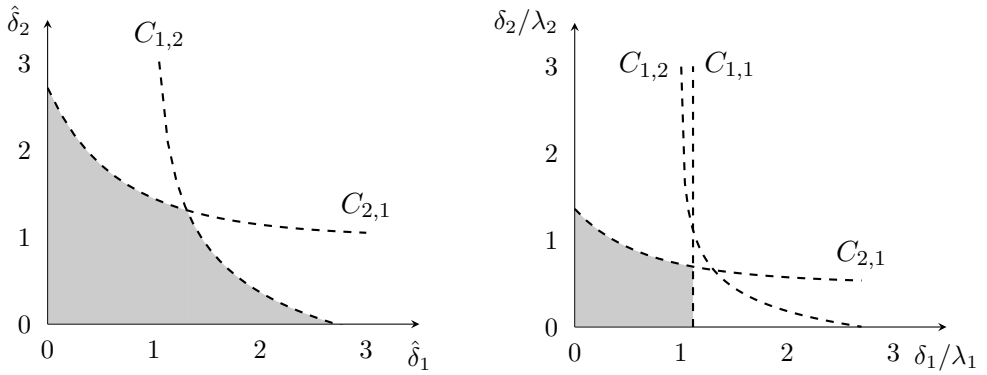


Figure 3: Strength and Herculean equilibrium in a linear model with two asymmetric firms and type-I extreme value distributions; λ_1 varies and $\delta_1 = 1$ and $\delta_2 = \lambda_2 = 5/4$.



(a) $n_1 = n_2 = 1$ **(b)** $n_1 = 2$ and $n_2 = 1$

Figure 4: Equilibrium uniqueness (shaded area) – linear model with asymmetric firms and type-I extreme value distributions.

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Appendix

A Omitted Proofs

Proof of Lemma 1. We show that s_i exists and that $\sigma_i(s)$ single crosses zero.
Existence: Observe that assumptions A4 and A2 jointly imply $\sigma_i(\underline{v}_i) < 0$. Similarly, assumption A4 and Lemma B.1 (see Appendix B) imply, $\sigma_i(\overline{v}_i) \geq \Pi_i(\overline{v}_i, a_{-i}) > 0$. Then, by the Intermediate value Theorem, there exist $\hat{s}$ such that $\sigma_i(\hat{s}) = 0$.
Uniqueness: By Lemma B.1 and the chain rule, we have that $\sigma_i'(s) > 0$. Thus, $\sigma_i(s)$ single crosses zero; i.e., there is a unique value s_i satisfying $\sigma_i(s_i) = 0$. ■

Proof of Proposition 1. This proof makes use of Lemma A.1, presented below.
Lemma A.1. *Under condition (4), two symmetric firms that best respond to each other must play the same cutoff strategy.*
Proof. Consider two symmetric firms, p and q, and fix *any* profile of cutoffs strategies $\mathbf{x}_{-p,q}$ for the rest of the firms. Define $\Pi_{p,q}(y, x) \equiv \Pi_p(x_p = x, x_q = y, \mathbf{x}_{-p,q})$ where $\Pi_p(\mathbf{x})$ is the function defined in (1). $\Pi_{p,q}(x, y)$ represents p's expected profit of entering the market under valuation x when q plays the entry cutoff y and all other firms play according to $\mathbf{x}_{-p,q}$. The equilibrium condition for firm p holds whenever there exists x and y such that $\Pi_{p,q}(x, y) = 0$. Define $b(x)$ to be the value of y such that $\Pi_{p,q}(x, b(x)) = 0$; i.e., $b(x)$ is q's best response to x. By Lemma B.2 in Appendix B, $b(x)$ exists and is uniquely defined for each x. To prove the Lemma we need to prove three claims.
Claim 1. There exists a unique equilibrium such that $x = y$.
Proof. Start by assuming that symmetric firms play symmetric cutoffs; i.e., $x = y = z$. Define $\hat{\sigma}(z) = \Pi_{p,q}(z, z)$ and observe that, by symmetry among firms, $\hat{\sigma}(z) = \Pi_{q,p}(z, z)$. Thus, if the equilibrium condition is satisfied by firm p, it is also satisfied by firm q. A p,q-symmetric equilibrium exists whenever $\hat{\sigma}(z) = 0$. We show that there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Following analogous steps to those in Lemma 1, it is easy to show $\hat{\sigma}(\underline{v}_p) < 0$ and $\hat{\sigma}(\overline{v}_p) > 0$; so that, there exists $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Using Lemma B.1, we can show that $\hat{\sigma}'(z) > 0$. Hence, the value $\hat{z}$ is unique. □
Claim 2. Under condition (4):[21] $0 > b'(x) > -\frac{f(x)}{F(x)} \frac{F(b(x))}{f(b(x))}$.
Proof. Let $\mathbf{x} = (x, b(x), \mathbf{x}_{-p,q})$. Using implicit differentiation and equations (B.1) and (B.2) from Lemma B.1, we obtain

$$b'(x) = -\frac{d\Pi_{q,p}(b(x), x)}{dx} \Big/ \frac{d\Pi_{q,p}(b(x), x)}{dy} = -\frac{f(x)}{F(x)} \frac{\Delta_{q,p}(\mathbf{x})}{\Pi_q'(\mathbf{x})}$$

which is negative as the denominator and numerator are positive. To obtain the lower bound for $b'(x)$ simply use condition (4). □
Claim 3. An increase in x, which q best responds by playing $b(x)$, leads firm p to strictly increase its profit; i.e., $\Pi_{p,q}(x, b(x))$ is increasing in x.
Proof. Differentiating $\Pi_{p,q}(x, b(x))$ with respect to x, using the chain rule, and equations

[21] For ease in notation we drop sub-indexes from F when referring to firms p and q.

(B.1) and (B.2) we obtain

$$\begin{aligned}\frac{d\Pi_{p,q}}{dx} &= \frac{\partial \Pi_{p,q}}{\partial x} + \frac{db(x)}{dx}\frac{\partial \Pi_{p,q}}{\partial y} \\ &= \Pi'_q(\mathbf{x}) + \frac{db(x)}{dx}\frac{f(b(x))}{F(b(x))}\Delta_{p,q}(\mathbf{x}) > \Pi'_2(\mathbf{x}) - \frac{f(x)}{F(x)}\Delta_{p,q}(\mathbf{x}) > 0,\end{aligned}$$

where $\mathbf{x} = (x, b(x), \mathbf{x}_{-p,q})$. The first inequality follows from Claim 2, whereas the second from condition (4); which proves the claim. □

We prove Lemma A.1 by contradiction. Recall that $\mathbf{x}_{-p,q}$ is fixed throughout the proof. Suppose, without loss of generality, that there exists $y < x$ constituting an equilibrium. By Claim 1 there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Suppose first $y < \hat{z} < x$. Because

$$\hat{\sigma}(\hat{z}) = \Pi_{p,q}(\hat{z}, \hat{z}) = \Pi_{p,q}(\hat{z}, b(\hat{z})) = 0,$$

Claim 3 implies that we must have $\Pi_{p,q}(x, b(x) = y) > 0$ as $x > \hat{z}$, which contradicts (x, y) being an equilibrium. Suppose now $y < x < \hat{z}$. Lemma B.1 and Claim 1 imply

$$0 = \hat{\sigma}(\hat{z}) > \hat{\sigma}(x) = \Pi_{p,q}(x, x) > \Pi_{p,q}(x, y)$$

which contradicts (x, y) being an equilibrium. Analogous argument can be constructed for the case $\hat{z} < y < x$, proving the Lemma. ■

To prove the proposition observe: (i) By Lemma 1, there exists a unique value of strength and, therefore, a unique symmetric equilibrium, which also corresponds to the unique herculean equilibrium. (ii) If firms are not playing a symmetric equilibrium, then there must exists two symmetric firms best-responding to each other but playing different cutoffs, contradicting Lemma A.1. ■

Proof of Theorem 1. *Proof preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume, without loss of generality, that $s_1 < s_2$. Let $\hat{\mathbf{x}} = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm $i \in G_1$ and let $\hat{\Pi}_1(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ represent a firm in group one's expected profit of entering under valuation x_1 when firms play group-symmetric strategies x_1 and x_2.Define $b_1(x)$ to be the function that solves $\hat{\Pi}_1(b_1(x), x) = 0$. Thus, $b_1(x)$ corresponds to group one's symmetric best response to group two playing the group-symmetric cutoff x. By Lemma B.2, the value $b_1(x)$ exists and is unique; i.e., $b_1(x)$ is well defined.

Claim 4. $b_1(s_1) = s_1$ and, under sufficient condition (7), $0 > b'_1(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(b_1(x))}{f_1(b_1(x))}$.

Proof. By definition of strength we know $\hat{\Pi}_1(s_1, s_1) = 0$, therefore $b_1(s_1) = s_1$. Using implicit differentiation, the chain rule, that groups member are symmetric, and Lemma B.1

$$b'_1(x) = -\frac{\frac{d\hat{\Pi}_1(b_1(x),x)}{dx_2}}{\frac{d\hat{\Pi}_1(b_1(x),x)}{dx_1}} = -\frac{n_2\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_{j\in G_2}}}{\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_1} + (n_1 - 1)\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_{j\in G_1}}} = -\frac{n_2\frac{f_2(x_2)}{F_2(x_2)}\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi'_1(\hat{\mathbf{x}}) + (n_1 - 1)\frac{f_1(x_1)}{F_1(x_1)}\Delta_{1,1}(\hat{\mathbf{x}})}$$

where $\Delta_{i,j}(\mathbf{x})$, defined in equation (3), represents the profit loss of firm i when a firm in group j enters the market with type x_j. Because numerator and denominator are positive, the equation above proves $b'_1(x) < 0$ for all x. For the lower bound of $b_1(x)$ observe that $\Delta_{1,1} > 0$. Take a lower bound for $b'_1(x)$ by making $\Delta_{1,1}$ zero. The lower

bound $b_1'(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(b_1(x))}{f_1(b_1(x))}$ follows by using sufficient condition (7). □

Existence of a herculean equilibrium: Define the function $h_2 : [s_1, \infty) \to \mathbb{R}$ by $h_2(x) = \hat{\Pi}_2(b_1(x), x)$. This function is continuous and corresponds to the expected profit of a firm in group 2 when it enters the market under valuation x, group two plays the group-symmetric cutoff x, and group one plays their group-symmetric best response $b_1(x)$. Define x_2 to be the value satisfying $h_2(x_2) = 0$ and let $x_1 = b_1(x_2)$. The next two claims prove that an herculean equilibrium ($x_1 < x_2$) exists, $x_1 < s_1$, and $x_2 > s_2$.

Claim 5. $x_2 \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$.

Proof. Because $b_1(x)$ is decreasing in x and $b_1(s_1) = s_1$, we have that $x_1 = b_1(x_2) < s_1 < x_2$ if and only if $x_2 > s_1$. □

Claim 6. $h_2(s_2) < 0$ and there exists $\hat{x} > s_2$ such that $h_2(\hat{x}) > 0$. Thus, by the intermediate value theorem, the herculean equilibrium cutoff $x_2 \in (s_2, \hat{x})$ exists.

Proof. Because group two is weak, and $b_1(x)$ is decreasing in x, we know that $b_1(s_2) < b_1(s_1) = s_1 < s_2$ (where Claim 4 was used in the equality). Lemma B.1 and the definition of strength implies $h_2(s_2) = \hat{\Pi}_2(b_1(s_2), s_2) < \hat{\Pi}_2(s_2, s_2) = 0$, proving $h_2(s_2) < 0$. For the second part of the claim, observe that, by Lemma B.1, $\hat{\Pi}_2(x_1, x_2)$ is increasing in x; then, $\hat{\Pi}_2(b_1(x), x) \geq \hat{\Pi}_2(a, x)$ for all x. Take $\hat{x} = \overline{v}_2$ and observe that, by assumption A4, $\hat{\Pi}_2(a, \hat{x}) > 0$, proving the result. □

Uniqueness of equilibrium: Start by observing that, under condition (6), Lemma A.1 applies. Therefore, it is without loss to restrict the analysis to group-symmetric strategies. To prove uniqueness, then, we need to show that no other herculean equilibrium exists and that we can not have an equilibrium where $x_2 < x_1$.

Claim 7. There exists a unique herculean equilibrium.

Proof. To prove uniqueness within the herculean class, we shown $h_2'(x) > 0$ so that $h_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (b_1(x), \ldots, b_1(x), x, \ldots, x)$. Differentiating $h_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain

$$\begin{aligned} h_2'(x) &= \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{\partial \Pi_2'(\hat{\mathbf{x}})}{\partial x_2} + b'(x) n_1 \frac{\partial \Pi_2'(\hat{\mathbf{x}})}{\partial x_1} \\ &> \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) - n_1 \frac{f_2(x)}{F_2(x)}\Delta_{2,1}(\hat{\mathbf{x}}) > (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) > 0, \end{aligned}$$

where the first inequality follows from using Lemma B.1 and the bound in Claim 4. The second inequality follows from sufficient condition (7). Proving that the derivative is positive and uniqueness within the herculean class. □

Claim 8. There is no group-symmetric equilibrium in which the strong group plays a higher cutoff than the weak group.

Proof. We show that no non-herculean equilibrium—i.e., $x_1 > x_2$ but $s_1 < s_2$—can exist. Define $b_2(x)$ to be the function that satisfies $\hat{\Pi}_2(x, b_2(x)) = 0$; $b_2(x)$ corresponds to group two's best response to the cutoff of group one when $x_1 = x$. As before, Lemma B.2 implies that $b_2(x)$ is well defined. Similarly, following the steps of Claim 4, it can be shown: $b_2(s_2) = s_2$, $b_2'(x) < 0$, and, under condition (7), $b_2'(x)$ is bounded below by $-\frac{f_1(x)F_2(b_2(x))}{F_1(x)f_2(b_2(x))}$.

Define the continuous function $h_1(x) = \hat{\Pi}_1(x, b_2(x))$ which corresponds to the expected profit of a firm in group one when entering the market under valuation x and its opponents play the pair of group-symmetric strategies $(x, b_2(x))$. We show that there is

no x satisfying $x_1 = x > b_2(x) = x_2$ and $h_1(x) = 0$; i.e., no non-herculean equilibrium exists. Start by observing that $x > b_2(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 1 we showed the function $\sigma_1(s) = \hat{\Pi}_1(s, s)$ is strictly increasing in s. Then, by the definition of strength and by firm two being weak ($s_1 < s_2$),

$$\sigma_1(s_1) = \hat{\Pi}_1(s_1, s_1) = 0 < \sigma_1(s_2) = \hat{\Pi}_1(s_2, s_2) = \hat{\Pi}_1(s_2, b_2(s_2)) = h_1(s_2),$$

showing that $h_1(s_2) > 0$. Following analogous steps to those in Claim 7, which requires using lower bound for $b_2'(x)$ and sufficient condition (7), we can show that $h_1'(x) > 0$. Then, because $h_1(s_2) > 0$ and $h_1'(x) > 0$ for all x, $h_1(x)$ never crosses zero when $x > s_2$ and the result follows. □ ■

Proof of Lemma 2. We start by showing the order in the context of quasi-symmetry in profit. Let s_i be the strength of firm i, using (2) we obtain

$$\begin{aligned} 0 = \sigma_i(s_i) &= \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi_i \left(s_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} \\ &> \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi_{i+1} \left(s_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} = \sigma_{i+1}(s_i), \end{aligned}$$

where in the inequality we used $\pi_i(v, \mathbf{v}_{n_e - 1}) > \pi_{i+1}(v, \mathbf{v}_{n_e - 1})$. In the last equality, after chancing the profit identity, we used $E_i = E_{i+1}$. Then, by Lemma 1, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$.

For quasi-symmetry in distribution, rewriting (2) we obtain

$$\begin{aligned} 0 = \sigma_i(s_i) &= \sum_{e \in E_i \cap E_{i+1}} \left\{ \left(\prod_{j \in O_i(e)} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi \left(s_i, v_{e\setminus i} \right) f_{i+1}(v_{i+1}) \phi(v_{e\setminus i, i+1}) d^{n_e - 1} v_{e\setminus i} \right\} \\ &+ \sum_{e \in E_i \setminus E_{i+1}} \left\{ \left(F_{i+1}(s_i) \prod_{j \in O_i(e)\setminus i+1} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi \left(s_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} \\ &> \sum_{e \in E_i \cap E_{i+1}} \left\{ \left(\prod_{j \in O_i(e)} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi \left(s_i, v_{e\setminus i} \right) f_i(v_i) \phi(v_{e\setminus i, i+1}) d^{n_e - 1} v_{e\setminus i} \right\} \\ &+ \sum_{e \in E_i \setminus E_{i+1}} \left\{ \left(F_i(s_i) \prod_{j \in O_i(e)\setminus i} F_j(s_i) \right) \int_{(s_i)_{j \in I_i(e)}}^{b} \pi \left(s_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} = \sigma_{i+1}(s_i), \end{aligned}$$

where the inequality uses two properties of FOSD. In the second term we used $F_i(v) < F_{i+1}(v)$. In the first term, we used $\int_{s_i}^{b} h(v) f_i(v) dv \leq \int_{s_i}^{b} h(v) f_j(v) dv$ for any non-increasing function $h(x)$. Then, by Lemma 1, $\sigma_{i+1}(s)$ is increasing in s and $s_{i+1} > s_i$. ■

Proof of Theorem 2. We present the proof when firms are quasi-symmetric in distributions. The proof when firms are quasi-symmetric in profit is, basically, identical but we can drop the subindices from the distribution functions. Using Lemma 2 we order firms using stochastic dominance, from stronger (firm 1) to weakest (firm n).

Existence of an herculean equilibrium. We prove existence by construction. For any vector of cutoff strategies $\mathbf{x}$ and $k \in \{2, \ldots, n\}$ let $\mathbf{x}^k = (x_k, x_{k+1}, \ldots, x_n)$. Construct as follows:

- **Firm** 1: Define $b_1^1(\mathbf{x}^2)$ to be firm's 1 best response to $\mathbf{x}^2$; i.e., $b_1^1(\mathbf{x}^2)$ satisfies

$$\Pi_1(b_1^1(\mathbf{x}^2), \mathbf{x}^2) = 0.$$

  where $\Pi_1(\mathbf{x})$ is defined in (1). By Lemma B.2 in the Auxiliary Result section, $b_1^1(\mathbf{x}^2)$ exists and (the best response) is unique.

- **Firm** 2: Let $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(b_1^1(\mathbf{x}^2), \mathbf{x}^2)$; that is, $\hat{\Pi}_2(\mathbf{x}^2)$ represents firm's 2 profit after incorporating that firm 1 is best responding to $\mathbf{x}^2$. Define $b_2^2(\mathbf{x}^3)$ to be a solution to $\hat{\Pi}_2(b_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. By Lemma B.2, $b_2^2(\mathbf{x}^3)$ exists. This function represents firm's 2 best response when firms 1 and 2 are mutually best responding to each other and to $\mathbf{x}^3$. For ease in notation, denote firm's 1 best response after incorporating firm's 2 best response as $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$.[22]

  Claim 9. For any $\mathbf{x}^3$, $b_2^2(\mathbf{x}^3) > b_1^2(\mathbf{x}^3)$.

  Proof. Fix $\mathbf{x}^3$ and find the value $\hat{x}$ that satisfies $\hat{x} = b_1^1(\hat{x}, \mathbf{x}^3)$. The value $\hat{x}$ exists by continuity of $b_1^1(\mathbf{x}^2)$ and by $b_1^1(\mathbf{x}^2)$ being bounded below and above by $\underline{v}_1$ and $\overline{v}_1$ respectively (by assumption A4). Then by Lemma B.3 in the auxiliary results section we have $\Pi_2(\hat{x}, \hat{x}, \mathbf{x}^3) < \Pi_1(\hat{x}, \hat{x}, \mathbf{x}^3) = 0$. Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma B.2) value that solves $\Pi_2(z_m, y_{m+1}, \mathbf{x}^3) = 0$ (i.e., y_{m+1} is firm's 2 best response to the cutoffs $(z_m, \mathbf{x}^3)$) and; (iii) $z_{m+1} = b_1^1(y_{m+1}, \mathbf{x}^3)$. By definition, z_{m+1} solves $\Pi_1(z_{m+1}, y_{m+1}, \mathbf{x}^3) = 0$ and, by Lemma B.2, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_2(\hat{x}, \hat{x}, \mathbf{x}^3) < 0$ and $\Pi_2(\mathbf{x})$ being strictly increasing in the 2nd dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.1) $\Pi_1(\mathbf{x})$ is also increasing in the 2nd dimension, we have $\Pi_1(z_1, y_2, \mathbf{x}^3) > 0$, which implies $z_2 = b_1^1(y_2, \mathbf{x}^3) < z_1 = b_1^1(y_1, \mathbf{x}^3)$. This, in turn, implies (by Lemma B.3)

$$\Pi_2(z_2, y_2, \mathbf{x}^3) < \Pi_1(z_2, y_2, \mathbf{x}^3) = 0;$$

  which implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A4, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_2$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_1$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = b_1^1(y_\infty, \mathbf{x}^3)$ and; (ii) $\Pi_2(z_\infty, y_\infty, \mathbf{x}^3) = \hat{\Pi}_2(y_\infty, \mathbf{x}^3) = 0$ (i.e., $y_\infty = b_2^2(\mathbf{x}^3)$). Thus, $b_1^1(y_\infty, \mathbf{x}^3) = b_1^2(\mathbf{x}^3)$ and, as $z_\infty < \hat{x} < y_\infty$, we have $b_2^2(\mathbf{x}^3) > b_1^2(\mathbf{x}^3)$. □

- **Firm** $k \leq n$: Suppose we have shown the existence of $b_\ell^\ell(\mathbf{x}^{\ell+1})$ for every $\ell \in \{1, \ldots, k-1\}$ and have recursively defined $b_j^\ell(\mathbf{x}^{\ell+1}) = b_j^{\ell-1}(b_\ell^\ell(\mathbf{x}^{\ell+1}), \mathbf{x}^{\ell+1})$ for $j \in \{1, \ldots, \ell\}$. Let $\hat{\Pi}_k(\mathbf{x}^k) = \Pi_k(b_1^{k-1}(\mathbf{x}^k), \ldots, b_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$ represent firm's k profit after incorporating that every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$. Define $b_k^k(\mathbf{x}^{k+1})$ (observe that $\mathbf{x}^{k+1}$ is empty—i.e., a number, not a function—if

[22] More generally, for $j < k$ the notation $b_j^k(\mathbf{x}^k)$ represents firm's j best response to $\mathbf{x}^k$ after incorporating the best response of every firm up to k.

$k = n$) to be a solution to $\hat{\Pi}_k(b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. By Lemma B.2, $b_k^k(\mathbf{x}^k)$ exists. This function represents firm's k best response to $\mathbf{x}^{k+1}$ when every firm $j \in \{1, \ldots, k-1\}$ is mutually best responding to each other and to $\mathbf{x}^k$.

Claim 10. For any $\mathbf{x}^{k+1}$, if firm $k-1$ is stronger than k the solution $b_k^k(\mathbf{x}^{k+1})$ satisfies $b_k^k(\mathbf{x}^{k+1}) > b_{k-1}^k(\mathbf{x}^{k+1})$.

Proof. Fix any $\mathbf{x}^{k+1}$ and let $b_k^k(\mathbf{x}^{k+1})$ be one of the solutions found in the previous step. Then define the vector of cutoffs $\mathbf{x} = \left(b_1^k(\mathbf{x}^{k+1}), \ldots, b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}\right)$. Throughout the proof, the vector of strategies for every firm except firm k and $k-1$, $\mathbf{x}_{-k,k-1}$, remains fixed (i.e., they are numbers not functions). Define $\hat{x}$ to be a value satisfying $\hat{x} = b_{k-1}^{k-1}(\hat{x}, \mathbf{x}^{k+1})$. The value $\hat{x}$ exists by continuity of $b_{k-1}^{k-1}(\mathbf{x}^k)$ and by $b_{k-1}^{k-1}(\mathbf{x}^k)$ being bounded below and above by $\underline{v}_{k-1}$ and $\overline{v}_{k-1}$ respectively (by assumption A4). By definition of best response $\hat{x}$ satisfies $\Pi_{k-1}(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) = 0$. Then, by Lemma B.3, we have

$$\Pi_k(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) < \Pi_{k-1}(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) = 0.$$

Define a pair of sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ satisfying: (i) $y_1 = z_1 = \hat{x}$; (ii) y_{m+1} is the unique (by Lemma B.2) value that solves $\Pi_k(z_m, y_{m+1}, \mathbf{x}_{-k,k-1}) = 0$ (i.e., y_{m+1} is firm's k best response to the cutoffs $(z_m, \mathbf{x}_{-k,k-1})$) and; (iii) $z_{m+1} = b_{k-1}^{k-1}(y_{m+1}, \mathbf{x}^{k+1})$. By definition, z_{m+1} solves $\Pi_{k-1}(z_{m+1}, y_{m+1}, \mathbf{x}_{-k,k-1}) = 0$ and, Lemma B.2, the value z_{m+1} is also unique. We show that $\{y_m\}_{m\in\mathbb{N}}$ is increasing and $\{z_m\}_{m\in\mathbb{N}}$ decreasing. Because $\Pi_k(\hat{x}, \hat{x}, \mathbf{x}_{-k,k-1}) < 0$ and $\Pi_k(\mathbf{x})$ being strictly increasing in the kth dimension, $y_2 > y_1 = \hat{x}$. Similarly, because (by Lemma B.1) $\Pi_{k-1}(\mathbf{x})$ is also increasing in the kth dimension, we have $\Pi_{k-1}(\hat{x}, y_2, \mathbf{x}_{-k,k-1}) > 0$, which implies $z_2 = b_{k-1}^{k-1}(y_2, \mathbf{x}^{k+1}) < b_{k-1}^{k-1}(y_1, \mathbf{x}^{k+1}) = z_1$. This, in turn, implies (by Lemma B.3)

$$\Pi_k(z_2, y_2, \mathbf{x}_{-k,k-1}) < \Pi_{k-1}(z_2, y_2, \mathbf{x}_{-k,k-1}) = 0,$$

which, in turns, implies $y_3 > y_2$. By induction, the argument generalizes to an arbitrary step m and the sequences $\{y_m, z_m\}_{m\in\mathbb{N}}$ are monotonically increasing and decreasing respectively. By assumption A4, $\{y_m\}_{m\in\mathbb{N}}$ is bounded above by $\overline{v}_k$ and $\{z_m\}_{m\in\mathbb{N}}$ is bounded below by $\underline{v}_{k-1}$. Thus, the sequences converge to y_∞ and z_∞, respectively. By convergence, we have: (i) $z_\infty = b_{k-1}^{k-1}(y_\infty, \mathbf{x}^{k+1})$ and; (ii) $\Pi_k(z_\infty, y_\infty, \mathbf{x}_{-k,k-1}) = \hat{\Pi}_k(y_\infty, \mathbf{x}_{-k,k-1}) = 0$ (i.e., $y_\infty = b_k^k(\mathbf{x}^{k+1})$). Thus, $b_{k-1}^{k-1}(y_\infty, \mathbf{x}^{k+1}) = b_{k-1}^k(\mathbf{x}^{k+1})$ and, as $z_\infty < \hat{x} < y_\infty$, we have $b_k^k(\mathbf{x}^{k+1}) > b_{k-1}^k(\mathbf{x}^{k+1})$. □

Thus, we have constructed an equilibrium vector $\mathbf{x} = (b_1^n(x_n), \ldots, b_{n-1}^n(x_n), x_n)$ with the property that $x_i < x_{i+1}$; i.e., a Herculean equilibrium.

Uniqueness within the herculean-equilibrium class: We show that at each step k of the previous construction there is a unique best response $x_k = b_k^k(\mathbf{x}^{k+1})$ to $\mathbf{x}^{k+1}$.

- **Firm** 1: The uniqueness of $b_1^1(\mathbf{x}^2)$ follows from Lemma B.2. The next result is needed for subsequent steps.

  Claim 11. Under condition (9), for every $j \in \{2, \ldots, n\}$, $db_1^1(\mathbf{x}^2)/dx_j$ satisfies:

$$0 > \frac{db_1^1(\mathbf{x}^2)}{dx_j} = -\frac{f_j(x_j)}{F_j(x_j)} \frac{\Delta_{1,j}(\mathbf{x})}{\Pi_1'(\mathbf{x})} > -\frac{f_j(x_j)}{F_j(x_j)} \frac{F_1(x_1)}{f_1(x_1)} \frac{1}{n-1}. \tag{A.1}$$

$$\frac{F_j(x_j)}{f_j(x_j)} \frac{db_1^1(\mathbf{x}^2)}{dx_j} < \frac{F_q(x_q)}{f_q(x_q)} \frac{db_1^1(\mathbf{x}^2)}{dx_q} \frac{1}{n-1} \quad \text{for } q \in \{2, \ldots, j-1\} \tag{A.2}$$

Proof. Let $\mathbf{x} = (b_1^1(\mathbf{x}^2), \mathbf{x}^2)$; using implicit differentiation and Lemma B.1 we obtain

$$\frac{db_1^1(\mathbf{x}^2)}{dx_j} = -\frac{\partial \Pi_1(\mathbf{x})/\partial x_j}{\partial \Pi_1(\mathbf{x})/\partial x_1} = -\frac{f_j(x_j)}{F_j(x_j)}\frac{\Delta_{1,j}(\mathbf{x})}{\Pi_1'(\mathbf{x})}, \tag{A.3}$$

which is negative as, $\Delta_{1,j}(\mathbf{x}) > 0$ and $\Pi_{1,j}(\mathbf{x}) > 0$ for every $\mathbf{x}$. The lower bound in equation (A.1) follows from applying condition (9) into equation (A.3). Property (A.2) follows from observing

$$\frac{F_q(x_q)}{f_q(x_q)}\frac{db_1^1(\mathbf{x}^2)}{dx_q}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_1^1(\mathbf{x}^2)}{dx_j} = \frac{1}{\Pi_1'(\mathbf{x})}\left(\Delta_{1,j}(\mathbf{x}) - \frac{\Delta_{1,q}(\mathbf{x})}{n-1}\right) > 0,$$

where the equality follows from substituting in equation (A.3), and the inequality follows from Lemma B.4 and the fact that $q \in \{2, \ldots, j-1\}$. □

- **Firm** 2: Fix $\mathbf{x}^3$ and let $\mathbf{x} = (b_1^1(\mathbf{x}^2), \mathbf{x}^2)$, we need to show that the best response $b_2^2(\mathbf{x}^3)$ is unique. We do this by showing that $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(b_1^1(\mathbf{x}^2), \mathbf{x}^2)$ is strictly increasing in x_2; so that, $\hat{\Pi}_2(x_2, \mathbf{x}^3)$ single crosses zero and there is a unique value $b_2^2(\mathbf{x}^3)$ satisfying $\hat{\Pi}_2(b_2^2(\mathbf{x}^3), \mathbf{x}^3) = 0$. Using the chain rule and equation (B.2)

$$\begin{aligned}\hat{\Pi}_2'(\mathbf{x}^2) = \Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{d\Pi_2}{dx_1} &= \Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{f_1(b_1^1(\mathbf{x}^2))}{F_1(b_1^1(\mathbf{x}^2))}\Delta_{2,1}(\mathbf{x}) \\ &> \Pi_2'(\mathbf{x}) - \frac{f_2(x_2)}{F_2(x_2)}\frac{\Delta_{2,1}(\mathbf{x})}{n-1} > 0,\end{aligned} \tag{A.4}$$

where in the first inequality follows from the lower bound in equation (A.1) and the second inequality follows from sufficient condition (9). This proves uniqueness of the best response. The next result is needed for the induction argument in the proof.

Claim 12. Let $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$. Under condition (9), for every $j \in \{3, \ldots, n\}$ and $\ell \in \{1, 2\}$, $db_\ell^2(\mathbf{x}^3)/dx_j$ satisfies:

$$\frac{db_2^2(\mathbf{x}^3)}{dx_j} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{2,j}(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_j}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})} \tag{A.5}$$

$$0 > \frac{db_\ell^2(\mathbf{x}^3)}{dx_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_\ell(x_\ell)}{f_\ell(x_\ell)}\frac{1}{n-1} \quad \text{and,} \tag{A.6}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{db_2^2(\mathbf{x}^3)}{dx_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{db_2^2(\mathbf{x}^3)}{dx_q}\frac{1}{n-1} \quad \text{for } q \in \{3, \ldots, j-1\} \tag{A.7}$$

Proof. To show equation (A.5) use implicit differentiation, the chain rule, and equation (B.2) to obtain

$$\frac{db_2^2(\mathbf{x}^3)}{dx_j} = -\frac{\frac{d\hat{\Pi}_2}{dx_j}}{\frac{d\hat{\Pi}_2}{dx_2}} = -\frac{\frac{d\Pi_2}{dx_j} + \frac{db_1^1(\mathbf{x}^2)}{dx_j}\frac{d\Pi_2}{dx_1}}{\Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{d\Pi_2}{dx_1}} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{2,j}(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_j}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}{\Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x})}.$$

Observe, by equation (A.4), that the denominator is positive. Using lower bound (A.1) and Lemma B.4 we can see that the numerator is also positive, implying that the derivative is negative; which proves the upper bound of (A.6) when $\ell = 2$. For

the lower bound of equation (A.6) when $\ell = 2$, using equation (A.5), observe that equation (A.6) holds if and only if the following expression is positive:

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\left(\frac{F_2(x_2)}{f_2(x_2)}\frac{\Pi_2'(\mathbf{x})}{n-1} - \Delta_{2,j}(\mathbf{x})\right) + \frac{f_1(x_1)}{F_1(x_1)}\left(\frac{F_2(x_2)}{f_2(x_2)}\frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_1^1(\mathbf{x}^2)}{dx_j}\right)\Delta_{2,1}(\mathbf{x})\right].$$

The first round bracket is positive by sufficient condition (9). The second round bracket is positive by property (A.2). Thus, the expression is indeed positive and the lower bound in equation (A.6) holds.

We now prove the bounds of (A.6) when $\ell = 1$. Using $b_1^2(\mathbf{x}^3) = b_1^1(b_2^2(\mathbf{x}^3), \mathbf{x}^3)$, observe

$$\frac{db_1^2(\mathbf{x}^3)}{dx_j} = \frac{db_1^1(\mathbf{x}^2)}{dx_j} + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{db_2^2(\mathbf{x}^3)}{dx_j}. \tag{A.8}$$

Using (A.3) to substitute for $db_1^1(\mathbf{x}^2)/dx_\ell$ with $\ell \in \{2, j\}$ and using the lower bound in equation (A.6) when $\ell = 2$, we obtain the following upper bound:

$$\frac{db_1^2(\mathbf{x}^3)}{dx_j} < \frac{f_j(x_j)}{F_j(x_j)}\frac{1}{\Pi_1'(\mathbf{x})}\left[\frac{\Delta_{1,2}(\mathbf{x})}{n-1} - \Delta_{1,j}(\mathbf{x})\right] < 0,$$

the inequality follows from Lemma B.4; proving the upper bound. The lower bound in equation (A.6) follows from using equation (A.8) and observing

$$\frac{db_1^2(\mathbf{x}^3)}{dx_j} > \frac{db_1^1(\mathbf{x}^2)}{dx_j} > -\frac{f_j(x_j)}{F(x_j)}\frac{F_1(x_1)}{f_1(x_1)}\frac{1}{n-1},$$

where the inequalities follow from $db_2^2(\mathbf{x}^3)/dx_j \cdot db_1^1(\mathbf{x}^2)/dx_2 > 0$ and equation (A.1), respectively.

Finally, to prove property (A.7) use equation (A.5) to write

$$\frac{F_q(x_q)}{f_q(x_q)}\frac{db_2^2(\mathbf{x}^3)}{dx_q}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_2^2(\mathbf{x}^3)}{dx_j} =$$
$$\frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + \frac{f_1(x_1)}{F_1(x_1)}\left(\frac{F_j(x_j)}{f_j(x_j)}\frac{db_1^1(\mathbf{x}^2)}{dx_j} - \frac{F_q(x_q)}{f_q(x_q)}\frac{db_1^1(\mathbf{x}^2)}{dx_q}\frac{1}{n-1}\right)\Delta_{2,1}(\mathbf{x})\right],$$

where $D_2 = \Pi_2'(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}\frac{f_1(x_1)}{F_1(x_1)}\Delta_{2,1}(\mathbf{x}) > 0$. We show that a lower bound of this expression is positive. Taking $-db_1^1(\mathbf{x}^2)/dx_q > 0$ to zero, we obtain

$$\frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} + \frac{f_1(x_1)}{F_1(x_1)}\frac{F_j(x_j)}{f_j(x_j)}\frac{db_1^1(\mathbf{x}^2)}{dx_j}\Delta_{2,1}(\mathbf{x})\right]$$
$$> \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{\Delta_{2,q}(\mathbf{x})}{n-1} - \frac{\Delta_{2,1}(\mathbf{x})}{n-1}\right] > \frac{1}{D_2}\left[\Delta_{2,j}(\mathbf{x}) - \frac{2\Delta_{2,q}(\mathbf{x})}{n-1}\right] > 0.$$

The first inequality follows from using lower bound (A.1). The other two inequalities follow from Lemma B.4 and the fact that $q \in \{2, \ldots, j-1\}$. □

- **Firm** $k \in \{3, \dots, n\}$: Suppose that, for every $p \in \{1, \dots, k-1\}$ and $j \in \{p+1, \dots, n\}$, we have proven that: $b_p^p(\mathbf{x}^{p+1})$ is unique;

$$0 > \frac{db_p^p(\mathbf{x}^k)}{dx_j} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{p,j}(\mathbf{x}) + \sum_{\ell=1}^{p-1} \frac{db_\ell^{p-1}(\mathbf{x}^p)}{dx_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{p,\ell}(\mathbf{x})}{\Pi_p'(\mathbf{x}) + \sum_{\ell=1}^{p-1} \frac{db_\ell^{p-1}(\mathbf{x}^p)}{dx_p}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{p,\ell}(\mathbf{x})}; \tag{A.9}$$

$$0 > \frac{db_q^p(\mathbf{x}^k)}{dx_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_q(x_q)}{f_q(x_q)}\frac{1}{n-1} \quad \text{for } q \in \{1, \dots, p\} \text{ and;} \tag{A.10}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{db_p^p(\mathbf{x}^{p+1})}{dx_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{db_p^p(\mathbf{x}^{p+1})}{dx_q}\frac{1}{n-1} \quad \text{for } q \in \{p+1, \dots, j-1\}. \tag{A.11}$$

Fix $\mathbf{x}^{k+1}$ and let $\mathbf{x} = (b_1^{k-1}(\mathbf{x}^k), \dots, b_{k-1}^{k-1}(\mathbf{x}^k), \mathbf{x}^k)$. We show that the best response $b_k^k(\mathbf{x}^{k+1})$ is unique by showing that $\hat{\Pi}_k(\mathbf{x}^k)$ is strictly increasing in x_k. Differentiating,

$$\begin{aligned}\hat{\Pi}_k'(\mathbf{x}^k) &= \Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1} \frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x}) \\ &> \Pi_k'(\mathbf{x}) - \frac{f_k(x_k)}{F_k(x_k)}\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1} > \Pi_k'(\mathbf{x}) - \frac{f_k(x_k)}{F_k(x_k)}\frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1} > 0,\end{aligned}$$

where the inequalities follow from lower bound (A.10), Lemma B.4 and, sufficient condition (9), respectively. This proves uniqueness of the best response. The next result completes the induction argument.

Claim 13. Under condition (9), for every $j \in \{k+1, \dots, m\}$ and $p \in \{1, \dots, k\}$, $db_p^k(\mathbf{x}^{k+1})/dx_j$ satisfies

$$\frac{db_k^k(\mathbf{x}^{k+1})}{dx_j} = -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1} \frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1} \frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})} \tag{A.12}$$

$$0 > \frac{db_p^k(\mathbf{x}^{k+1})}{dx_j} > -\frac{f_j(x_j)}{F_j(x_j)}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1} \quad \text{and,} \tag{A.13}$$

$$\frac{F_j(x_j)}{f_j(x_j)}\frac{db_k^k(\mathbf{x}^{k+1})}{dx_j} < \frac{F_q(x_q)}{f_q(x_q)}\frac{db_k^k(\mathbf{x}^{k+1})}{dx_q}\frac{1}{n-1} \quad \text{for } q \in \{k+1, \dots, j-1\} \tag{A.14}$$

Proof. To show equation (A.12) use the implicit differentiation, the chain rule, and equation (B.2) to obtain

$$\begin{aligned}\frac{db_k^k(\mathbf{x}^{k+1})}{dx_j} &= -\frac{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_j}{\partial\hat{\Pi}_k(\mathbf{x})/\partial x_k} = -\frac{\frac{d\Pi_k(\mathbf{x})}{dx_j} + \sum_{\ell=1}^{k-1}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}\frac{d\Pi_k(\mathbf{x})}{dx_\ell}}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{d\Pi_k(\mathbf{x})}{dx_\ell}}. \\ &= -\frac{\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}{\Pi_k'(\mathbf{x}) + \sum_{\ell=1}^{k-1}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})}.\end{aligned}$$

We already showed that the denominator is positive. We show that a lower bound of the numerator is positive, which immediately implies the upper bound in equation

(A.13) for the case when $p = k$. Using equation (A.10) a lower bound for the numerator is

$$\frac{f_j(x_j)}{F_j(x_j)}\Delta_{k,j}(\mathbf{x}) - \frac{f_j(x_j)}{F_j(x_j)}\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1} > \frac{f_j(x_j)}{F_j(x_j)}\left[\Delta_{k,j}(\mathbf{x}) - \frac{(k-1)\Delta_{k,k-1}(\mathbf{x})}{n-1}\right] > 0,$$

where both inequalities follows from Lemma B.4. Thus, the numerator is positive.

For the lower bound in equation (A.13) in the case $p = k$, replace (A.12) into (A.13) and observe that the inequality holds if and only if the following expression is positive

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{\Pi_k'(\mathbf{x})}{n-1} - \Delta_{k,j}(\mathbf{x})\right) + \sum_{\ell=1}^{k-1}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}\right)\Delta_{k,\ell}(\mathbf{x})\right]. \quad \text{(A.15)}$$

The first term in round brackets is positive due to sufficient condition (9). We now work with the summation and show that it is also positive. Before doing this, observe that, by definition, for every $\ell \in \{1, \dots, k-1\}$

$$b_\ell^k(\mathbf{x}^{k+1}) = b_\ell^\ell(b_{\ell+1}^k(\mathbf{x}^{k+1}), b_{\ell+2}^k(\mathbf{x}^{k+1}), \dots, b_k^k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}).$$

Then, for any $j \in \{k+1, \dots, m\}$

$$\frac{db_\ell^k(\mathbf{x}^{k+1})}{dx_j} = \frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_j} + \sum_{q=\ell+1}^{k}\frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_q}\frac{db_q^k(\mathbf{x}^{k+1})}{dx_j}. \quad \text{(A.16)}$$

For a given ℓ in the summation in equation (A.15), we use equation (A.16) to write the round bracket as

$$\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_j}\right) + \sum_{q=\ell+1}^{k-1}\frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_q}\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{db_q^{k-1}(\mathbf{x}^k)}{dx_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_q^{k-1}(\mathbf{x}^k)}{dx_j}\right). \quad \text{(A.17)}$$

Substitute equation (A.17) when $\ell = 1$ into the summation in equation (A.15) to obtain

$$\sum_{\ell=2}^{k-1}\left(\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x}) + \frac{db_1^1(\mathbf{x}^2)}{dx_2}a_1\right)\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}\right) + a_1\left(\frac{F_k(x_k)}{f_k(x_k)}\frac{db_1^1(\mathbf{x}^k)}{dx_k}\frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)}\frac{db_1^1(\mathbf{x}^k)}{dx_j}\right), \quad \text{(A.18)}$$

where $a_1 = \Delta_{k,1}(\mathbf{x})\frac{f_1(x_1)}{F_1(x_1)} > 0$. Then, substituting (in increasing order) into equation (A.18) the expression (A.17) for $\ell = 2$, $\ell = 3$ until $\ell = k-1$, we obtain that the

summation in equation (A.15) is equal to

$$\sum_{\ell=1}^{k-1} a_\ell \left(\frac{F_k(x_k)}{f_k(x_k)} \frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_k} \frac{1}{n-1} - \frac{F_j(x_j)}{f_j(x_j)} \frac{db_\ell^\ell(\mathbf{x}^{\ell+1})}{dx_j} \right) > 0, \tag{A.19}$$

where

$$a_\ell = \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \Delta_{k,\ell}(\mathbf{x}) + \sum_{p=1}^{\ell-1} \frac{db_p^p(\mathbf{x}^{p+1})}{dx_\ell} a_p \tag{A.20}$$

is defined recursively. The parenthesis in equation (A.19) is positive by equation (A.11). We show that each a_ℓ is positive, which proves the lower bound in equation (A.13) when $p = k$. By induction, suppose that for every $p \in \{1, \ldots, \ell - 1\}$ we have shown that $(f_p(x_p)/F_p(x_p))\Delta_{k,p}(\mathbf{x}) \leq a_p > 0$ (we already showed this for a_1). We need to show that the same inequalities hold for equation (A.20). First, because $db_p^p(\mathbf{x}^{p+1})/dx_\ell < 0$ and $a_p > 0$ (by induction hypothesis) it is easy to see that $a_\ell < (f_\ell(x_\ell)/F_\ell(x_\ell))\Delta_{k,\ell}(\mathbf{x})$. Using the lower bound in equation (A.10) and the upper bound for a_p we obtain the following lower bound for equation (A.20)

$$a_l > \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \left[\Delta_{k,\ell}(\mathbf{x}) - \sum_{p=1}^{\ell-1} \frac{\Delta_{k,p}(\mathbf{x})}{n-1} \right] > \frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \left[1 - \frac{(\ell-1)}{n-1} \right] \Delta_{k,\ell}(\mathbf{x}) > 0,$$

where the second inequality follows from Lemma B.4; which proves the result.

To prove the upper bound in equation (A.13) for $p \in \{1, \ldots, k-1\}$ we proceed by induction downwards. Suppose that for every firm $\ell \in \{p+1, \ldots, k\}$ we have proven

$$0 > \frac{db_\ell^k(\mathbf{x}^{k+1})}{dx_j} > -\frac{f_j(x_j)}{F_j(x_j)} \frac{F_\ell(x_\ell)}{f_\ell(x_\ell)} \frac{1}{n-1} \tag{A.21}$$

we prove that equation (A.13) holds for p. Observing that, in (A.16), $db_p^p(\mathbf{x}^{p+1})/dx_\ell < 0$ we can construct an upper bound for $db_p^k(\mathbf{x}^{k+1})/dx_j$ using the induction hypothesis (A.21)

$$\frac{db_p^k(\mathbf{x}^{k+1})}{dx_j} < \frac{db_p^p(\mathbf{x}^{p+1})}{dx_j} - \sum_{\ell=p+1}^{k} \frac{db_p^p(\mathbf{x}^{p+1})}{dx_\ell} \frac{f_j(x_j)}{F_j(x_j)} \frac{F_\ell(x_\ell)}{f_\ell(x_\ell)} \frac{1}{n-1}$$

Using equation (A.9), the upper bound for $db_p^k(\mathbf{x}^{k+1})/dx_j$ is equal to

$$\begin{aligned} \frac{1}{D_p} \frac{f_j(x_j)}{F_j(x_j)} \sum_{\ell=p+1}^{k} \left(\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)} \Delta_{p,\ell}(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{db_q^{p-1}(\mathbf{x}^p)}{dx_\ell} \frac{f_q(x_q)}{F_q(x_q)} \Delta_{p,q}(\mathbf{x}) \right) \frac{F_\ell(x_\ell)}{f_\ell(x_\ell)} \frac{1}{n-1} \\ - \frac{1}{D_p} \frac{f_j(x_j)}{F_j(x_j)} \left(\Delta_{p,j}(\mathbf{x}) + \frac{F_j(x_j)}{f_j(x_j)} \sum_{q=1}^{p-1} \frac{db_q^{p-1}(\mathbf{x}^p)}{dx_j} \frac{F_q(x_q)}{f_q(x_q)} \Delta_{p,q}(\mathbf{x}) \right) \end{aligned}$$

where $D_p = \Pi_p'(\mathbf{x}) + \sum_{q=1}^{p-1} \frac{db_q^{p-1}(\mathbf{x}^p)}{dx_p} \frac{f_q(x_q)}{F_q(x_q)} \Delta_{p,q}(\mathbf{x}) > 0$. Taking $db_q^{p-1}(\mathbf{x}^p)/dx_\ell < 0$ equal to zero and $db_q^{p-1}(\mathbf{x}^p)/dx_j < 0$ to the lower bound in equation (A.10), we build the following upper bound for the previous expression (and omitting D_p, as it does

not affect the sign)

$$\frac{f_j(x_j)}{F_j(x_j)}\left[\sum_{\ell=p+1}^{k}\frac{\Delta_{p,\ell}(\mathbf{x})}{n-1}+\sum_{q=1}^{p-1}\frac{\Delta_{p,q}(\mathbf{x})}{n-1}-\Delta_{p,j}(\mathbf{x})\right]<\frac{f_j(x_j)}{F_j(x_j)}\left[\frac{k-1}{n-1}-1\right]\Delta_{p,j}(\mathbf{x})\leq 0.$$

The inequality follows from equation Lemma B.4; proving $db_p^k(\mathbf{x}^{k+1})/dx_j<0$.

The lower bound for $db_p^k(\mathbf{x}^{k+1})/dx_j$ follows from equation (A.16) and observing

$$\frac{db_p^k(\mathbf{x}^{k+1})}{dx_j}>\frac{db_p^p(\mathbf{x}^{p+1})}{dx_j}>-\frac{f_j(x_j)}{F(x_j)}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1}$$

where the first inequality follows from $(db_p^p(\mathbf{x}^{k+1})/dx_\ell)\cdot(db_\ell^k(\mathbf{x}^{k+1})/dx_j)>0$ for every ℓ, and the second from the lower bound in equation (A.10).

Finally, we prove equation (A.14) using equation (A.12) to write

$$\begin{aligned}\frac{F_q(x_q)}{f_q(x_q)}\frac{db_k^k(\mathbf{x}^{k+1})}{dx_q}\frac{1}{n-1}-\frac{F_j(x_j)}{f_j(x_j)}\frac{db_k^k(\mathbf{x}^{k+1})}{dx_j}=\frac{1}{D_k}\Bigg[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}+\\ \sum_{\ell=1}^{k-1}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\left(\frac{F_j(x_j)}{f_j(x_j)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_j}-\frac{F_q(x_q)}{f_q(x_q)}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_q}\frac{1}{n-1}\right)\Delta_{k,\ell}(\mathbf{x})\Bigg],\end{aligned}$$

where $D_k=\Pi_k'(\mathbf{x})+\sum_{\ell=1}^{k-1}\frac{db_\ell^{k-1}(\mathbf{x}^k)}{dx_k}\frac{f_\ell(x_\ell)}{F_\ell(x_\ell)}\Delta_{k,\ell}(\mathbf{x})>0$. We show that a lower bound of this expression is positive. Taking $-db_\ell^{k-1}(\mathbf{x}^k)/dx_q>0$ to zero and $db_\ell^{k-1}(\mathbf{x}^k)/dx_j<0$ to the lower bound in equation (A.10), we obtain

$$\frac{1}{D_k}\left[\Delta_{k,j}(\mathbf{x})-\frac{\Delta_{k,q}(\mathbf{x})}{n-1}-\sum_{\ell=1}^{k-1}\frac{\Delta_{k,\ell}(\mathbf{x})}{n-1}\right]>\frac{1}{D}\left[\Delta_{k,j}(\mathbf{x})-\frac{k\Delta_{k,q}(\mathbf{x})}{n-1}\right]>0.$$

The inequalities follow from Lemma B.4 and the fact that $q\in\{k,\ldots,j-1\}$. □

Because at each step, best responses are unique and at $k=n$ the firm has only one best response when every firm $k<n$ is best responding to x_n and to each other, there is a unique Herculean equilibrium within the herculean class.

No non-herculean equilibria exists: By contradiction. Suppose $\mathbf{x}$ represents a non-herculean equilibrium. Order firms from smallest cutoff x_1 to largest, x_n. Let p be the first instance (smallest cutoff) that a strength reversal occurs. That is, $x_p<x_{p+1}$ but $s_p>s_{p+1}$. Because every firm $k\in\{1,\ldots,p\}$ is ordered by strength, they satisfy conditions (A.12), (A.10), and (A.11). We show that x_{p+1} cannot lie above x_p (i.e, a contradiction). Fix $\mathbf{x}$ and let $\hat{x}$ be the value that satisfies $b_p(\hat{x},\mathbf{x}_{-p,p+1})=\hat{x}$, where $b_p(\mathbf{x}_{-p})$ is firm's p best response to $\mathbf{x}_{-p}$. This best response exists (and is unique) by Lemma B.2. The value $\hat{x}$ exists because $b_p(\mathbf{x}_{-p})$ is continuously decreasing in x_{p+1}. In addition, following analogous steps to those in Claim 11, we can show that $db_p(\mathbf{x}_{-p})/dx_{p+1}>-\frac{f_{p+1}(x_{p+1})}{F_{p+1}(x_{p+1})}\frac{F_p(x_p)}{f_p(x_p)}\frac{1}{n-1}$. Then, by Lemma B.3, $\Pi_p(\hat{x},\hat{x},\mathbf{x}_{-p,p+1})=0<\Pi_{p+1}(\hat{x},\hat{x},\mathbf{x}_{-p,p+1})$. Also, letting $\hat{\mathbf{x}}=(b_p(\mathbf{x}_{-p}),\mathbf{x}_{-p})$ observe that

$$\frac{d\Pi_{p+1}(\hat{\mathbf{x}})}{dx_{p+1}}=\Pi_{p+1}'(\hat{\mathbf{x}})+\frac{db_p(\mathbf{x}_{-p})}{dx_{p+1}}\frac{\partial\Pi_{p+1}(\hat{\mathbf{x}})}{\partial x_p}>\Pi_{p+1}'(\hat{\mathbf{x}})-\frac{f_{p+1}(x_{p+1})}{F_{p+1}(x_{p+1})}\frac{\Delta_{p+1,p}(\hat{\mathbf{x}})}{n-1}>0$$

Thus, $\Pi_{p+1}(\hat{\mathbf{x}})$ is strictly increasing in x_{p+1} which implies that $\Pi_{p+1}(\hat{\mathbf{x}}) > 0$ for every $x_{p+1} \geq \hat{x}$, which implies that no $x_{p+1} > b_p(\mathbf{x}_{-p}) = x_p$ exists. ■

B Auxiliary Results

Lemma B.1. *$\Pi_i(\mathbf{x})$ is strictly increasing in every dimension of $\mathbf{x}$.*

Proof of Lemma B.1. Start with the derivative of Π_i with respect to i, then

$$\frac{d\Pi_i}{dx_i} \equiv \Pi_i'(\mathbf{x}) = \sum_{e \in E_i} \left\{ \left(\prod_{j \in O_i(e)} F_j(x_j) \right) \int_{\{x_j\}_{j \in I_i(e)}}^{\infty} \pi_i' \left(x_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\} > 0, \quad \text{(B.1)}$$

which is positive as, by assumption A4, there is a positive probability that firm i is the sole entrant. For the derivative of Π_i with respect to x_j, pick a market structure e such that $j \in O_i(e)$. Conditional on e, the derivative of Π_i with respect to x_j is equal to

$$f_j(x_j) \left(\prod_{k \in O_i(e)\setminus j} F_k(x_k) \right) \int_{\{x_k\}_{k \in I_i(e)}}^{\infty} \pi_i \left(x_i, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i}.$$

Now take market structure e from above and, using Leibnitz differentiation, compute the derivative of Π_i with respect to x_j conditional on market structure $e \cup j$; i.e., entry decisions by every firm remain the same as in e except that of firm j, which now enters

$$-f_j(x_j) \left(\prod_{k \in O_i(e)\setminus j} F_k(x_k) \right) \int_{\{x_k\}_{k \in I_i(e)}}^{\infty} \pi_i \left(x_i, x_j, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i},$$

where we used $O_i(e)\setminus j = O_i(e \cup j)$ and $I_i(e) = I_i(e \cup j)\setminus j$. Observe that both expressions from above only differ in sign and in the profit function that is integrated over. Summing both equations delivers an expression where the integral is over $\delta_{i,j}(x_i, x_j, v_{e\setminus i}) \geq 0$, which is defined in equation (3). Summing across every market structure we obtain

$$\begin{aligned} \frac{d\Pi_i}{dx_j} &= \frac{f_j(x_j)}{F_j(x_j)} \sum_{e \in E_i \setminus E_j} \left\{ \left(\prod_{k \in O_i(e)} F_k(x_k) \right) \int_{(x_k)_{k \in I_i(e)}}^{\infty} \delta_{i,j} \left(x_i, x_j, v_{e\setminus i} \right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\}. \\ &= \frac{f_j(x_j)}{F_j(x_j)} \Delta_{i,j}(\mathbf{x}) > 0 \end{aligned} \quad \text{(B.2)}$$

Thus, the derivative is positive. ■

Lemma B.2. *Let Π_i be defined by (1). Let A and B be disjoint sets of k and r firms, such that $i \in A$. Define $f : [a,b]^{k+r} \to [a,b]^{n-k-r}$ to be any continuous function and let $\mathbf{x}_B$ be any vector of cutoff strategies for firm in B. Then, there exist a value $\hat{x}$ such that the symmetric k-dimensional vector $\hat{x}_A = (\hat{x})_{i \in A}$ satisfies $\Pi_i(\hat{x}_A, f(\hat{x}_A, \mathbf{x}_B), \mathbf{x}_B) = 0$. When the function f is constant—i.e., when $\mathbf{x}_{-A} = (f(\hat{x}_A, \mathbf{x}_B), \mathbf{x}_B)$ is equal to an exogenously given vector—the value of $\hat{x}$ is unique.*

Proof. Fix $\mathbf{x}_B$, because f is continuous, the function $\Pi_i \left(x_A, f(x_A, \mathbf{x}_B), \mathbf{x}_B \right)$ is continuous in the input value x of the symmetric vector x_A. Let $\underline{v}_A = (\underline{v}_i)_{i \in A}$ and

$\overline{v}_A = (\overline{v}_i)_{i\in A}$. Observe that assumptions A4 A2 jointly imply $\Pi_i\left(\underline{v}_A, f(\underline{v}_A, \mathbf{x}_B), \mathbf{x}_B\right) \leq \pi_i(\underline{v}_i) < 0$. Similarly, assumption A4 and Lemma B.1 imply, $\Pi_i\left(\overline{v}_A, f(\overline{v}_A, \mathbf{x}_B), \mathbf{x}_B\right) \geq \Pi_i\left(\overline{v}_i, a_{-i}\right) > 0$. Then, by the Intermediate Value Theorem, there exist $\hat{x} \in (\underline{v}_i, \overline{v}_i)$ such that $\Pi_i\left(\hat{x}_A, f(\hat{x}_A, \mathbf{x}_B), \mathbf{x}_B\right) = 0$. For uniqueness when f is constant, by the chain rule, $d\Pi_i/dx_i = \sum_{k\in A} \partial\Pi_i/\partial x_k > 0$ where the inequality follows from Lemma B.1. Therefore $\Pi_i\left(x_A, f(x_A, \mathbf{x}_B), \mathbf{x}_B\right)$, as a function of the value x for the symmetric vector x_A, crosses zero once. ■

Lemma B.3. *Suppose firms are quasi-symmetric. Order them by strength, then for any firm $i < j$ and vector of strategies for the other firms $\mathbf{x}_{-i,j}$, we have*

$$\Pi_i(y, y, \mathbf{x}_{-i,j}) > \Pi_j(y, y, \mathbf{x}_{-i,j}).$$

Proof. If firms are quasi-symmetric in profit, the inequality follows from definition. Recall $\phi(v_e) = \prod_{j\in I(e)} f_j(v_j)$ and let $\phi_{-k}(v_e) = \prod_{j\in I(e)\setminus k} f_j(v_j)$. For games quasi-symmetric in distribution, observe

$$\begin{aligned}
\Pi_i(y, y, \mathbf{x}_{-i,j}) = &\sum_{e\in E_i\setminus E_j} \left\{ \left(F_j(y) \prod_{k\in O_i(e)\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in I_i(e)}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
&\sum_{e\in E_i\cap E_j} \left\{ \left(\prod_{k\in O_i(e)} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in I_i(e)\setminus j}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi_{-k}(v_{e\setminus i}) f_j(v) d^{n_e-1} v_{e\setminus i} \right\} \\
> &\sum_{e\in E_i\setminus E_j} \left\{ \left(F_i(y) \prod_{k\in O_i(e)\setminus j} F_k(x_k) \right) \int_{(x_k)_{k\in I_i(e)}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\} + \\
&\sum_{e\in E_i\cap E_j} \left\{ \left(\prod_{k\in O_i(e)} F_k(x_k) \right) \int_y^b \int_{(x_k)_{k\in I_i(e)\setminus j}}^{b} \pi_i\left(x_i, v_{e\setminus i}\right) \phi_{-k}(v_{e\setminus i}) f_i(v) d^{n_e-1} v_{e\setminus i} \right\} \\
= &\Pi_j(y, y, \mathbf{x}_{-i,j})
\end{aligned}$$

where the inequality uses two properties of FOSD. The first term uses that $F_i(x) \leq F_j(x)$ for all x. The second term uses that $\int_y^b h(x) f_i(x) dx \leq \int_y^b h(x) f_j(x) dx$ for any non-increasing function $h(x)$. ■

Lemma B.4. *Let firm k be stronger than firm k. Suppose the firms play cutoffs $x_k < x_j$; then, for any firm i, $\Delta_{i,j}(\mathbf{x}) \geq \Delta_{i,k}(\mathbf{x})$ if: (i) The firms are quasi-symmetric in profits or (ii) The firms are quasi-symmetric in distribution and satisfy the profit loss only depends on the number of entrants.*

Proof. Start by observing that, in the expression for $\Delta_{i,j}(\mathbf{x})$ (see equation (3)), the sum over market structures can be divided into two disjoints sets: $e \in E_i \setminus (E_j \cap E_k)$ and $e \in (E_i \cap E_k) \setminus E_j$. Using these sets subtract $\Delta_{i,j}(\mathbf{x}) - \Delta_{i,k}(\mathbf{x})$ to obtain

$$\sum_{e\in E_i\setminus(E_j\cap E_k)} \left\{ \left(\prod_{\ell\in O_i(e)} F_\ell(x_\ell) \right) \int_{(x_\ell)_{\ell\in I_i(e)}}^{\infty} \left(\delta_i(x_i, x_j, v_{e\setminus i}) - \delta_i(x_i, x_k, v_{e\setminus i}) \right) \phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\}$$

$$+\sum_{e\in(E_i\cap E_k)\setminus E_j}\left\{\left(F_j(x_j)\prod_{\ell\in O_i(e)\setminus j}F_\ell(x_\ell)\right)\int_{x_k}^{\infty}\int_{(x_\ell)_{\ell\in I_i(e)\setminus k}}^{\infty}\delta_i(x_i,x_j,v_{e\setminus i})\phi_{-k}(v_{e\setminus i})f_k(v_k)d^{n_e-1}v_{e\setminus i}\right\}$$
$$-\sum_{e\in(E_i\cap E_j)\setminus E_k}\left\{\left(F_k(x_k)\prod_{\ell\in O_i(e)\setminus k}F_\ell(x_\ell)\right)\int_{x_j}^{\infty}\int_{(x_\ell)_{\ell\in I_i(e)\setminus j}}^{\infty}\delta_i(x_i,x_k,v_{e\setminus i})\phi_{-j}(v_{e\setminus i})f_j(v_j)d^{n_e-1}v_{e\setminus i}\right\} \tag{B.3}$$

where, by profits being anonymous, we dropped the second sub index from $\delta_i(x_i,x_j,v_{e\setminus i})$ and $\phi_{-k}(v_e)=\prod_{j\in I(e)\setminus k}f_j(v_j)$. Observe that the first term is non-negative as

$$\delta_i(x_i,x_j,v_{e\setminus i})-\delta_i(x_i,x_k,v_{e\setminus i})=\pi_i(x_i,x_k,v_{e\setminus i})-\pi_i(x_i,x_j,v_{e\setminus i})\geq 0$$

where the last inequality follow from assumption A2 and $x_k<x_j$. We now show that the subtraction of the second and third terms in equation (B.3) is non-negative. Observe that for each market structure $e\in(E_i\cap E_k)\setminus E_j$, firm k participates but firm j does not. We construct a lower bound for the second term in equation (B.3) by reversing the participation roles of k and j in e. We then show that the lower bound is equal to the third term. Thus, the subtraction is non-negative.

When the game is quasi-symmetric in profit, we can drop the sub-index from the distributions of type. Bounding the second term

$$\sum_{e\in(E_i\cap E_k)\setminus E_j}\left\{\left(F(x_j)\prod_{\ell\in O_i(e)\setminus j}F(x_\ell)\right)\int_{x_k}^{\infty}\int_{(x_\ell)_{\ell\in I_i(e)\setminus k}}^{\infty}\delta_i(x_i,x_j,v_{e\setminus i})\phi_{-k}(v_{e\setminus i})f(v_k)d^{n_e-1}v_{e\setminus i}\right\}$$
$$>\sum_{e\in(E_i\cap E_k)\setminus E_j}\left\{\left(F(x_k)\prod_{\ell\in O_i(e)\setminus j}F(x_\ell)\right)\int_{x_j}^{\infty}\int_{(x_\ell)_{\ell\in I_i(e)\setminus k}}^{\infty}\delta_i(x_i,x_k,v_{e\setminus i})\phi_{-k}(v_{e\setminus i})f(v_k)d^{n_e-1}v_{e\setminus i}\right\}$$
$$=\sum_{e\in(E_i\cap E_j)\setminus E_k}\left\{\left(F(x_k)\prod_{\ell\in O_i(e)\setminus k}F(x_\ell)\right)\int_{x_j}^{\infty}\int_{(x_\ell)_{\ell\in I_i(e)\setminus j}}^{\infty}\delta_i(x_i,x_k,v_{e\setminus i})\phi_{-k}(v_{e\setminus i})f(v_k)d^{n_e-1}v_{e\setminus i}\right\}$$

where in the inequality we used $x_j>x_k$ in the probability of firm j being out of the market and in the domain of the integration over k's types, $\delta_i(x_i,x_j,v_{e\setminus i})\geq 0$ (so that integrating over a smaller domain decreases the value of the integral) and $\delta_i(x_i,s,v_{e\setminus i})$ being increasing in s (by assumption A2). The equality follows by re-arranging indexes, noticing that we simply inverted the roles of firm k and j; i.e., the lower bound above is identical to the third term in equation (B.3), proving the result.

When the game is quasi-symmetric in distribution and the profit loss only depends on the number of entrants, the second term becomes

$$\sum_{e\in(E_i\cap E_k)\setminus E_j}\left\{\left(F_j(x_j)\prod_{\ell\in O_i(e)\setminus j}F_\ell(x_\ell)\right)\left((1-F_k(x_k))\prod_{\ell\in I_i(e)\setminus k}(1-F_\ell(x_\ell))\right)\delta_i(x_i,n_e)\right\}$$
$$>\sum_{e\in(E_i\cap E_k)\setminus E_j}\left\{\left(F_k(x_k)\prod_{\ell\in O_i(e)\setminus j}F_\ell(x_\ell)\right)\left((1-F_j(x_j))\prod_{\ell\in I_i(e)\setminus k}(1-F_\ell(x_\ell))\right)\delta_i(x_i,n_e)\right\}$$

where the first inequality uses stochastic dominance and the fact that $x_k < x_j$ (so that $F_j(x_j) \geq F_j(x_k) \geq F_k(x_k)$). The equality follows by re-arranging indexes, noticing that the lower bound above is identical to the third term in equation (B.3), proving the result. ■