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ALGORITHM IS EXPERIMENT:
MACHINE LEARNING, MARKET DESIGN, AND POLICY ELIGIBILITY RULES

By

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Abstract

Algorithms produce a growing portion of decisions and recommendations both in policy and business. Such algorithmic decisions are natural experiments (conditionally quasi-randomly assigned instruments) since the algorithms make decisions based only on observable input variables. We use this observation to develop a treatment-effect estimator for a class of stochastic and deterministic algorithms. Our estimator is shown to be consistent and asymptotically normal for well-defined causal effects. A key special case of our estimator is a high-dimensional regression discontinuity design. The proofs use tools from differential geometry and geometric measure theory, which may be of independent interest.

The practical performance of our method is first demonstrated in a high-dimensional simulation resembling decision-making by machine learning algorithms. Our estimator has smaller mean squared errors compared to alternative estimators. We finally apply our estimator to evaluate the effect of Coronavirus Aid, Relief, and Economic Security (CARES) Act, where more than $10 billion worth of relief funding is allocated to hospitals via an algorithmic rule. The estimates suggest that the relief funding has little effects on COVID-19-related hospital activity levels. Naive OLS and IV estimates exhibit substantial selection bias.

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1 Introduction

Today's society increasingly resorts to machine learning ("AI") algorithms for decision-making and resource allocation. For example, judges in the US make legal judgements aided by predictions from supervised machine learning (descriptive regression). Supervised learning is also used by governments to detect potential criminals and terrorists, and finance companies (such as banks and insurance companies) to screen potential customers. Tech companies like Facebook, Microsoft, and Netflix allocate digital content by reinforcement learning and bandit algorithms. Uber and other ride sharing services adjust prices using their surge pricing algorithms to take into account local demand and supply information. Retailers and e-commerce platforms engage in algorithmic pricing. Similar algorithms are encroaching on increasingly high-stakes settings, such as in healthcare and the military.

Other types of algorithms also loom large. School districts, college admissions systems, and labor markets use matching algorithms for position and seat allocations. Objects worth astronomical sums of money change hands every day in algorithmically run auctions – not only household objects, art and antiquities, but also securities, energy, and public procurements. Many public policy domains like Medicaid often decide who are eligible based on algorithmic rules.

All of the above, diverse examples share a common trait: a decision-making algorithm makes decisions based only on its observable input variables. Conditional on the observable variables, therefore, algorithmic treatment decisions are (quasi-)randomly assigned in the sense that they are independent of any potential outcome or unobserved heterogeneity. This property turns algorithm-based treatment decisions into instrumental variables (IVs) that can be used for measuring the causal effect of the final treatment assignment. The algorithm-based instrument may produce regression-discontinuity-style local variation (e.g., machine judges), stratified randomization (e.g., several bandit and reinforcement learning algorithms), or some combination of the two.

Based on the above observation, this paper shows how to use data obtained from algorithmic decision-making to identify and estimate causal effects. In our framework, the analyst observes a random sample $\{(Y_i, X_i, D_i, Z_i)\}_{i=1}^n$, where Y_i is the outcome of interest, $X_i \in \mathbb{R}^p$ is a vector of pre-treatment covariates (algorithm's input variables), D_i is the binary treatment assignment, possibly made by humans, and Z_i is the binary treatment recommendation made by some algorithm. The treatment recommendation Z_i is randomly determined based on the known probability $ML(X_i) = \Pr(Z_i = 1|X_i)$ independently of everything else conditional on X_i. The central assumption is that *the analyst knows function ML and is able to simulate it.* That is, the analyst is able to compute the recommendation probability $ML(x)$ given any input value $x \in \mathbb{R}^p$. The algorithm's recommendation Z_i may influence the final treatment assignment D_i, determined as $D_i = Z_i D_i(1) + (1 - Z_i)D_i(0)$, where $D_i(z)$ is the potential treatment assignment that would be realized if $Z_i = z$. Finally, the observed outcome Y_i is determined as $Y_i = D_i Y_i(1) + (1 - D_i)Y_i(0)$, where $Y_i(1)$ and $Y_i(0)$ are potential outcomes that would be realized if the individual were treated and not treated, respectively. This setup is an IV model where the IV satisfies the conditional independence condition but may not satisfy the overlap

(full-support) condition. There appears to be no standard estimator for this setup.

Example. There is a rapidly growing trend of development and real-world implementation of automated disease detection algorithms (Gulshan *et al.*, 2016). Machine learning, in particular deep learning, is used to detect various diseases and to predict patients at risk. Using our framework described above, a detection algorithm predicts whether an individual i has a certain disease ($Z_i = 1$) or not ($Z_i = 0$) based on a digital image $X_i \in \mathbb{R}^p$ of a part of the individual's body, where each $X_{ij} \in \mathbb{R}$ denotes the intensity value of a pixel in the image. The algorithm uses training data and machine learning (e.g., deep learning) to construct a binary classifier $ML : \mathbb{R}^p \to \{0, 1\}$. The classifier takes an image of individual i as input and makes a binary prediction of whether the individual has the disease:

$$Z_i \equiv ML(X_i).$$

The algorithm's diagnosis Z_i may influence the doctor's treatment decision for the individual, denoted by $D_i \in \{0, 1\}$. We are interested in how the treatment decision D_i affects the individual's outcome Y_i.

Within this framework, we first characterize the whole sources of causal-effect identification (quasi-experimental variation) for a class of algorithms, nesting both stochastic and deterministic ones. This class includes all of the aforementioned examples, thus nesting existing insights on quasi-experimental variation in particular algorithms, such as surge pricing (Cohen, Hahn, Hall, Levitt and Metcalfe, 2016), bandit (Li, Chu, Langford and Schapire, 2010), reinforcement learning (Precup, 2000), supervised learning (Cowgill, 2018; Bundorf, Polyakova and Tai-Seale, 2019), and market-design algorithms (Abdulkadiroğlu, Angrist, Narita and Pathak, 2017, Forthcoming; Narita, 2020, 2021). Our framework also reveals new sources of identification for algorithms that, at first sight, do not appear to produce a natural experiment.

The sources of causal-effect identification turn out to be summarized by a suitable modification of the Propensity Score (Rosenbaum and Rubin, 1983), which we call the Quasi Propensity Score (QPS). The Quasi Propensity Score at covariate value x is the average probability of a treatment recommendation in a shrinking neighborhood around x, defined as

$$p^{ML}(x) \equiv \lim_{\delta \to 0} \frac{\int_{B(x,\delta)} ML(x^*) dx^*}{\int_{B(x,\delta)} dx^*},$$

where $B(x, \delta)$ is a p-dimensional ball with radius δ centered at x. Conditional on the Quasi Propensity Score, algorithmic decisions are quasi-randomly assigned. The Quasi Propensity Score provides an easy-to-check condition for what causal effects the data from an algorithm allow us to identify; non-degeneracy of the Quasi Propensity Score is both sufficient and necessary for identification of average causal effects conditional on covariates. In particular, we show that the conditional local average treatment effect (LATE; Imbens and Angrist, 1994) is identified for the subpopulation with nondegenerate Quasi Propensity Score.

Based on the identification analysis, we offer a way of estimating treatment effects using the algorithm-produced data. The treatment effects can be estimated by two-stage least squares

(2SLS) where we regress the outcome on the treatment with the algorithm's recommendation as an IV.[1] To make the algorithmic recommendation a conditionally independent IV, we need to control for appropriate variables. Motivated by the fact that algorithmic decision IVs are quasi-randomly assigned conditional on the Quasi Propensity Score, we propose controlling for the Quasi Propensity Score as follows.

1. Standardize each characteristic X_{ij} to have mean zero and variance one for each $j = 1, ..., p$, where p is the number of input characteristics.

2. For small bandwidth $\delta > 0$ and a large number of simulation draws S, compute

$$p^s(X_i; \delta) = \frac{1}{S} \sum_{s=1}^{S} ML(X^*_{i,s}),$$

   where $X^*_{i,1}, ..., X^*_{i,S}$ are S independent simulation draws from the uniform distribution on $B(X_i, \delta)$.[2]

3. Using the observations with $p^s(X_i; \delta) \in (0, 1)$, run the following 2SLS IV regression:

$$D_i = \gamma_0 + \gamma_1 Z_i + \gamma_2 p^s(X_i; \delta) + \nu_i \text{ (First Stage)}$$
$$Y_i = \beta_0 + \beta_1 D_i + \beta_2 p^s(X_i; \delta) + \epsilon_i \text{ (Second Stage)}.$$

   Let $\hat{\beta}^s_1$ be the estimated coefficient on D_i.

As the main theoretical result, we prove the 2SLS estimator $\hat{\beta}^s_1$ is a consistent and asymptotically normal estimator of a well-defined causal effect (weighted average of conditional local average treatment effects). We also show that inference based on the conventional 2SLS heteroskedasticity-robust standard errors is asymptotically valid as long as the bandwidth δ goes to zero at an appropriate rate. We prove the asymptotic properties by exploiting results from differential geometry and geometric measure theory. There appears to be no existing estimator with these asymptotic properties even for a multidimensional RDD, a special case of our framework where the decision-making algorithm is deterministic and uses multiple input (running) variables for determining treatment recommendations. Our estimator can therefore be considered as one of the few consistent and asymptotically normal estimators for a high-dimensional RDD. Moreover, our asymptotic result applies to much more general settings with stochastic algorithms, deterministic algorithms, and combinations of the two.

[1] Recent empirical studies document that algorithmic treatment recommendations have impacts on final treatment assignment by humans (Cowgill, 2018; Bundorf *et al.*, 2019).

[2] For the bandwidth δ, we suggest to the analyst to consider several different values and check if the 2SLS estimates are robust to bandwidth changes, as we often do in regression discontinuity design (RDD) applications. It is hard to pick δ in a data-driven way. Common methods for bandwidth selection in univariate RDDs include Imbens and Kalyanaraman (2012) and Calonico, Cattaneo and Titiunik (2014), who estimate the bandwidth that minimizes the asymptotic mean squared error (AMSE). It is not straightforward to estimate the AMSE-optimal bandwidth in our setting with many running variables and complex IV assignment, since it requires nonparametric estimation of functions on the high-dimensional covariate space such as conditional mean functions, their derivatives, the curvature of the RDD boundary, etc.

The practical value of our estimator is demonstrated through simulation and an original application. We first conduct a Monte Carlo simulation mimicking real-world decision-making based on machine learning. We consider a data-generating process combining stochastic and deterministic algorithms. Treatment recommendations are randomly assigned for a small experimental segment of the population and are determined by a deterministic machine learning algorithm for the rest of the population. The deterministic algorithm uses high-dimensional predictors. Our estimator is shown to have smaller mean squared errors compared to alternative estimators.

Our empirical application is an analysis of COVID-19 relief funding. The Coronavirus Aid, Relief, and Economic Security (CARES) Act and Paycheck Protection Program designated $175 billion for COVID-19 response efforts and reimbursement to health care entities for expenses or lost revenues (Kakani, Chandra, Mullainathan and Obermeyer, 2020), as "*financially insecure hospitals may be less capable of investing in COVID-19 response efforts*" (Khullar, Bond and Schpero, 2020). We ask whether this problem is alleviated by the relief funding to hospitals.

We identify the causal effects of the relief funding by exploiting the funding eligibility rule. The government employs an algorithmic rule to decide which hospitals are eligible for funding. This fact allows us to apply our 2SLS with Quasi-Propensity-Score controls to estimate the effect of relief funding. Specifically, 2SLS estimators use eligibility status as an instrumental variable for funding amounts, while controlling for the Quasi Propensity Score induced by the eligibility-determining algorithm. The resulting estimates suggest that COVID-19 relief funding has little effect on outcomes, such as the number of COVID-19 patients hospitalized and in ICU at each hospital. The estimated effects of causal relief funding are much smaller and less significant than the naive ordinary least squares (OLS) or 2SLS estimates with no controls. This finding contributes to emerging work on how healthcare providers respond to financial shocks (Duggan, 2000; Dranove, Garthwaite and Ody, 2017).

Related Literature

Theoretically, our framework integrates the classic propensity-score (selection-on-observables) scenario with a multidimensional extension of the RDD. We analyze this integrated setup in the IV world with noncompliance. Our estimator applies to this general setting, which allows for both stochastic IV assignment (the propensity-score scenario) and deterministic IV assignment (the high-dimensional RDD). This general setting appears to have no prior established estimator. Armstrong and Kolesár (2020) provide an estimator for a related setting with perfect compliance.[3]

When we adapt our estimator to the sharp multidimensional RDD case, our estimator has three features. First, it is a consistent and asymptotically normal estimator of a well-interpreted causal effect (average of conditional treatment effects along the RDD boundary) even if treatment effects are heterogeneous. Second, it uses observations near all the boundary points as opposed to using only observations near one specific boundary point, which avoids variance explosion

[3]Building on their prior work (Armstrong and Kolesár, 2018), Armstrong and Kolesár (2020) consider estimation and inference on average treatment effects under the assumption that the final treatment assignment is independent of potential outcomes conditional on observables. Their estimator is not applicable to the IV world we consider. Their method and our method also achieve different goals; their goal lies in finite-sample optimality and asymptotically valid inference while our goal is to obtain consistency and asymptotic normality.

even when X_i is high dimensional. Third, it can be easily implemented even in cases with high-dimensional data and complex algorithms (RDD boundaries), where identifying the decision boundary from a general decision algorithm is hard. No prior estimator appears to have all of these properties (Papay, Willett and Murnane, 2011; Zajonc, 2012; Wong, Steiner and Cook, 2013; Keele and Titiunik, 2015; Cattaneo, Titiunik, Vazquez-Bare and Keele, 2016; Imbens and Wager, 2019). In Appendix A.1, we provide a detailed review of the most closely related papers on the multidimensional RDD.

The Quasi Propensity Score used in this paper also shares its spirit with the local random assignment interpretation of the RDD, discussed by Hahn, Todd and van der Klaauw (2001), Frölich (2007), Cattaneo, Frandsen and Titiunik (2015), Cattaneo, Titiunik and Vazquez-Bare (2017), Frandsen (2017), Sekhon and Titiunik (2017), Frölich and Huber (2019) and Abdulkadiroğlu *et al.* (Forthcoming). These papers provide special cases of this paper's framework.

Substantively, there are heated discussions about whether algorithmic decisions are "better" than human decisions, where "better" is in terms of fairness and efficiency (Hoffman, Kahn and Li, 2017; Horton, 2017; Kleinberg, Lakkaraju, Leskovec, Ludwig and Mullainathan, 2017). In this study, we take a complementary perspective in that we take a decision algorithm as given, no matter whether it is good or bad, and study how to use its produced data for impact evaluation.

This paper also relates to the emerging literature on the integration of machine learning, causal inference, and the social sciences. While we are interested in machine learning as a data-*production* tool, the existing literature (except the above mentioned strand) focuses on machine learning as a data-*analysis* tool. For example, a set of predictive studies applies machine learning to make predictions important for social policy questions (Kleinberg *et al.*, 2017; Einav, Finkelstein, Mullainathan and Obermeyer, 2018). Another set of causal and structural work repurposes machine learning to aid with causal inference and structural econometrics (Athey and Imbens, 2017; Belloni, Chernozhukov, Fernández-Val and Hansen, 2017; Bonhomme, Lamadon and Manresa, 2019; Mullainathan and Spiess, 2017). We supplement these studies by highlighting the role of machine learning as a data-production tool.

2 Framework

Our framework is a mix of the conditional independence, high-dimensional RDD, and instrumental variable scenarios. We are interested in the effect of some binary treatment $D_i \in \{0, 1\}$ on some outcome of interest $Y_i \in \mathbb{R}$ in the setup in the introduction. As is standard in the literature, we impose the exclusion restriction that the treatment recommendation $Z_i \in \{0, 1\}$ does not affect the observed outcome other than through the treatment assignment D_i. This allows us to define the potential outcomes indexed against the treatment assignment D_i alone.[4]

We consider algorithms that make treatment recommendations based solely on individual i's predetermined, observable covariates $X_i = (X_{i1}, ..., X_{ip})' \in \mathbb{R}^p$. Let the function $ML : \mathbb{R}^p \to [0, 1]$ represent the decision algorithm, where $ML(X_i) = \Pr(Z_i = 1|X_i)$ is the probability that

[4]Formally, let $Y_i(d, z)$ denote the potential outcome that would be realized if i's treatment assignment and recommendation were d and z, respectively. The exclusion restriction assumes that $Y_i(d, 1) = Y_i(d, 0)$ for $d \in \{0, 1\}$ (Imbens and Angrist, 1994).

the treatment is recommended for individual i with covariates X_i.[5] We assume that the analyst knows the algorithm ML and is able to simulate it. That is, the analyst is able to compute the recommendation probability $ML(x)$ given any input value $x \in \mathbb{R}^p$. In typical machine-learning scenarios, an algorithm first applies machine learning on X_i to make some prediction and then uses the prediction to output the recommendation probability $ML(X_i)$. The treatment recommendation Z_i for individual i is then randomly determined based on the probability $ML(X_i)$ independently of everything else. Consequently, the following conditional independence property holds regardless of how the algorithm is constructed and how the algorithm computes a recommendation probability.

Property 1 (Conditional Independence)**.** $Z_i \perp\!\!\!\perp (Y_i(1), Y_i(0), D_i(1), D_i(0)) | X_i$.

Let Y_{zi} be defined as $Y_{zi} \equiv D_i(z)Y_i(1) + (1 - D_i(z))Y_i(0)$ for $z \in \{0, 1\}$. Y_{zi} is the potential outcome when the treatment recommendation is $Z_i = z$. It follows from Property 1 that $Z_i \perp\!\!\!\perp (Y_{1i}, Y_{0i}) | X_i$.

Note that the codomain of ML contains 0 and 1, allowing for deterministic treatment assignments conditional on X_i. Our framework therefore nests the RDD as a special case.[6] Another special case of our framework is the classic conditional independence scenario with the common support condition ($ML(X_i) \in (0, 1)$ almost surely). In addition to these simple settings, this framework nests many other situations, such as multidimensional RDDs and complex machine learning and market-design algorithms, as illustrated in Section 7.

We put a few assumptions on the covariates X_i and the ML algorithm. To simplify the exposition, the main text assumes that the distribution of X_i is absolutely continuous with respect to the Lebesgue measure. In practice, the input variables of an algorithm often include discrete variables. Appendix A.3 extends the analysis and proposed method to the case where some covariates in X_i are discrete. Let $\mathcal{X}$ be the support of X_i, $\mathcal{X}_0 = \{x \in \mathcal{X} : ML(x) = 0\}$, $\mathcal{X}_1 = \{x \in \mathcal{X} : ML(x) = 1\}$, $\mathcal{L}^p$ be the Lebesgue measure on $\mathbb{R}^p$, and $\text{int}(A)$ denote the interior of a set $A \subset \mathbb{R}^p$.

Assumption 1.

(a) (Almost Everywhere Continuity of ML) *ML is continuous almost everywhere with respect to the Lebesgue measure.*

(b) (Measure Zero Boundaries of $\mathcal{X}_0$ and $\mathcal{X}_1$) $\mathcal{L}^p(\mathcal{X}_k) = \mathcal{L}^p(\text{int}(\mathcal{X}_k))$ *for* $k = 0, 1$.

Assumption 1 (a) allows the function ML to be discontinuous on a set of points with the Lebesgue measure zero. For example, ML is allowed to be a discontinuous step function as long as it is continuous almost everywhere. Assumption 1 (b) holds if the Lebesgue measures of the boundaries of $\mathcal{X}_0$ and $\mathcal{X}_1$ are zero.

[5] We assume that the function ML is supported on $\mathbb{R}^p$ irrespective of the support of X_i.

[6] Most of the existing studies on RDDs define the potential treatment assignment indexed against the running variable like $D_i(x)$, which represents the counterfactual treatment assignment the individual i would have received if her running variable had been set to x. Unlike prior work, we define it indexed against the treatment recommendation z.

3 Identification

What causal effects can be learned from data (Y_i, X_i, D_i, Z_i) generated by the ML algorithm? A key step toward answering this question is what we call the *Quasi Propensity Score* (QPS). To define it, let:

$$p^{ML}(x;\delta) \equiv \frac{\int_{B(x,\delta)} ML(x^*)dx^*}{\int_{B(x,\delta)} dx^*},$$

where $B(x,\delta) = \{x^* \in \mathbb{R}^p : \|x - x^*\| < \delta\}$ is the (open) δ-ball around $x \in \mathcal{X}$.[7] Here, $\|\cdot\|$ denotes the Euclidean distance on $\mathbb{R}^p$. To make common δ for all dimensions reasonable, we normalize X_{ij} to have mean zero and variance one for each $j = 1, ..., p$.[8] We assume that ML is a $\mathcal{L}^p$-measurable function so that the integrals exist. We then define QPS as follows:

$$p^{ML}(x) \equiv \lim_{\delta \to 0} p^{ML}(x;\delta).$$

QPS at x is the average probability of a treatment recommendation in a shrinking ball around x.[9] We call this the *Quasi* Propensity Score, since this score modifies the standard propensity score $ML(X_i)$ to incorporate local variation in the score. We discuss when QPS exists in Section 3.1.

Figure 1 illustrates QPS. In the example, X_i is two dimensional, and the support of X_i is divided into three sets depending on the value of ML. For the interior points of each set, QPS is equal to ML (as formally implied by Part 2 of Corollary 2 below). On the border of any two sets, QPS is the average of the ML values in the two sets. Thus, $p^{ML}(x) = \frac{1}{2}(0 + 0.5) = 0.25$ for any x in the open line segment AB, $p^{ML}(x) = \frac{1}{2}(0.5 + 1) = 0.75$ for any x in the open line segment BC, and $p^{ML}(x) = \frac{1}{2}(0 + 1) = 0.5$ for any x in the open line segment BD.

QPS provides an easy-to-check condition for whether an algorithm allows us to identify causal effects. Here we say that a causal effect is *identified* if it is uniquely determined by the joint distribution of (Y_i, X_i, D_i, Z_i). Our identification analysis uses the following continuity condition.

Assumption 2 (Local Mean Continuity)**.** *For $z \in \{0,1\}$, the conditional expectation functions $E[Y_{zi}|X_i]$ and $E[D_i(z)|X_i]$ are continuous at any point $x \in \mathcal{X}$ such that $p^{ML}(x) \in (0,1)$ and $ML(x) \in \{0,1\}$.*

[7] Whether we use an open ball or closed ball does not affect $p^{ML}(x;\delta)$. When we instead use a rectangle, ellipsoid, or any standard kernel function to define $p^{ML}(x;\delta)$, the limit $\lim_{\delta\to 0} p^{ML}(x;\delta)$ may be different at some points (e.g., at discontinuity points of ML), but the same identification results hold under suitable conditions. We use a ball for simplicity and practicality.

[8] This normalization is without loss of generality in the following sense. Take a vector X_i^* of any continuous random variables and $ML^* : \mathbb{R}^p \to [0,1]$. The normalization induces the random vector $X_i = A(X_i^* - E[X_i^*])$, where A is a diagonal matrix with diagonal entries $\frac{1}{\mathrm{Var}(X_{i1}^*)^{1/2}}, ..., \frac{1}{\mathrm{Var}(X_{ip}^*)^{1/2}}$. Let $ML(x) = ML^*(A^{-1}x + E[X_i^*])$. Then (X_i^*, ML^*) is equivalent to (X_i, ML) in the sense that $ML(X_i) = ML^*(X_i^*)$ for any individual i.

[9] The idea behind QPS shares some of its spirit with the local randomization interpretation of RDDs (Frölich, 2007; Cattaneo *et al.*, 2015, 2017): the treatment assignment is considered as good as randomly assigned in a neighborhood of the cutoff.

Assumption 2 is a natural multivariate extension of the local mean continuity condition that is frequently assumed in the RDD.[10] $ML(x) \in \{0,1\}$ means that the treatment recommendation Z_i is deterministic conditional on $X_i = x$. If QPS at the point x is nondegenerate ($p^{ML}(x) \in (0,1)$), however, there exists a point close to x that has a different value of ML from x's, which creates variation in the treatment recommendation near x. For any such point x, Assumption 2 requires that the points close to x have similar conditional means of the outcome Y_{zi} and treatment assignment $D_i(z)$.[11] Note that Assumption 2 does not require continuity of the conditional means at x for which $ML(x) \in (0,1)$, since the identification of the conditional means at such points follows from Property 1 without continuity.

Under the above assumptions, the following identification result holds.

Proposition 1 (Identification)**.** *Under Assumptions 1 and 2:*

(a) $E[Y_{1i} - Y_{0i}|X_i = x]$ *and* $E[D_i(1) - D_i(0)|X_i = x]$ *are identified for every* $x \in \text{int}(\mathcal{X})$ *such that* $p^{ML}(x) \in (0,1)$.[12]

(b) Let A *be any open subset of* $\mathcal{X}$ *such that* $p^{ML}(x)$ *exists for all* $x \in A$*. Then either* $E[Y_{1i} - Y_{0i}|X_i \in A]$ *or* $E[D_i(1) - D_i(0)|X_i \in A]$ *or both are identified only if* $p^{ML}(x) \in (0,1)$ *for almost every* $x \in A$ *(with respect to the Lebesgue measure).*[13]

Proof. See Appendix C.1. □

Proposition 1 characterizes a necessary and sufficient condition for identification. Part (a) says that the average effects of the treatment recommendation Z_i on the outcome Y_i and on the treatment assignment D_i for the individuals with $X_i = x$ are both identified if QPS at x is neither 0 nor 1. Non-degeneracy of QPS at x implies that there are both types of individuals who receive $Z_i = 1$ and $Z_i = 0$ among those whose X_i is close to x. Assumption 2 ensures that these individuals are similar in terms of average potential outcomes and treatment assignments. We can therefore identify the average effects conditional on $X_i = x$. In Figure 1, $p^{ML}(x) \in (0,1)$ holds for any x in the shaded region (the union of the minor circular segment made by the chord AC and the line segment BD).

[10] In the RDD with a single running variable, the point x for which $p^{ML}(x) \in (0,1)$ and $ML(x) \in \{0,1\}$ is the cutoff point at which the treatment probability discontinuously changes.

[11] In the context of the RDD with a single running variable, one sufficient condition for continuity of $E[Y_{zi}|X_i]$ is a local independence condition in the spirit of Hahn *et al.* (2001): $(Y_i(1), Y_i(0), D_i(1), D_i(0))$ is independent of X_i near x. A weaker sufficient condition, which allows such dependence, is that $E[Y_i(d)|D_i(1) = d_1, D_i(0) = d_0, X_i]$ and $\Pr(D_i(1) = d_1, D_i(0) = d_0|X_i)$ are continuous at x for every $d \in \{0,1\}$ and $(d_1, d_0) \in \{0,1\}^2$ (Dong, 2018). This assumes that the conditional means of the potential outcomes for each of the four types determined based on the potential treatment assignment $D_i(z)$ and the conditional probabilities of those types are continuous at the cutoff. These two sets of conditions are sufficient for continuity of $E[Y_{zi}|X_i]$ regardless of the dimension of X_i, accommodating multidimensional RDDs.

[12] The causal effects may not be identified at a boundary point x of $\mathcal{X}$ for which $p^{ML}(x) \in (0,1)$. For example, if $ML(x^*) = 1$ for all $x^* \in B(x,\delta) \cap \mathcal{X}$ and $ML(x^*) = 0$ for all $x^* \in B(x,\delta) \setminus \mathcal{X}$ for any sufficiently small $\delta > 0$, $p^{ML}(x) \in (0,1)$ but the causal effects are not identified at x since $\Pr(Z_i = 0|X_i \in B(x,\delta)) = 0$.

[13] We assume that p^{ML} is a $\mathcal{L}^p$-measurable function so that $\{x \in A : p^{ML}(x) = 0\}$ and $\{x \in A : p^{ML}(x) = 1\}$ are $\mathcal{L}^p$-measurable.

A consequence of Part (a) is that it is possible to identify $\int_{\{x^* \in \text{int}(\mathcal{X}): p^{ML}(x^*) \in (0,1)\}} \omega(x) E[Y_{1i} - Y_{0i} | X_i = x] d\mu(x)$ and $\int_{\{x^* \in \text{int}(\mathcal{X}): p^{ML}(x^*) \in (0,1)\}} \omega(x) E[D_i(1) - D_i(0) | X_i = x] d\mu(x)$ for any known or identified function $\omega : \mathbb{R}^p \to \mathbb{R}$ and any measure μ provided that the integrals exist.

Part (a) nests two well-known identification results as special cases. First, suppose that $ML(x) \in (0,1)$ for every $x \in \mathcal{X}$. This corresponds to the classic conditional independence setting (or stratified randomization setting) with nondegenerate assignment probability, in which conditional average causal effects are identified (see for example Angrist and Pischke (2008)). Second, suppose that $ML(x) \in \{0,1\}$ for all $x \in \mathcal{X}$ but the value of ML discontinuously changes at some point x^* so that $p^{ML}(x^*) \in (0,1)$. This case corresponds to an RDD, in which the average causal effect at a boundary point is identified under continuity of conditional expectation functions of potential outcomes (Hahn *et al.*, 2001; Keele and Titiunik, 2015).

Part (b) provides a necessary condition for identification. It says that if the average effect of the treatment recommendation conditional on X_i being in some open set A is identified, then we must have $p^{ML}(x) \in (0,1)$ for almost every $x \in A$. If, to the contrary, there is a subset of A of nonzero measure for which $p^{ML}(x) = 1$ (or $p^{ML}(x) = 0$), then Z_i has no variation in the subset, which makes it impossible to identify the average effect for the subset.

Proposition 1 concerns causal effects of treatment *recommendation*, not of treatment *assignment*. The proposition implies that the conditional average treatment effects and the conditional local average treatment effects (LATEs) are identified under additional assumptions.

Corollary 1 (Perfect and Imperfect Compliance)**.** *Under Assumptions 1 and 2:*

(a) The average treatment effect conditional on $X_i = x$, $E[Y_i(1) - Y_i(0) | X_i = x]$, is identified for every $x \in \text{int}(\mathcal{X})$ such that $p^{ML}(x) \in (0,1)$ and $\Pr(D_i(1) > D_i(0) | X_i = x) = 1$ (perfect compliance).

(b) Let A be any open subset of $\mathcal{X}$ such that $p^{ML}(x)$ exists for all $x \in A$, and $\Pr(D_i(1) > D_i(0) | X_i \in A) = 1$. Then $E[Y_i(1) - Y_i(0) | X_i \in A]$ is identified only if $p^{ML}(x) \in (0,1)$ for almost every $x \in A$.

(c) The local average treatment effect conditional on $X_i = x$, $E[Y_i(1) - Y_i(0) | D_i(1) \neq D_i(0), X_i = x]$, is identified for every $x \in \text{int}(\mathcal{X})$ such that $p^{ML}(x) \in (0,1)$, $\Pr(D_i(1) \geq D_i(0) | X_i = x) = 1$ (monotonicity), and $\Pr(D_i(1) \neq D_i(0) | X_i = x) > 0$ (existence of compliers).

(d) Let A be any open subset of $\mathcal{X}$ such that $p^{ML}(x)$ exists for all $x \in A$, $\Pr(D_i(1) \geq D_i(0) | X_i \in A) = 1$, and $\Pr(D_i(1) \neq D_i(0) | X_i \in A) > 0$. Then $E[Y_i(1) - Y_i(0) | D_i(1) \neq D_i(0), X_i \in A]$ is identified only if $p^{ML}(x) \in (0,1)$ for almost every $x \in A$.

Proof. See Appendix C.2. □

Non-degeneracy of QPS $p^{ML}(x)$ therefore summarizes what causal effects the data from ML identify. Note that the key condition ($p^{ML}(x) \in (0,1)$) holds for some points x for every standard algorithm except trivial algorithms that always recommend a treatment with probability 0 or 1. Therefore, the data from almost every algorithm identify some causal effect.

3.1 Existence of the Quasi Propensity Score

The above results assume that QPS exists, but is it fair to assume so? In general, QPS may fail to exist; we provide such an example in Appendix A.2. Nevertheless, it exists for most covariate points and typical ML algorithms. For each $x \in \mathcal{X}$ and each $q \in \text{Supp}(ML(X_i))$, define

$$\mathcal{U}_{x,q} \equiv \{u \in B(\mathbf{0}, 1) : \lim_{\delta \to 0} ML(x + \delta u) = q\},$$

where $\mathbf{0} \in \mathbb{R}^p$ is a vector of zeros. $\mathcal{U}_{x,q}$ is the set of vectors in $B(\mathbf{0}, 1)$ such that the value of ML approaches q as we approach x from the direction of the vector. With this notation, we obtain a sufficient condition for the existence of QPS at a point x.

Proposition 2. *Take any $x \in \mathcal{X}$. If there exists a countable set $Q \subset \text{Supp}(ML(X_i))$ such that $\mathcal{L}^p(\cup_{q \in Q} \mathcal{U}_{x,q}) = \mathcal{L}^p(B(\mathbf{0}, 1))$ and $\mathcal{U}_{x,q}$ is $\mathcal{L}^p$-measurable for all $q \in Q$, then $p^{ML}(x)$ exists and is given by*

$$p^{ML}(x) = \frac{\sum_{q \in Q} q \mathcal{L}^p(\mathcal{U}_{x,q})}{\mathcal{L}^p(B(\mathbf{0}, 1))}.$$

Proof. See Appendix C.3. □

If almost every point in $B(\mathbf{0}, 1)$ is contained by one of countably many $\mathcal{U}_{x,q}$'s, therefore, QPS exists and is equal to the weighted average of the values of q with the weight proportional to the hypervolume of $\mathcal{U}_{x,q}$. This result implies that QPS exists in practically important cases.

Corollary 2.

1. (Continuity points) *If ML is continuous at $x \in \mathcal{X}$, then $p^{ML}(x)$ exists and $p^{ML}(x) = ML(x)$.*

2. (Interior points) *Let $\mathcal{X}_q = \{x \in \mathcal{X} : ML(x) = q\}$ for some $q \in [0, 1]$. Then, for any interior point $x \in \text{int}(\mathcal{X}_q)$, $p^{ML}(x)$ exists and $p^{ML}(x) = q$.*

3. (Smooth boundary points) *Suppose that $\{x \in \mathcal{X} : ML(x) = q_1\} = \{x \in \mathcal{X} : f(x) \geq 0\}$ and $\{x \in \mathcal{X} : ML(x) = q_2\} = \{x \in \mathcal{X} : f(x) < 0\}$ for some $q_1, q_2 \in [0, 1]$, where $f : \mathbb{R}^p \to \mathbb{R}$. Let $x \in \mathcal{X}$ be a boundary point such that $f(x) = 0$, and suppose that f is continuously differentiable in a neighborhood of x with $\nabla f(x) = (\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_p})' \neq \mathbf{0}$. In this case, $p^{ML}(x)$ exists and $p^{ML}(x) = \frac{1}{2}(q_1 + q_2)$.*

4. (Intersection points under CART and random forests) *Let $p = 2$, and suppose that $\{x \in \mathcal{X} : ML(x) = q_1\} = \{(x_1, x_2)' \in \mathcal{X} : x_1 \leq 0 \text{ or } x_2 \leq 0\}$, $\{x \in \mathcal{X} : ML(x) = q_2\} = \{(x_1, x_2)' \in \mathcal{X} : x_1 > 0, x_2 > 0\}$, and $\mathbf{0} = (0, 0)' \in \mathcal{X}$. This is an example in which tree-based algorithms such as Classification And Regression Tree (CART) and random forests are used to create ML. In this case, $p^{ML}(\mathbf{0})$ exists and $p^{ML}(\mathbf{0}) = \frac{3}{4}q_1 + \frac{1}{4}q_2$.*

Proof. See Appendix C.4. □

4 Estimation

The sources of quasi-random assignment characterized in Proposition 1 suggest a way of estimating causal effects of the treatment. In view of Proposition 1, it is possible to nonparametrically estimate conditional average causal effects $E[Y_{1i} - Y_{0i}|X_i = x]$ and $E[D_i(1) - D_i(0)|X_i = x]$ for points x such that $p^{ML}(x) \in (0, 1)$. This approach is hard to use in practice, however, when X_i is high dimensional.

We instead seek an estimator that aggregates conditional effects at different points into a single average causal effect. Proposition 1 suggests that conditioning on QPS makes algorithm-based treatment recommendation quasi-randomly assigned. This motivates the use of an algorithm's recommendation as an instrument conditional on QPS, which we operationalize as follows.

4.1 Two-Stage Least Squares Meets QPS

Suppose that we observe a random sample $\{(Y_i, X_i, D_i, Z_i)\}_{i=1}^n$ of size n from the population whose data generating process is described in the introduction and Section 2. Consider the following 2SLS regression using the observations with $p^{ML}(X_i; \delta_n) \in (0, 1)$:

$$D_i = \gamma_0 + \gamma_1 Z_i + \gamma_2 p^{ML}(X_i; \delta_n) + \nu_i \tag{1}$$

$$Y_i = \beta_0 + \beta_1 D_i + \beta_2 p^{ML}(X_i; \delta_n) + \epsilon_i, \tag{2}$$

where bandwidth δ_n shrinks toward zero as the sample size n increases. Let $I_{i,n} = 1\{p^{ML}(X_i; \delta_n) \in (0, 1)\}$, $\mathbf{D}_{i,n} = (1, D_i, p^{ML}(X_i; \delta_n))'$, and $\mathbf{Z}_{i,n} = (1, Z_i, p^{ML}(X_i; \delta_n))'$. The 2SLS estimator $\hat{\beta}$ is then given by

$$\hat{\beta} = (\sum_{i=1}^n \mathbf{Z}_{i,n}\mathbf{D}'_{i,n}I_{i,n})^{-1} \sum_{i=1}^n \mathbf{Z}_{i,n}Y_i I_{i,n}.$$

Let $\hat{\beta}_1$ denote the 2SLS estimator of β_1 in the above regression.

The above regression uses true QPS $p^{ML}(X_i; \delta_n)$, but it may be difficult to analytically compute if ML is complex. In such a case, we propose to approximate $p^{ML}(X_i; \delta_n)$ using brute force simulation. We draw a value of x from the uniform distribution on $B(X_i, \delta_n)$ a number of times, compute $ML(x)$ for each draw, and take the average of $ML(x)$ over the draws.[14] Formally, let $X^*_{i,1}, ..., X^*_{i,S_n}$ be S_n independent draws from the uniform distribution on $B(X_i, \delta_n)$, and calculate

$$p^s(X_i; \delta_n) = \frac{1}{S_n} \sum_{s=1}^{S_n} ML(X^*_{i,s}).$$

We compute $p^s(X_i; \delta_n)$ for each $i = 1, ..., n$ independently across i so that $p^s(X_1; \delta_n), ..., p^s(X_n; \delta_n)$ are independent of each other. For fixed n and X_i, the approximation error relative to true $p^{ML}(X_i; \delta_n)$ has a $1/\sqrt{S_n}$ rate of convergence.[15] This rate does not depend on the dimension of X_i, so the simulation error can be made negligible even when X_i is high dimensional.

[14] See Appendix A.5 for how to efficiently sample from the uniform distribution on a p-dimensional ball.

[15] More precisely, we have $|p^s(X_i; \delta_n) - p^{ML}(X_i; \delta_n)| = O_{p^s}(1/\sqrt{S_n})$, where O_{p^s} indicates the stochastic boundedness in terms of the probability distribution of the S_n simulation draws.

Now consider the following simulation version of the 2SLS regression using the observations with $p^s(X_i;\delta_n) \in (0,1)$:

$$D_i = \gamma_0 + \gamma_1 Z_i + \gamma_2 p^s(X_i;\delta_n) + \nu_i \tag{3}$$

$$Y_i = \beta_0 + \beta_1 D_i + \beta_2 p^s(X_i;\delta_n) + \epsilon_i. \tag{4}$$

Let $\hat{\beta}_1^s$ denote the 2SLS estimator of β_1 in the simulation-based regression. This regression is the same as the 2SLS regression (1) and (2) except that we use the simulated QPS $p^s(X_i;\delta_n)$ in place of $p^{ML}(X_i;\delta_n)$.[16]

4.2 Consistency and Asymptotic Normality

We establish the consistency and asymptotic normality of the 2SLS estimators $\hat{\beta}_1$ and $\hat{\beta}_1^s$. Our consistency and asymptotic normality result uses the following assumptions.

Assumption 3.

(a) (Finite Moment) $E[Y_i^4] < \infty$.

(b) (Nonzero First Stage) *There exists a constant* $c > 0$ *such that* $E[D_i(1) - D_i(0)|X_i = x] > c$ *for every* $x \in \mathcal{X}$ *such that* $p^{ML}(x) \in (0,1)$.

(c) (Nonzero Conditional Variance) *If* $\Pr(ML(X_i) \in (0,1)) > 0$, *then* $\mathrm{Var}(ML(X_i)|ML(X_i) \in (0,1)) > 0$.

If $\Pr(ML(X_i) \in (0,1)) = 0$, *then the following conditions (d)–(g) hold.*

(d) (Nonzero Variance) $\mathrm{Var}(ML(X_i)) > 0$.

For a set $A \subset \mathbb{R}^p$, *let* $\mathrm{cl}(A)$ *denote the closure of* A *and let* ∂A *denote the boundary of* A, *i.e.,* $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A)$.

(e) (C^2 Boundary of Ω^*) *There exists a partition* $\{\Omega_1^*, ..., \Omega_M^*\}$ *of* $\Omega^* = \{x \in \mathbb{R}^p : ML(x) = 1\}$ *(the set of the covariate points whose* ML *value is one) such that*

(i) $\mathrm{dist}(\Omega_m^*, \Omega_{m'}^*) > 0$ *for any* $m, m' \in \{1, ..., M\}$ *such that* $m \neq m'$. *Here* $\mathrm{dist}(A, B) = \inf_{x \in A, y \in B} \|x - y\|$ *is the distance between two sets* A *and* $B \subset \mathbb{R}^p$;

(ii) Ω_m^* *is nonempty, bounded, open, connected and twice continuously differentiable for each* $m \in \{1, ..., M\}$. *Here we say that a bounded open set* $A \subset \mathbb{R}^p$ *is twice continuously differentiable if for every* $x \in A$, *there exists a ball* $B(x, \epsilon)$ *and a one-to-one mapping*

[16] In many industry and policy applications, the analyst is only able to change the algorithm's recommendation Z_i by redesigning the algorithm. In this case, the effect of recommendation Z_i on outcome Y_i may also be of interest to the practitioner. We can estimate the effect of recommendation by running the following ordinary least squares (OLS) regression using the observations with $p^s(X_i;\delta) \in (0,1)$:

$$Y_i = \alpha_0 + \alpha_1 Z_i + \alpha_2 p^s(X_i;\delta) + u_i.$$

The estimated coefficient on Z_i, $\hat{\alpha}_1^s$, is our preferred estimator of the recommendation effect.

ψ from $B(x,\epsilon)$ onto an open set $D \subset \mathbb{R}^p$ such that ψ and ψ^{-1} are twice continuously differentiable, $\psi(B(x,\epsilon) \cap A) \subset \{(x_1,...,x_p) \in \mathbb{R}^p : x_p > 0\}$ and $\psi(B(x,\epsilon) \cap \partial A) \subset \{(x_1,...,x_p) \in \mathbb{R}^p : x_p = 0\}$.

Let f_X denote the probability density function of X_i and let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^p$.[17]

(f) (Regularity of Deterministic ML)

(i) *$\mathcal{H}^{p-1}(\partial\Omega^*) < \infty$, and $\int_{\partial\Omega^*} f_X(x) d\mathcal{H}^{p-1}(x) > 0$.*

(ii) *There exists $\delta > 0$ such that $ML(x) = 0$ for almost every $x \in N(\mathcal{X},\delta) \setminus \Omega^*$, where $N(A,\delta) = \{x \in \mathbb{R}^p : \|x - y\| < \delta \text{ for some } y \in A\}$ for a set $A \subset \mathbb{R}^p$ and $\delta > 0$.*

(g) (Conditional Moments and Density near $\partial\Omega^*$) *There exists $\delta > 0$ such that*

(i) *$E[Y_{1i}|X_i]$, $E[Y_{0i}|X_i]$, $E[D_i(1)|X_i]$, $E[D_i(0)|X_i]$ and f_X are continuously differentiable and have bounded partial derivatives on $N(\partial\Omega^*,\delta)$;*

(ii) *$E[Y_{1i}^2|X_i]$, $E[Y_{0i}^2|X_i]$, $E[Y_{1i}D_i(1)|X_i]$ and $E[Y_{0i}D_i(0)|X_i]$ are continuous on $N(\partial\Omega^*,\delta)$;*

(iii) *$E[Y_i^4|X_i]$ is bounded on $N(\partial\Omega^*,\delta)$.*

Assumption 3 is a set of conditions for establishing consistency. Assumption 3 (b) assumes that, conditional on each value of X_i for which QPS is nondegenerate, more individuals would change their treatment assignment status from 0 to 1 in response to treatment recommendation than would change it from 1 to 0.[18] Under this assumption, the estimated first-stage coefficient on Z_i converges to a positive quantity. Note that, if there exists $c < 0$ such that $E[D_i(1)-D_i(0)|X_i = x] < c$ for every $x \in \mathcal{X}$ with $p^{ML}(x) \in (0,1)$, changing the labels of treatment recommendation makes Assumption 3 (b) hold.

Assumption 3 (c) rules out potential multicollinearity. If the support of $ML(X_i)$ contains only one value in $(0,1)$, $p^{ML}(X_i;\delta_n)$ is asymptotically constant and equal to $ML(X_i)$ conditional on $p^{ML}(X_i;\delta_n) \in (0,1)$, resulting in the multicollinearity between $p^{ML}(X_i;\delta_n)$ and the constant term. Although dropping the constant term from the 2SLS regression solves this issue, Assumption 3 (c) allows us to only consider the regression with a constant for the purpose of simplifying the presentation. In Appendix C.6, we provide 2SLS estimators that are consistent and asymptotically normal even if we do not know whether Assumption 3 (c) holds.

Assumption 3 (d)–(g) are a set of conditions we require for proving consistency and asymptotic normality of $\hat{\beta}_1$ when ML is deterministic and produces only multidimensional regression-discontinuity variation. Assumption 3 (d) says that ML produces variation in the treatment recommendation.

[17] The k-dimensional Hausdorff measure on $\mathbb{R}^p$ is defined as follows. Let Σ be the Lebesgue σ-algebra on $\mathbb{R}^p$ (the set of all Lebesgue measurable sets on $\mathbb{R}^p$). For $A \in \Sigma$ and $\delta > 0$, let $\mathcal{H}^k_\delta(A) = \inf\{\sum_{j=1}^\infty d(E_j)^k : A \subset \cup_{j=1}^\infty E_j, d(E_j) < \delta, E_j \subset \mathbb{R}^p \text{ for all } j\}$, where $d(E) = \sup\{\|x - y\| : x, y \in E\}$. The k-dimensional Hausdorff measure of A on $\mathbb{R}^p$ is $\mathcal{H}^k(A) = \lim_{\delta\to 0} \mathcal{H}^k_\delta(A)$.

[18] At the cost of making the presentation more complex, the assumption can be relaxed so that the sign of $E[D_i(1) - D_i(0)|X_i = x]$ is allowed to vary over x with $p^{ML}(x) \in (0,1)$.

Assumption 3 (e) imposes the differentiability of the boundary of $\Omega^* = \{x \in \mathbb{R}^p : ML(x) = 1\}$. The conditions are satisfied if, for example, $\Omega^* = \{x \in \mathbb{R}^p : f(x) \geq 0\}$ for some twice continuously differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ such that $\nabla f(x) \neq \mathbf{0}$ for all $x \in \mathbb{R}^p$ with $f(x) = 0$. Ω^* takes this form, for example, when the conditional treatment effect $E[Y_i(1) - Y_i(0)|X]$ is predicted by supervised learning based on smooth models such as lasso and ridge regressions, and treatment is recommended to individuals who are estimated to experience nonnegative treatment effects.

In general, the differentiability of Ω^* may not hold. For example, if tree-based algorithms such as Classification And Regression Tree (CART) and random forests are used to predict the conditional treatment effect, the predicted conditional treatment effect function is not differentiable at some points. Although the resulting Ω^* does not exactly satisfy Assumption 3 (e), the assumptions approximately hold in that Ω^* is arbitrarily well approximated by a set that satisfies the differentiability condition.[19]

Part (i) of Assumption 3 (f) says that the boundary of Ω^* is $(p-1)$ dimensional and that the boundary has nonzero density. Part (ii) puts a weak restriction on the values ML takes on outside the support of X_i. It requires that $ML(x) = 0$ for almost every $x \notin \Omega^*$ that is outside $\mathcal{X}$ but is in the neighborhood of $\mathcal{X}$. $ML(x)$ may take on any value if x is not close to $\mathcal{X}$. These conditions hold in practice.[20]

Assumption 3 (g) imposes continuity, continuous differentiability and boundedness on the conditional moments of potential outcomes and the probability density near the boundary of Ω^*.

When ML is stochastic, asymptotic normality requires additional assumptions. Let

$$C^* = \{x \in \mathbb{R}^p : ML \text{ is continuously differentiable at } x\},$$

and let $D^* = \mathbb{R}^p \setminus C^*$ be the set of points at which ML is not continuously differentiable.

Assumption 4. *If* $\Pr(ML(X_i) \in (0,1)) > 0$*, then the following conditions (a)–(c) hold.*

(a) (Probability of Neighborhood of D^*) $\Pr(X_i \in N(D^*, \delta)) = O(\delta)$.

(b) (Bounded Partial Derivatives of ML) *The partial derivatives of ML are bounded on C^*.*

(c) (Bounded Conditional Mean) $E[Y_i|X_i]$ *is bounded on $\mathcal{X}$.*

Assumption 4 is required for proving asymptotic normality of $\hat{\beta}_1$ when ML is stochastic. To explain the role of Assumption 4 (a), consider a path of covariate points $x_\delta \in N(D^*, \delta) \cap C^*$ indexed by $\delta > 0$. Since ML is continuous at x_δ, $p^{ML}(x_\delta) = ML(x_\delta)$ as implied by Part 1 of Corollary 2. However, $p^{ML}(x_\delta; \delta)$ does not necessarily get sufficiently close to $ML(x_\delta)$ even

[19] Consider the example in Part 4 of Corollary 2 with $q_1 = 0$ and $q_2 = 1$. In this example, $\Omega^* = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^2$, where $\Omega_k = \{x \in \mathbb{R}^2 : x_2 \geq \frac{1}{kx_1}, x_1 > 0\}$ for each k. Ω_k is twice continuously differentiable for all k, and well approximates Ω^* for a large k in that $d_H(\Omega^*, \Omega_k) \to 0$ as $k \to \infty$, where $d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}$ is the Hausdorff distance between two sets A and $B \subset \mathbb{R}^p$.

[20] The boundary of Ω^* fails to be $(p-1)$ dimensional, for example, when the covariate space is three dimensional ($p = 3$) and Ω^* is a straight line, not a set with nonzero volume nor even a plane. In this example, the boundary is the same as Ω^*, and its two-dimensional Hausdorff measure is zero.

as $\delta \to 0$, since x_δ is in the δ-neighborhood of D^* and hence ML may discontinuously change within the δ-ball $B(x_\delta, \delta)$. Assumption 4 (a) requires that the probability of X_i being in the δ-neighborhood of D^* shrinks to zero at the rate of δ, which makes the points in the neighborhood negligible.

Assumption 4 (a) often holds in practice. If ML is continuously differentiable on $\mathcal{X}$, then $D^* \cap \mathcal{X} = \emptyset$, so this condition holds. If, for example, the treatment recommendation is randomly assigned based on a stratified randomized experiment or on the ϵ-Greedy algorithm (see Part 2 (a) of Example 1 in Section 7), D^* is the boundary at which the recommendation probability changes discontinuously. For any boundary of standard shape, the probability of X_i being in the δ-neighborhood of the boundary vanishes at the rate of δ, and the required condition is satisfied. We provide a sufficient condition for this condition in Appendix A.4.

Assumption 4 (b) and (c) are regularity conditions, imposing the boundedness of the partial derivatives of ML and of the conditional mean of the outcome.

Assumption 5 (The Number of Simulation Draws)**.** *$n^{-1/2} S_n \to \infty$, and $\Pr(p^{ML}(X_i; \delta_n) \in (0, \gamma \frac{\log n}{S_n}) \cup (1 - \gamma \frac{\log n}{S_n}, 1)) = o(n^{-1/2} \delta_n^{1/2})$ for some $\gamma > \frac{1}{2}$.*

Assumption 5 is the key to proving asymptotic normality of the simulation-based estimator $\hat{\beta}_1^s$. Assumption 5 says that we need to choose the number of simulation draws S_n so that it grows to infinity faster than $n^{1/2}$, and that the probability that $p^{ML}(X_i; \delta_n)$ lies on the tails $(0, \gamma \frac{\log n}{S_n}) \cup (1 - \gamma \frac{\log n}{S_n}, 1)$ vanishes faster than $n^{-1/2} \delta_n^{1/2}$. This condition makes the bias caused by using $p^s(X_i; \delta_n)$ instead of $p^{ML}(X_i; \delta_n)$ asymptotically negligible. To illustrate how the second part of this assumption restricts the rate at which S_n goes to infinity, consider an example where $\Pr(p^{ML}(X_i; \delta_n) \in (0, 1)) = O(\delta_n)$, and $p^{ML}(X_i; \delta_n)$ is approximately uniformly distributed on the tails $(0, \gamma \frac{\log n}{S_n}) \cup (1 - \gamma \frac{\log n}{S_n}, 1)$. In this case, $\Pr(p^{ML}(X_i; \delta_n) \in (0, \gamma \frac{\log n}{S_n}) \cup (1 - \gamma \frac{\log n}{S_n}, 1)) = O(\delta_n \frac{\log n}{S_n})$, and the second part of Assumption 5 requires that S_n grow sufficiently fast so that $\frac{n^{1/2} \delta_n^{1/2} \log n}{S_n} = o(1)$. One choice of S_n satisfying this is $S_n = \alpha n^\kappa \delta_n^{1/2}$ for some $\alpha > 0$ and $\kappa > \frac{1}{2}$, in which case $\frac{n^{1/2} \delta_n^{1/2} \log n}{S_n} = \frac{\log n}{\alpha n^{\kappa - 1/2}} = o(1)$.

Under the above conditions, the 2SLS estimators $\hat{\beta}_1$ and $\hat{\beta}_1^s$ are consistent and asymptotically normal estimators of a weighted average treatment effect.

Theorem 1 (Consistency and Asymptotic Normality)**.** *Suppose that Assumptions 1 and 3 hold, and that $\delta_n \to 0$, $n\delta_n \to \infty$ and $S_n \to \infty$ as $n \to \infty$. Then the 2SLS estimators $\hat{\beta}_1$ and $\hat{\beta}_1^s$ converge in probability to*

$$\beta_1 \equiv \lim_{\delta \to 0} E[\omega_i(\delta)(Y_i(1) - Y_i(0))],$$

where

$$\omega_i(\delta) = \frac{p^{ML}(X_i; \delta)(1 - p^{ML}(X_i; \delta))(D_i(1) - D_i(0))}{E[p^{ML}(X_i; \delta)(1 - p^{ML}(X_i; \delta))(D_i(1) - D_i(0))]}.$$

Suppose, in addition, that Assumptions 4 and 5 hold and that $n\delta_n^2 \to 0$ as $n \to \infty$. Then

$$\hat{\sigma}_n^{-1}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0, 1),$$
$$(\hat{\sigma}_n^s)^{-1}(\hat{\beta}_1^s - \beta_1) \xrightarrow{d} \mathcal{N}(0, 1),$$

where we define $\hat{\sigma}_n^{-1}$ *and* $(\hat{\sigma}_n^s)^{-1}$ *as follows. Let*

$$\hat{\mathbf{\Sigma}}_n = (\sum_{i=1}^{n} \mathbf{Z}_{i,n}\mathbf{D}_{i,n}' I_{i,n})^{-1}(\sum_{i=1}^{n} \hat{\epsilon}_{i,n}^2 \mathbf{Z}_{i,n}\mathbf{Z}_{i,n}' I_{i,n})(\sum_{i=1}^{n} \mathbf{D}_{i,n}\mathbf{Z}_{i,n}' I_{i,n})^{-1},$$

where

$$\hat{\epsilon}_{i,n} = Y_i - \mathbf{D}_{i,n}'\hat{\beta}.$$

$\hat{\mathbf{\Sigma}}_n$ *is the conventional heteroskedasticity-robust estimator for the variance of the 2SLS estimator.* $\hat{\sigma}_n^2$ *is the second diagonal element of* $\hat{\mathbf{\Sigma}}_n$*.* $(\hat{\sigma}_n^s)^2$ *is the analogously-defined estimator for the variance of* $\hat{\beta}_1^s$ *from the simulation-based regression.*

Proof. See Appendix C.6. □

Theorem 1 says that the 2SLS estimators converge to a weighted average of causal effects for the subpopulation whose QPS is nondegenerate ($p^{ML}(X_i;\delta) \in (0,1)$) and who would switch their treatment status in response to the treatment recommendation ($D_i(1) \neq D_i(0)$).[21] The limit $\lim_{\delta\to 0} E[\omega_i(\delta)(Y_i(1) - Y_i(0))]$ always exists under the assumptions of Theorem 1. It also shows that inference based on the conventional 2SLS heteroskedasticity-robust standard errors is asymptotically valid if δ_n goes to zero at an appropriate rate. The convergence rate of $\hat{\beta}_1$ is $O_p(1/\sqrt{n})$ if $\Pr(ML(X_i) \in (0,1)) > 0$ and is $O_p(1/\sqrt{n\delta_n})$ if $\Pr(ML(X_i) \in (0,1)) = 0$.

Our consistency result requires that δ_n goes to zero slower than n^{-1}. The rate condition ensures that, when $\Pr(ML(X_i) \in (0,1)) = 0$, we have sufficiently many observations in the δ_n-neighborhood of the boundary of Ω^*. Importantly, the rate condition does not depend on the dimension of X_i, unlike other bandwidth-based estimation methods such as kernel methods. This is because we use all the observations in the δ-neighborhood of the boundary, and the number of those observations is of order $n\delta_n$ regardless of the dimension of X_i if the dimension of the boundary is one less than the dimension of X_i, i.e., $(p-1)$.

The asymptotic normality result requires that δ_n goes to zero sufficiently quickly. When $\Pr(ML(X_i) \in (0,1)) > 0$, we need to use a small enough δ_n so that $p^{ML}(X_i;\delta_n)$ converges to $p^{ML}(X_i)$ at a fast rate and δ_n-neighborhood of D^* is asymptotically small enough. When $\Pr(ML(X_i) \in (0,1)) = 0$, the asymptotic normality is based on undersmoothing, which eliminates the asymptotic bias by using the observations sufficiently close to the boundary of Ω^*.

Whether or not $\Pr(ML(X_i) \in (0,1)) = 0$, when we use simulated QPS, the consistency result requires that the number of simulation draws S_n goes to infinity as n increases while the asymptotic normality result requires a sufficiently fast growth rate of S_n to make the bias caused by using $p^s(X_i;\delta_n)$ negligible.

Finally, note that the weight $\omega_i(\delta)$ given in Theorem 1 is negative if $D_i(1) < D_i(0)$, so $E[\omega_i(\delta)(Y_i(1) - Y_i(0))]$ may not be a causally interpretable convex combination of treatment effects $Y_i(1) - Y_i(0)$. This can happen because the treatment effect of those whose treatment assignment switches from 1 to 0 in response to the treatment recommendation (defiers) negatively contributes to $E[\omega_i(\delta)(Y_i(1) - Y_i(0))]$. Additional assumptions prevent this problem. If the

[21] In principle, it is possible to estimate other weighted averages and the unweighted average by reweighting different observations appropriately.

treatment effect is constant, for example, the 2SLS estimators are consistent for the treatment effect.

Corollary 3. *Suppose that Assumptions 1 and 3 hold, that the treatment effect is constant, i.e.,* $Y_i(1) - Y_i(0) = b$ *for some constant* b*, and that* $\delta_n \to 0$*,* $n\delta_n \to \infty$*, and* $S_n \to \infty$ *as* $n \to \infty$*. Then the 2SLS estimators* $\hat{\beta}_1$ *and* $\hat{\beta}_1^s$ *converge in probability to* b*.*

Another approach is to impose monotonicity (Imbens and Angrist, 1994). If $\Pr(D_i(1) \geq D_i(0)|X_i = x) = 1$, we have

$$E[(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))|X_i = x] = E[D_i(1) - D_i(0)|X_i = x]LATE(x),$$

where $LATE(x) = E[Y_i(1) - Y_i(0)|D_i(1) \neq D_i(0), X_i = x]$ is the local average treatment effect (LATE) conditional on $X_i = x$. The 2SLS estimators are then consistent for a weighted average of conditional LATEs with all weights nonnegative.

Corollary 4. *Suppose that Assumptions 1 and 3 hold, that* $\Pr(D_i(1) \geq D_i(0)|X_i = x) = 1$ *for any* $x \in \mathcal{X}$ *with* $p^{ML}(x) \in (0,1)$*, and that* $\delta_n \to 0$*,* $n\delta_n \to \infty$ *and* $S_n \to \infty$ *as* $n \to \infty$*. Then the 2SLS estimators* $\hat{\beta}_1$ *and* $\hat{\beta}_1^s$ *converge in probability to*

$$\lim_{\delta \to 0} E[\omega(X_i; \delta)LATE(X_i)],$$

where

$$\omega(x;\delta) = \frac{p^{ML}(x;\delta)(1 - p^{ML}(x;\delta))E[D_i(1) - D_i(0)|X_i = x]}{E[p^{ML}(X_i;\delta)(1 - p^{ML}(X_i;\delta))(D_i(1) - D_i(0))]}.$$

The probability limit of the 2SLS estimators is a weighted average of conditional LATEs over all values of X_i with nondegenerate QPS $p^{ML}(X_i;\delta_n)$. The weights are proportional to $p^{ML}(X_i;\delta_n)(1 - p^{ML}(X_i;\delta_n))$, and to the proportion of compliers, $E[D_i(1) - D_i(0)|X_i]$.

4.3 Special Cases

The result in Theorem 1 holds whether ML is stochastic ($\Pr(ML(X_i) \in (0,1)) > 0$) or deterministic ($\Pr(ML(X_i) \in (0,1)) = 0$). As shown in the proof of Theorem 1, if we consider these two underlying cases separately, the probability limit of the 2SLS estimators has a more specific expression. If $\Pr(ML(X_i) \in (0,1)) > 0$,

$$\operatorname{plim} \hat{\beta}_1 = \operatorname{plim} \hat{\beta}_1^s = \frac{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))]}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]}. \tag{5}$$

The 2SLS estimators converge to a weighted average of treatment effects for the subpopulation with nondegenerate $ML(X_i)$. To relate this result to existing work, consider the following 2SLS regression with the (standard) propensity score $ML(X_i)$ control:

$$D_i = \gamma_0 + \gamma_1 Z_i + \gamma_2 ML(X_i) + \nu_i \tag{6}$$

$$Y_i = \beta_0 + \beta_1 D_i + \beta_2 ML(X_i) + \epsilon_i. \tag{7}$$

Existing results show that under conditional independence, the 2SLS estimator from this regression converges in probability to the treatment-variance weighted average of treatment effects in (5) (Angrist and Pischke, 2008; Hull, 2018).[22] Not surprisingly, for this selection-on-observables case, our result shows that the 2SLS estimator is consistent for the same treatment effect whether we use as a control the propensity score, QPS, or simulated QPS.

Importantly, using QPS as a control allows us to consistently estimate a causal effect even if ML is deterministic and produces high-dimensional regression-discontinuity variation.[23] If $\Pr(ML(X_i) \in (0,1)) = 0$,

$$\operatorname{plim} \hat{\beta}_1 = \operatorname{plim} \hat{\beta}_1^s = \frac{\int_{\partial\Omega^*} E[(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))|X_i = x] f_X(x) d\mathcal{H}^{p-1}(x)}{\int_{\partial\Omega^*} E[D_i(1) - D_i(0)|X_i = x] f_X(x) d\mathcal{H}^{p-1}(x)}. \tag{9}$$

The 2SLS estimators converge to a weighted average of treatment effects for the subpopulation who are on the boundary of the treated region.

Recall that the 2SLS regression uses the observations with $p^{ML}(X_i; \delta_n) \in (0,1)$ (or $p^s(X_i; \delta_n) \in (0,1)$ when we use simulated QPS) only. By definition, if $p^{ML}(X_i; \delta) \in (0,1)$, X_i must be in the δ-neighborhood of the boundary of Ω^*. Therefore, to derive the probability limit of $\hat{\beta}_1$, it is necessary to derive the limits of the integrals of relevant variables over the δ-neighborhood (e.g., $\int_{N(\partial\Omega^*,\delta)} E[Y_i|X_i = x] f_X(x) dx$) as δ shrinks to zero. We take an approach drawing on change of variables techniques from differential geometry and geometric measure theory.[24] In this approach, we first use the coarea formula (Lemma B.3 in Appendix B.3) to write the integral of an integrable function g over $N(\partial\Omega^*, \delta)$ in terms of the iterated integral over the levels sets of the signed distance function of Ω^*:

$$\int_{N(\partial\Omega^*,\delta)} g(x) dx = \int_{-\delta}^{\delta} \int_{\{x' \in \mathbb{R}^p : d^s_{\Omega^*}(x') = \lambda\}} g(x) d\mathcal{H}^{p-1}(x) d\lambda, \tag{10}$$

[22] Precisely speaking, Angrist and Pischke (2008) consider the OLS regression of Y_i (or D_i) on Z_i controlling a dummy variable for every value taken on by X_i (i.e., the model is saturated in X_i) when X_i is a discrete variable:

$$Y_i = \alpha_1 Z_i + \sum_{x \in \mathcal{X}} \alpha_{2,x} 1\{X_i = x\} + u_i. \tag{8}$$

By the Frisch-Waugh Theorem, the population coefficient on Z_i from (8) is given by $\alpha_1 = \frac{E[(Z_i - E[Z_i|X_i])Y_i]}{E[(Z_i - E[Z_i|X_i])^2]}$. Angrist and Pischke (2008) show that this expression is reduced to the treatment-variance weighted average of treatment effects $\frac{E[ML(X_i)(1-ML(X_i))(Y_{1i}-Y_{0i})]}{E[ML(X_i)(1-ML(X_i))}$ under the conditional independence assumption. Their derivation follows even when X_i is continuous and we control the propensity score linearly.

[23] In the standard RDD with a single running variable X_i and cutoff c, $p^{ML}(X_i; \delta_n) = \frac{X_i - c}{2\delta_n} + \frac{1}{2}$ if $X_i \in [c - \delta_n, c + \delta_n]$ and $p^{ML}(X_i; \delta_n) \in \{0, 1\}$ otherwise. Since $p^{ML}(X_i; \delta_n)$ is linear in the running variable X_i if $p^{ML}(X_i; \delta_n) \in (0,1)$, the estimator $\hat{\beta}_1$ becomes a local regression estimator with the box kernel that places the same slope coefficient of X_i on both sides of the cutoff. Under our assumptions, $\hat{\beta}_1$ and standard local linear estimators are shown to have the same fastest possible convergence rate.

[24] Our approach using geometric theory shows that $\hat{\beta}_1$ converges to an integral of the conditional treatment effect over boundary points with respect to the Hausdorff measure. In constrast, prior studies on multidimensional RDDs express treatment effect estimands in terms of expectations conditional on X_i being in the boundary like $E[Y_{1i} - Y_{0i}|X_i \in \partial\Omega^*]$ (Zajonc, 2012). However, those conditional expectations are, formally, not well-defined, since $\mathcal{L}^p(\partial\Omega^*) = 0$ and hence $\Pr(X_i \in \partial\Omega^*) = 0$. We therefore prefer our expression in terms of an integral with respect to the Hausdorff measure to any expressions in terms of conditional expectations on the boundary. Arias, Rubio-Ramírez and Waggoner (2018), Bornn, Shephard and Solgi (2019), and Qiao (2021) use similar tools from differential geometry and geometric measure theory, but for different purposes.

where $d^s_{\Omega^*}$ is the signed distance function of Ω^* (see Appendix B.2 for the definition). The set $\{x' \in \mathbb{R}^p : d^s_{\Omega^*}(x') = \lambda\}$ is a level set of $d^s_{\Omega^*}$, which collects the points in Ω^* when $\lambda > 0$ and the points in $\mathbb{R}^p \setminus \Omega^*$ when $\lambda < 0$ whose distance to the boundary $\partial\Omega^*$ is $|\lambda|$. Figure 2a shows a visual illustration.

We then use the area formula (Lemma B.4 in Appendix B.3) to write the integral over each level set in terms of the integral over the boundary $\partial\Omega^*$:

$$\int_{\{x' \in \mathbb{R}^p : d^s_{\Omega^*}(x') = \lambda\}} g(x) d\mathcal{H}^{p-1}(x) = \int_{\partial\Omega^*} g(x^* + \lambda \nu_{\Omega^*}(x^*)) J^{\partial\Omega^*}_{p-1} \psi_{\Omega^*}(x^*, \lambda) d\mathcal{H}^{p-1}(x^*), \tag{11}$$

where $\nu_{\Omega^*}(x^*)$ is the inward unit normal vector of $\partial\Omega^*$ at x^* (the unit vector orthogonal to all vectors in the tangent space of $\partial\Omega^*$ at x^* that points toward the inside of Ω^*), and $J^{\partial\Omega^*}_{p-1}\psi_{\Omega^*}(x^*, \lambda)$ is the Jacobian of the transformation $\psi_{\Omega^*}(x^*, \lambda) = x^* + \lambda\nu_{\Omega^*}(x^*)$. Figure 2b illustrates this change of variables formula. Finally, combining (10) and (11) and proceeding with further analysis, we prove in Appendix C.6.3 that when g is continuous,

$$\int_{N(\partial\Omega^*, \delta)} g(x) dx = \delta \left(\int_{\partial\Omega^*} g(x) d\mathcal{H}^{p-1}(x) + o(1) \right).$$

Thus, the integral over the δ-neighborhood of $\partial\Omega^*$ scaled up by δ^{-1} converges to the integral over boundary points with respect to the $(p-1)$-dimensional Hausdorff measure. This result is used to derive the expression of the probability limit of $\hat{\beta}_1$ given by (9).

5 Machine Learning Simulation

We conduct a Monte Carlo experiment to assess the feasibility and performance of our method. Consider a tech company that conducts a randomized experiment (randomized controlled trial; RCT) using a small segment of the population and, at the same time, applies a deterministic decision algorithm to the rest of the population. We generate a random sample $\{(Y_i, X_i, D_i, Z_i)\}_{i=1}^n$ of size $n = 10,000$ as follows. There are 100 covariates ($p = 100$), and $X_i \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$. $Y_i(0)$ is generated as $Y_i(0) = 0.75 X_i' \alpha_0 + 0.25 \epsilon_{0i}$, where $\alpha_0 \in \mathbb{R}^{100}$, and $\epsilon_{0i} \sim \mathcal{N}(0, 1)$. We consider two models for $Y_i(1)$, one in which the the treatment effect $Y_i(1) - Y_i(0)$ does not depend on X_i and one in which the effect depends on X_i.

Model A. $Y_i(1) = Y_i(0) + \epsilon_{1i}$, where $\epsilon_{1i} \sim \mathcal{N}(0, 1)$.

Model B. $Y_i(1) = Y_i(0) + X_i' \alpha_1$, where $\alpha_1 \in \mathbb{R}^{100}$.

The choice of parameters $\mathbf{\Sigma}$, α_0 and α_1 is explained in Appendix D. $D_i(0)$ and $D_i(1)$ are generated as $D_i(0) = 0$ and $D_i(1) = 1\{Y_i(1) - Y_i(0) > u_i\}$, where $u_i \sim \mathcal{N}(0, 1)$. To generate Z_i, let $q_{0.495}$ and $q_{0.505}$ be the 49.5th and 50.5th (empirical) quantiles of the first covariate X_{1i}, and let $\tau_{pred}(X_i)$ be a real-valued function of X_i, which we regard as a prediction of the effect of recommendation on the outcome for individual i obtained from past data. We will explain how we construct τ_{pred}

in the next paragraph. Z_i is then generated as

$$Z_i = \begin{cases} Z_i^* \sim \text{Bernoulli}(0.5) & \text{if } X_{1i} \in [q_{0.495}, q_{0.505}] \\ 1 & \text{if } X_{1i} \notin [q_{0.495}, q_{0.505}] \text{ and } \tau_{pred}(X_i) \geq 0 \\ 0 & \text{if } X_{1i} \notin [q_{0.495}, q_{0.505}] \text{ and } \tau_{pred}(X_i) < 0. \end{cases}$$

The first case corresponds to the RCT segment while the latter two cases to the deterministic algorithm segment. The function ML is given by

$$ML(x) = \begin{cases} 0.5 & \text{if } x_1 \in [q_{0.495}, q_{0.505}] \\ 1 & \text{if } x_1 \notin [q_{0.495}, q_{0.505}] \text{ and } \tau_{pred}(x) \geq 0 \\ 0 & \text{if } x_1 \notin [q_{0.495}, q_{0.505}] \text{ and } \tau_{pred}(x) < 0. \end{cases}$$

Finally, D_i and Y_i are generated as $D_i = Z_i D_i(1) + (1 - Z_i) D_i(0)$ and $Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0)$, respectively.

We simulate $1,000$ hypothetical samples from the above data-generating process. Before obtaining $1,000$ samples, we construct τ_{pred} using an independent sample $\{(\tilde{Y}_i, \tilde{X}_i, \tilde{D}_i, \tilde{Z}_i)\}_{i=1}^{\tilde{n}}$ of size $\tilde{n} = 2,000$. The distribution of $(\tilde{Y}_i, \tilde{X}_i, \tilde{D}_i, \tilde{Z}_i)$ is the same as that of (Y_i, X_i, D_i, Z_i) except (1) that $\tilde{Y}_i(1)$ is generated as $\tilde{Y}_i(1) = \tilde{Y}_i(0) + 0.5\tilde{X}_i'\alpha_1 + 0.5\epsilon_{1i}$, where $\epsilon_{1i} \sim \mathcal{N}(0, 1)$ and (2) that $\tilde{Z}_i \sim \text{Bernoulli}(0.5)$. This can be viewed as data from a past randomized experiment conducted to construct an algorithm. We then use random forests separately for the subsamples with $\tilde{Z}_i = 1$ and $\tilde{Z}_i = 0$ to make a prediction of $\tilde{Y}_i$ from $\tilde{X}_i$. Let $\mu_z(x)$ be the trained prediction model, and set $\tau_{pred}(x) = \mu_1(x) - \mu_0(x)$.

This mimics a situation in which the decision maker first conducts an experiment that randomly assigns Z_i to predict the conditional average effect of Z_i and then constructs an algorithm that greedily chooses the treatment predicted to perform better based on the predicted effect. We generate the sample $\{(\tilde{Y}_i, \tilde{X}_i, \tilde{D}_i, \tilde{Z}_i)\}_{i=1}^{\tilde{n}}$ and construct τ_{pred} only once, and we use it for all of the $1,000$ samples. The distribution of the sample $\{(Y_i, X_i, D_i, Z_i)\}_{i=1}^{n}$ is thus held fixed for all simulations.

5.1 Estimators and Estimands

We use the data $\{(Y_i, X_i, D_i, Z_i)\}_{i=1}^{n}$ to estimate treatment effect parameters. Our main approach is 2SLS with QPS controls in Theorem 1. To compute QPS, we use $S = 400$ simulation draws for each observation.

We compare our approach with two alternatives. The first alternative is 2SLS with ML controls. This method uses the observations with $ML(X_i) \in (0, 1)$ to run the 2SLS regression of Y_i on a constant, D_i, and $ML(X_i)$ using Z_i as an instrument for D_i (see (6) and (7) in Section 4.3) and reports the coefficient on D_i. The second alternative is OLS of Y_i on a constant and D_i (i.e., the difference in the sample mean of Y_i between the treated group and untreated group) using all observations.

We consider four parameters as target estimands: ATE $\equiv E[Y_i(1) - Y_i(0)]$, ATE(RCT) $\equiv E[Y_i(1) - Y_i(0) | X_{1i} \in [q_{0.495}, q_{0.505}]]$, LATE $\equiv E[Y_i(1) - Y_i(0) | D_i(1) \neq D_i(0)]$, and LATE(RCT) $\equiv E[Y_i(1) - Y_i(0) | D_i(1) \neq D_i(0), X_{1i} \in [q_{0.495}, q_{0.505}]]$. In the case where the treatment effect does

not depend on X_i (Model A), the conditional effects are homogeneous, and ATE and LATE are the same as ATE(RCT) and LATE(RCT), respectively. In the case where the treatment effect depends on X_i (Model B), the conditional effects are heterogeneous. However, since the RCT segment consists of those in the middle of the distribution of X_{1i}, the average effect for the RCT segment is close to the unconditional average effect. As a result, ATE is equal to ATE(RCT) and LATE is similar to LATE(RCT) under this data-generating process.

For both models, the 2SLS estimator converges in probability to LATE(RCT) whether we control for QPS or ML.[25] However, 2SLS with ML controls uses only the individuals for the RCT segment while 2SLS with QPS controls additionally uses the individuals near the decision boundary of the deterministic algorithm (i.e., the boundary of the region for which $\tau_{pred}(x) \geq 0$). Therefore, 2SLS with QPS controls is expected to produce a more precise estimate than 2SLS with ML controls if the conditional effects for those near the boundary are not far from the target estimand.

5.2 Results

Table 1 reports the bias, standard deviation (SD), and root mean squared error (RMSE) of each estimator. Panels A and B present the results for the cases where the conditional effects are homogeneous and heterogeneous, respectively. Note first that OLS with no controls is significantly biased, showing the importance of correcting for omitted variable bias. 2SLS with QPS achieves this goal, as suggested by its smaller biases across all possible treatment effect models, target parameters, and values of δ. 2SLS with QPS controls shows a consistent pattern; as the bandwidth δ grows, the bias increases while the variance declines. For several values of δ, 2SLS with QPS controls outperforms 2SLS with ML controls in terms of the RMSE. This finding implies that exploiting individuals near the high-dimensional decision boundary of the deterministic algorithm can lead to better performance than using only the individuals in the RCT segment.

To evaluate our inference procedure based on Theorem 1, we also report the coverage probabilities of the 95% confidence intervals for LATE(RCT) constructed from the 2SLS estimates and their heteroskedasticity-robust standard errors. The confidence intervals still offer nearly correct coverage when δ is small, which supports the implication of Theorem 1 that the inference procedure is valid when we use a sufficiently small δ.

6 Empirical Policy Application

6.1 Hospital Relief Funding during the COVID-19 Pandemic

We also provide a real-world empirical application. As part of the 3-phase Coronavirus Aid, Relief, and Economic Security (CARES) Act, the government has distributed tens of billions of dollars of relief funding to hospitals since April 2020. We focus on an initial portion of this funding ($10 billion), which was allocated to hospitals that qualified as "safety net hospitals" according to a specific eligibility criterion. This eligibility criterion intends to focus on hospitals that

[25] The 2SLS estimators converge in probability to the right-hand side of Eq. (5), which is the same as LATE(RCT) under the data-generating process of this simulation.

"*disproportionately provide care to the most vulnerable, and operate on thin margins.*" Specifically, an acute care hospital was deemed eligible for funding if the following conditions hold:

- Medicare Disproportionate Patient Percentage (DPP) of 20.2% or greater. DPP is equal to the sum of the percentage of Medicare inpatient days attributable to patients eligible for both Medicare Part A and Supplemental Security Income (SSI), and the percentage of total inpatient days attributable to patients eligible for Medicaid but not Medicare Part A.[26]

- Annual Uncompensated Care (UCC) of at least $\$25,000$ per bed. UCC is a measure of hospital care provided for which no payment was received from the patient or insurer. It is the sum of a hospital's bad debt and the financial assistance it provides.[27]

- Profit Margin (Net income/(Net patient revenue + Total other income)) of 3.0% or less.

Hospitals that do not qualify on any of the three dimensions are funding ineligible. Figure 3 visualizes how the three dimensions determine safety net eligibility. As the bottom two-dimensional planes show, eligibility discontinuously changes as hospitals cross the eligibility boundary in the space of the three characteristics. This setting is a three-dimensional RDD, falling under our framework.

The final funding amount is calculated as follows. Each eligible hospital is assigned an individual facility score, which is calculated as the product of DPP and the number of beds in that hospital. This facility score determines the share of funding allocated to the hospital, out of the total $10 billion. The share received by each hospital is determined by the ratio of the hospital's facility score to the sum of facility scores across all eligible hospitals. The amount of funding that can be received is bounded below at $5 million and capped above at $50 million.

We use publicly available data from the Healthcare Cost Report Information System (HCRIS)[28], to replicate[29] the funding eligibility status as well as the amount of funding received. To obtain outcome measures of interest, we use the publicly available COVID-19 Reported Patient Impact and Hospital Capacity by Facility dataset. This provides facility-level data on hospital utilization aggregated on a weekly basis, from July 31st 2020 onwards.[30] Summary measures of outcome variables and hospital characteristics are documented in Table 2. Safety net hospitals have higher levels of inpatient beds and ICU beds occupied by patients who have lab-confirmed or suspected COVID-19. They also have a higher number of employees and beds and shorter lengths of inpatient stay.

[26] Source: `https://www.cms.gov/Medicare/Medicare-Fee-for-Service-Payment/AcuteInpatientPPS/dsh`

[27] Source: `https://www.aha.org/fact-sheets/2020-01-06-fact-sheet-uncompensated-hospital-care-cost`

[28] We use the RAND cleaned version of this dataset which can be accessed at `https://www.hospitaldatasets.org/`

[29] We use the methodology detailed in the CARE ACT website to project funding based on 2018 financial year cost reports.

[30] Source: `https://healthdata.gov/Hospital/COVID-19-Reported-Patient-Impact-and-Hospital-Capa/anag-cw7u`

6.2 Covariate Balance Estimates

Using the above data, we study the effect of safety net funding on relevant hospital outcomes, such as the total number of inpatient beds and the number of staffed ICU beds occupied by adult COVID patients reported between July 31st 2020 and August 6th 2020.

We first evaluate the balancing property of QPS conditioning using QPS-controlled differences in covariate means for hospitals who are and are not deemed eligible for safety net funding. Specifically, we run the following OLS regression of hospital-level characteristics on the eligibility status using observations with $p^s(X_i;\delta_n) \in (0,1)$:

$$W_i = \gamma_0 + \gamma_1 Z_i + \gamma_2 p^s(X_i;\delta_n) + \eta_i,$$

where W_i is one of the predetermined financial and utilization characteristics of the hospital, Z_i is a funding eligibility dummy, X_i is a vector of three input variables (DPP, UCC, and profit margin) that determine the funding eligibility, and $p^s(X_i;\delta_n)$ is the simulated QPS. The estimated coefficient on Z_i is the QPS-controlled difference in the mean of the covariate between eligible and ineligible hospitals. For comparison, we also run the OLS regression of hospital characteristics on the eligibility status with no controls using the whole sample.

Table 3 reports the covariate balance estimates. Column 1 shows that, without controlling for QPS, eligible hospitals are significantly different from ineligible hospitals. We find that all the relevant hospital eligibility characteristics are strongly associated with eligibility.

Once we control for QPS, eligible and ineligible hospitals have similar financial and utilization characteristics, as reported in columns 2–6 of Table 3. These estimates are consistent with our theoretical results, establishing the empirical relevance of QPS controls.

6.3 2SLS Estimates

Causal effects of safety net funding are estimated by 2SLS using funding eligibility as an instrument for the amount of funding received. We run the following 2SLS regression on four different hospital-level outcome variables:

$$\begin{aligned} D_i &= \gamma_0 + \gamma_1 Z_i + \gamma_2 p^s(X_i;\delta_n) + v_i \\ Y_i &= \beta_0 + \beta_1 D_i + \beta_2 p^s(X_i;\delta_n) + \epsilon_i, \end{aligned}$$

where Y_i is a hospital-level outcome and D_i is the amount of relief funding received.[31] We also run the OLS and 2SLS regressions with no controls, computed using the sample of all hospitals, as benchmark estimators.

The first stage effects of safety net eligibility on funding amount (in millions), shown in columns 2–9 of Table 4, suggest that safety net eligibility boosts funding significantly. For example, in column 2 of Table 4, we can see that being eligible for the funding increases the safety net funding by approximately 14 million dollars.

[31] This specification uses a continuous treatment, unlike our theoretical framework with a binary treatment. We obtain similar results when the treatment is a binary transformation of the amount of relief funding received (e.g., a dummy indicating whether the amount exceeds a certain value). Results are available upon request.

OLS estimates of funding effects, reported as the benchmark in column 1 of Table 4, indicate that safety net funding is associated with a higher number of adult inpatient beds and higher number of staffed ICU beds utilized by patients who have lab-confirmed or suspected COVID. The estimates indicate that a million dollar increase in funding is associated with 5.52 more adult inpatient beds occupied by patients with lab-confirmed or suspected COVID. The corresponding increase in total adult inpatient beds occupied by those who have lab-confirmed COVID is 4.50 and the increase in staffed ICU beds occupied by those who have lab-confirmed or suspected COVID is 1.66. The estimated increase in staffed ICU beds occupied by lab-confirmed COVID patients is 1.50. Naive 2SLS estimates with no controls produce similar results.

In contrast with the OLS or uncontrolled 2SLS estimates, the 2SLS estimates with QPS controls in columns 3–9 show a different picture. The gains in number of inpatient beds and staffed ICU beds occupied by suspected and lab-confirmed COVID patients become much smaller and lose significance across all bandwidth specifications. These results suggest that QPS reveals important selection bias in the estimated effects of safety net funding. Once we control for QPS to eliminate the bias, the safety net relief funding has little to no effect on the hospital utilization level by COVID-19 patients.

7 Other Examples

Here we give real-world examples and discuss the applicability of our framework.

Example 1 (Bandit and Reinforcement Learning)**.** We are constantly exposed to digital information (movie, music, news, search results, advertisements, and recommendations) through a variety of devices and platforms. Tech companies allocate these pieces of content by using bandit and reinforcement learning algorithms. Our method is applicable to many popular bandit and reinforcement learning algorithms. For simplicity, assume that individuals perfectly comply with the treatment recommendation ($D_i = Z_i$).

1. (Bandit Algorithms) The algorithms below first use past data and supervised learning to estimate the conditional means and variances of potential outcomes, $E[Y_i(z)|X_i]$ and $\text{Var}(Y_i(z)|X_i)$, for each $z \in \{0, 1\}$. Let μ_z and σ_z^2 denote the estimated functions. The algorithms use $\mu_z(X_i)$ and $\sigma_z^2(X_i)$ to determine the treatment assignment for individual i.

(a) (Thompson Sampling Using Gaussian Priors) The algorithm first samples potential outcomes from the normal distribution with mean $(\mu_0(X_i), \mu_1(X_i))$ and variance-covariance matrix $\text{diag}(\sigma_0^2(X_i), \sigma_1^2(X_i))$. The algorithm then chooses the treatment with the highest sampled potential outcome:

$$Z_i^{TS} \equiv \underset{z\in\{0,1\}}{\arg\max}\, y(z), \quad ML^{TS}(X_i) = E[\underset{z\in\{0,1\}}{\arg\max}\, y(z)|X_i],$$

where $y(z) \sim \mathcal{N}(\mu_z(X_i), \sigma_z^2(X_i))$ independently across z. These algorithms often induce quasi-experimental variation in treatment assignment, as a strand of the computer science

literature has observed (Precup, 2000; Li *et al.*, 2010; Narita, Yasui and Yata, 2019; Saito, Aihara, Matsutani and Narita, 2021). The function ML has an analytical expression:

$$ML^{TS}(x) = 1 - \Phi\left(\frac{\mu_0(x) - \mu_1(x)}{\sqrt{\sigma_0^2(x) + \sigma_1^2(x)}}\right),$$

where Φ is the cumulative distribution function of a standard normal distribution. Suppose that the functions μ_0, μ_1, σ_0^2 and σ_1^2 are continuous. QPS for this case is given by

$$p^{TS}(x) = 1 - \Phi\left(\frac{\mu_0(x) - \mu_1(x)}{\sqrt{\sigma_0^2(x) + \sigma_1^2(x)}}\right).$$

This QPS is nondegenerate, meaning that the data from the algorithm allow for causal-effect identification.

(b) (Upper Confidence Bound, UCB) Unlike the above stochastic one, the UCB algorithm (Li *et al.*, 2010) is a deterministic algorithm, producing a less obvious example of our framework. This algorithm chooses the treatment with the highest upper confidence bound for the potential outcome:

$$Z_i^{UCB} \equiv \arg\max_{z=0,1}\{\mu_z(X_i) + \alpha\sigma_z(X_i)\}, \quad ML^{UCB}(x) = \arg\max_{z=0,1}\{\mu_z(x) + \alpha\sigma_z(x)\},$$

where α is chosen so that $|\mu_z(x) - E[Y_i(z)|X_i = x]| \leq \alpha\sigma_z(x)$ at least with some probability, for example, 0.95, for every x. Suppose that the function $g = \mu_1 - \mu_0 + \alpha(\sigma_1 - \sigma_0)$ is continuous on $\mathcal{X}$ and is continuously differentiable in a neighborhood of x with $\nabla g(x) \neq \mathbf{0}$ for any $x \in \mathcal{X}$ such that $g(x) = 0$. QPS for this case is given by

$$p^{UCB}(x) = \begin{cases} 0 & \text{if } \mu_1(x) + \alpha\sigma_1(x) < \mu_0(x) + \alpha\sigma_0(x) \\ 0.5 & \text{if } \mu_1(x) + \alpha\sigma_1(x) = \mu_0(x) + \alpha\sigma_0(x) \\ 1 & \text{if } \mu_1(x) + \alpha\sigma_1(x) > \mu_0(x) + \alpha\sigma_0(x). \end{cases}$$

This means that the UCB algorithm produces potentially complicated quasi-experimental variation along the boundary in the covariate space where the algorithm's treatment recommendation changes from one to the other. It is possible to identify and estimate causal effects across the boundary.

2. (Reinforcement Learning Algorithms) Extending bandit algorithms to dynamically changing environments, reinforcement learning algorithms optimize decisions in dynamic environments, where the state (the set of observables that the agent receives from the environment) and action in the current period can affect the future states and outcomes. Let $\{(X_{ti}, Z_{ti}, Y_{ti})\}_{t=0}^{\infty}$ denote the trajectory of the states, treatment assignments, and outcomes in periods $t = 0, 1, 2, \cdots$ for individual i. For simplicity, we assume that the trajectory follows a Markov decision process.[32]

[32]Under a Markov decision process, the distribution of the state X_{ti} only depends on the last state and treatment assignment $(X_{t-1,i}, Z_{t-1,i})$, the distribution of the outcome Y_{ti} only depends on the current state and treatment assignment (X_{ti}, Z_{ti}), and these distributions are stationary over periods.

Let $Y_{ti}(1)$ and $Y_{ti}(0)$ represent the potential outcomes in period t. Let $Q : \mathcal{X} \times \{0,1\} \to \mathbb{R}$ be the optimal state-action value function, called the *Q-function*: for $(x,z) \in \mathcal{X} \times \{0,1\}$,

$$Q(x,z) \equiv \max_{\pi:\mathcal{X}\to[0,1]} E\left[\sum_{t=0}^{\infty} \gamma^t (Y_{ti}(1)\pi(X_{ti}) + Y_{ti}(0)(1-\pi(X_{ti}))|X_{0i} = x, Z_{0i} = z\right],$$

where $\gamma \in [0,1)$ is a discount factor, and π is a policy function that assigns the probability of treatment to each possible state.

(a) (ϵ-Greedy) This algorithm first uses past data to yield $\hat{Q}$, an estimate of the Q-function. For example, the fitted Q iteration (Ernst, Geurts and Wehenkel, 2005) is used to estimate Q.[33] The algorithm then chooses the best treatment based on $\hat{Q}(X_{ti}, z)$ with probability $1 - \frac{\epsilon}{2}$ and chooses the other treatment with probability $\frac{\epsilon}{2}$: for each t,

$$Z_{ti}^{\epsilon} \equiv \begin{cases} \arg\max_{z=0,1} \hat{Q}(X_{ti}, z) & \text{with probability } 1 - \frac{\epsilon}{2} \\ 1 - \arg\max_{z=0,1} \hat{Q}(X_{ti}, z) & \text{with probability } \frac{\epsilon}{2}, \end{cases}$$

$$ML^{\epsilon}(x) = \begin{cases} \frac{\epsilon}{2} & \text{if } \hat{Q}(x,1) < \hat{Q}(x,0) \\ 1 - \frac{\epsilon}{2} & \text{if } \hat{Q}(x,1) > \hat{Q}(x,0). \end{cases}$$

Suppose that the function $g(\cdot) = \hat{Q}(\cdot,1) - \hat{Q}(\cdot,0)$ is continuous on $\mathcal{X}$ and is continuously differentiable in a neighborhood of x with $\nabla g(x) \neq \mathbf{0}$ for any $x \in \mathcal{X}$ such that $g(x) = 0$. QPS for this case is given by

$$p^{\epsilon}(x) = \begin{cases} \frac{\epsilon}{2} & \text{if } \hat{Q}(x,1) < \hat{Q}(x,0) \\ 0.5 & \text{if } \hat{Q}(x,1) = \hat{Q}(x,0) \\ 1 - \frac{\epsilon}{2} & \text{if } \hat{Q}(x,1) > \hat{Q}(x,0). \end{cases}$$

(b) (Policy Gradient Methods) Policy gradient methods such as REINFORCE (Williams, 1992) and Actor-Critic approximate the optimal policy function by parametrization and learn the parameter using stochastic gradient ascent. Let $\pi(x;\theta)$ be a parametrization of the policy function that is differentiable with respect to θ.[34] Suppose that we have collected a set of L trajectories $\{(x_t^l, z_t^l, y_t^l)_{t=0}^{T_l} : l = 1, ..., L\}$ by running the policy $\pi(x;\theta^0)$ for L individuals. Policy gradient methods use the trajectories to update the policy parameter

[33] Suppose that we have collected a set of L four-tuples $\{(x_{t_l}^l, z_{t_l}^l, y_{t_l}^l, x_{t_l+1}^l)\}_{l=1}^L$ as a result of the agent interacting with the dynamic environment. Given the dataset and an initial approximation $\hat{Q}$ of Q (e.g., $\hat{Q}(x,z) = 0$ for all (x,z)), we repeat the following steps until some stopping condition is reached: 1. For each $l = 1, ..., L$, calculate $q^l = y_{t_l}^l + \gamma \max_{z\in\{0,1\}} \hat{Q}(x_{t_l+1}^l, z)$; 2. Use $\{(x_{t_l}^l, z_{t_l}^l, q^l)\}_{l=1}^L$ and a supervised learning method to train a model that predicts q from (x,z). Let the model be a new approximation $\hat{Q}$ of Q. Possible supervised learning methods used in the second step include tree-based methods, neural networks (Neural Fitted Q Iteration) and deep neural networks (Deep Fitted Q Iteration).

[34] For example, π might be a softmax function with a linear index: $\pi(x;\theta) = \frac{\exp(x'\theta)}{1+\exp(x'\theta)}$. Another example is a neural network whose input is a representation of the state x, whose output is the treatment assignment probability, and whose weights are represented by the parameter θ.

to θ^1 by stochastic gradient ascent. The algorithms then use the updated policy function $\pi(x;\theta^1)$ to determine the treatment assignment for new episodes. For each t,

$$Z_{ti}^{PG} \equiv \begin{cases} 1 & \text{with probability } \pi(X_{ti};\theta^1) \\ 0 & \text{with probability } 1-\pi(X_{ti};\theta^1), \end{cases} \qquad ML^{TG}(x) = \pi(x;\theta^1).$$

Suppose that the function $\pi(\cdot;\theta^1)$ is continuous. QPS for this case is given by

$$p^{TG}(x) = \pi(x;\theta^1).$$

Example 2 (Unsupervised Learning)**.** Customer segmentation is a core marketing practice that divides a company's customers into groups based on their characteristics and behavior so that the company can effectively target marketing activities at each group. Many businesses today use unsupervised learning algorithms, clustering algorithms in particular, to perform customer segmentation. Using our notation, assume that a company decides whether it targets a campaign at customer i ($Z_i = 1$) or not ($Z_i = 0$). The company first uses a clustering algorithm such as K-means clustering or Gaussian mixture model clustering to divide customers into K groups, making a partition $\{S_1, ..., S_K\}$ of the covariate space $\mathbb{R}^p$. The company then conducts the campaign targeted at some of the groups:

$$Z_i^{CL} \equiv 1\{X_i \in \cup_{k\in T} S_k\}, \quad ML^{CL}(x) = 1\{x \in \cup_{k\in T} S_k\},$$

where $T \subset \{1, .., K\}$ is the set of the indices of the target groups.

For example, suppose that the company uses K-means clustering, which creates a partition in which a covariate value x belongs to the group with the nearest centroid. Let $c_1, ..., c_K$ be the centroids of the K groups, and define a set-valued function $C : \mathbb{R}^p \to 2^{\{1,...,K\}}$, where $2^{\{1,...,K\}}$ is the power set of $\{1, ..., K\}$, as $C(x) \equiv \arg\min_{k\in\{1,...,K\}} \|x - c_k\|$. If $C(x)$ is a singleton, x belongs to the unique group in $C(x)$. If $C(x)$ contains more than one indices, the group to which x belongs is arbitrarily determined. QPS for this case is given by

$$p^{CL}(x) = \begin{cases} 0 & \text{if } C(x) \cap T = \emptyset \\ 0.5 & \text{if } |C(x)| = 2,\ x \in \partial(\cup_{k\in T} S_k) \\ 1 & \text{if } C(x) \subset T \end{cases}$$

and $p^{CL}(x) \in (0, 1)$ if $|C(x)| \geq 3$ and $x \in \partial(\cup_{k\in T} S_k)$, where $|C(x)|$ is the number of elements in $C(x)$.[35] Thus, it is possible to identify causal effects across the boundary $\partial(\cup_{k\in T} S_k)$.

Example 3 (Supervised Learning)**.** Millions of times each year, judges make jail-or-release decisions that hinge on a prediction of what a defendant would do if released. Many judges now use proprietary algorithms (like COMPAS criminal risk score) to make such predictions and use the predictions to support jail-or-release decisions. Using our notation, assume that a

[35] If $|C(x)| = 2$ and $x \in \partial(\cup_{k\in T} S_k)$, x is on a linear boundary between one target group and one non-target group, and hence QPS is 0.5. If $|C(x)| \geq 3$ and $x \in \partial(\cup_{k\in T} S_k)$, x is a common endpoint of several group boundaries, and QPS is determined by the angles at which the boundaries intersect.

criminal risk algorithm recommends jailing ($Z_i = 1$) or releasing ($Z_i = 0$) for each defendant i. The algorithm uses defendant i's observable characteristics X_i, including criminal history and demographics. The algorithm first translates X_i into a risk score $r(X_i)$, where $r : \mathbb{R}^p \to \mathbb{R}$ is a function estimated by supervised leaning based on past data and assumed to be fixed. For example, Kleinberg *et al.* (2017) construct a version of $r(X_i)$ using gradient boosted decision trees. The algorithm then uses the risk score to make the final recommendation:

$$Z_i^{SL} \equiv 1\{r(X_i) > c\}, \quad ML^{SL}(x) = 1\{r(x) > c\},$$

where $c \in \mathbb{R}$ is a constant threshold that is set ex ante.[36] A similar procedure applies to the screening of potential borrowers by banks and insurance companies based on credit scores estimated by supervised learning (Agarwal, Chomsisengphet, Mahoney and Stroebel, 2017).

A widely-used approach to identifying and estimating treatment effects in these settings is to use the score $r(X_i)$ as a continuous univariate running variable and apply a univariate RDD method (Cowgill, 2018). However, whether $r(X_i)$ is continuously distributed or not depends on how the function r is constructed. For example, suppose that r is constructed by a tree-based algorithm and is the following simple regression tree with three terminal nodes:

$$r(x) = \begin{cases} r_1 & \text{if } x_1 \leq 0 \\ r_2 & \text{if } x_1 > 0, x_2 \leq 0 \\ r_3 & \text{if } x_1 > 0, x_2 > 0, \end{cases}$$

where $r_1 < r_2 < c < r_3$.[37] In this case, the score $r(X_i)$ is a discrete variable, and hence it may not be suitable to apply a standard univariate RDD method.

Our approach is applicable to this case as long as at least one of the original multi-dimensional covariates X_i are continuously distributed. QPS for this case is given by

$$p^{SL}(x) = \begin{cases} 0 & \text{if } x_1 < 0 \text{ or } x_2 < 0 \\ 0.25 & \text{if } x_1 = x_2 = 0 \\ 0.5 & \text{if } (x_1 = 0, x_2 > 0) \text{ or } (x_1 > 0, x_2 = 0) \\ 1 & \text{if } x_1 > 0, x_2 > 0. \end{cases}$$

It is therefore possible to identify causal effects across the boundary $\{x \in \mathcal{X} : (x_1 = 0, x_2 \geq 0) \text{ or } (x_1 > 0, x_2 = 0)\}$.

Example 4 (Policy Eligibility Rules)**.** Medicaid and other welfare policies often decide who are eligible based on algorithmic rules, as studied by Currie and Gruber (1996) and Brown, Kowalski and Lurie (2020).[38] Using our notation, the state government determines whether each

[36]The algorithm sometimes discretizes the original risk score $r(X_i)$ into $d(r(X_i))$, where $d : \mathbb{R} \to \mathbb{N}$ (Cowgill, 2018). In this case, the algorithm uses the discretized risk score to make the final recommendation: $Z_i^{SL} \equiv 1\{d(r(X_i)) > c\}$.

[37]If the regression tree is larger, or ensemble methods such as random forests and gradient boosted decision trees are used to construct r, r is of similar form but has a more complicated expression.

[38]These papers estimate the effect of Medicaid eligibility by exploiting variation in the eligibility rule across states and over time (simulated instrumental variable method). In contrast, our method exploits local variation in the eligibility status across different individuals given a fixed eligibility rule.

individual i is eligible ($Z_i = 1$) or not ($Z_i = 0$) for Medicare. The state government's eligibility rule $ML^{Medicaid}$ maps individual characteristics X_i (e.g. income, family composition) into an eligibility decision $Z_i^{Medicare}$. A similar procedure also applies to bankruptcy laws (Mahoney, 2015). These policy eligibility rules produce quasi-experimental variation as in Example 3.

Example 5 (Mechanism Design: Matching and Auction)**.** Centralized economic mechanisms such as matching and auction are also suitable examples, as summarized below:

	Matching (e.g., School Choice)	Auction
i	Student	Bidder
X_i	Preference/Priority/Tie-breaker	Bid
Z_i	Whether student i is assigned treatment school	Whether bidder i wins the good
D_i	Whether student i attends treatment school	same as Z_i
Y_i	Student i's future test score	Bidder i's future economic performance

In mechanism design and other algorithms with capacity constraints, the treatment recommendation for individual i may depend not only on X_i but also on the characteristics of others. These interactive situations can be accommodated by our framework if we consider the following large market setting.[39] Suppose that there is a continuum of individuals $i \in [0,1]$ and that the recommendation probability for individual i with covariate X_i is determined by a function M as follows:

$$\Pr(Z_i = 1 | X_i; F_{X_{-i}}) = M(X_i; F_{X_{-i}}).$$

Here $F_{X_{-i}} = \Pr(\{j \in [0,1] \setminus \{i\} : X_j \leq x\})$ is the distribution of X among all individuals $j \in [0,1] \setminus \{i\}$. The function $M : \mathbb{R}^p \times \mathcal{F} \to [0,1]$, where $\mathcal{F}$ is a set of distributions on $\mathbb{R}^p$, gives the recommendation probability for each individual in the market. With a continuum of individuals, for any $i \in [0,1]$, $F_{X_{-i}}$ is the same as the distribution of X in the whole market, denoted by F_X. Therefore, the data generated by the mechanism M are equivalent to the data generated by the algorithm $ML : \mathbb{R}^p \to [0,1]$ such that $ML(x) \equiv M(x; F_X)$ for all $x \in \mathbb{R}^p$. Our framework is applicable to this large-market interactive setting.

The above discussions can be summarized as follows.

Corollary 5. *In all the above examples, there exists $x \in \text{int}(\mathcal{X})$ such that $p^{ML}(x) \in (0,1)$. Therefore, a causal effect is identified under Assumptions 1 and 2.*

8 Conclusion

As algorithmic decisions become the new norm, the world becomes a mountain of natural experiments and instruments. These instruments enable us to estimate causal treatment effects,

[39] The approach proposed by Borusyak and Hull (2020) is applicable to finite-sample settings if the treatment recommendation probability, which may depend on all individuals' characteristics, is nondegenerate for multiple individuals.

as we formalize and illustrate in this paper. Our analysis clarifies a few implications for policy and management practices around algorithmic decision-making. It is important to record the implementation of algorithms in a replicable, simulatable way, including what input variables X_i are used to make algorithmic recommendation Z_i. Another key point is to record an algorithm's recommendation Z_i even if they are superseded by a human decision D_i. These data retention efforts would go a long way to exploit the full potential of algorithms as natural experiments.

In addition to estimating treatment effects, instruments induced by algorithms can also help inform the improvement of algorithms. To see this, suppose some algorithm ML_1 is in use. As we characterize in this paper, this algorithm ML_1 produces instrument IV_1. We can then use instrument IV_1 to make counterfactual predictions about what would happen if we change ML_1 to another algorithm ML_2. We'd then switch to ML_2 if it is predicted to be better than the previous algorithm. This algorithm change in turn would produce another cycle of natural experiments and improvements:

$$ML_1 \to IV_1 \to Algorithm\ Improvement_1 \to ML_2 \to IV_2 \to Algorithm\ Improvement_2 ...$$

This cycle of natural experiments and improvements may provide an alternative to well-established A/B testing (randomized experiment). A/B testing is often technically, politically, or managerially infeasible, since deploying a new algorithm is time- and money-consuming, and entails a risk of failure and ethical concerns (Narita, 2021). This difficulty with randomized experiment may be alleviated by additionally making use of algorithms as natural experiments.

Our agenda for future research includes a formalization of such optimal policy (algorithm) learning. Another important topic is data-driven bandwidth selection. This work needs to extend Imbens and Kalyanaraman (2012) and Calonico *et al.* (2014)'s bandwidth selection methods in the univariate RDD to our setting. Inference on treatment effects in our framework relies on conventional large sample reasoning. It seems natural to additionally consider permutation or randomization inference. It will also be challenging but interesting to develop finite-sample optimal estimation and inference strategies such as those recently introduced by Armstrong and Kolesár (2018, 2020) and Imbens and Wager (2019). Finite-sample bias is also a related important topic for further work (Narita, 2020). Finally, we look forward to empirical applications of our method in a variety of business, policy, and scientific domains.

References

Abdulkadiroğlu, A., Angrist, J. D., Narita, Y. and Pathak, P. A. (2017). Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation. *Econometrica*, **85** (5), 1373–1432.

—, —, — and Pathak, P. A. (Forthcoming). Breaking Ties: Regression Discontinuity Design Meets Market Design. *Econometrica*.

Agarwal, S., Chomsisengphet, S., Mahoney, N. and Stroebel, J. (2017). Do Banks Pass Through Credit Expansions to Consumers Who Want to Borrow? *Quarterly Journal of Economics*, **133** (1), 129–190.

Angrist, J. D. and Pischke, J.-S. (2008). *Mostly Harmless Econometrics: An Empiricist's Companion*. Princeton University Press.

Arias, J. E., Rubio-Ramírez, J. F. and Waggoner, D. F. (2018). Inference Based on Structural Vector Autoregressions Identified with Sign and Zero Restrictions: Theory and Applications. *Econometrica*, **86** (2), 685–720.

Armstrong, T. B. and Kolesár, M. (2018). Optimal Inference in a Class of Regression Models. *Econometrica*, **86** (2), 655–683.

— and Kolesár, M. (2020). Finite-Sample Optimal Estimation and Inference on Average Treatment Effects Under Unconfoundedness. *Econometrica*.

Athey, S. and Imbens, G. W. (2017). The State of Applied Econometrics: Causality and Policy Evaluation. *Journal of Economic Perspectives*, **31** (2), 3–32.

Belloni, A., Chernozhukov, V., Fernández-Val, I. and Hansen, C. (2017). Program Evaluation and Causal Inference with High-Dimensional Data. *Econometrica*, **85** (1), 233–298.

Bonhomme, S., Lamadon, T. and Manresa, E. (2019). Discretizing Unobserved Heterogeneity. *University of Chicago, Becker Friedman Institute for Economics Working Paper No. 2019-16*, unpublished Manuscript, University of Chicago.

Bornn, L., Shephard, N. and Solgi, R. (2019). Moment conditions and bayesian nonparametrics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **81** (1), 5–43.

Borusyak, K. and Hull, P. (2020). Non-Random Exposure to Exogenous Shocks: Theory and Applications. *NBER Working Paper No. 27845*, unpublished Manuscript, University of Chicago.

Brown, D., Kowalski, A. E. and Lurie, I. Z. (2020). Long-Term Impacts of Childhood Medicaid Expansions on Outcomes in Adulthood. *The Review of Economic Studies*, **87** (2), 729–821.

Bundorf, K., Polyakova, M. and Tai-Seale, M. (2019). How Do Humans Interact with Algorithms? Experimental Evidence from Health Insurance. *NBER Working Paper No. 25976*.

Calonico, S., Cattaneo, M. D. and Titiunik, R. (2014). Robust Nonparametric Confidence Intervals for Regression-Discontinuity Designs. *Econometrica*, **82** (6), 2295–2326.

Cattaneo, M. D., Frandsen, B. R. and Titiunik, R. (2015). Randomization Inference in the Regression Discontinuity Design: An Application to Party Advantages in the US Senate. *Journal of Causal Inference*, **3** (1), 1–24.

—, Titiunik, R. and Vazquez-Bare, G. (2017). Comparing Inference Approaches for RD Designs: A Reexamination of the Effect of Head Start on Child Mortality. *Journal of Policy Analysis and Management*, **36** (3), 643–681.

—, —, — and Keele, L. (2016). Interpreting Regression Discontinuity Designs with Multiple Cutoffs. *Journal of Politics*, **78** (4), 1229–1248.

Cohen, P., Hahn, R., Hall, J., Levitt, S. and Metcalfe, R. (2016). Using Big Data to Estimate Consumer Surplus: The Case of Uber. *NBER Working Paper No. 22627*, unpublished Manuscript, University of Chicago.

Cowgill, B. (2018). The Impact of Algorithms on Judicial Discretion: Evidence from Regression Discontinuities. Unpublished Manuscript, Columbia Business School.

Crasta, G. and Malusa, A. (2007). The Distance Function from the Boundary in a Minkowski Space. *Transactions of the American Mathematical Society*, **359**, 5725–5759.

Currie, J. and Gruber, J. (1996). Health Insurance Eligibility, Utilization of Medical Care, and Child Health. *Quarterly Journal of Economics*, **111** (2), 431–466.

Dong, Y. (2018). Alternative Assumptions to Identify LATE in Fuzzy Regression Discontinuity Designs. *Oxford Bulletin of Economics and Statistics*, **80** (5), 1020–1027.

Dranove, D., Garthwaite, C. and Ody, C. (2017). How do nonprofits respond to negative wealth shocks? the impact of the 2008 stock market collapse on hospitals. *RAND Journal of Economics*, **48** (2), 485–525.

Duggan, M. G. (2000). Hospital Ownership and Public Medical Spending. *Quarterly Journal of Economics*, **115** (4), 1343–1373.

Einav, L., Finkelstein, A., Mullainathan, S. and Obermeyer, Z. (2018). Predictive Modeling of U.S. Health Care Spending in Late Life. *Science*, **360** (6396), 1462–1465.

Ernst, D., Geurts, P. and Wehenkel, L. (2005). Tree-Based Batch Mode Reinforcement Learning. *Journal of Machine Learning Research*, **6**, 503–556.

Frandsen, B. R. (2017). Party Bias in Union Representation Elections: Testing for Manipulation in the Regression Discontinuity Design When the Running Variable is Discrete. In

Regression Discontinuity Designs: Theory and Applications, Emerald Publishing Limited, pp. 281–315.

Frölich, M. (2007). Regression Discontinuity Design with Covariates. *IZA Discussion Paper No. 3024*, unpublished Manuscript, University of St. Gallen.

Frölich, M. and Huber, M. (2019). Including Covariates in the Regression Discontinuity Design. *Journal of Business and Economic Statistics*, **37** (4), 736–748.

Gulshan, V. *et al.* (2016). Development and Validation of a Deep Learning Algorithm for Detection of Diabetic Retinopathy in Retinal Fundus Photographs. *Journal of the American Medical Association*, **316** (22), 2402–2410.

Hahn, J., Todd, P. and van der Klaauw, W. (2001). Identification and Estimation of Treatment Effects with a Regression-Discontinuity Design. *Econometrica*, **69** (1), 201–209.

Hoffman, M., Kahn, L. B. and Li, D. (2017). Discretion in Hiring. *Quarterly Journal of Economics*, **133** (2), 765–800.

Horton, J. J. (2017). The Effects of Algorithmic Labor Market Recommendations: Evidence from a Field Experiment. *Journal of Labor Economics*, **35** (2), 345–385.

Hull, P. (2018). Subtracting the Propensity Score in Linear Models. Unpublished Manuscript, MIT.

Imbens, G. and Kalyanaraman, K. (2012). Optimal Bandwidth Choice for the Regression Discontinuity Estimator. *The Review of Economic Studies*, **79** (3), 933–959.

— and Wager, S. (2019). Optimized regression discontinuity designs. *Review of Economics and Statistics*, **101** (2), 264–278.

Imbens, G. W. and Angrist, J. D. (1994). Identification and Estimation of Local Average Treatment Effects. *Econometrica*, **62** (2), 467–475.

Kakani, P., Chandra, A., Mullainathan, S. and Obermeyer, Z. (2020). Allocation of COVID-19 Relief Funding to Disproportionately Black Counties. *Journal of the American Medical Association (JAMA)*, **324** (10), 1000–1003.

Keele, L. J. and Titiunik, R. (2015). Geographic Boundaries as Regression Discontinuities. *Political Analysis*, **23** (1), 127–155.

Khullar, D., Bond, A. M. and Schpero, W. L. (2020). COVID-19 and the Financial Health of US Hospitals. *Journal of the American Medical Association (JAMA)*, **323** (21), 2127–2128.

Kleinberg, J., Lakkaraju, H., Leskovec, J., Ludwig, J. and Mullainathan, S. (2017). Human Decisions and Machine Predictions. *Quarterly Journal of Economics*, **133** (1), 237–293.

Krantz, S. G. and Parks, H. R. (2008). *Geometric Integration Theory*. Birkhäuser Basel.

Li, L., Chu, W., Langford, J. and Schapire, R. E. (2010). A Contextual-Bandit Approach to Personalized News Article Recommendation. *Proceedings of the 19th international conference on World Wide Web (WWW)*, pp. 661–670.

Li, S. (2011). Concise Formulas for the Area and Volume of a Hyperspherical Cap. *Asian Journal of Mathematics and Statistics*, **4**, 66–70.

Mahoney, N. (2015). Bankruptcy as Implicit Health Insurance. *American Economic Review*, **105** (2), 710–46.

Mullainathan, S. and Spiess, J. (2017). Machine learning: An applied econometric approach. *Journal of Economic Perspectives*, **31** (2), 87–106.

Narita, Y. (2020). A Theory of Quasi-Experimental Evaluation of School Quality. *Management Science*.

— (2021). Incorporating ethics and welfare into randomized experiments. *Proceedings of the National Academy of Sciences*, **118** (1).

—, Yasui, S. and Yata, K. (2019). Efficient Counterfactual Learning from Bandit Feedback. *Proceedings of the 33rd AAAI Conference on Artificial Intelligence*, pp. 4634–4641.

Papay, J. P., Willett, J. B. and Murnane, R. J. (2011). Extending the Regression-Discontinuity Approach to Multiple Assignment Variables. *Journal of Econometrics*, **161** (2), 203–207.

Precup, D. (2000). Eligibility Traces for Off-Policy Policy Evaluation. *Proceedings of the Seventeenth International Conference on Machine Learning*, pp. 759–766.

Qiao, W. (2021). Nonparametric Estimation of Surface Integrals on Level Sets. *Bernoulli*, **27** (1), 155–191.

Rosenbaum, P. R. and Rubin, D. B. (1983). The Central Role of the Propensity Score in Observational Studies for Causal Effects. *Biometrika*, **70** (1), 41–55.

Saito, Y., Aihara, S., Matsutani, M. and Narita, Y. (2021). Open bandit dataset and pipeline: Towards realistic and reproducible off-policy evaluation. Unpublished Manuscript, Tokyo Institute of Technology.

Sekhon, J. S. and Titiunik, R. (2017). On Interpreting the Regression Discontinuity Design as a Local Experiment. In *Regression Discontinuity Designs: Theory and Applications*, Emerald Publishing Limited, pp. 1–28.

Stein, E. M. and Shakarchi, R. (2005). *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton Lectures in Analysis, Princeton, NJ: Princeton Univ. Press.

Voelker, A. R., Gosmann, J. and Stewart, T. C. (2017). Efficiently Sampling Vectors and Coordinates from the n-Sphere and n-Ball. *Centre for Theoretical Neuroscience - Technical Report*.

Williams, R. J. (1992). Simple Statistical Gradient-Following Algorithms for Connectionist Reinforcement Learning. *Machine Learning*, **8**, 229–256.

Wong, V. C., Steiner, P. M. and Cook, T. D. (2013). Analyzing Regression-Discontinuity Designs with Multiple Assignment Variables: A Comparative Study of Four Estimation Methods. *Journal of Educational and Behavioral Statistics*, **38** (2), 107–141.

Zajonc, T. (2012). Regression Discontinuity Design with Multiple Forcing Variables. *Essays on Causal Inference for Public Policy*, pp. 45–81.

Figure 1: Example of the Quasi Propensity Score

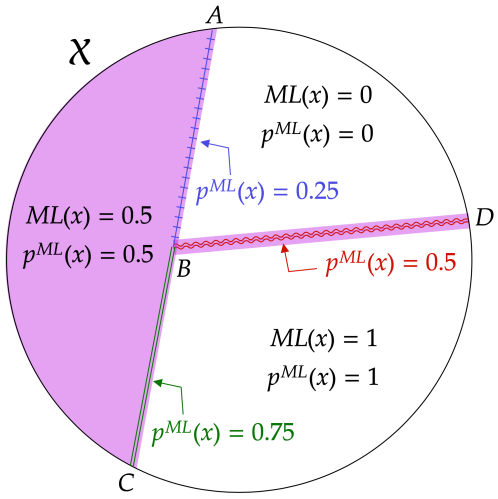
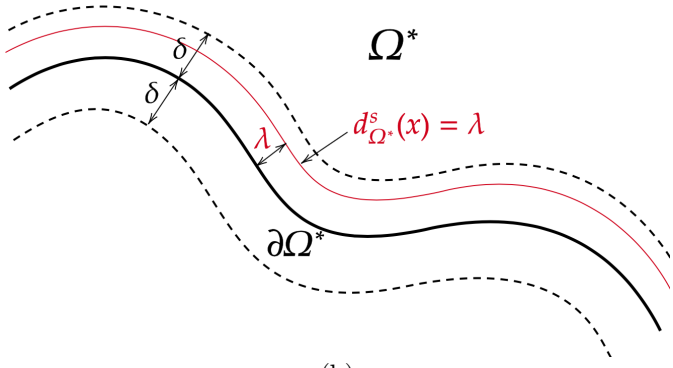


Figure 2: Illustration of the Change of Variables Techniques

(a)



(b)

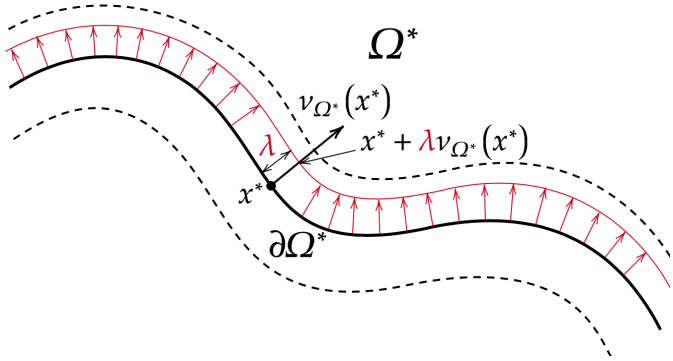
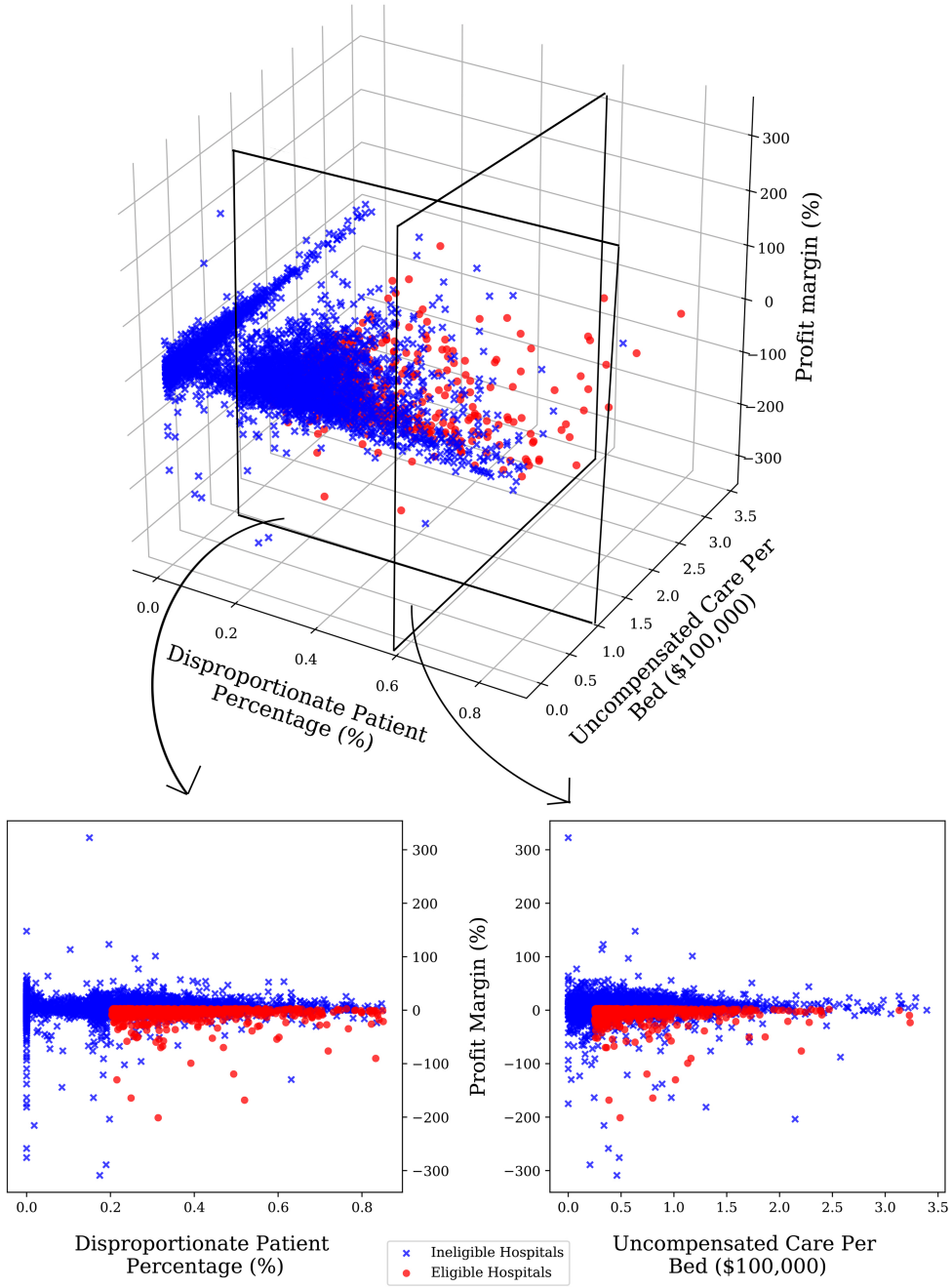


Table 1: Bias, SD, and RMSE of Estimators and Coverage of 95% Confidence Intervals

	Our Method: 2SLS with Quasi Propensity Score Controls						2SLS with ML Controls	OLS with No Controls
	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$		
Panel A: Homogeneous Conditional Effects								
Estimand: ATE = ATE(RCT) = 0								
Bias	.603	.634	.644	.659	.684	.740	.572	.754
SD	.304	.205	.157	.110	.078	.061	.372	.024
RMSE	.675	.667	.663	.668	.689	.842	.683	.754
Estimand: LATE = LATE(RCT) = 0.564								
Bias	.039	.070	.080	.095	.120	.176	.008	.190
SD	.304	.205	.157	.110	.078	.061	.372	.024
RMSE	.306	.217	.176	.145	.143	.186	.372	.191
Coverage	94.8%	92.8%	92.9%	84.6%	69.6%	18.6%	—	—
Avg N	235	727	1275	2567	3995	5561	100	10000
Panel B: Heterogeneous Conditional Effects								
Estimand: ATE = ATE(RCT) = 0								
Bias	.568	.587	.589	.604	.636	.709	.545	1.192
SD	.331	.222	.170	.118	.083	.063	.399	.025
RMSE	.657	.628	.613	.615	.642	.712	.676	1.193
Estimand: LATE = 0.564								
Bias	.004	.023	.025	.040	.072	.145	−.019	.628
SD	.331	.222	.170	.118	.083	.063	.399	.025
RMSE	.331	.223	.172	.125	.110	.158	.399	.629
Estimand: LATE(RCT) = 0.559								
Bias	.009	.028	.030	.045	.077	.150	−.014	.633
SD	.331	.222	.170	.118	.083	.063	.399	.025
RMSE	.331	.224	.173	.127	.114	.163	.399	.634
Coverage	95.9%	94.8%	95.0%	93.2%	87.1%	37.4%	—	—
Avg N	235	723	1274	2567	3993	5561	100	10000

Notes: This table shows the bias, the standard deviation (SD) and the root mean squared error (RMSE) of 2SLS with Quasi Propensity Score controls, 2SLS with ML controls, and OLS with no controls. These statistics are computed with the estimand set to ATE, ATE(RCT), LATE, or LATE(RCT). The row "Coverage" in each panel shows the probabilities that the 95% confidence intervals of the form $[\hat{\beta}_1^s - 1.96\hat{\sigma}_n^s, \hat{\beta}_1^s + 1.96\hat{\sigma}_n^s]$ contains LATE(RCT), where $\hat{\beta}_1^s$ is the 2SLS estimate with Quasi Propensity Score controls and $\hat{\sigma}_n^s$ is its heteroskedasticity-robust standard error. We use $1,000$ replications of a size $10,000$ simulated sample to compute these statistics. We use several possible values of δ to compute the Quasi Propensity Score. All Quasi Propensity Scores are computed by averaging 400 simulation draws of the ML value. Panel A reports the results under the model in which the treatment effect does not depend on X_i. Panel B reports the results under the model in which the treatment effect depends on X_i. The bottom row "Avg N" in each panel shows the average number of observations used for estimation (i.e., the average number of observations for which the Quasi Propensity Score or the ML value is strictly between 0 and 1).

Figure 3: Three-dimensional Regression Discontinuity in Hospital Funding Eligibility.



Notes: We remove hospitals above the 99th percentile of disproportionate patient percentage and uncompensated care per bed, for visibility purposes. The top figure visualizes the three dimensions that determine safety net eligibility. The bottom figures show the data points plotted along 2 out of 3 dimensions. The bottom left panel plots disproportionate patient percentage against profit margin, while the bottom right panel plots uncompensated care per bed against profit margin.

Table 2: Outcome Variables and Hospital Characteristics

	All	Ineligible Hospitals	Eligible Hospitals
Panel A: Outcome Variable Means			
Patients in adult inpatient beds with lab-confirmed or suspected COVID	105.37	98.32	135.60
Patients in adult inpatient beds with lab-confirmed COVID (including those with both lab-confirmed COVID and influenza)	80.12	73.90	107.65
Patients in adult ICU beds with lab-confirmed or suspected COVID	31.40	28.93	42.17
Patients in adult ICU beds who have lab-confirmed COVID (including those with both lab-confirmed COVID and influenza)	26.67	24.43	36.71
Observations	4,008	3,291	717
Panel B: Hospital Characteristics			
Beds	143.66	134.60	188.35
Interns and residents (full-time equivalents) per bed	.06	.05	.11
Adult and pediatric hospital beds	120.26	113.29	154.66
Ownership: Proprietary (for-profit)	.19	.20	.18
Ownership: Governmental	.22	.22	.23
Ownership: Voluntary (non-profit)	.58	.58	.59
Inpatient length of stay	9.21	10.14	4.66
Employees on payroll (full-time equivalents)	973.90	897.31	1351.57
Observations	4,633	3,852	781

Notes: This table reports averages of outcome variables and hospital characteristics by safety net eligibility. A safety net hospital is defined as any acute care hospital with disproportionate patient percentage of 20.2% or greater, annual uncompensated care of at least $25,000 per bed and profit margin of 3.0% or less. Panel A reports the outcome variable means. Outcome variable estimates are 7 day sums for the week spanning July 31st 2020 to August 6th 2020. Inpatient bed totals also include observation beds. Panel B reports the means for hospital characteristics for the financial year 2018.

Table 3: Covariate Balance Regressions

	No Controls	Our Method with Quasi Propensity Score Controls							Mean
		$\delta = 0.01$	$\delta = 0.025$	$\delta = 0.05$	$\delta = 0.075$	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Beds	53.75***	204.96	28.85	9.92	0.42	4.22	16.01	8.95	134.60
	(7.05)	(106.65)	(67.20)	(47.17)	(38.63)	(33.69)	(20.11)	(14.36)	
	N=4633	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
Costs per discharge	−49.95**	4.12	3.52*	1.72	−6.76	−0.43	5.68	6.33	66.28
(in thousands)	(17.93)	(2.12)	(1.51)	(1.24)	(8.34)	(2.06)	(4.25)	(4.80)	
	N=3539	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
Disproportionate	0.21***	−0.09	−0.09	−0.09	−0.08	−0.09	−0.06*	−0.07***	.18
payment percent	(0.01)	(0.09)	(0.07)	(0.07)	(0.05)	(0.05)	(0.02)	(0.02)	
	N=4633	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
Full time employees	454.26***	2,841.76	307.37	127.92	27.38	−11.29	200.42	114.27	897.32
	(69.23)	(1,729.87)	(1,009.69)	(652.56)	(491.24)	(428.97)	(218.73)	(141.57)	
	N=4626	N=89	N=234	N=472	N=655	N=851	N=1696	N=2336	
Medicare net revenue	18.36***	37.35	−9.10	−4.61	−2.60	0.05	3.59	−0.28	20.04
(in millions)	(2.39)	(30.38)	(18.55)	(14.19)	(11.80)	(10.77)	(6.66)	(4.62)	
	N=4511	N=88	N=234	N=471	N=653	N=848	N=1659	N=2295	
Occupancy	0.07***	0.19	0.07	−0.00	0.01	0.01	0.03	0.04**	.44
	(0.01)	(0.10)	(0.06)	(0.04)	(0.04)	(0.03)	(0.02)	(0.01)	
	N=4624	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
Operating margin	−0.11***	−0.04	−0.01	0.03	0.02	0.03	0.06***	0.07***	.02
	(0.01)	(0.06)	(0.05)	(0.03)	(0.03)	(0.03)	(0.02)	(0.01)	
	N=4541	N=88	N=234	N=465	N=646	N=841	N=1651	N=2285	
Profit margin	−0.11***	−0.03	−0.01	0.02	0.01	0.02	0.04**	0.06***	.04
	(0.01)	(0.06)	(0.04)	(0.03)	(0.03)	(0.02)	(0.01)	(0.01)	
	N=4633	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
Uncompensated	19,540.28***	3,654.68	11,010.77	−4,644.86	−10,167.80	−11,096.86	−7,850.91	−6,018.15	56,556.02
care per bed	(3,827.22)	(12,124.80)	(10,352.08)	(8,868.36)	(7,606.21)	(7,274.64)	(4,520.95)	(3,638.71)	
	N=4633	N=89	N=235	N=473	N=656	N=852	N=1699	N=2339	
p-value for joint significance	0	.697	.439	.565	.738	.236	.001	0	

Notes: This table shows the results of the covariate balance regressions at the hospital level. The dependent variables for these regressions are drawn from the Healthcare Cost Report Information System for the financial year 2018. Disproportionate patient percentage, profit margin and uncompensated care per bed are used to determine the hospital's safety net funding eligibility. Other dependent variables shown indicate the financial health and utilization of the hospitals. In column 1, we regress the dependent variables on the safety net eligibility of the hospital with no controls. In columns 2–8, we regress the dependent variables on funding eligibility controlling for the Quasi Propensity Score with different values of bandwidth δ. All Quasi Propensity Scores are computed by averaging 1,000 simulation draws. Column 9 shows the mean of dependent variables for hospitals that are ineligible to receive safety net funding. Robust standard errors are reported in the parenthesis and number of observations are reported separately for each regression. The last row reports the p-value of the joint significance test.

Table 4: Estimated Effects of Safety Net Funding on Hospital Utilization

	OLS with No Controls	2SLS with No Controls	Our Method: 2SLS with Quasi Propensity Score Controls						
			$\delta =$ 0.01	$\delta =$ 0.025	$\delta =$ 0.05	$\delta =$ 0.075	$\delta = 0.1$	$\delta =$ 0.25	$\delta = 0.5$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
7 day sum of patient currently hospitalized in an adult inpatient bed (including observation beds) who have lab-confirmed or suspected COVID									
First stage		13.79***	15.07*	13.81***	14.61***	14.02***	13.85***	13.84***	13.11***
(in millions)		(0.49)	(5.79)	(3.56)	(2.29)	(1.86)	(1.64)	(1.02)	(0.73)
$1mm of funding	5.52***	2.70***	−1.16	−1.24	−2.40	−4.55	−3.16	0.15	−0.28
	(0.68)	(0.58)	(5.36)	(5.11)	(4.79)	(4.64)	(3.70)	(1.64)	(1.22)
Observations	3544	3544	74	192	386	535	698	1375	1934
7 day sum of patient currently hospitalized in an adult inpatient bed (including observation beds) who have lab-confirmed COVID (including those with both lab-confirmed COVID and influenza)									
First stage		13.91***	16.70**	14.81***	15.32***	14.62***	14.41***	14.01***	13.24***
(in millions)		(0.50)	(6.10)	(3.68)	(2.34)	(1.91)	(1.67)	(1.04)	(0.74)
$1mm of funding	4.50***	2.43***	−0.09	−1.77	1.79	−0.21	−0.07	−0.08	−0.52
	(0.63)	(0.50)	(4.09)	(3.70)	(2.17)	(2.00)	(1.74)	(1.17)	(0.97)
Observations	3572	3572	71	188	378	527	689	1355	1911
7 day sum of patient currently hospitalized in a designated adult ICU bed who have lab-confirmed or suspected COVID									
First stage		13.92***	14.70*	13.89***	16.02***	15.12***	14.68***	14.26***	13.27***
(in millions)		(0.50)	(5.58)	(3.50)	(2.34)	(1.92)	(1.69)	(1.06)	(0.75)
$1mm of funding	1.66***	0.95***	0.89	0.87	0.47	−0.16	0.06	0.03	−0.26
	(0.21)	(0.18)	(1.39)	(1.18)	(0.73)	(0.70)	(0.61)	(0.42)	(0.36)
Observations	3452	3452	72	183	367	507	659	1300	1832
7 day sum of patient currently hospitalized in a designated adult ICU bed who have lab-confirmed COVID (including those with both lab-confirmed COVID and influenza)									
First stage		13.93***	15.78*	14.29***	16.06***	15.34***	14.92***	14.37***	13.50***
(in millions)		(0.50)	(6.07)	(3.73)	(2.42)	(2.00)	(1.75)	(1.09)	(0.76)
$1mm of funding	1.50***	0.88***	0.49	0.08	0.28	−0.10	0.04	−0.10	−0.29
	(0.21)	(0.17)	(1.45)	(1.26)	(0.70)	(0.64)	(0.57)	(0.40)	(0.34)
Observations	3510	3510	67	178	363	503	648	1305	1853

Notes: In this table we regress relevant outcomes at the hospital level on safety net funding. Column 1 presents the results of OLS regression of the outcome variables on safety net funding without any controls. In columns 2–9, we instrument safety net funding with eligibility to receive this funding and present the results of 2SLS regressions. In columns 2–9, the first stage shows the effect of being deemed eligible on the amount of relief funding received by hospitals, in millions of dollars. Column 2 shows the results of a 2SLS regression with no controls. In columns 3–9, we run this regression controlling for the Quasi Propensity Score with different values of bandwidth δ on the sample with nondegenerate Quasi Propensity Score. All Quasi Propensity Scores are computed by averaging 1,000 simulation draws. Robust standard errors are reported in parentheses.

A Extensions and Discussions

A.1 Related Literature: Details

In this section, we discuss the related methodological literature on the multidimensional RDD in detail. Imbens and Wager (2019) propose the finite-sample-minimax linear estimator of the form $\sum_{i=1}^{n} \gamma_i Y_i$ and uniform confidence intervals for treatment effects in the multidimensional RDD. One version of their approach constructs a linear estimator by choosing the weight $(\gamma_i)_{i=1}^{n}$ greedily to make the inference as precise as possible. Although their estimator is favorable in terms of precision, it is not obvious what estimand the estimator estimates, without assuming a constant treatment effect. The other version of Imbens and Wager (2019)'s approach and some other existing approaches (Zajonc, 2012; Keele and Titiunik, 2015) consider nonparametric estimation of the conditional average treatment effect $E[Y_i(1) - Y_i(0)|X_i = x]$ for a specified boundary point x. The estimand has a clear interpretation, but "when curvature is nonnegligible, equation (6) can effectively make use of only data near the specified focal point c, thus resulting in relatively long confidence intervals" (Imbens and Wager, 2019, p. 268), where equation (6) defines their estimator.

To obtain more precise estimates while keeping interpretability, several papers studying a two-dimensional RDD, including Zajonc (2012) and Keele and Titiunik (2015), propose to estimate an integral of conditional average treatment effects over the boundary. Their approach first nonparametrically estimates $E[Y_i(1) - Y_i(0)|X_i = x]$ and the density of X_i for a large number of points x in the boundary and then computes the weighted average of the estimated conditional average treatment effects with the weight set to the estimated density.

The above approach is difficult to implement, however, when X_i is high dimensional or the decision algorithm is a complex, black box function of X_i, for the following reasons. First, it is computationally demanding to estimate $E[Y_i(1) - Y_i(0)|X_i = x]$ for numerous points in the boundary such that the weighted average well approximates the integral of $E[Y_i(1) - Y_i(0)|X_i = x]$ over the boundary. Second, identifying boundary points from a general decision algorithm itself is hard unless it has a known analytical form. By contrast, we develop an estimator that uses observations near all the boundary points without tracing out the boundary or knowing its analytical form, thus alleviating the limitations of existing estimators.

A.2 QPS May Not Exist But Does Exist for Almost All x

Figure 4 shows an example where QPS does not exist at $\mathbf{0}$. In this example, X_i is two dimensional, and

$$ML(x) = \begin{cases} 1 & \text{if } 3(\frac{1}{2})^{k-1} < \|x\| \leq 4(\frac{1}{2})^{k-1} \text{ for some } k = 1, 2, \cdots \\ 0 & \text{if } 2(\frac{1}{2})^{k-1} < \|x\| \leq 3(\frac{1}{2})^{k-1} \text{ for some } k = 1, 2, \cdots. \end{cases}$$

It is shown that

$$p^{ML}(\mathbf{0}; \delta) = \begin{cases} \frac{7}{12} & \text{if } \delta = 4(\frac{1}{2})^{k-1} \text{ for some } k = 1, 2, \cdots \\ \frac{7}{27} & \text{if } \delta = 3(\frac{1}{2})^{k-1} \text{ for some } k = 1, 2, \cdots. \end{cases}$$

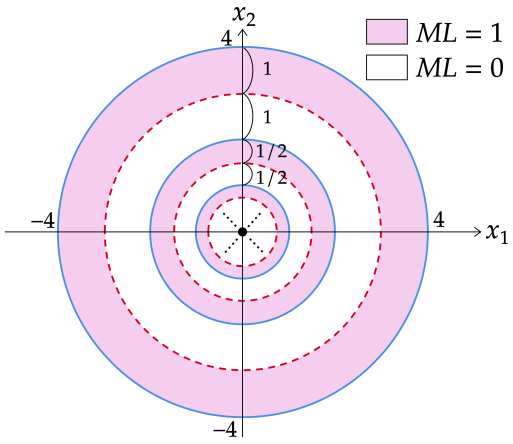


Figure 4: An example of the ML algorithm for which the Quasi Propensity Score fails to exist

Therefore, $\lim_{\delta \to 0} p^{ML}(\mathbf{0}; \delta)$ does not exist.

Nevertheless, the Quasi Propensity Score exists for almost every x, as shown in the following proposition.

Proposition A.1. *$p^{ML}(x)$ exists and is equal to $ML(x)$ for almost every $x \in \mathcal{X}$ (with respect to the Lebesgue measure).*

Proof. See Appendix C.5. □

A.3 Discrete Covariates

In this section, we provide the definition of QPS and identification and consistency results when X_i includes discrete covariates. Suppose that $X_i = (X_{di}, X_{ci})$, where $X_{di} \in \mathbb{R}^{p_d}$ is a vector of discrete covariates, and $X_{ci} \in \mathbb{R}^{p_c}$ is a vector of continuous covariates. Let $\mathcal{X}_d$ denote the support of X_{di} and be assumed to be finite. We also assume that X_{ci} is continuously distributed conditional on X_{di}, and let $\mathcal{X}_c(x_d)$ denote the support of X_{ci} conditional on $X_{di} = x_d$ for each $x_d \in \mathcal{X}_d$. Let $\mathcal{X}_{c,0}(x_d) = \{x_c \in \mathcal{X}_c(x_d) : ML(x_d, x_c) = 0\}$ and $\mathcal{X}_{c,1}(x_d) = \{x_c \in \mathcal{X}_c(x_d) : ML(x_d, x_c) = 1\}$.

Define QPS as follows: for each $x = (x_d, x_c) \in \mathcal{X}$,

$$p^{ML}(x; \delta) \equiv \frac{\int_{B(x_c, \delta)} ML(x_d, x_c^*) dx_c^*}{\int_{B(x_c, \delta)} dx_c^*},$$
$$p^{ML}(x) \equiv \lim_{\delta \to 0} p^{ML}(x; \delta),$$

where $B(x_c, \delta) = \{x_c^* \in \mathbb{R}^{p_c} : \|x_c - x_c^*\| \leq \delta\}$ is the δ-ball around $x_c \in \mathbb{R}^{p_c}$. In other words, we take the average of the $ML(x_d, x_c^*)$ values when x_c^* is uniformly distributed on $B(x_c, \delta)$ holding x_d fixed, and let $\delta \to 0$. Below, we assume that Assumptions 1, 2, 3 and 4 hold conditional on X_{di}.

Assumption A.1 (Almost Everywhere Continuity of ML)**.**

(a) For every $x_d \in \mathcal{X}_d$, $ML(x_d, \cdot)$ is continuous almost everywhere with respect to the Lebesgue measure $\mathcal{L}^{p_c}$.

(b) For every $x_d \in \mathcal{X}_d$, $\mathcal{L}^{p_c}(\mathcal{X}_{c,k}(x_d)) = \mathcal{L}^{p_c}(\text{int}(\mathcal{X}_{c,k}(x_d)))$ for $k = 0, 1$.

A.3.1 Identification

Assumption A.2 (Local Mean Continuity)**.** *For every $x_d \in \mathcal{X}_d$ and $z \in \{0, 1\}$, the conditional expectation functions $E[Y_{zi}|X_i = (x_d, x_c)]$ and $E[D_i(z)|X_i = (x_d, x_c)]$ are continuous in x_c at any point $x_c \in \mathcal{X}_c(x_d)$ such that $p^{ML}(x_d, x_c) \in (0, 1)$ and $ML(x_d, x_c) \in \{0, 1\}$.*

Let $\text{int}_c(\mathcal{X}) = \{(x_d, x_c) \in \mathcal{X} : x_c \in \text{int}(\mathcal{X}_c(x_d))\}$. We say that a set $A \subset \mathbb{R}^p$ *is open relative to* $\mathcal{X}$ if there exists an open set $U \subset \mathbb{R}^p$ such that $A = U \cap \mathcal{X}$. For a set $A \subset \mathbb{R}^p$, let $\mathcal{X}_d^A = \{x_d \in \mathcal{X}_d : (x_d, x_c) \in A \text{ for some } x_c \in \mathbb{R}^{p_c}\}$ and $\mathcal{X}_c^A(x_d) = \{x_c \in \mathcal{X}_c : (x_d, x_c) \in A\}$ for each $x_d \in \mathcal{X}_d^A$.

Proposition A.2. *Under Assumptions A.1 and A.2:*

(a) $E[Y_{1i} - Y_{0i}|X_i = x]$ and $E[D_i(1) - D_i(0)|X_i = x]$ are identified for every $x \in \text{int}_c(\mathcal{X})$ such that $p^{ML}(x) \in (0, 1)$.

(b) Let A be any subset of $\mathcal{X}$ open relative to $\mathcal{X}$ such that $p^{ML}(x)$ exists for all $x \in A$. Then either $E[Y_{1i} - Y_{0i}|X_i \in A]$ or $E[D_i(1) - D_i(0)|X_i \in A]$, or both are identified only if $p^{ML}(x) \in (0, 1)$ for almost every $x_c \in \mathcal{X}_c^A(x_d)$ for every $x_d \in \mathcal{X}_d^A$.

Proof. See Appendix C.7. □

A.3.2 Estimation

For each $x_d \in \mathcal{X}_d$, let $\Omega^*(x_d) = \{x_c \in \mathbb{R}^{p_c} : ML(x_d, x_c) = 1\}$. Also, let $\mathcal{X}_d^* = \{x_d \in \mathcal{X}_d : \text{Var}(ML(X_i)|X_{di} = x_d) > 0\}$, and let $f_{X_c|X_d}$ denote the probability density function of X_{ci} conditional on X_{di}. In addition, for each $x_d \in \mathcal{X}_d$, let

$$C^*(x_d) = \{x_c \in \mathbb{R}^{p_c} : ML(x_d, \cdot) \text{ is continuously differentiable at } x_c\},$$

and let $D^*(x_d) = \mathbb{R}^{p_c} \setminus C^*(x_d)$.

Assumption A.3.

(a) (Finite Moments) $E[Y_i^4] < \infty$.

(b) (Nonzero First Stage) *There exists a constant $c > 0$ such that $E[D_i(1) - D_i(0)|X_i = x] > c$ for every $x \in \mathcal{X}$ such that $p^{ML}(x) \in (0, 1)$.*

(c) (Nonzero Conditional Variance) *If* $\Pr(ML(X_i) \in (0, 1)) > 0$, *then* $\text{Var}(ML(X_i)|ML(X_i) \in (0, 1)) > 0$.

If $\Pr(ML(X_i) \in (0,1)) = 0$*, then the following conditions (d)–(g) hold.*

(d) (Nonzero Variance) $\mathcal{X}_d^* \neq \emptyset$.

(e) (C^2 Boundary of $\Omega^*(x_d)$) *For each* $x_d \in \mathcal{X}_d^*$*, there exists a partition* $\{\Omega_1^*(x_d), ..., \Omega_M^*(x_d)\}$ *of* $\Omega^*(x_d)$ *such that*

(i) $\text{dist}(\Omega_m^*(x_d), \Omega_{m'}^*(x_d)) > 0$ *for any* $m, m' \in \{1, ..., M\}$ *such that* $m \neq m'$*;*

(ii) $\Omega_m^*(x_d)$ *is nonempty, bounded, open, connected and twice continuously differentiable for each* $m \in \{1, ..., M\}$.

(f) (Regularity of Deterministic ML)

(i) For each $x_d \in \mathcal{X}_d^*$*,* $\mathcal{H}^{p_c-1}(\partial\Omega^*(x_d)) < \infty$*, and* $\int_{\partial\Omega^*(x_d)} f_{X_c|X_d}(x_c|x_d) d\mathcal{H}^{p_c-1}(x_c) > 0$.

(ii) There exists $\delta > 0$ *such that* $ML(x_d, x_c) = 0$ *for almost every* $x_c \in N(\mathcal{X}_c(x_d), \delta) \setminus \Omega^*(x_d)$.

(g) (Conditional Means and Density near $\partial\Omega^*(x_d)$) *For each* $x_d \in \mathcal{X}_d^*$*, there exists* $\delta > 0$ *such that*

(i) $E[Y_{1i}|X_i = (x_d, \cdot)]$*,* $E[Y_{0i}|X_i = (x_d, \cdot)]$*,* $E[D_i(1)|X_i = (x_d, \cdot)]$*,* $E[D_i(0)|X_i = (x_d, \cdot)]$ *and* $f_{X_c|X_d}(\cdot|x_d)$ *are continuously differentiable and have bounded partial derivatives on* $N(\partial\Omega^*(x_d), \delta)$*;*

(ii) $E[Y_{1i}^2|X_i = (x_d, \cdot)]$*,* $E[Y_{0i}^2|X_i = (x_d, \cdot)]$*,* $E[Y_{1i}D_i(1)|X_i = (x_d, \cdot)]$ *and* $E[Y_{0i}D_i(0)|X_i = (x_d, \cdot)]$ *are continuous on* $N(\partial\Omega^*(x_d), \delta)$*;*

(iii) $E[Y_i^4|X_i = (x_d, \cdot)]$ *is bounded on* $N(\partial\Omega^*(x_d), \delta)$.

Assumption A.4. *If* $\Pr(ML(X_i) \in (0,1)) > 0$*, then the following conditions (a)–(c) hold.*

(a) (Probability of Neighborhood of $D^*(x_d)$) *For each* $x_d \in \mathcal{X}_d^*$*,* $\Pr(X_i \in N(D^*(x_d), \delta)) = O(\delta)$.

(b) (Bounded Partial Derivatives of ML) *For each* $x_d \in \mathcal{X}_d^*$*, the partial derivatives of* $ML(x_d, \cdot)$ *are bounded on* $C^*(x_d)$.

(c) (Bounded Conditional Mean) *For each* $x_d \in \mathcal{X}_d^*$*,* $E[Y_i|X_i = (x_d, \cdot)]$ *is bounded on* $\mathcal{X}_c(x_d)$.

Theorem A.1. *Suppose that Assumptions A.1 and A.3 hold, and that* $\delta_n \to 0$*,* $n\delta_n \to \infty$ *and* $S_n \to \infty$ *as* $n \to \infty$*. Then the 2SLS estimators* $\hat{\beta}_1$ *and* $\hat{\beta}_1^s$ *converge in probability to*

$$\beta_1 \equiv \lim_{\delta \to 0} E[\omega_i(\delta)(Y_i(1) - Y_i(0))],$$

where

$$\omega_i(\delta) = \frac{p^{ML}(X_i;\delta)(1 - p^{ML}(X_i;\delta))(D_i(1) - D_i(0))}{E[p^{ML}(X_i;\delta)(1 - p^{ML}(X_i;\delta))(D_i(1) - D_i(0))]}.$$

Suppose, in addition, that Assumptions A.4 and 5 hold and that $n\delta_n^2 \to 0$ *as* $n \to \infty$*. Then*

$$\hat{\sigma}_n^{-1}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0,1),$$
$$(\hat{\sigma}_n^s)^{-1}(\hat{\beta}_1^s - \beta_1) \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. See Appendix C.8. □

As in the case in which all covariates are continuous, the probability limit of the 2SLS estimators has more specific expressions depending on whether $\Pr(ML(X_i) \in (0,1)) > 0$ or not. If $\Pr(ML(X_i) \in (0,1)) > 0$,

$$\operatorname{plim} \hat{\beta}_1 = \operatorname{plim} \hat{\beta}_1^s = \frac{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))]}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]}.$$

If $\Pr(ML(X_i) \in (0,1)) = 0$,

$$\begin{aligned}
&\operatorname{plim} \hat{\beta}_1 \\
&= \operatorname{plim} \hat{\beta}_1^s \\
&= \frac{\sum_{x_d \in \mathcal{X}_d^*} \Pr(X_{di} = x_d) \int_{\partial\Omega^*(x_d)} E[(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))|X_i = x] f_{X_c|X_d}(x_c|x_d) d\mathcal{H}^{p_c-1}(x_c)}{\sum_{x_d \in \mathcal{X}_d^*} \Pr(X_{di} = x_d) \int_{\partial\Omega^*(x_d)} E[D_i(1) - D_i(0)|X_i = x] f_{X_c|X_d}(x_c|x_d) d\mathcal{H}^{p_c-1}(x_c)}.
\end{aligned}$$

A.4 A Sufficient Condition for Assumption 4 (a)

We provide a sufficient condition for Assumption 4 (a).

Assumption A.5.

(a) (Twice Continuous Differentiability of D^*) *There exist* $C_1^*, ..., C_M^* \subset \mathbb{R}^p$ *such that*

(i) $\partial(\tilde{C}^*) = D^*$, *where* $\tilde{C}^* \equiv \cup_{m=1}^M C_m^*$;

(ii) $\operatorname{dist}(C_m^*, C_{m'}^*) > 0$ *for any* $m, m' \in \{1, ..., M\}$ *such that* $m \neq m'$;

(iii) C_m^* *is nonempty, bounded, open, connected and twice continuously differentiable for each* $m \in \{1, ..., M\}$.

(b) (Regularity of D^*) $\mathcal{H}^{p-1}(D^*) < \infty$.

(c) (Bounded Density near D^*) *There exists* $\delta > 0$ *such that* f_X *is bounded on* $N(D^*, \delta)$.

The key condition is the twice continuous differentiability of D^*. This condition holds if, for example, the ϵ-Greedy algorithm described in Part 2 (a) of Example 1 in Section 7 uses an estimated Q-function that is twice continuously differentiable in x.

Under Assumption A.5 (a), by Lemma B.4 in Appendix B.3 and with change of variables $v = \frac{\lambda}{\delta}$, for any sufficiently small $\delta > 0$,

$$\begin{aligned}
\Pr(X_i \in N(D^*, \delta)) &= \int_{-\delta}^{\delta} \int_{D^*} f_X(u + \lambda \nu_{\tilde{C}^*}(u)) J_{p-1}^{D^*} \psi_{\tilde{C}^*}(u, \lambda) d\mathcal{H}^{p-1}(u) d\lambda \\
&= \delta \int_{-1}^{1} \int_{D^*} f_X(u + \delta v \nu_{\tilde{C}^*}(u)) J_{p-1}^{D^*} \psi_{\tilde{C}^*}(u, \delta v) d\mathcal{H}^{p-1}(u) dv.
\end{aligned}$$

(See Appendix B for the notation.) If f_X is bounded on $N(D^*, \delta)$ and $\mathcal{H}^{p-1}(D^*) < \infty$, the right-hand side is $O(\delta)$.

A.5 Sampling from Uniform Distribution on p-Dimensional Ball

When we calculate QPS by simulation, we need to uniformly sample from $B(X_i, \delta)$. We introduce three existing methods to uniformly sample from a p-dimensional unit ball $B(\mathbf{0}, 1)$. By multiplying the sampled vector by δ and adding X_i to it, we can sample from a uniform distribution on $B(X_i, \delta)$.

Method 1.
1. Sample $x_1, ..., x_p$ independently from the uniform distribution on $[-1, 1]$.
2. Accept the vector $x = (x_1, ..., x_p)$ if $\sum_{k=1}^{p} x_k^2 \leq 1$ and reject it otherwise.

Method 1 is a practical choice when p is small (e.g. $p = 2, 3$), but is inefficient for higher dimensions, since the acceptance rate decreases to zero quickly as p increases. The conventional method used for higher dimensions is the following.

Method 2.
1. Sample $x_1^*, ..., x_p^*$ independently from the standard normal distribution, and compute the vector $s = (x_1^*, ..., x_p^*)/\sqrt{\sum_{k=1}^{p}(x_k^*)^2}$.
2. Sample u from the uniform distribution on $[0, 1]$.
3. Return the vector $x = u^{1/p}s$.

There is yet another method efficient for higher dimensions, which is recently proposed by Voelker, Gosmann and Stewart (2017).

Method 3.
1. Sample $x_1^*, ..., x_{p+2}^*$ independently from the standard normal distribution, and compute the vector $s = (x_1^*, ..., x_{p+2}^*)/\sqrt{\sum_{k=1}^{p+2}(x_k^*)^2}$.
2. Return the vector $x = (s_1, ..., s_p)$.

B Notation and Lemmas

B.1 Basic Notations

For a scalar-valued differentiable function $f : A \subset \mathbb{R}^n \to \mathbb{R}$, let $\nabla f : A \to \mathbb{R}^n$ be a gradient of f: for every $x \in A$,

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n} \right)'.$$

Also, when the second-order partial derivatives of f exist, let $D^2 f(x)$ be the Hessian matrix:

$$D^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

for each $x \in A$.

Let $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ be a function such that its first-order partial derivatives exist. For each $x \in A$, let $Jf(x)$ be the Jacobian matrix of f at x:

$$Jf(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_m} \end{bmatrix}.$$

For a positive integer n, let I_n denote the $n \times n$ identity matrix.

B.2 Differential Geometry

We provide some concepts and facts from differential geometry of twice continuously differentiable sets, following Crasta and Malusa (2007). Let $A \subset \mathbb{R}^p$ be a twice continuously differentiable set. For each $x \in \partial A$, we denote by $\nu_A(x) \in \mathbb{R}^p$ the inward unit normal vector of ∂A at x, that is, the unit vector orthogonal to all vectors in the tangent space of ∂A at x that points toward the inside of A. For a set $A \subset \mathbb{R}^p$, let $d_A^s : \mathbb{R}^p \to \mathbb{R}$ be the signed distance function of A, defined by

$$d_A^s(x) = \begin{cases} d(x, \partial A) & \text{if } x \in \text{cl}(A) \\ -d(x, \partial A) & \text{if } x \in \mathbb{R}^p \setminus \text{cl}(A), \end{cases}$$

where $d(x, B) = \inf_{y \in B} \|y - x\|$ for any $x \in \mathbb{R}^p$ for a set $B \subset \mathbb{R}^p$. Note that we can write $N(\partial A, \delta) = \{x \in \mathbb{R}^p : -\delta < d_A^s(x) < \delta\}$ for $\delta > 0$. Lastly, let $\Pi_{\partial A}(x) = \{y \in \partial A : \|y - x\| = d(x, \partial A)\}$ be the set of projections of x on ∂A.

Lemma B.1 (Corollary of Theorem 4.16, Crasta and Malusa (2007))**.** *Let $A \subset \mathbb{R}^p$ be nonempty, bounded, open, connected and twice continuously differentiable. Then the function d_A^s is twice continuously differentiable on $N(\partial A, \mu)$ for some $\mu > 0$. In addition, for every $x_0 \in \partial A$, $\Pi_{\partial A}(x_0 + t\nu_A(x_0)) = \{x_0\}$ for every $t \in (-\mu, \mu)$. Furthermore, for every $x \in N(\partial A, \mu)$, $\Pi_{\partial A}(x)$ is a singleton, $\nabla d_A^s(x) = \nu_A(y)$ and $x = y + d_A^s(x)\nu_A(y)$ for $y \in \Pi_{\partial A}(x)$, and $\|\nabla d_A^s(x)\| = 1$.*

Proof. We apply results from Crasta and Malusa (2007). Let $K = \{x \in \mathbb{R}^p : \|x\| \leq 1\}$. K is nonempty, compact, convex subset of $\mathbb{R}^p$ with the origin as an interior point. The polar body of K, defined as $K_0 = \{y \in \mathbb{R}^p : y \cdot x \leq 1 \text{ for all } x \in K\}$, is K itself. The gauge functions $\rho_K, \rho_{K_0} : \mathbb{R}^p \to [0, \infty]$ of K and K_0 are given by

$$\rho_K(x) \equiv \inf\{t \geq 0 : x \in tK\} = \|x\|,$$
$$\rho_{K_0}(x) \equiv \inf\{t \geq 0 : x \in tK_0\} = \|x\|.$$

Given ρ_{K_0}, the Minkowski distance from a set $S \subset \mathbb{R}^p$ is defined as

$$\delta_S(x) \equiv \inf_{y \in S} \rho_{K_0}(x - y), \quad x \in \mathbb{R}^p.$$

Note that we can write

$$d_A^s(x) = \begin{cases} \delta_{\partial A}(x) & \text{if } x \in \text{cl}(A) \\ -\delta_{\partial A}(x) & \text{if } x \in \mathbb{R}^p \setminus \text{cl}(A). \end{cases}$$

It then follows from Theorem 4.16 of Crasta and Malusa (2007) that d_A^s is twice continuously differentiable on $N(\partial A, \mu)$ for some $\mu > 0$, and for every $x_0 \in \partial A$,

$$\nabla d_A^s(x_0) = \frac{\nu_A(x_0)}{\rho_K(\nu_A(x_0))} = \frac{\nu_A(x_0)}{\|\nu_A(x_0)\|} = \nu_A(x_0),$$

where the last equality follows since $\nu_A(x_0)$ is a unit vector. It then follows that $\|\nabla d_A^s(x_0)\| = \|\nu_A(x_0)\| = 1$ for every $x_0 \in \partial A$. Also, it is obvious that, for every $x_0 \in \partial A$, $\Pi_{\partial A}(x_0) = \{x_0\}$ and $x_0 = x_0 + d_A^s(x_0)\nu_A(x_0)$, since $d_A^s(x_0) = 0$. In addition, as stated in the proof of Theorem 4.16 of Crasta and Malusa (2007), μ is chosen so that (4.7) in Proposition 4.6 of Crasta and Malusa (2007) holds for every $x_0 \in \partial A$ and every $t \in (-\mu, \mu)$. That is, $\Pi_{\partial A}(x_0 + t\nabla\rho_K(\nu_A(x_0))) = \{x_0\}$ for every $x_0 \in \partial A$ and every $t \in (-\mu, \mu)$. Since $\nabla\rho_K(\nu_A(x_0)) = \frac{\nu_A(x_0)}{\|\nu_A(x_0)\|} = \nu_A(x_0)$, $\Pi_{\partial A}(x_0 + t\nu_A(x_0)) = \{x_0\}$ for every $x_0 \in \partial A$ and every $t \in (-\mu, \mu)$.

Furthermore, for every $x \in N(\partial A, \mu) \setminus \partial A$, $\Pi_{\partial A}(x)$ is a singleton as shown in the proof of Theorem 4.16 of Crasta and Malusa (2007). Let $\pi_{\partial A}(x)$ be the unique element in $\Pi_{\partial A}(x)$. By Lemma 4.3 of Crasta and Malusa (2007), for every $x \in N(\partial A, \mu) \setminus \partial A$,

$$\nabla d_A^s(x) = \frac{\nu_A(\pi_{\partial A}(x))}{\rho_K(\nu_A(\pi_{\partial A}(x)))} = \frac{\nu_A(\pi_{\partial A}(x))}{\|\nu_A(\pi_{\partial A}(x))\|} = \nu_A(\pi_{\partial A}(x)),$$

where the last equality follows since $\nu_A(\pi_{\partial A}(x))$ is a unit vector. It then follows that $\|\nabla d_A^s(x)\| = \|\nu_A(\pi_{\partial A}(x))\| = 1$ for every $x \in N(\partial A, \mu) \setminus \partial A$.

Lastly, note that

$$\delta_{\partial A}(x) = \begin{cases} d_A^s(x) & \text{if } x \in N(\partial A, \mu) \cap \operatorname{int}(A) \\ -d_A^s(x) & \text{if } x \in N(\partial A, \mu) \setminus \operatorname{cl}(A), \end{cases}$$

and

$$\nabla\delta_{\partial A}(x) = \begin{cases} \nabla d_A^s(x) & \text{if } x \in N(\partial A, \mu) \cap \operatorname{int}(A) \\ -\nabla d_A^s(x) & \text{if } x \in N(\partial A, \mu) \setminus \operatorname{cl}(A), \end{cases}$$

so $\delta_{\partial A}(x)\nabla\delta_{\partial A}(x) = d_A^s(x)\nabla d_A^s(x) = d_A^s(x)\nu_A(\pi_{\partial A}(x))$ for every $x \in N(\partial A, \mu) \setminus \partial A$. By Proposition 3.3 (i) of Crasta and Malusa (2007), for every $x \in N(\partial A, \mu) \setminus \partial A$,

$$\nabla\rho_K(\nabla\delta_{\partial A}(x)) = \frac{x - \pi_{\partial A}(x)}{\delta_{\partial A}(x)},$$

which implies that

$$\begin{aligned} x &= \pi_{\partial A}(x) + \delta_{\partial A}(x)\nabla\rho_K(\nabla\delta_{\partial A}(x)) \\ &= \pi_{\partial A}(x) + \delta_{\partial A}(x)\frac{\nabla\delta_{\partial A}(x)}{\|\nabla\delta_{\partial A}(x)\|} = \pi_{\partial A}(x) + d_A^s(x)\nu_A(\pi_{\partial A}(x)). \end{aligned}$$

□

We say that a set $A \subset \mathbb{R}^n$ is a *m-dimensional C^1 submanifold of* $\mathbb{R}^n$ if for every point $x \in A$, there exist an open neighborhood $V \subset \mathbb{R}^n$ of x and a one-to-one continuously differentiable function ϕ from an open set $U \subset \mathbb{R}^m$ to $\mathbb{R}^n$ such that the Jacobian matrix $J\phi(u)$ is of rank m for all $u \in U$, and $\phi(U) = V \cap A$.

Lemma B.2. *Let $A \subset \mathbb{R}^p$ be nonempty, bounded, open, connected and twice continuously differentiable. Then ∂A is a $(p-1)$-dimensional C^1 submanifold of $\mathbb{R}^p$,*

Proof. Fix any $x^* \in \partial A$. By Lemma B.1, $\nabla d_A^s(x^*)$ is nonzero. Without loss of generality, let $\frac{\partial d_A^s(x^*)}{\partial x_p} \neq 0$. Let $\psi : \mathbb{R}^p \to \mathbb{R}^p$ be the function such that $\psi(x) = (x_1, ..., x_{p-1}, d_A^s(x))$. ψ is continuously differentiable, and the Jacobian matrix of ψ at x^* is given by

$$J\psi(x^*) = \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1}(x^*) & \cdots & \frac{\partial \psi_1}{\partial x_p}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_p}{\partial x_1}(x^*) & \cdots & \frac{\partial \psi_p}{\partial x_p}(x^*) \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & I_{p-1} & & \vdots \\ & & & 0 \\ \frac{\partial d_A^s(x^*)}{\partial x_1} & \cdots & \frac{\partial d_A^s(x^*)}{\partial x_{p-1}} & \frac{\partial d_A^s(x^*)}{\partial x_p} \end{pmatrix}.$$

Since $\frac{\partial d_A^s(x^*)}{\partial x_p} \neq 0$, the Jacobian matrix is invertible. By the Inverse Function Theorem, there exist an open set V containing x^* and an open set W containing $\psi(x^*)$ such that $\psi : V \to W$ has an inverse function $\psi^{-1} : W \to V$ that is continuously differentiable. We make V small enough so that $\frac{\partial d_A^s(x)}{\partial x_p} \neq 0$ for every $x \in V$. The Jacobian matrix of ψ^{-1} is given by $J\psi^{-1}(y) = J\psi(\psi^{-1}(y))^{-1}$ for all $y \in W$.

Now note that $\psi(x) = (x_1, ..., x_{p-1}, 0)$ for all $x \in V \cap \partial A$ by the definition of d_A^s. Let $U = \{(x_1, ..., x_{p-1}) \in \mathbb{R}^{p-1} : x \in V \cap \partial A\}$ and $\phi : U \to \mathbb{R}^p$ be a function such that $\phi(u) = \psi^{-1}((u, 0))$ for all $u \in U$. Below we verify that ϕ is one-to-one and continously differentiable, that $J\phi(u)$ is of rank $p-1$ for all $u \in U$, that $\phi(U) = V \cap \partial A$, and that U is open.

First, ϕ is one-to-one, since ψ^{-1} is one-to-one, and $(u, 0) \neq (u', 0)$ if $u \neq u'$. Second, ϕ is continuously differentiable, since ψ^{-1} is so. The Jacobian matrix of ϕ at $u \in U$ is by definition

$$J\phi(u) = \begin{pmatrix} \frac{\partial \psi_1^{-1}}{\partial y_1}((u,0)) & \cdots & \frac{\partial \psi_1^{-1}}{\partial y_{p-1}}((u,0)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_p^{-1}}{\partial y_1}((u,0)) & \cdots & \frac{\partial \psi_p^{-1}}{\partial y_{p-1}}((u,0)) \end{pmatrix}.$$

Note that this is the left $p \times (p-1)$ submatrix of $J\psi^{-1}((u, 0))$. Since $J\psi^{-1}((u, 0))$ has full rank, $J\phi(u)$ is of rank $p-1$. Moreover,

$$\begin{aligned} \phi(U) &= \{\psi^{-1}((u, 0)) : u \in U\} \\ &= \{\psi^{-1}((x_1, ..., x_{p-1}, 0)) : x \in V \cap \partial A\} \\ &= \{\psi^{-1}(\psi(x)) : x \in V \cap \partial A\} \\ &= V \cap \partial A. \end{aligned}$$

Lastly, we show that U is open. Pick any $\bar{u} \in U$. Then, there exists $\bar{x}_p \in \mathbb{R}$ such that $(\bar{u}, \bar{x}_p) \in V \cap \partial A$. As $(\bar{u}, \bar{x}_p) \in V \cap \partial A$, $d_A^s((\bar{u}, \bar{x}_p)) = 0$. Since $\frac{\partial d_A^s((\bar{u}, \bar{x}_p))}{\partial x_p} \neq 0$, it follows by the Implicit Function Theorem that there exist an open set $S \subset \mathbb{R}^{p-1}$ containing $\bar{u}$ and a continuously differentiable function $g : S \to \mathbb{R}$ such that $g(\bar{u}) = \bar{x}_p$ and $d_A^s(u, g(u)) = 0$ for all $u \in S$. Since g is continuous, $(\bar{u}, g(\bar{u})) \in V$ and V is open, there exists an open set $S' \subset S$ containing $\bar{u}$ such that $(u, g(u)) \in V$ for all $u \in S'$. By the definition of d_A^s, $d_A^s(x) = 0$ if and only if $x \in \partial A$. Therefore, if $u \in S'$, $(u, g(u))$ must be contained by ∂A, for otherwise $d_A^s(u, g(u)) \neq 0$, which is a contradiction. Thus, $(u, g(u)) \in V \cap \partial A$ and hence $u \in U$ for all $u \in S'$. This implies that S' is an open subset of U containing $\bar{u}$, which proves that U is open. □

B.3 Geometric Measure Theory

We provide some concepts and facts from geometric measure theory, following Krantz and Parks (2008). Recall that for a function $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$ and a point $x \in A$ at which f is differentiable, $Jf(x)$ denotes the Jacobian matrix of f at x.

Lemma B.3 (Coarea Formula, Lemma 5.1.4 and Corollary 5.2.6 of Krantz and Parks (2008))**.** *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \geq n$, then*

$$\int_A g(x) J_n f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \int_{\{x' \in A : f(x') = y\}} g(x) d\mathcal{H}^{m-n}(x) d\mathcal{L}^n(y)$$

for every Lebesgue measurable subset A of $\mathbb{R}^m$ and every $\mathcal{L}^m$-measurable function $g : A \to \mathbb{R}$, where for each $x \in \mathbb{R}^m$ at which f is differentiable,

$$J_n f(x) = \sqrt{\det((Jf(x))(Jf(x))')}.$$

Let A be an m-dimensional C^1 submanifold of $\mathbb{R}^n$. Let $x \in A$ and let $\phi : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be as in the definition of m-dimensional C^1 submanifold. We denote by $T_A(x)$ the tangent space of A at x, $\{J\phi(u)v : v \in \mathbb{R}^m\}$, where $u = \phi^{-1}(x)$.

Lemma B.4 (Area Formula, Lemma 5.3.5 and Theorem 5.3.7 of Krantz and Parks (2008))**.** *Suppose $m \leq \nu$ and $f : \mathbb{R}^n \to \mathbb{R}^\nu$ is Lipschitz. If A is an m-dimensional C^1 submanifold of $\mathbb{R}^n$, then*

$$\int_A g(x) J_m^A f(x) d\mathcal{H}^m(x) = \int_{\mathbb{R}^\nu} \sum_{x \in A : f(x) = y} g(x) d\mathcal{H}^m(y)$$

for every $\mathcal{H}^m$-measurable function $g : A \to \mathbb{R}$, where for each $x \in \mathbb{R}^n$ at which f is differentiable,

$$J_m^A f(x) = \frac{\mathcal{H}^m(\{Jf(x)y : y \in P\})}{\mathcal{H}^m(P)}$$

for an arbitrary m-dimensional parallelepiped P contained in $T_A(x)$.

Let $A \subset \mathbb{R}^p$. For each $x \in \mathbb{R}^p$ at which d_A^s is differentiable and for each $\lambda \in \mathbb{R}$, let $\psi_A(x, \lambda) = x + \lambda \nabla d_A^s(x)$.

Lemma B.5. *Let $\Omega \subset \mathbb{R}^p$, and suppose that there exists a partition $\{\Omega_1, ..., \Omega_M\}$ of Ω such that*

(i) $\text{dist}(\Omega_m, \Omega_{m'}) > 0$ *for any $m, m' \in \{1, ..., M\}$ such that $m \neq m'$;*

(ii) *Ω_m is nonempty, bounded, open, connected and twice continuously differentiable for each $m \in \{1, ..., M\}$.*

Then there exists $\mu > 0$ such that d_Ω^s is twice continuously differentiable on $N(\partial\Omega, \mu)$ and that

$$\int_{N(\partial\Omega, \delta)} g(x) dx = \int_{-\delta}^{\delta} \int_{\partial\Omega} g(u + \lambda \nu_\Omega(u)) J_{p-1}^{\partial\Omega} \psi_\Omega(u, \lambda) d\mathcal{H}^{p-1}(u) d\lambda$$

for every $\delta \in (0, \mu)$ and every function $g : \mathbb{R}^p \to \mathbb{R}$ that is integrable on $N(\partial\Omega, \delta)$, where for each fixed $\lambda \in (-\mu, \mu)$, $J_{p-1}^{\partial\Omega} \psi_\Omega(\cdot, \lambda)$ is calculated by applying the operation $J_{p-1}^{\partial\Omega}$ to the function $\psi_\Omega(\cdot, \lambda)$. Futhermore, $J_{p-1}^{\partial\Omega} \psi_\Omega(x, \cdot)$ is continuously differentiable in λ and $J_{p-1}^{\partial\Omega} \psi_\Omega(x, 0) = 1$ for every $x \in \partial\Omega$, and $J_{p-1}^{\partial\Omega} \psi_\Omega(\cdot, \cdot)$ and $\frac{\partial J_{p-1}^{\partial\Omega} \psi_\Omega(\cdot, \cdot)}{\partial \lambda}$ are bounded on $\partial\Omega \times (-\mu, \mu)$.

Proof. Let $\bar{\mu} = \frac{1}{2}\min_{m,m' \in \{1,...,M\}, m \neq m'} \text{dist}(\Omega_m^*, \Omega_{m'})$ so that $\{N(\partial\Omega_m, \bar{\mu})\}_{m=1}^M$ is a partition of $N(\partial\Omega, \bar{\mu})$. Note that for every $m \in \{1, ..., M\}$, $d_\Omega^s(x) = d_{\Omega_m}^s(x)$ for every $x \in N(\partial\Omega_m, \bar{\mu})$. By Lemma B.1, for every $m \in \{1, ..., M\}$, there exists $\bar{\mu}_m > 0$ such that $d_{\Omega_m}^s$ is twice continuously differentiable on $N(\partial\Omega_m, \bar{\mu}_m)$. Letting $\mu \in (0, \min\{\bar{\mu}, \bar{\mu}_1, ..., \bar{\mu}_M\})$, we have that d_Ω^s is twice continuously differentiable on $N(\partial\Omega, \mu)$. This implies that d_Ω^s is Lipschitz on $N(\partial\Omega, \mu)$. For every $\delta \in (0, \mu)$ and every function $g : \mathbb{R}^p \to \mathbb{R}$ that is integrable on $N(\partial\Omega, \delta)$,

$$\begin{aligned}
\int_{N(\partial\Omega,\delta)} g(x)dx &= \int_{\{x' \in \mathbb{R}^p : d_\Omega^s(x') \in (-\delta,\delta)\}} g(x)\sqrt{\det(\|\nabla d_\Omega^s(x)\|)}dx \\
&= \int_{\{x' \in \mathbb{R}^p : d_\Omega^s(x') \in (-\delta,\delta)\}} g(x)\sqrt{\det(\nabla d_\Omega^s(x)'\nabla d_\Omega^s(x))}dx \\
&= \int_{\{x' \in \mathbb{R}^p : d_\Omega^s(x') \in (-\delta,\delta)\}} g(x)\sqrt{\det((Jd_\Omega^s(x))(Jd_\Omega^s(x))')}dx \\
&= \int_{\mathbb{R}} \int_{\{x' \in \mathbb{R}^p : d_\Omega^s(x') \in (-\delta,\delta), d_\Omega^s(x') = \lambda\}} g(x)d\mathcal{H}^{p-1}(x)d\lambda \\
&= \int_{-\delta}^{\delta} \int_{\{x' \in \mathbb{R}^p : d_\Omega^s(x') = \lambda\}} g(x)d\mathcal{H}^{p-1}(x)d\lambda, \qquad (12)
\end{aligned}$$

where the first equality follows since $\|\nabla d_\Omega^s(x)\| = 1$ for every $x \in N(\partial\Omega, \delta)$ by Lemma B.1, the third equality follows from the definition of the Jacobian matrix, and the fourth equality follows from Lemma B.3.

Let $\Gamma(\lambda) = \{x \in \mathbb{R}^p : d_\Omega^s(x) = \lambda\}$ for each $\lambda \in (-\mu, \mu)$. Since ∇d_Ω^s is differentiable on $N(\partial\Omega, \mu)$, $\psi_\Omega(x, \lambda)$ is defined on $N(\partial\Omega, \mu) \times \mathbb{R}$. We show that $\{\psi_\Omega(x_0, \lambda) : x_0 \in \partial\Omega\} \subset \Gamma(\lambda)$ for every $\lambda \in (-\mu, \mu)$. By Lemma B.1, for every $x_0 \in \partial\Omega$, $\psi_\Omega(x_0, \lambda) = x_0 + \lambda\nu_\Omega(x_0)$ and

$$\Pi_{\partial\Omega}(\psi_\Omega(x_0, \lambda)) = \Pi_{\partial\Omega}(x_0 + \lambda\nu_\Omega(x_0)) = \{x_0\}.$$

Hence,

$$d(\psi_\Omega(x_0, \lambda), \partial\Omega) = \|\psi_\Omega(x_0, \lambda) - x_0\| = \|\lambda\nu_\Omega(x_0)\| = |\lambda|.$$

Since $\nu_\Omega(x_0)$ is an inward normal vector, $\psi_\Omega(x_0, \lambda) \in \text{cl}(A)$ if $0 \leq \lambda < \mu$, and $\psi_\Omega(x, \lambda_0) \in \mathbb{R}^p \setminus \text{cl}(A)$ if $-\mu < \lambda < 0$. It follows that

$$\begin{aligned}
d_A^s(\psi_\Omega(x_0, \lambda)) &= \begin{cases} |\lambda| & \text{if } 0 \leq \lambda < \mu \\ -|\lambda| & \text{if } \mu < \lambda < 0 \end{cases} \\
&= \lambda,
\end{aligned}$$

so $\{\psi_\Omega(x_0, \lambda) : x_0 \in \partial\Omega\} \subset \Gamma(\lambda)$. It also holds that $\Gamma(\lambda) \subset \{\psi_\Omega(x_0, \lambda) : x_0 \in \partial\Omega\}$, since by Lemma B.1, for every $x \in \Gamma(\lambda)$,

$$\psi_\Omega(\pi_{\partial\Omega}(x), \lambda) = \pi_{\partial\Omega}(x) + \lambda\nabla d_\Omega^s(\pi_{\partial\Omega}(x)) = \pi_{\partial\Omega}(x) + d_\Omega^s(x)\nu_\Omega(\pi_{\partial\Omega}(x)) = x,$$

where $\pi_{\partial\Omega}(x)$ is the unique element in $\Pi_{\partial\Omega}(x)$. Thus, $\{\psi_\Omega(x_0, \lambda) : x_0 \in \partial\Omega\} = \Gamma(\lambda)$.

Now note that $\{\partial\Omega_m\}_{m=1}^M$ is a partition of $\partial\Omega$, since $\text{dist}(\Omega_m, \Omega_{m'}) > 0$ for any $m, m' \in \{1, ..., M\}$ such that $m \neq m'$. By Lemma B.2, $\partial\Omega_m$ is a $(p-1)$-dimensional C^1 submanifold of $\mathbb{R}^p$ for every $m \in \{1, ..., M\}$, and hence $\partial\Omega$ is a $(p-1)$-dimensional C^1 submanifold of $\mathbb{R}^p$. Furthermore, since ∇d_Ω^s is continuously differentiable on $N(\partial\Omega, \mu)$, $\psi_\Omega(\cdot, \lambda)$ is continuously differentiable on $N(\partial\Omega, \mu)$, which implies that $\psi_\Omega(\cdot, \lambda)$ is Lipschitz on $N(\partial\Omega, \mu)$ for every $\lambda \in \mathbb{R}$. Applying Lemma B.4, we have that for every $\lambda \in (-\mu, \mu)$,

$$\begin{aligned}\int_{\partial\Omega} g(u + \lambda\nu_\Omega(u)) J_{p-1}^{\partial\Omega}\psi_\Omega(u, \lambda) d\mathcal{H}^{p-1}(u) &= \int_{\partial\Omega} g(\psi_\Omega(u, \lambda)) J_{p-1}^{\partial\Omega}\psi_\Omega(u, \lambda) d\mathcal{H}^{p-1}(u) \\ &= \int_{\mathbb{R}^p} \sum_{u\in\partial\Omega:\psi_\Omega(u,\lambda)=x} g(\psi_\Omega(u, \lambda)) d\mathcal{H}^{p-1}(x). \qquad (13)\end{aligned}$$

If $x \notin \{\psi_\Omega(u, \lambda) : u \in \partial\Omega\}$, $\{u \in \partial\Omega : \psi_\Omega(u, \lambda) = x\} = \emptyset$. If $x \in \{\psi_\Omega(u, \lambda) : u \in \partial\Omega\}$, there exists $u \in \partial\Omega$ such that $x = \psi_\Omega(u, \lambda)$. Since $\Pi_{\partial\Omega}(x) = \Pi_{\partial\Omega}(u + \lambda\nabla d_\Omega^s(u)) = \Pi_{\partial\Omega}(u + \lambda\nu_\Omega(u)) = \{u\}$ by Lemma B.1, such u is unique, and hence $\{u \in \partial\Omega : \psi_\Omega(u, \lambda) = x\}$ is a singleton. It follow that

$$\begin{aligned}\int_{\mathbb{R}^p} \sum_{u\in\partial\Omega:\psi_\Omega(u,\lambda)=x} g(\psi_\Omega(u, \lambda)) d\mathcal{H}^{p-1}(x) &= \int_{\{\psi_\Omega(u,\lambda):u\in\partial\Omega\}} g(x) d\mathcal{H}^{p-1}(x) \\ &= \int_{\Gamma(\lambda)} g(x) d\mathcal{H}^{p-1}(x), \qquad (14)\end{aligned}$$

where the last equality holds since $\{\psi_\Omega(u, \lambda) : u \in \partial\Omega\} = \Gamma(\lambda)$. Combining (12), (13) and (14), we obtain

$$\int_{N(\partial\Omega,\delta)} g(x)dx = \int_{-\delta}^{\delta}\int_{\partial\Omega} g(u + \lambda\nu_\Omega(u)) J_{p-1}^{\partial\Omega}\psi_\Omega(u, \lambda) d\mathcal{H}^{p-1}(u)d\lambda.$$

We next show that $J_{p-1}^{\partial\Omega}\psi_\Omega(x, \cdot)$ is continuously differentiable in λ and $J_{p-1}^{\partial\Omega}\psi_\Omega(x, 0) = 1$ for every $x \in \partial\Omega$. Fix an $x \in \partial\Omega$, and let $V_\Omega(x)$ be an arbitrary $p \times (p-1)$ matrix whose columns $v_1(x), ..., v_{p-1}(x) \in \mathbb{R}^p$ form an orthonormal basis of $T_{\partial\Omega}(x)$. Let $P(x) \subset T_{\partial\Omega}(x)$ be a parallelepiped determined by $v_1(x), ..., v_{p-1}(x)$, that is, let $P(x) = \{\sum_{k=1}^{p-1} c_k v_k(x) : 0 \leq c_k \leq 1 \text{ for } k = 1, ..., p-1\}$. Since $v_1(x), ..., v_{p-1}(x)$ are linearly independent, $P(x)$ is a $(p-1)$-dimensional parallelepiped. It follows that for each fixed $\lambda \in \mathbb{R}$,

$$\begin{aligned}\{J\psi_\Omega(x, \lambda)y : y \in P(x)\} &= \{J\psi_\Omega(x, \lambda)\sum_{k=1}^{p-1} c_k v_k(x) : 0 \leq c_k \leq 1 \text{ for } k = 1, ..., p-1\} \\ &= \{\sum_{k=1}^{p-1} c_k J\psi_\Omega(x, \lambda)v_k(x) : 0 \leq c_k \leq 1 \text{ for } k = 1, ..., p-1\} \\ &= \{\sum_{k=1}^{p-1} c_k w_k(x, \lambda) : 0 \leq c_k \leq 1 \text{ for } k = 1, ..., p-1\},\end{aligned}$$

where $w_k(x, \lambda) = J\psi_\Omega(x, \lambda)v_k(x)$ for $k = 1, ..., p-1$. Since $J\psi_\Omega(x, \lambda)v_k(x)$ is the k-th column of $J\psi_\Omega(x, \lambda)V_\Omega(x)$, $\{J\psi_\Omega(x, \lambda)y : y \in P(x)\}$ is the parallelepiped determined by the columns of

$J\psi_\Omega(x,\lambda)V_\Omega(x)$. By Proposition 5.1.2 of Krantz and Parks (2008), we have that

$$\begin{aligned}
J^{\partial\Omega}_{p-1}\psi_\Omega(x,\lambda) &= \frac{\mathcal{H}^{p-1}(\{\sum_{k=1}^{p-1} c_k w_k(x,\lambda) : 0 \le c_k \le 1 \text{ for } k=1,...,p-1\})}{\mathcal{H}^{p-1}(P(x))} \\
&= \frac{\sqrt{\det((J\psi_\Omega(x,\lambda)V_\Omega(x))'(J\psi_\Omega(x,\lambda)V_\Omega(x)))}}{\sqrt{\det(V_\Omega(x)'V_\Omega(x))}} \\
&= \frac{\sqrt{\det((V_\Omega(x)+\lambda D^2 d^s_\Omega(x)V_\Omega(x))'(V_\Omega(x)+\lambda D^2 d^s_\Omega(x)V_\Omega(x)))}}{\sqrt{\det(I_{p-1})}} \\
&= \sqrt{\det(V_\Omega(x)'V_\Omega(x) + 2V_\Omega(x)'\lambda D^2 d^s_\Omega(x)V_\Omega(x) + V_\Omega(x)'(\lambda D^2 d^s_\Omega(x))^2 V_\Omega(x))} \\
&= \sqrt{\det(I_{p-1} + \lambda V_\Omega(x)'(2D^2 d^s_\Omega(x) + \lambda(D^2 d^s_\Omega(x))^2)V_\Omega(x)))} \\
&= \sqrt{\det(I_p + \lambda V_\Omega(x)V_\Omega(x)'(2D^2 d^s_\Omega(x) + \lambda(D^2 d^s_\Omega(x))^2))},
\end{aligned}$$

where we use the fact that $V_\Omega(x)'V_\Omega(x) = I_{p-1}$ and the fact that $\det(I_m + AB) = \det(I_n + BA)$ for an $m \times n$ matrix A and an $n \times m$ matrix B (the Weinstein-Aronszajn identity). For every $x \in \partial\Omega$, $J^{\partial\Omega}_{p-1}\psi_\Omega(x,\cdot)$ is continuously differentiable in λ, and $J^{\partial\Omega}_{p-1}\psi_\Omega(x,0) = \sqrt{\det(I_p)} = 1$.

Lastly, we show that $J^{\partial\Omega}_{p-1}\psi_\Omega(\cdot,\cdot)$ and $\frac{\partial J^{\partial\Omega}_{p-1}\psi_\Omega(\cdot,\cdot)}{\partial\lambda}$ are bounded on $\partial\Omega \times (-\mu,\mu)$. Let $f, h : \partial\Omega \times \mathbb{R}^{p\times(p-1)} \to \mathbb{R}^{p\times p}$ be functions such that

$$\begin{aligned}
f(x,A) &= 2AA'D^2 d^s_\Omega(x), \\
h(x,A) &= AA'(D^2 d^s_\Omega(x))^2.
\end{aligned}$$

Also, let $k : \partial\Omega \times \mathbb{R} \times \mathbb{R}^{p\times(p-1)} \to \mathbb{R}$ be a function such that

$$k(x,\lambda,A) = \sqrt{\det(I_p + \lambda f(x,A) + \lambda^2 h(x,A))}.$$

Observe that

$$J^{\partial\Omega}_{p-1}\psi_\Omega(x,\lambda) = k(x,\lambda,V_\Omega(x))$$

and that

$$\begin{aligned}
&\frac{\partial J^{\partial\Omega}_{p-1}\psi_\Omega(x,\lambda)}{\partial\lambda} \\
&= \left.\frac{\partial k(x,\lambda,A)}{\partial\lambda}\right|_{A=V_\Omega(x)} \\
&= \left.\frac{1}{2k(x,\lambda,A)} \sum_{i,j} \frac{\partial\det(I_p + \lambda f(x,A) + \lambda^2 h(x,A))}{\partial b_{ij}} (f_{ij}(x,A) + 2\lambda h_{ij}(x,A))\right|_{A=V_\Omega(x)},
\end{aligned}$$

where $\frac{\partial\det(B)}{\partial b_{ij}}$ denotes the partial derivative of the function $\det : \mathbb{R}^{p\times p} \to \mathbb{R}$ with respect to the (i,j) entry of B.

Note that $k(\cdot,\cdot,\cdot)$ and $\frac{\partial k(\cdot,\cdot,\cdot)}{\partial\lambda}$ are continuous on $\partial\Omega \times \mathbb{R} \times \mathbb{R}^{p\times(p-1)}$ (except at the points for which $k(x,\lambda,A) = 0$), since det is infinitely differentiable, and f and h are continuous on

$\partial\Omega \times \mathbb{R}^{p\times(p-1)}$. Let $S = \{(x,\lambda,A) \in \partial\Omega \times [-\mu,\mu] \times \mathbb{R}^{p\times(p-1)} : \|a_j\| = 1 \text{ for } k = 1,...,p-1\}$, where a_j denotes the jth column of A. Since $k(\cdot,\cdot,\cdot)$ and $\frac{\partial k(\cdot,\cdot,\cdot)}{\partial\lambda}$ are continuous and S is closed and bounded, $\bar{k} = \max_{(x,\lambda,A)\in S} |k(x,\lambda,A)|$ and $\bar{k}' = \max_{(x,\lambda,A)\in S} |\frac{\partial k(x,\lambda,A)}{\partial\lambda}|$ exist. Since $(x,\lambda,V_\Omega(x)) \in S$ for every $(x,\lambda) \in \partial\Omega \times (-\mu,\mu)$, it follows that $|J_{p-1}^{\partial\Omega}\psi_\Omega(x,\lambda)| \leq \bar{k}$ and $|\frac{\partial J_{p-1}^{\partial\Omega}\psi_\Omega(x,\lambda)}{\partial\lambda}| \leq \bar{k}'$ for every $(x,\lambda) \in \partial\Omega \times (-\mu,\mu)$. □

B.4 Other Lemmas

Lemma B.6. *Let $\{V_i\}_{i=1}^\infty$ be i.i.d. random variables such that $E[V_i^2] < \infty$. If Assumption 1 holds, then for $l \geq 0$ and $m = 0, 1$,*

$$E[V_i p^{ML}(X_i;\delta)^l 1\{p^{ML}(X_i;\delta) \in (0,1)\}^m] \to E[V_i ML(X_i)^l 1\{ML(X_i) \in (0,1)\}^m]$$

as $\delta \to 0$. Moreover, if, in addition, $\delta_n \to 0$ as $n \to \infty$, then for $l \geq 0$,

$$\frac{1}{n}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n} \xrightarrow{p} E[V_i ML(X_i)^l 1\{ML(X_i) \in (0,1)\}]$$

as $n \to \infty$.

Proof. Note that $E[\frac{1}{n}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n}] = E[V_i p^{ML}(X_i;\delta_n)^l 1\{p^{ML}(X_i;\delta_n) \in (0,1)\}]$. We show that

$$E[V_i p^{ML}(X_i;\delta)^l 1\{p^{ML}(X_i;\delta) \in (0,1)\}^m] \to E[V_i ML(X_i)^l 1\{ML(X_i) \in (0,1)\}^m]$$

for $l \geq 0$ and $m = 0, 1$ as $\delta \to 0$, and that

$$\text{Var}(\frac{1}{n}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n}) \to 0$$

for $l \geq 0$ as $n \to \infty$. For the first part, we have

$$E[V_i p^{ML}(X_i;\delta)^l 1\{p^{ML}(X_i;\delta) \in (0,1)\}^m] = \int_{\mathcal{X}} E[V_i|X_i = x] p^{ML}(x;\delta)^l 1\{p^{ML}(x;\delta) \in (0,1)\}^m f_X(x)dx.$$

Suppose ML is continuous at x and $ML(x) \in (0,1)$. Then $\lim_{\delta\to 0} p^{ML}(x;\delta) = ML(x)$ by Part 1 of Corollary 2, and hence $p^{ML}(x;\delta) \in (0,1)$ for sufficiently small $\delta > 0$. It follows that $1\{p^{ML}(x;\delta) \in (0,1)\} \to 1 = 1\{ML(x) \in (0,1)\}$ as $\delta \to 0$. Suppose $x \in \text{int}(\mathcal{X}_0) \cup \text{int}(\mathcal{X}_1)$. Then $B(x,\delta) \subset \mathcal{X}_0$ or $B(x,\delta) \subset \mathcal{X}_1$ for sufficiently small $\delta > 0$ by the fact that $\text{int}(\mathcal{X}_0)$ and $\text{int}(\mathcal{X}_1)$ are open, and hence $1\{p^{ML}(x;\delta) \in (0,1)\} \to 0 = 1\{ML(x) \in (0,1)\}$ as $\delta \to 0$. Therefore, $\lim_{\delta\to 0} p^{ML}(x;\delta) = ML(x)$ and $\lim_{\delta\to 0} 1\{p^{ML}(x;\delta) \in (0,1)\} = 1\{ML(x) \in (0,1)\}$ for almost every $x \in \mathcal{X}$, since ML is continuous at x for almost every $x \in \mathcal{X}$ by Assumption 1 (a), and either $ML(x) \in (0,1)$ or $x \in \text{int}(\mathcal{X}_0) \cup \text{int}(\mathcal{X}_1)$ for almost every $x \in \mathcal{X}$ by Assumption 1 (b). By the Dominated Convergence Theorem,

$$\begin{aligned} E[V_i p^{ML}(X_i;\delta)^l 1\{p^{ML}(X_i;\delta) \in (0,1)\}^m] &\to \int_{\mathcal{X}} E[V_i|X_i = x] ML(x)^l 1\{ML(x) \in (0,1)\}^m f_X(x)dx \\ &= E[V_i ML(X_i)^l 1\{ML(X_i) \in (0,1)\}^m] \end{aligned}$$

as $\delta \to 0$. As for variance,

$$\begin{aligned}\mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} V_i p^{ML}(X_i;\delta_n)^l I_{i,n}) &\leq \frac{1}{n} E[V_i^2 p^{ML}(X_i;\delta_n)^{2l}(I_{i,n})^2] \\ &\leq \frac{1}{n} E[V_i^2] \\ &\to 0\end{aligned}$$

as $n \to \infty$. □

Lemma B.7. *Let $\{(\delta_n, S_n)\}_{n=1}^{\infty}$ be any sequence of positive numbers and positive integers. Fix $x \in \mathcal{X}$, and let $X_1^*, ..., X_{S_n}^*$ be S_n independent draws from the uniform distribution on $B(x, \delta_n)$ so that*

$$p^s(x;\delta_n) = \frac{1}{S_n}\sum_{s=1}^{S_n} ML(X_s^*).$$

Then,

$$\begin{aligned}E[p^s(x;\delta_n) - p^{ML}(x;\delta_n)] &= 0, \\ E[(p^s(x;\delta_n) - p^{ML}(x;\delta_n))^2] &\leq \frac{1}{S_n}, \\ |E[p^s(x;\delta_n)^2 - p^{ML}(x;\delta_n)^2]| &\leq \frac{1}{S_n}, \\ E[(p^s(x;\delta_n)^2 - p^{ML}(x;\delta_n)^2)^2] &\leq \frac{4}{S_n}, \\ \Pr(p^s(x;\delta_n) \in \{0,1\}) &\leq (1 - p^{ML}(x;\delta_n))^{S_n} + p^{ML}(x;\delta_n)^{S_n}.\end{aligned}$$

Moreover, for any $\epsilon > 0$,

$$E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)|] \leq \frac{1}{S_n\epsilon^2} + \epsilon,$$

and if $S_n \to \infty$, then

$$E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)|] \to 0$$

as $n \to \infty$.

Proof. By construction, $E[ML(X_s^*)] = p^{ML}(x;\delta_n)$, so

$$\begin{aligned}E[p^s(x;\delta_n) - p^{ML}(x;\delta_n)] &= E[\frac{1}{S_n}\sum_{s=1}^{S_n} ML(X_s^*)] - p^{ML}(x;\delta_n) \\ &= E[ML(X_s^*)] - p^{ML}(x;\delta_n) \\ &= 0.\end{aligned}$$

We have

$$\begin{aligned}
E[(p^s(x;\delta_n) - p^{ML}(x;\delta_n))^2] &= \mathrm{Var}(p^s(x;\delta_n)) \\
&= \mathrm{Var}(\frac{1}{S_n}\sum_{s=1}^{S_n} ML(X_s^*)) \\
&= \frac{1}{S_n}\mathrm{Var}(ML(X_s^*)) \\
&\leq \frac{1}{S_n}E[ML(X_s^*)^2] \\
&\leq \frac{1}{S_n},
\end{aligned}$$

$$\begin{aligned}
|E[p^s(x;\delta_n)^2 - p^{ML}(x;\delta_n)^2]| &= |\mathrm{Var}(p^s(x;\delta_n)) + (E[p^s(x;\delta_n)])^2 - p^{ML}(x;\delta_n)^2| \\
&\leq \frac{1}{S_n} + |(p^{ML}(x;\delta_n))^2 - p^{ML}(x;\delta_n)^2| \\
&= \frac{1}{S_n},
\end{aligned}$$

and

$$\begin{aligned}
&E[(p^s(x;\delta_n)^2 - p^{ML}(x;\delta_n)^2)^2] \\
&= E[(p^s(x;\delta_n) + p^{ML}(x;\delta_n))^2(p^s(x;\delta_n) - p^{ML}(x;\delta_n))^2] \\
&\leq 4E[(p^s(x;\delta_n) - p^{ML}(x;\delta_n))^2] \\
&\leq \frac{4}{S_n}.
\end{aligned}$$

Now note that we have the following bounds on $\Pr(ML(X_s^*) = 0)$ and $\Pr(ML(X_s^*) = 1)$:

$$\begin{aligned}
0 \leq \Pr(ML(X_s^*) = 0) \leq 1 - p^{ML}(x;\delta_n), \\
0 \leq \Pr(ML(X_s^*) = 1) \leq p^{ML}(x;\delta_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &\leq \Pr(p^s(x;\delta_n) \in \{0,1\}) \\
&= \Pr(ML(X_s^*) = 0)^{S_n} + \Pr(ML(X_s^*) = 1)^{S_n} \\
&\leq (1 - p^{ML}(x;\delta_n))^{S_n} + p^{ML}(x;\delta_n)^{S_n}.
\end{aligned}$$

Lastly, for any $\epsilon > 0$,

$$\begin{aligned}
&E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)|] \\
&= E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)||\,|p^s(x;\delta_n) - p^{ML}(x;\delta_n)| \geq \epsilon]\Pr(|p^s(x;\delta_n) - p^{ML}(x;\delta_n)| \geq \epsilon) \\
&\quad + E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)||\,|p^s(x;\delta_n) - p^{ML}(x;\delta_n)| < \epsilon]\Pr(|p^s(x;\delta_n) - p^{ML}(x;\delta_n)| < \epsilon) \\
&< 1 \cdot \frac{\mathrm{Var}(p^s(x;\delta_n))}{\epsilon^2} + \epsilon \cdot 1 \\
&\leq \frac{1}{S_n\epsilon^2} + \epsilon,
\end{aligned}$$

where we use Chebyshev's inequality for the first inequality. We can make $E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)|]$ arbitrarily close to zero by taking sufficiently small $\epsilon > 0$ and sufficiently large S_n, which implies that $E[|p^s(x;\delta_n) - p^{ML}(x;\delta_n)|] = o(1)$ if $S_n \to \infty$.

□

Lemma B.8. *Let $I^s_{i,n} = 1\{p^s(X_i;\delta_n) \in (0,1)\}$, and let $\{V_i\}_{i=1}^\infty$ be i.i.d. random variables such that $E[V_i^2] < \infty$. If Assumption 1 holds, $S_n \to \infty$, and $\delta_n \to 0$, then*

$$\frac{1}{n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I^s_{i,n} - \frac{1}{n}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n} = o_p(1)$$

for $l = 0,1,2,3,4$. If, in addition, Assumption 5 holds, and $E[V_i|X_i]$ is bounded, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I^s_{i,n} - \frac{1}{\sqrt{n}}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n} = o_p(1)$$

for $l = 0,1,2$.

Proof. We have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I^s_{i,n} - \frac{1}{n}\sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_{i,n} \\
=&\frac{1}{n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l (I^s_{i,n} - I_{i,n}) + \frac{1}{n}\sum_{i=1}^n V_i (p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l) I_{i,n}.
\end{aligned}$$

We first consider $\frac{1}{n}\sum_{i=1}^n V_i (p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l) I_{i,n}$. By Lemma B.7, for $l = 0,1,2$,

$$\begin{aligned}
&|E[\frac{1}{n}\sum_{i=1}^n V_i (p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l) I_{i,n}]| \\
&= |E[V_i (p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l) I_{i,n}]| \\
&= E[|E[V_i|X_i]||E[p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l|X_i]|I_{i,n}] \\
&\le \frac{1}{S_n} E[|E[V_i|X_i]|I_{i,n}] \\
&= O(S_n^{-1}).
\end{aligned}$$

Also, by Lemma B.7,

$$\begin{aligned}
&|E[\frac{1}{n}\sum_{i=1}^n V_i (p^s(X_i;\delta_n)^3 - p^{ML}(X_i;\delta_n)^3) I_{i,n}]| \\
&= |E[V_i (p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n)^2 + p^s(X_i;\delta_n)p^{ML}(X_i;\delta_n) + p^{ML}(X_i;\delta_n)^2) I_{i,n}]| \\
&\le E[|E[V_i|X_i]||E[(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n)^2 + p^s(X_i;\delta_n)p^{ML}(X_i;\delta_n) + p^{ML}(X_i;\delta_n)^2)|X_i]|I_{i,n}] \\
&\le 3E[|E[V_i|X_i]|E[|p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n)||X_i]I_{i,n}] \\
&= o(1),
\end{aligned}$$

and

$$
\begin{aligned}
&|E[\frac{1}{n}\sum_{i=1}^{n} V_i(p^s(X_i;\delta_n)^4 - p^{ML}(X_i;\delta_n)^4)I_{i,n}]| \\
&= |E[V_i(p^s(X_i;\delta_n)^2 + p^{ML}(X_i;\delta_n)^2)(p^s(X_i;\delta_n) + p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))I_{i,n}]| \\
&\leq E[|E[V_i|X_i]||E[(p^s(X_i;\delta_n)^2 + p^{ML}(X_i;\delta_n)^2)(p^s(X_i;\delta_n) + p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))|X_i]|I_{i,n}] \\
&\leq 4E[|E[V_i|X_i]|E[|p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n)||X_i]I_{i,n}] \\
&= o(1).
\end{aligned}
$$

As for variance, for $l = 0, 1, 2$,

$$
\begin{aligned}
\mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_{i,n}) &\leq \frac{1}{n}E[V_i^2(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2 I_{i,n}] \\
&\leq \frac{1}{n}E[E[V_i^2|X_i]E[(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2|X_i]I_{i,n}] \\
&\leq \frac{4}{nS_n}E[E[V_i^2|X_i]I_{i,n}] \\
&= O((nS_n)^{-1}),
\end{aligned}
$$

and for $l = 3, 4$,

$$
\begin{aligned}
\mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_{i,n}) &\leq \frac{1}{n}E[V_i^2(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2 I_{i,n}] \\
&\leq \frac{1}{n}E[V_i^2 I_{i,n}] \\
&= o(1).
\end{aligned}
$$

Therefore, $\frac{1}{n}\sum_{i=1}^{n} V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_{i,n} = o_p(1)$ if $S_n \to \infty$ for $l = 0, 1, 2, 3, 4$, and $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_{i,n} = o_p(1)$ if $n^{-1/2}S_n \to \infty$ for $l = 0, 1, 2$.

We next show that $\frac{1}{n}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I_{i,n}^s - I_{i,n}) = o_p(1)$ if $S_n \to \infty$ and $\delta_n \to 0$ for $l \geq 0$. We have

$$
\begin{aligned}
|E[\frac{1}{n}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I_{i,n}^s - I_{i,n})]| &= |E[V_i p^s(X_i;\delta_n)^l(I_{i,n}^s - I_{i,n})]| \\
&\leq E[|E[V_i|X_i]||E[p^s(X_i;\delta_n)^l(I_{i,n}^s - I_{i,n})|X_i]|] \\
&\leq E[|E[V_i|X_i]|E[|I_{i,n}^s - I_{i,n}||X_i]].
\end{aligned}
$$

Note that by construction, $1\{p^s(X_i;\delta_n) \in (0,1)\} \leq 1\{p^{ML}(X_i;\delta_n) \in (0,1)\}$ with probability one conditional on $X_i = x$, so that

$$
E[|I_{i,n}^s - I_{i,n}||X_i = x] = -E[I_{i,n}^s - I_{i,n}|X_i = x].
$$

Suppose ML is continuous at x and $ML(x) \in (0,1)$. Then $\lim_{\delta\to 0} p^{ML}(x;\delta) = ML(x) \in (0,1)$ by Part 1 of Corollary 2, and hence $p^{ML}(x;\delta_n) \in [\epsilon, 1-\epsilon]$ for sufficiently small $\delta_n > 0$ for some

constant $\epsilon \in (0, 1/2)$. It follows that

$$\begin{aligned} E[I^s_{i,n}|X_i = x] &= 1 - \Pr(p^s(x;\delta_n) \in \{0,1\}) \\ &\geq 1 - (1 - p^{ML}(x;\delta_n))^{S_n} - p^{ML}(x;\delta_n)^{S_n} \\ &\geq 1 - 2(1-\epsilon)^{S_n} \\ &\to 1 \end{aligned}$$

as $S_n \to \infty$, where the first inequality follows from Lemma B.7. This implies that $E[I^s_{i,n} - I_{i,n}|X_i = x] \to 0$ as $n \to \infty$. Suppose $x \in \text{int}(\mathcal{X}_0) \cup \text{int}(\mathcal{X}_1)$. Then $B(x, \delta_n) \subset \mathcal{X}_0$ or $B(x, \delta_n) \subset \mathcal{X}_1$ for sufficiently small $\delta_n > 0$ by the fact that $\text{int}(\mathcal{X}_0)$ and $\text{int}(\mathcal{X}_1)$ are open, and hence $p^{ML}(x;\delta_n) \in \{0,1\}$ and $p^s(x;\delta_n) \in \{0,1\}$ for sufficiently small $\delta_n > 0$, so that $E[I^s_{i,n} - I_{i,n}|X_i = x] \to 0$ as $n \to \infty$. Therefore, $E[I^s_{i,n} - I_{i,n}|X_i = x] \to 0$ for almost every $x \in \mathcal{X}$, since ML is continuous at x for almost every $x \in \mathcal{X}$ by Assumption 1 (a), and either $ML(x) \in (0,1)$ or $x \in \text{int}(\mathcal{X}_0) \cup \text{int}(\mathcal{X}_1)$ for almost every $x \in \mathcal{X}$ by Assumption 1 (b). By the Dominated Convergence Theorem,

$$-E[|E[V_i|X_i]|E[I^s_{i,n} - I_{i,n}|X_i]] \to 0$$

as $n \to \infty$.

As for variance,

$$\begin{aligned} \text{Var}(\frac{1}{n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l (I^s_{i,n} - I_{i,n})) &\leq \frac{1}{n} E[V_i^2 p^s(X_i;\delta_n)^{2l}(I^s_{i,n} - I_{i,n})^2] \\ &\leq \frac{1}{n} E[V_i^2] \\ &\to 0. \end{aligned}$$

Lastly, we show that, for $l \geq 0$, $\frac{1}{\sqrt{n}}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l (I^s_{i,n} - I_{i,n}) = o_p(1)$ if Assumption 5 holds, and $E[V_i|X_i]$ is bounded. Let $\eta_n = \gamma \frac{\log n}{S_n}$, where γ is the one satisfying Assumption 5. We have

$$\begin{aligned} &|E[\frac{1}{\sqrt{n}}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l (I^s_{i,n} - I_{i,n})]| \\ &\leq \sqrt{n} E[|E[V_i|X_i]|E[|I^s_{i,n} - I_{i,n}||X_i]] \\ &= -\sqrt{n} E[|E[V_i|X_i]|E[I^s_{i,n} - 1|X_i] I_{i,n}] \\ &\leq \sqrt{n} E[|E[V_i|X_i]|((1 - p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n}) I_{i,n}] \\ &= \sqrt{n} E[|E[V_i|X_i]|((1 - p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n}) 1\{p^{ML}(X_i;\delta_n) \in (0,\eta_n) \cup (1-\eta_n, 1)\}] \\ &\quad + \sqrt{n} E[|E[V_i|X_i]|((1 - p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n}) 1\{p^{ML}(X_i;\delta_n) \in [\eta_n, 1-\eta_n]\}] \\ &\leq (\sup_{x \in \mathcal{X}} |E[V_i|X_i = x]|)(\sqrt{n}\Pr(p^{ML}(X_i;\delta_n) \in (0,\eta_n) \cup (1-\eta_n, 1)) + 2\sqrt{n}(1-\eta_n)^{S_n}), \end{aligned}$$

where the second equality follows from the fact that $I^s_{i,n} \leq I_{i,n}$ with strict inequality only if $I_{i,n} = 1$. By Assumption 5, $\sqrt{n}\Pr(p^{ML}(X_i;\delta_n) \in (0,\eta_n) \cup (1-\eta_n, 1)) = o(1)$. As for $\sqrt{n}(1-\eta_n)^{S_n}$, first observe that $\eta_n = \gamma \frac{\log n}{S_n} = \gamma \frac{\log n}{n^{1/2}} \frac{1}{n^{-1/2} S_n} \to 0$, since $n^{-1/2} S_n \to \infty$ and $\frac{\log n}{n^{1/2}} \to 0$. Using the

fact that $e^t \geq 1+t$ for every $t \in \mathbb{R}$, we have

$$\begin{aligned}\sqrt{n}(1-\eta_n)^{S_n} &\leq \sqrt{n}(e^{-\eta_n})^{S_n} \\ &= \sqrt{n}e^{-\eta_n S_n} \\ &= \sqrt{n}e^{-\gamma \log n} \\ &= \sqrt{n}n^{-\gamma} \\ &= n^{1/2-\gamma} \\ &\to 0,\end{aligned}$$

since $\gamma > 1/2$. As for variance,

$$\begin{aligned}\mathrm{Var}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I^s_{i,n}-I_{i,n})) &\leq E[V_i^2 p^s(X_i;\delta_n)^{2l}(I^s_{i,n}-I_{i,n})^2] \\ &\leq E[V_i^2|I^s_{i,n}-I_{i,n}|] \\ &= E[E[V_i^2|X_i]E[|I^s_{i,n}-I_{i,n}||X_i]] \\ &= o(1).\end{aligned}$$

□

C Proofs

C.1 Proof of Proposition 1

Suppose that Assumptions 1 and 2 hold. Here, we only show that

(a) $E[Y_{1i}-Y_{0i}|X_i=x]$ is identified for every $x \in \mathrm{int}(\mathcal{X})$ such that $p^{ML}(x) \in (0,1)$.

(b) Let A be any open subset of $\mathcal{X}$ such that $p^{ML}(x)$ exists for all $x \in A$. Then $E[Y_{1i}-Y_{0i}|X_i \in A]$ is identified only if $p^{ML}(x) \in (0,1)$ for almost every $x \in A$.

The results for $E[D_i(1)-D_i(0)|X_i=x]$ and $E[D_i(1)-D_i(0)|X_i \in A]$ are obtained by a similar argument.

Proof of Part (a). Pick an $x \in \mathrm{int}(\mathcal{X})$ such that $p^{ML}(x) \in (0,1)$. If $ML(x) \in (0,1)$, $E[Y_{1i}-Y_{0i}|X_i=x]$ and $E[D_i(1)-D_i(0)|X_i=x]$ are trivially identified by Property 1:

$$E[Y_i|X_i=x,Z_i=1]-E[Y_i|X_i=x,Z_i=0]=E[Y_{1i}-Y_{0i}|X_i=x].$$

We next consider the case where $ML(x) \in \{0,1\}$. Since $x \in \mathrm{int}(\mathcal{X})$, $B(x,\delta) \subset \mathcal{X}$ for any sufficiently small $\delta > 0$. Moreover, since $p^{ML}(x) = \lim_{\delta \to 0} p^{ML}(x;\delta) \in (0,1)$, $p^{ML}(x;\delta) \in (0,1)$ for any sufficiently small $\delta > 0$. This implies that we can find points $x_{0,\delta}, x_{1,\delta} \in B(x,\delta)(\subset \mathcal{X})$ such that $ML(x_{0,\delta}) < 1$ and $ML(x_{1,\delta}) > 0$ for any sufficiently small $\delta > 0$, for otherwise $p^{ML}(x;\delta) \in \{0,1\}$. Noting that $x_{0,\delta} \to x$ and $x_{1,\delta} \to x$ as $\delta \to 0$,

$$\begin{aligned}\lim_{\delta \to 0}(E[Y_i|X_i=x_{1,\delta},Z_i=1]-E[Y_i|X_i=x_{0,\delta},Z_i=0]) &= \lim_{\delta \to 0}(E[Y_{i1}|X_i=x_{1,\delta}]-E[Y_{i0}|X_i=x_{0,\delta}]) \\ &= E[Y_{1i}-Y_{0i}|X_i=x],\end{aligned}$$

where the first equality follows from Property 1, and the second from Assumption 2. □

Proof of Part (b).

Suppose to the contrary that $\mathcal{L}^p(\{x \in A : p^{ML}(x) \in \{0,1\}\}) > 0$. Without loss of generality, assume $\mathcal{L}^p(\{x \in A : p^{ML}(x) = 1\}) > 0$. The proof proceeds in four steps.

Step C.1.1. $\mathcal{L}^p(A \cap \mathcal{X}_1) > 0$.

Proof. By Assumption 1, ML is continuous almost everywhere. Part 1 of Cororally 2 then implies that $p^{ML}(x) = ML(x)$ for almost every $x \in \{x^* \in A : p^{ML}(x^*) = 1\}$. Since $\mathcal{L}^p(\{x \in A : p^{ML}(x) = 1\}) > 0$, $\mathcal{L}^p(\{x \in A : p^{ML}(x) = 1, p^{ML}(x) = ML(x)\}) > 0$, and hence $\mathcal{L}^p(A \cap \mathcal{X}_1) > 0$. □

Step C.1.2. $A \cap \text{int}(\mathcal{X}_1) \neq \emptyset$.

Proof. Suppose that $A \cap \text{int}(\mathcal{X}_1) = \emptyset$. Then, we must have that $A \cap \mathcal{X}_1 \subset \mathcal{X}_1 \setminus \text{int}(\mathcal{X}_1)$. It then follows that $\mathcal{L}^p(A \cap \mathcal{X}_1) \leq \mathcal{L}^p(\mathcal{X}_1 \setminus \text{int}(\mathcal{X}_1)) = \mathcal{L}^p(\mathcal{X}_1) - \mathcal{L}^p(\text{int}(\mathcal{X}_1)) = 0$, where the last equality holds by Assumption 1. But this is a contradiction to the result from Step C.1.1. □

Step C.1.3. $p^{ML}(x) = 1$ *for any* $x \in \text{int}(\mathcal{X}_1)$.

Proof. Pick any $x \in \text{int}(\mathcal{X}_1)$. By the definition of interior, $B(x, \delta) \subset \mathcal{X}_1$ for any sufficiently small $\delta > 0$. Therefore, $p^{ML}(x; \delta) = 1$ for any sufficiently small $\delta > 0$. □

Step C.1.4. $E[Y_{1i} - Y_{0i} | X_i \in A]$ *is not identified.*

Proof. We first introduce some notation. Let $\mathbf{Q}$ be the set of all distributions of $(Y_{1i}, Y_{0i}, X_i, Z_i)$ satisfying Property 1 and Assumptions 1 and 2. Let $\mathbf{P}$ be the set of all distributions of (Y_i, X_i, Z_i). Let $T : \mathbf{Q} \to \mathbf{P}$ be a function such that, for $Q \in \mathbf{Q}$, $T(Q)$ is the distribution of $(Z_i Y_{1i} + (1 - Z_i) Y_{0i}, X_i, Z_i)$, where the distribution of $(Y_{1i}, Y_{0i}, X_i, Z_i)$ is Q. Let Q_0 and P_0 denote the true distributions of $(Y_{1i}, Y_{0i}, X_i, Z_i)$ and (Y_i, X_i, Z_i), respectively. Given P_0, the identified set of $E[Y_{1i} - Y_{0i} | X_i \in A]$ is given by $\{E_Q[Y_{1i} - Y_{0i} | X_i \in A] : P_0 = T(Q), Q \in \mathbf{Q}\}$, where $E_Q[\cdot]$ is the expectation operator under distribution Q. We show that this set contains two distinct values. In what follows, $\Pr(\cdot)$ and $E[\cdot]$ without a subscript denote the probability and expectation under the true distributions Q_0 and P_0 as up until now.

Now pick any $x^* \in A \cap \text{int}(\mathcal{X}_1)$. Since A and $\text{int}(\mathcal{X}_1)$ are open, there is some $\delta > 0$ such that $B(x^*, \delta) \subset A \cap \text{int}(\mathcal{X}_1)$. Let $\epsilon = \frac{\delta}{2}$, and consider a function $f : \mathcal{X} \to \mathbb{R}$ such that $f(x) = E[Y_{0i} | X = x]$ for all $x \in \mathcal{X} \setminus B(x^*, \epsilon)$ and $f(x) = E[Y_{0i} | X = x] - 1$ for all $x \in B(x^*, \epsilon)$. Below, we show that f is continuous at any point $x \in \mathcal{X}$ such that $p^{ML}(x) \in (0,1)$ and $ML(x) \in \{0,1\}$. Pick any $x \in \mathcal{X}$ such that $p^{ML}(x) \in (0,1)$ and $ML(x) \in \{0,1\}$. Since $B(x^*, \delta) \subset \text{int}(\mathcal{X}_1)$ and $\text{int}(\mathcal{X}_1) \subset \{x' \in \mathcal{X} : p^{ML}(x') = 1\}$ by Step C.1.3, $x \notin B(x^*, \delta)$. Hence, $B(x, \epsilon) \subset \mathcal{X} \setminus B(x^*, \epsilon)$. By Assumption 2 and the definition of f, f is continuous at x.

Now take any random vector $(Y_{1i}^*, Y_{0i}^*, X_i^*, Z_i^*)$ that is distributed according to the true distribution Q_0. Let Q be the distribution of $(Y_{1i}^Q, Y_{0i}^Q, X_i^Q, Z_i^Q)$, where $(Y_{1i}^Q, X_i^Q, Z_i^Q) = (Y_{1i}^*, X_i^*, Z_i^*)$, and

$$Y_{0i}^Q = \begin{cases} Y_{0i}^* & \text{if } X_i^* \in \mathcal{X} \setminus B(x^*, \epsilon) \\ Y_{0i}^* - 1 & \text{if } X_i^* \in B(x^*, \epsilon) \end{cases}$$

Note first that $Q \in \mathbf{Q}$, since $E_Q[Y_{1i}^Q|X_i^Q = x] = E[Y_{1i}^*|X_i^* = x]$ and $E_Q[Y_{0i}^Q|X_i^Q = x] = f(x)$, where $E[Y_{1i}^*|X_i^*]$ and f are both continuous at any point $x \in \mathcal{X}$ such that $p^{ML}(x) \in (0,1)$ and $ML(x) \in \{0,1\}$. Also, $Z_i^Q = Z_i^* = 1$ if $X_i^* \in B(x^*, \epsilon)$. It then follows that

$$\begin{aligned} Y_i^Q &= Z_i^Q Y_{1i}^Q + (1 - Z_i^Q) Y_{0i}^Q \\ &= \begin{cases} Z_i^* Y_{1i}^* + (1 - Z_i^*) Y_{0i}^* & \text{if } X_i^* \in \mathcal{X} \setminus B(x^*, \epsilon) \\ Z_i^* Y_{1i}^* & \text{if } X_i^* \in B(x^*, \epsilon) \end{cases} \end{aligned}$$

and

$$\begin{aligned} Y_i^* &= Z_i^* Y_{1i}^* + (1 - Z_i^*) Y_{0i}^* \\ &= \begin{cases} Z_i^* Y_{1i}^* + (1 - Z_i^*) Y_{0i}^* & \text{if } X_i^* \in \mathcal{X} \setminus B(x^*, \epsilon) \\ Z_i^* Y_{1i}^* & \text{if } X_i^* \in B(x^*, \epsilon). \end{cases} \end{aligned}$$

Thus, $Y_i^Q = Y_i^*$, and hence $T(Q) = T(Q_0) = P_0$.

Using $E_Q[Y_{1i}^Q|X_i^Q = x] = E[Y_{1i}^*|X_i^* = x]$ and $E_Q[Y_{0i}^Q|X_i^Q = x] = f(x)$, we have

$$\begin{aligned} & E_Q[Y_{1i}^Q - Y_{0i}^Q | X_i^Q \in A] \\ &= E_Q[E_Q[Y_{1i}^Q|X_i^Q]|X_i^Q \in A] \\ &\quad - E_Q[E_Q[Y_{0i}^Q|X_i^Q]|X_i^Q \in A, X_i^Q \notin B(x^*, \epsilon)] \Pr_Q(X_i^Q \notin B(x^*, \epsilon)|X_i^Q \in A) \\ &\quad - E_Q[E_Q[Y_{0i}^Q|X_i^Q]|X_i^Q \in B(x^*, \epsilon)] \Pr_Q(X_i^Q \in B(x^*, \epsilon)|X_i^Q \in A) \\ &= E[E[Y_{1i}^*|X_i^*]|X_i^* \in A] - E[f(X_i^*)|X_i^* \in A, X_i^* \notin B(x^*, \epsilon)] \Pr(X_i^* \notin B(x^*, \epsilon)|X_i^* \in A) \\ &\quad - E[f(X_i^*)|X_i^* \in B(x^*, \epsilon)] \Pr(X_i^* \in B(x^*, \epsilon)|X_i^* \in A) \\ &= E[Y_{1i}^*|X_i^* \in A] - E[Y_{0i}^*|X_i^* \in A, X_i^* \notin B(x^*, \epsilon)] \Pr(X_i^* \notin B(x^*, \epsilon)|X_i^* \in A) \\ &\quad - E[Y_{0i}^* - 1|X_i^* \in B(x^*, \epsilon)] \Pr(X_i^* \in B(x^*, \epsilon)|X_i^* \in A) \\ &= E[Y_{1i}^* - Y_{0i}^*|X_i^* \in A] + \Pr(X_i^* \in B(x^*, \epsilon)|X_i^* \in A). \end{aligned}$$

By the definition of support, $\Pr(X_i^* \in B(x^*, \epsilon)) > 0$. Since $T(Q) = T(Q_0) = P_0$ but $E_Q[Y_{1i}^Q - Y_{0i}^Q|X_i^Q \in A] \neq E[Y_{1i}^* - Y_{0i}^*|X_i^* \in A]$, $E[Y_{1i} - Y_{0i}|X_i \in A]$ is not identified. □

□

□

C.2 Proof of Corollary 1

If $\Pr(D_i(1) - D_i(0) = 1|X_i = x) = 1$, $\Pr(Y_{1i} - Y_{0i} = Y_i(1) - Y_i(0)|X_i = x) = 1$, and hence $E[Y_{1i} - Y_{0i}|X_i = x] = E[Y_i(1) - Y_i(0)|X_i = x]$. Then, Parts (a) and (b) follow from Proposition 1. If $\Pr(D_i(1) \geq D_i(0)|X_i = x) = 1$, we have

$$\begin{aligned} E[Y_{1i} - Y_{0i}|X_i = x] &= E[(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))|X_i = x] \\ &= \Pr(D_i(1) \neq D_i(0)|X_i = x) E[Y_i(1) - Y_i(0)|D_i(1) \neq D_i(0), X_i = x]. \end{aligned}$$

If in addition $\Pr(D_i(1) \neq D_i(0)|X_i = x) > 0$, we obtain

$$\begin{aligned} E[Y_i(1) - Y_i(0)|D_i(1) \neq D_i(0), X_i = x] &= \frac{E[Y_{1i} - Y_{0i}|X_i = x]}{\Pr(D_i(1) \neq D_i(0)|X_i = x)} \\ &= \frac{E[Y_{1i} - Y_{0i}|X_i = x]}{E[D_i(1) - D_i(0)|X_i = x]}. \end{aligned}$$

Then, Part (c) follows from Proposition 1 (a). Part (d) is established by following the procedure used to show Proposition 1 (b). □

C.3 Proof of Proposition 2

With change of variables $u = \frac{x^* - x}{\delta}$, we have

$$\begin{aligned} p^{ML}(x;\delta) &= \frac{\int_{B(x,\delta)} ML(x^*)dx^*}{\int_{B(x,\delta)} dx^*} \\ &= \frac{\delta^p \int_{B(\mathbf{0},1)} ML(x+\delta u)du}{\delta^p \int_{B(\mathbf{0},1)} du} \\ &= \frac{\int_{\cup_{q\in Q}\mathcal{U}_{x,q}} ML(x+\delta u)du + \int_{B(\mathbf{0},1)\setminus\cup_{q\in Q}\mathcal{U}_{x,q}} ML(x+\delta u)du}{\int_{B(\mathbf{0},1)} du} \\ &= \frac{\sum_{q\in Q}\int_{\mathcal{U}_{x,q}} ML(x+\delta u)du}{\int_{B(\mathbf{0},1)} du}, \end{aligned}$$

where the last equality follows from the assumption that $\mathcal{L}^p(\cup_{q\in Q}\mathcal{U}_{x,q}) = \mathcal{L}^p(B(\mathbf{0},1))$. By the definition of $\mathcal{U}_{x,q}$, for each $q \in Q$, $\lim_{\delta\to 0} ML(x+\delta u) = q$ for any $u \in \mathcal{U}_{x,q}$. By the Dominated Convergence Theorem,

$$\begin{aligned} p^{ML}(x) &= \lim_{\delta\to 0} p^{ML}(x;\delta) \\ &= \frac{\sum_{q\in Q} q\mathcal{L}^p(\mathcal{U}_{x,q})}{\mathcal{L}^p(B(\mathbf{0},1))}. \end{aligned}$$

The numerator exists, since $q \leq 1$ for all $q \in Q$ and $\sum_{q\in Q}\mathcal{L}^p(\mathcal{U}_{x,q}) = \mathcal{L}^p(B(\mathbf{0},1))$. □

C.4 Proof of Corollary 2

1. Suppose that ML is continuous at $x \in \mathcal{X}$, and let $q = ML(x)$. Then, by definition, $\mathcal{U}_{x,q} = B(\mathbf{0},1)$. By Proposition 2, $p^{ML}(x)$ exists, and $p^{ML}(x) = q$. □

2. Pick any $x \in \text{int}(\mathcal{X}_q)$. ML is continuous at x, since there exists $\delta > 0$ such that $B(x,\delta) \subset \mathcal{X}_q$ by the definition of interior. By the previous result, $p^{ML}(x)$ exists, and $p^{ML}(x) = q$. □

3. Let $\mathcal{N}$ be the neighborhood of x on which f is continuously differentiable. By the mean value theorem, for any sufficiently small $\delta > 0$,

$$\begin{aligned} f(x+\delta u) &= f(x) + \nabla f(\tilde{x}_\delta)\cdot\delta u \\ &= \nabla f(\tilde{x}_\delta)\cdot\delta u \end{aligned}$$

for some $\tilde{x}_\delta$ which is on the line segment connecting x and $x+\delta u$. Since $\tilde{x}_\delta \to x$ as $\delta \to 0$ and ∇f is continuous on $\mathcal{N}$, $\nabla f(\tilde{x}_\delta)\cdot u \to \nabla f(x)\cdot u$ as $\delta \to 0$. Therefore, if $\nabla f(x)\cdot u > 0$, then $f(x+\delta u) = \nabla f(\tilde{x}_\delta)\cdot \delta u > 0$ for any sufficiently small $\delta > 0$, and if $\nabla f(x)\cdot u < 0$, then $f(x+\delta u) = \nabla f(\tilde{x}_\delta)\cdot \delta u < 0$ for any sufficiently small $\delta > 0$. We then have

$$\mathcal{U}_x^+ \equiv \{u \in B(\mathbf{0},1) : \nabla f(x)\cdot u > 0\} \subset \mathcal{U}_{x,q_1}$$
$$\mathcal{U}_x^- \equiv \{u \in B(\mathbf{0},1) : \nabla f(x)\cdot u < 0\} \subset \mathcal{U}_{x,q_2}.$$

Let V be the Lebesgue measure of a half p-dimensional unit ball. Since $V = \mathcal{L}^p(\mathcal{U}_x^+) \leq \mathcal{L}^p(\mathcal{U}_{x,q_1})$, $V = \mathcal{L}^p(\mathcal{U}_x^-) \leq \mathcal{L}^p(\mathcal{U}_{x,q_2})$, and $\mathcal{L}^p(\mathcal{U}_{x,q_1}) + \mathcal{L}^p(\mathcal{U}_{x,q_2}) \leq \mathcal{L}^p(B(\mathbf{0},1)) = 2V$, it follows that $\mathcal{L}^p(\mathcal{U}_{x,q_1}) = \mathcal{L}^p(\mathcal{U}_{x,q_2}) = V$. By Proposition 2, $p^{ML}(x)$ exists, and $p^{ML}(x) = \frac{1}{2}(q_1+q_2)$. □

4. We have that $\mathcal{U}_{\mathbf{0},q_1} = \{(u_1,u_2)' \in B(\mathbf{0},1) : u_1 \leq 0 \text{ or } u_2 \leq 0\}$ and $\mathcal{U}_{\mathbf{0},q_2} = \{(u_1,u_2)' \in B(\mathbf{0},1) : u_1 > 0, u_2 > 0\}$. By Proposition 2, $p^{ML}(x)$ exists, and $p^{ML}(x) = \frac{q_1\mathcal{L}^2(\mathcal{U}_{\mathbf{0},q_1}) + q_2\mathcal{L}^2(\mathcal{U}_{\mathbf{0},q_2})}{\mathcal{L}^2(B(\mathbf{0},1))} = \frac{3}{4}q_1 + \frac{1}{4}q_2$. □

C.5 Proof of Proposition A.1

Since ML is a $\mathcal{L}^p$-measurable and bounded function, ML is locally integrable with respect to the Lebesgue measure, i.e., for every ball $B \subset \mathbb{R}^p$, $\int_B ML(x)dx$ exists. An application of the Lebesgue differentiation theorem (see e.g. Theorem 1.4 in Chapter 3 of Stein and Shakarchi (2005)) to the function ML shows that

$$\lim_{\delta \to 0} \frac{\int_{B(x,\delta)} ML(x^*)dx^*}{\int_{B(x,\delta)} dx^*} = ML(x)$$

for almost every $x \in \mathbb{R}^p$. □

C.6 Proof of Theorem 1

We prove consistency and asymptotic normality of the following estimators without imposing Assumption 3 (c). These estimators are aymptotically equivalent to the estimators defined in Section 4.1 if Assumption 3 (c) holds.

First, consider the following 2SLS regression using the observations with $p^{ML}(X_i;\delta_n) \in (0,1)$:

$$D_i = \gamma_0(1-\mathbf{I}_n) + \gamma_1 Z_i + \gamma_2 p^{ML}(X_i;\delta_n) + \nu_i \tag{15}$$
$$Y_i = \beta_0(1-\mathbf{I}_n) + \beta_1 D_i + \beta_2 p^{ML}(X_i;\delta_n) + \epsilon_i. \tag{16}$$

Here $\mathbf{I}_n$ is a dummy random variable which equals one if there exists a constant $q \in (0,1)$ such that $ML(X_i) \in \{0,q,1\}$ for all $i \in \{1,...,n\}$. $\mathbf{I}_n$ is the indicator that $ML(X_i)$ takes on only one nondegenerate value *in the sample*. If the support of $ML(X_i)$ (in the population) contains only one value in $(0,1)$, $p^{ML}(X_i;\delta_n)$ is asymptotically constant conditional on $p^{ML}(X_i;\delta_n) \in (0,1)$. To avoid the multicollinearity between asymptotically constant $p^{ML}(X_i;\delta_n)$ and a constant, we do not include the constant term if $\mathbf{I}_n = 1$. Let $I_{i,n} = 1\{p^{ML}(X_i;\delta_n) \in (0,1)\}$,

$\mathbf{D}_{i,n} = (1, D_i, p^{ML}(X_i;\delta_n))'$, $\mathbf{Z}_{i,n} = (1, Z_i, p^{ML}(X_i;\delta_n))'$, $\mathbf{D}^{nc}_{i,n} = (D_i, p^{ML}(X_i;\delta_n))'$, and $\mathbf{Z}^{nc}_{i,n} = (Z_i, p^{ML}(X_i;\delta_n))'$. The 2SLS estimator $\hat{\beta}$ from this regression is then given by

$$\hat{\beta} = \begin{cases} (\sum_{i=1}^n \mathbf{Z}_{i,n}\mathbf{D}'_{i,n} I_{i,n})^{-1} \sum_{i=1}^n \mathbf{Z}_{i,n} Y_i I_{i,n} & \text{if } \mathbf{I}_n{=}0 \\ (\sum_{i=1}^n \mathbf{Z}^{nc}_{i,n}(\mathbf{D}^{nc}_{i,n})' I_{i,n})^{-1} \sum_{i=1}^n \mathbf{Z}^{nc}_{i,n} Y_i I_{i,n} & \text{if } \mathbf{I}_n{=}1. \end{cases}$$

Let $\hat{\beta}_1$ denote the 2SLS estimator of β_1 in the above regression.

Similarly, consider the following simulation version of the 2SLS regression using the observations with $p^s(X_i;\delta_n) \in (0,1)$:

$$D_i = \gamma_0(1 - \mathbf{I}_n) + \gamma_1 Z_i + \gamma_2 p^s(X_i;\delta_n) + \nu_i \tag{17}$$

$$Y_i = \beta_0(1 - \mathbf{I}_n) + \beta_1 D_i + \beta_2 p^s(X_i;\delta_n) + \epsilon_i. \tag{18}$$

Let $\hat{\beta}^s_1$ denote the 2SLS estimator of β_1 in the simulation-based regression.

Below, we prove the following result.

Theorem C.1. *Suppose that Assumptions 1 and 3 hold except Assumption 3 (c), and that $\delta_n \to 0$, $n\delta_n \to \infty$ and $S_n \to \infty$ as $n \to \infty$. Then the 2SLS estimators $\hat{\beta}_1$ and $\hat{\beta}^s_1$ converge in probability to*

$$\beta_1 \equiv \lim_{\delta \to 0} E[\omega_i(\delta)(Y_i(1) - Y_i(0))],$$

where

$$\omega_i(\delta) = \frac{p^{ML}(X_i;\delta)(1 - p^{ML}(X_i;\delta))(D_i(1) - D_i(0))}{E[p^{ML}(X_i;\delta)(1 - p^{ML}(X_i;\delta))(D_i(1) - D_i(0))]}.$$

Suppose, in addition, that Assumptions 4 and 5 hold and that $n\delta_n^2 \to 0$ as $n \to \infty$. Then

$$\hat{\sigma}_n^{-1}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0,1),$$
$$(\hat{\sigma}^s_n)^{-1}(\hat{\beta}^s_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0,1).$$

where we define $\hat{\sigma}_n^{-1}$ and $(\hat{\sigma}^s_n)^{-1}$ as follows: let

$$\hat{\mathbf{\Sigma}}_n = \begin{cases} (\sum_{i=1}^n \mathbf{Z}_{i,n}\mathbf{D}'_{i,n} I_{i,n})^{-1} (\sum_{i=1}^n \hat{\epsilon}^2_{i,n} \mathbf{Z}_{i,n}\mathbf{Z}'_{i,n} I_{i,n}) (\sum_{i=1}^n \mathbf{D}_{i,n}\mathbf{Z}'_{i,n} I_{i,n})^{-1} & \text{if } \mathbf{I}_n = 0 \\ (\sum_{i=1}^n \mathbf{Z}^{nc}_{i,n}(\mathbf{D}^{nc}_{i,n})' I_{i,n})^{-1} (\sum_{i=1}^n \hat{\epsilon}^2_{i,n} \mathbf{Z}^{nc}_{i,n}(\mathbf{Z}^{nc}_{i,n})' I_{i,n}) (\sum_{i=1}^n \mathbf{D}^{nc}_{i,n}(\mathbf{Z}^{nc}_{i,n})' I_{i,n})^{-1} & \text{if } \mathbf{I}_n = 1, \end{cases}$$

where

$$\hat{\epsilon}_{i,n} = \begin{cases} Y_i - \mathbf{D}'_{i,n}\hat{\beta} & \text{if } \mathbf{I}_n = 0 \\ Y_i - (\mathbf{D}^{nc}_{i,n})'\hat{\beta} & \text{if } \mathbf{I}_n = 1. \end{cases}$$

Let $\hat{\sigma}^2_n$ denote the estimator for the variance of $\hat{\beta}_1$. That is, $\hat{\sigma}^2_n$ is the second diagonal element of $\hat{\mathbf{\Sigma}}_n$ when $\mathbf{I}_n = 0$ and is the first diagonal element of $\hat{\mathbf{\Sigma}}_n$ when $\mathbf{I}_n = 1$. $(\hat{\sigma}^s_n)^2$ is the analogously-defined estimator for the variance of $\hat{\beta}^s_1$ from the simulation-based regression.

Throughout the proof, we omit the subscript n from $I_{i,n}$, $\mathbf{D}_{i,n}$, $\mathbf{Z}_{i,n}$, $\hat{\epsilon}_{i,n}$, $\hat{\mathbf{\Sigma}}_n$, $\hat{\sigma}_n$, etc. for notational brevity. We provide proofs separately for the two cases, the case in which $\Pr(ML(X_i) \in (0,1)) > 0$ and the case in which $\Pr(ML(X_i) \in (0,1)) = 0$. For each case, we first prove consistency and asymptotic normality of $\hat{\beta}_1$, and then prove consistency and asymptotic normality of $\hat{\beta}^s_1$.

C.6.1 Consistency and Asymptotic Normality of $\hat{\beta}_1$ When $\Pr(ML(X_i) \in (0,1)) > 0$

By Lemma B.6,

$$\lim_{\delta \to 0} E[p^{ML}(X_i;\delta)(1-p^{ML}(X_i;\delta))(D_i(1)-D_i(0))] = E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))].$$

When $\Pr(ML(X_i) \in (0,1)) > 0$, $E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))] = E[p^{ML}(X_i)(1-p^{ML}(X_i))(D_i(1)-D_i(0))]$, since $p^{ML}(x) = ML(x)$ for almost every $x \in \mathcal{X}$ by Proposition A.1. Under Assumption 3 (b), $E[p^{ML}(X_i)(1-p^{ML}(X_i))(D_i(1)-D_i(0))] > 0$. Again by Lemma B.6,

$$\lim_{\delta \to 0} E[\omega_i(\delta)(Y_i(1)-Y_i(0))] = \frac{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))(Y_i(1)-Y_i(0)]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]}.$$

Let $\beta_1 = \frac{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))(Y_i(1)-Y_i(0)]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]}$. Let

$$\hat{\beta}^c = (\sum_{i=1}^n \mathbf{Z}_i \mathbf{D}_i' I_i)^{-1} \sum_{i=1}^n \mathbf{Z}_i Y_i I_i$$
$$\hat{\beta}^{nc} = (\sum_{i=1}^n \mathbf{Z}_i^{nc} (\mathbf{D}_i^{nc})' I_i)^{-1} \sum_{i=1}^n \mathbf{Z}_i^{nc} Y_i I_i,$$

and let $\hat{\beta}_1^c = (0,1,0)\hat{\beta}^c$ and $\hat{\beta}_1^{nc} = (1,0)\hat{\beta}^{nc}$. $\hat{\beta}_1$ is given by

$$\hat{\beta}_1 = \hat{\beta}_1^c(1-\mathbf{I}_n) + \hat{\beta}_1^{nc}\mathbf{I}_n.$$

Also, let $\tilde{\mathbf{D}}_i = (1, D_i, ML(X_i))'$, $\tilde{\mathbf{Z}}_i = (1, Z_i, ML(X_i))'$, $\tilde{\mathbf{D}}_i^{nc} = (D_i, ML(X_i))'$, $\tilde{\mathbf{Z}}_i^{nc} = (Z_i, ML(X_i))'$, and $I_i^{ML} = 1\{ML(X_i) \in (0,1)\}$.

We claim that $\Pr(\mathbf{I}_n = 1) \to 0$ when $\mathrm{Var}(ML(X_i)|I_i^{ML}=1) > 0$, and that $\Pr(\mathbf{I}_n = 1) \to 1$ when $\mathrm{Var}(ML(X_i)|I_i^{ML}=1) = 0$. To show the first claim, observe that $\mathbf{I}_n = 1$ if and only if $\hat{V}_n = 0$, where

$$\hat{V}_n = \frac{\sum_{i=1}^n (ML(X_i) - \frac{\sum_{i=1}^n ML(X_i) I_i^{ML}}{\sum_{i=1}^n I_i^{ML}})^2 I_i^{ML}}{\sum_{i=1}^n I_i^{ML}}$$

is the sample variance of $ML(X_i)$ conditional on $I_i^{ML} = 1$. When $\mathrm{Var}(ML(X_i)|I_i^{ML}=1) > 0$,

$$\begin{aligned}
\Pr(\mathbf{I}_n = 1) &= \Pr(\hat{V}_n = 0) \\
&\leq \Pr(|\hat{V}_n - \mathrm{Var}(ML(X_i)|I_i^{ML}=1)| \geq \mathrm{Var}(ML(X_i)|I_i^{ML}=1)) \\
&\to 0,
\end{aligned}$$

where the convergence follows since $\hat{V}_n \xrightarrow{p} \mathrm{Var}(ML(X_i)|I_i^{ML}=1) > 0$.

To show the second claim, note that, when $\mathrm{Var}(ML(X_i)|I_i^{ML}=1) = 0$, there exists $q \in (0,1)$ such that $\Pr(ML(X_i) = q | I_i^{ML} = 1) = 1$. It follows that

$$\begin{aligned}
\Pr(\mathbf{I}_n = 0) &= \Pr(ML(X_i) \in \{0,1\} \text{ for all } i = 1,...,n) \\
&\quad + \Pr(ML(X_i) = q' \text{ and } ML(X_j) = q'' \text{ for some } q', q'' \in (0,1) \text{ with } q' \neq q'' \\
&\qquad\qquad \text{for some } i,j \in \{1,...,n\}) \\
&= \Pr(ML(X_i) \in \{0,1\} \text{ for all } i = 1,...,n) \\
&= (1 - \Pr(ML(X_i) \in (0,1)))^n,
\end{aligned}$$

which converges to zero as $n \to \infty$, since $\Pr(ML(X_i) \in (0,1)) > 0$.

The above claims imply that $\hat{\beta}_1 = \hat{\beta}_1^c$ with probability approaching one when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$, and that $\hat{\beta}_1 = \hat{\beta}_1^{nc}$ with probability approaching one when $\text{Var}(ML(X_i)|I_i^{ML} = 1) = 0$. Therefore, to prove consistency and asymptotic normality of $\hat{\beta}_1$, it suffices to show those of $\hat{\beta}_1^c$ when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$ and those of $\hat{\beta}_1^{nc}$ when $\text{Var}(ML(X_i)|I_i^{ML} = 1) = 0$.

Below we first show that, if Assumptions 1 and 3 hold and $\delta_n \to 0$ as $n \to \infty$, then $\hat{\beta}_1 \xrightarrow{p} \beta_1$. We then show that, if, in addition, Assumption 4 holds and $n\delta_n^2 \to 0$ as $n \to \infty$, then $\hat{\sigma}^{-1}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$.

Proof of Consistency. To prove consistency of $\hat{\beta}_1$, we first show that $\hat{\beta}_1^c \xrightarrow{p} \beta_1$ when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$. We then show that $\hat{\beta}_1^{nc} \xrightarrow{p} \beta_1$ whether or not $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$. By Lemma B.6,

$$\hat{\beta}^c = (\sum_{i=1}^n \mathbf{Z}_i \mathbf{D}_i' I_i)^{-1} \sum_{i=1}^n \mathbf{Z}_i Y_i I_i \xrightarrow{p} (E[\tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i^{ML}])^{-1} E[\tilde{\mathbf{Z}}_i Y_i I_i^{ML}]$$

provided that $E[\tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i^{ML}]$ is invertible. After a few lines of algebra, we have

$$\begin{aligned}
&\det(E[\tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i^{ML}]) \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[D_i(Z_i - ML(X_i)) I_i^{ML}] \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[(Z_i D_i(1) + (1 - Z_i) D_i(0))(Z_i - ML(X_i)) I_i^{ML}] \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[((Z_i - Z_i ML(X_i)) D_i(1) - (1 - Z_i) ML(X_i) D_i(0)) I_i^{ML}] \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[((ML(X_i) - ML(X_i)^2) D_i(1) - (1 - ML(X_i)) ML(X_i) D_i(0)) I_i^{ML}] \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0)) I_i^{ML}] \\
&= \Pr(I_i^{ML} = 1)^2 \text{Var}(ML(X_i)|I_i^{ML} = 1) E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))],
\end{aligned}$$

where the fourth equality follows from Property 1. Therefore, $E[\tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i^{ML}]$ is invertible when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$. Another few lines of algebra gives

$$(E[\tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i^{ML}])^{-1} = \frac{1}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]} \begin{bmatrix} * & * & * \\ 0 & 1 & -1 \\ * & * & * \end{bmatrix}$$

when $\text{Var}(ML(X_i)|I_i^{ML}=1)>0$. Therefore, when $\text{Var}(ML(X_i)|I_i^{ML}=1)>0$,

$$\begin{aligned}
\hat{\beta}_1^c &\xrightarrow{p} \frac{E[Z_iY_iI_i^{ML}]-E[ML(X_i)Y_iI_i^{ML}]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]} \\
&= \frac{E[Z_iY_{1i}I_i^{ML}]-E[ML(X_i)(Z_iY_{1i}+(1-Z_i)Y_{0i})I_i^{ML}]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]} \\
&= \frac{E[ML(X_i)Y_{1i}I_i^{ML}]-E[ML(X_i)(ML(X_i)Y_{1i}+(1-ML(X_i))Y_{0i})I_i^{ML}]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]} \\
&= \frac{E[ML(X_i)(1-ML(X_i))(Y_{1i}-Y_{0i})I_i^{ML}]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]} \\
&= \frac{E[ML(X_i)(1-ML(X_i))((D_i(1)-D_i(0))(Y_i(1)-Y_i(0))]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]} \\
&= \beta_1,
\end{aligned}$$

where the third line follows from Property 1, and the second last follows from the definitions of Y_{1i} and Y_{0i}.

We next consider $\hat{\beta}_1^{nc}$. By Lemma B.6,

$$\hat{\beta}^{nc}=(\sum_{i=1}^n \mathbf{Z}_i^{nc}(\mathbf{D}_i^{nc})'I_i)^{-1}\sum_{i=1}^n \mathbf{Z}_i^{nc}Y_iI_i \xrightarrow{p} (E[\tilde{\mathbf{Z}}_i^{nc}(\tilde{\mathbf{D}}_i^{nc})'I_i^{ML}])^{-1}E[\tilde{\mathbf{Z}}_i^{nc}Y_iI_i^{ML}]$$

provided that $E[\tilde{\mathbf{Z}}_i^{nc}(\tilde{\mathbf{D}}_i^{nc})'I_i^{ML}]$ is invertible. After a few lines of algebra, we have

$$\begin{aligned}
\det(E[\tilde{\mathbf{Z}}_i^{nc}(\tilde{\mathbf{D}}_i^{nc})'I_i^{ML}]) &= E[ML(X_i)^2I_i^{ML}]E[D_i(Z_i-ML(X_i))I_i^{ML}] \\
&= E[ML(X_i)^2I_i^{ML}]E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))] \\
&> 0.
\end{aligned}$$

Another few lines of algebra gives

$$(E[\tilde{\mathbf{Z}}_i^{nc}(\tilde{\mathbf{D}}_i^{nc})'I_i^{ML}])^{-1}=\frac{1}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]}\begin{bmatrix}1 & -1\\ * & *\end{bmatrix}.$$

Therefore,

$$\hat{\beta}_1^{nc} \xrightarrow{p} \frac{E[Z_iY_iI_i^{ML}]-E[ML(X_i)Y_iI_i^{ML}]}{E[ML(X_i)(1-ML(X_i))(D_i(1)-D_i(0))]}=\beta_1.$$

□

Proof of Asymptotic Normality. Let $(\hat{\sigma}^c)^2$ be the second diagonal element of

$$\hat{\mathbf{\Sigma}}^c=(\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)^{-1}(\sum_{i=1}^n \hat{\epsilon}_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i)(\sum_{i=1}^n \mathbf{D}_i\mathbf{Z}_i'I_i)^{-1}$$

and $(\hat{\sigma}^{nc})^2$ be the first diagonal element of

$$\hat{\mathbf{\Sigma}}^{nc}=(\sum_{i=1}^n \mathbf{Z}_{i,n}^{nc}(\mathbf{D}_{i,n}^{nc})'I_i)^{-1}(\sum_{i=1}^n \hat{\epsilon}_{i,n}^2\mathbf{Z}_{i,n}^{nc}(\mathbf{Z}_{i,n}^{nc})'I_i)(\sum_{i=1}^n \mathbf{D}_{i,n}^{nc}(\mathbf{Z}_{i,n}^{nc})'I_i)^{-1}.$$

We only show that $(\hat{\sigma}^c)^{-1}(\hat{\beta}_1^c - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$ when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$. We can show that $(\hat{\sigma}^{nc})^{-1}(\hat{\beta}_1^{nc} - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$ by an analogous argument. The proof proceeds in six steps.

Step C.6.1.1. *Let $\tilde{\beta}_n = (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i])^{-1}E[\tilde{\mathbf{Z}}_i Y_i I_i]$, and let $\tilde{\beta}_{1,n}$ denote the second element of $\tilde{\beta}_n$. Then $\tilde{\beta}_{1,n} = \beta_1$ for any choice of $\delta_n > 0$.*

Proof. Note first that, for every $\delta > 0$, $p^{ML}(x;\delta) \in (0,1)$ for almost every $x \in \{x' \in \mathcal{X} : ML(x') \in (0,1)\}$, since by almost everywhere continuity of ML, for almost every $x \in \{x' \in \mathcal{X} : ML(x') \in (0,1)\}$, there exists an open ball $B \subset B(x,\delta)$ such that $ML(x') \in (0,1)$ for every $x' \in B$. After a few lines of algebra, we have

$$\begin{aligned}
\det(E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i]) &= \Pr(I_i = 1)^2 \text{Var}(ML(X_i)|I_i = 1) E[D_i(Z_i - ML(X_i))I_i] \\
&= \Pr(I_i = 1)^2 \text{Var}(ML(X_i)|I_i = 1) E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))I_i] \\
&= \Pr(I_i = 1)^2 \text{Var}(ML(X_i)|I_i = 1) E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))],
\end{aligned}$$

where the last equality holds since $p^{ML}(x;\delta) \in (0,1)$ for almost every $x \in \{x' \in \mathcal{X} : ML(x') \in (0,1)\}$. By the law of total conditional variance,

$$\begin{aligned}
&\text{Var}(ML(X_i)|I_i = 1) \\
&= E[\text{Var}(ML(X_i)|I_i = 1, I_i^{ML})|I_i = 1] + \text{Var}(E[ML(X_i)|I_i = 1, I_i^{ML}]|I_i = 1) \\
&\geq \sum_{t \in \{0,1\}} \text{Var}(ML(X_i)|I_i = 1, I_i^{ML} = t) \Pr(I_i^{ML} = t|I_i = 1) \\
&\geq \text{Var}(ML(X_i)|I_i = 1, I_i^{ML} = 1) \Pr(I_i^{ML} = 1|I_i = 1) \\
&= \text{Var}(ML(X_i)|I_i^{ML} = 1) \Pr(I_i^{ML} = 1|I_i = 1) \\
&> 0.
\end{aligned}$$

Therefore, $E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i]$ is invertible. Another few lines of algebra gives

$$(E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i])^{-1} = \frac{1}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]} \begin{bmatrix} * & * & * \\ 0 & 1 & -1 \\ * & * & * \end{bmatrix}.$$

It follows that

$$\begin{aligned}
\tilde{\beta}_{1,n} &= \frac{E[Z_i Y_i I_i] - E[ML(X_i)Y_i I_i]}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]} \\
&= \frac{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))I_i]}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]} \\
&= \frac{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))(Y_i(1) - Y_i(0))]}{E[ML(X_i)(1 - ML(X_i))(D_i(1) - D_i(0))]} \\
&= \beta_1.
\end{aligned}$$

□

We can write

$$\sqrt{n}(\hat{\beta}^c - \tilde{\beta}_n) = \underbrace{(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i - (\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i Y_i I_i}_{=(A)}$$
$$+ \underbrace{(\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i Y_i I_i - (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i])^{-1}\sqrt{n}E[\tilde{\mathbf{Z}}_i Y_i I_i]}_{=(B)}.$$

We first consider (B). Let $\tilde{\epsilon}_{i,n} = Y_i - \tilde{\mathbf{D}}_i'\tilde{\beta}_n$ so that

$$E[\tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i] = E[\tilde{\mathbf{Z}}_i(Y_i - \tilde{\mathbf{D}}_i'\tilde{\beta}_n)I_i] = E[\tilde{\mathbf{Z}}_i Y_i I_i] - E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i]\tilde{\beta}_n = 0.$$

Then

$$\begin{aligned}
(B) &= (\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i(\tilde{\mathbf{D}}_i'\tilde{\beta}_n + \tilde{\epsilon}_{i,n})I_i - (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i])^{-1}\sqrt{n}E[\tilde{\mathbf{Z}}_i(\tilde{\mathbf{D}}_i'\tilde{\beta}_n + \tilde{\epsilon}_{i,n})I_i] \\
&= \sqrt{n}(\tilde{\beta}_n - \tilde{\beta}_n) + (\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i - (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i])^{-1}\sqrt{n}E[\tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i] \\
&= (\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i.
\end{aligned}$$

Step C.6.1.2.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i \xrightarrow{d} \mathcal{N}(0, E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i^{ML}]).$$

Proof. We use the triangular-array Lyapunov CLT and the Cramér-Wold device. Pick a nonzero $\lambda \in \mathbb{R}^p$, and let $V_{i,n} = \frac{1}{\sqrt{n}}\lambda'\tilde{\mathbf{Z}}_i\tilde{\epsilon}_{i,n}I_i$. First, we have

$$\sum_{i=1}^n E[V_{i,n}^2] = \lambda' E[\tilde{\epsilon}_{i,n}^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i]\lambda.$$

By Lemma B.6,

$$\tilde{\beta}_n \to (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i^{ML}])^{-1}E[\tilde{\mathbf{Z}}_i Y_i I_i^{ML}]$$

as $n \to \infty$. Let $\beta = (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i^{ML}])^{-1}E[\tilde{\mathbf{Z}}_i Y_i I_i^{ML}]$ and $\tilde{\epsilon}_i = Y_i - \tilde{\mathbf{D}}_i'\beta$. We have

$$\begin{aligned}
E[\tilde{\epsilon}_{i,n}^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] &= E[(Y_i - \tilde{\mathbf{D}}_i'\tilde{\beta}_n)^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] \\
&= E[(\tilde{\epsilon}_i - \tilde{\mathbf{D}}_i'(\tilde{\beta}_n - \beta))^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] \\
&= E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] - 2E[\tilde{\epsilon}_i((\tilde{\beta}_{0,n} - \beta_0) + D_i(\tilde{\beta}_{1,n} - \beta_1) + ML(X_i)(\tilde{\beta}_{2,n} - \beta_2))\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] \\
&\quad + E[((\tilde{\beta}_{0,n} - \beta_0) + D_i(\tilde{\beta}_{1,n} - \beta_1) + ML(X_i)(\tilde{\beta}_{2,n} - \beta_2))^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i] \\
&\to E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i^{ML}]
\end{aligned}$$

as $n \to \infty$, where the convergence follows from Lemma B.6 and from the fact that $\tilde{\beta}_n \to \beta$. Therefore,

$$\sum_{i=1}^{n} E[V_{i,n}^2] \to \lambda' E[\tilde{\epsilon}_i^2 \tilde{\mathbf{Z}}_i \tilde{\mathbf{Z}}_i' I_i^{ML}]\lambda.$$

We next verify the Lyapunov condition: for some $t > 0$,

$$\sum_{i=1}^{n} E[|V_{i,n}|^{2+t}] \to 0.$$

We have

$$\sum_{i=1}^{n} E[|V_{i,n}|^4] = \frac{1}{n} E[|\lambda' \tilde{\mathbf{Z}}_i \tilde{\epsilon}_{i,n} I_i|^4].$$

We use the c_r-inequality: $E[|X+Y|^r] \leq 2^{r-1} E[|X|^r + |Y|^r]$ for $r \geq 1$. Repeating using the c_r-inequality gives

$$\begin{aligned} E[|\lambda' \tilde{\mathbf{Z}}_i \tilde{\epsilon}_{i,n} I_i|^4] &= E[|\lambda' \tilde{\mathbf{Z}}_i (Y_i - \tilde{\beta}_{0,n} - \tilde{\beta}_{1,n} D_i - \tilde{\beta}_{2,n} ML(X_i))|^4 I_i] \\ &\leq 2^{3c} E[(|\lambda' \tilde{\mathbf{Z}}_i|^4)(|Y_i|^4 + |\tilde{\beta}_{0,n}|^4 + |\tilde{\beta}_{1,n}|^4 D_i + |\tilde{\beta}_{2,n}|^4 ML(X_i)^4) I_i] \\ &\leq 2^{3c} (\lambda_1 + \lambda_2 + \lambda_3)^4 (E[Y_i^4] + \tilde{\beta}_{0,n}^4 + \tilde{\beta}_{1,n}^4 + \tilde{\beta}_{2,n}^4) \end{aligned}$$

for some finite constant c, and the right-hand side converges to

$$2^{3c} (\lambda_1 + \lambda_2 + \lambda_3)^4 (E[Y_i^4] + \tilde{\beta}_0^4 + \tilde{\beta}_1^4 + \tilde{\beta}_2^4),$$

which is finite under Assumption 3 (a). Therefore,

$$\sum_{i=1}^{n} E[|V_{i,n}|^4] \to 0,$$

and the conclusion follows from the Lyapunov CLT and the Cramér-Wold device. □

We next consider (A). We can write

$$\begin{aligned} (A) = &(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Z}_i \mathbf{D}_i' I_i)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbf{Z}_i Y_i I_i - \tilde{\mathbf{Z}}_i Y_i I_i) \\ &- (\frac{1}{n} \sum_{i=1}^{n} \mathbf{Z}_i \mathbf{D}_i' I_i)^{-1} [\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbf{Z}_i \mathbf{D}_i' I_i - \tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i)] (\frac{1}{n} \sum_{i=1}^{n} \tilde{\mathbf{Z}}_i \tilde{\mathbf{D}}_i' I_i)^{-1} \frac{1}{n} \sum_{i=1}^{n} \tilde{\mathbf{Z}}_i Y_i I_i. \end{aligned}$$

Step C.6.1.3. *Let $\{V_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $E[|V_i|] < \infty$ and that $E[V_i|X_i]$ is bounded on $N(D^*, \delta') \cap \mathcal{X}$ for some $\delta' > 0$. Then,*

$$E[V_i p^{ML}(X_i;\delta)^l (p^{ML}(X_i;\delta) - ML(X_i)) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] = O(\delta)$$

for $l = 0, 1$.

Proof. For every $x \notin N(D^*, \delta)$, $B(x, \delta) \cap D^* = \emptyset$, so ML is continuously differentiable on $B(x, \delta)$. By the mean value theorem, for every $x \notin N(D^*, \delta)$ and $a \in B(\mathbf{0}, \delta)$,

$$ML(x + a) = ML(x) + \nabla ML(y(x, a))' a$$

for some point $y(x, a)$ on the line segment connecting x and $x + a$. For every $x \notin N(D^*, \delta)$,

$$\begin{aligned} p^{ML}(x; \delta) &= \frac{\int_{B(\mathbf{0},1)} ML(x + \delta u) du}{\int_{B(\mathbf{0},1)} du} \\ &= \frac{\int_{B(\mathbf{0},1)} (ML(x) + \delta \nabla ML(y(x, \delta u))' u) du}{\int_{B(\mathbf{0},1)} du} \\ &= ML(x) + \delta \frac{\int_{B(\mathbf{0},1)} \nabla ML(y(x, \delta u))' u du}{\int_{B(\mathbf{0},1)} du}. \end{aligned}$$

Now, we can write

$$\begin{aligned} & E[V_i p^{ML}(X_i; \delta)^l (p^{ML}(X_i; \delta) - ML(X_i)) 1\{p^{ML}(X_i; \delta) \in (0, 1)\}] \\ = \; & E[V_i p^{ML}(X_i; \delta)^l (p^{ML}(X_i; \delta) - ML(X_i)) 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \notin N(D^*, \delta)\}] \\ & + E[V_i p^{ML}(X_i; \delta)^l (p^{ML}(X_i; \delta) - ML(X_i)) 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \in N(D^*, \delta)\}]. \end{aligned}$$

For the first term,

$$\begin{aligned} & |E[V_i p^{ML}(X_i; \delta)^l (p^{ML}(X_i; \delta) - ML(X_i)) 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \notin N(D^*, \delta)\}]| \\ = \; & \delta |E[V_i p^{ML}(X_i; \delta)^l \frac{\int_{B(\mathbf{0},1)} \nabla ML(y(X_i, \delta u))' u du}{\int_{B(\mathbf{0},1)} du} 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \notin N(D^*, \delta)\}]| \\ \leq \; & \delta E[|V_i| p^{ML}(X_i; \delta)^l \frac{\int_{B(\mathbf{0},1)} \sum_{k=1}^p |\frac{\partial ML(y(X_i, \delta u))}{\partial x_k}| |u_k| du}{\int_{B(\mathbf{0},1)} du} 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \notin N(D^*, \delta)\}] \\ \leq \; & \delta E[|V_i|] \sum_{k=1}^p \sup_{x \in C^*} \left| \frac{\partial ML(x)}{\partial x_k} \right| \frac{\int_{B(\mathbf{0},1)} |u_k| du}{\int_{B(\mathbf{0},1)} du} \\ = \; & O(\delta), \end{aligned}$$

where we use the assumption that the partial derivatives of ML is bounded on C^*. For the second term, for sufficiently small $\delta > 0$,

$$\begin{aligned} & |E[V_i p^{ML}(X_i; \delta)^l (p^{ML}(X_i; \delta) - ML(X_i)) 1\{p^{ML}(X_i; \delta) \in (0, 1)\} 1\{X_i \in N(D^*, \delta)\}]| \\ \leq \; & E[|E[V_i|X_i]| 1\{X_i \in N(D^*, \delta)\}] \\ \leq \; & C E[1\{X_i \in N(D^*, \delta)\}] \\ = \; & C \Pr(X_i \in N(D^*, \delta)) \\ = \; & O(\delta), \end{aligned}$$

where C is some constant, the second inequality follows from the assumption that $E[V_i|X_i]$ is bounded on $N(D^*, \delta') \cap \mathcal{X}$ for some $\delta' > 0$, and the last equality follows from Assumption 4 (a). □

Step C.6.1.4. $\frac{1}{\sqrt{n}}\sum_{i=1}^n(\mathbf{Z}_iY_iI_i - \tilde{\mathbf{Z}}_iY_iI_i) = o_p(1)$ *and* $\frac{1}{\sqrt{n}}\sum_{i=1}^n(\mathbf{Z}_i\mathbf{D}_i'I_i - \tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i'I_i) = o_p(1)$.

Proof. We only show that $\frac{1}{\sqrt{n}}\sum_{i=1}^n(p^{ML}(X_i;\delta_n)^2 - ML(X_i)^2)I_i = o_p(1)$. The proofs for the other elements are similar. As for bias,

$$\begin{aligned}
&E[\frac{1}{\sqrt{n}}\sum_{i=1}^n(p^{ML}(X_i;\delta_n)^2 - ML(X_i)^2)I_i] \\
&= \sqrt{n}E[(p^{ML}(X_i;\delta_n)^2 - ML(X_i)^2)I_i] \\
&= \sqrt{n}E[(p^{ML}(X_i;\delta_n) + ML(X_i))(p^{ML}(X_i;\delta_n) - ML(X_i))I_i] \\
&= \sqrt{n}O(\delta_n) \\
&= 0,
\end{aligned}$$

where the third equality follows from Step C.6.1.3 and the last from the assumption that $n\delta_n^2 \to 0$. As for variance, by Lemma B.6,

$$\begin{aligned}
&\mathrm{Var}(\frac{1}{\sqrt{n}}\sum_{i=1}^n(p^{ML}(X_i;\delta_n)^2 - ML(X_i)^2)I_i) \\
&\leq E[(p^{ML}(X_i;\delta_n)^2 - ML(X_i)^2)^2I_i] \\
&= E[(p^{ML}(X_i;\delta_n)^4 - 2p^{ML}(X_i;\delta_n)^2ML(X_i)^2 + ML(X_i)^4)I_i] \\
&\to E[(ML(X_i)^4 - 2ML(X_i)^2ML(X_i)^2 + ML(X_i)^4)I_i^{ML}] \\
&= 0.
\end{aligned}$$

□

Step C.6.1.5. $n\hat{\mathbf{\Sigma}}^c \xrightarrow{p} (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i'I_i^{ML}])^{-1}E[\hat{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}](E[\tilde{\mathbf{D}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}])^{-1}$.

Proof. Let $\epsilon_i = Y_i - \mathbf{D}_i'\beta$. We have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n\hat{\epsilon}_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i =& \frac{1}{n}\sum_{i=1}^n(Y_i - \mathbf{D}_i'\hat{\beta}^c)^2\mathbf{Z}_i\mathbf{Z}_i'I_i \\
=& \frac{1}{n}\sum_{i=1}^n(\epsilon_i - \mathbf{D}_i'(\hat{\beta}^c - \beta))^2\mathbf{Z}_i\mathbf{Z}_i'I_i \\
=& \frac{1}{n}\sum_{i=1}^n\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i \\
&- \frac{2}{n}\sum_{i=1}^n(Y_i - \mathbf{D}_i'\beta)((\hat{\beta}_0^c - \beta_0) + D_i(\hat{\beta}_1^c - \beta_1) + p^{ML}(X_i;\delta_n)(\hat{\beta}_2^c - \beta_2))\mathbf{Z}_i\mathbf{Z}_i'I_i \\
&+ \frac{1}{n}\sum_{i=1}^n((\hat{\beta}_0^c - \beta_0) + D_i(\hat{\beta}_1^c - \beta_1) + p^{ML}(X_i;\delta_n)(\hat{\beta}_2^c - \beta_2))^2\mathbf{Z}_i\mathbf{Z}_i'I_i \\
=& \frac{1}{n}\sum_{i=1}^n\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i + o_p(1)O_p(1),
\end{aligned}$$

where the last equality follows from the result that $\hat{\beta}^c - \beta = o_p(1)$ and from Lemma B.6. Again by Lemma B.6,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i &= \frac{1}{n}\sum_{i=1}^{n}(Y_i^2 - 2Y_i\mathbf{D}_i'\beta + \beta'\mathbf{D}_i\mathbf{D}_i'\beta)\mathbf{Z}_i\mathbf{Z}_i'I_i \\ &\xrightarrow{p} E[(Y_i^2 - 2Y_i\tilde{\mathbf{D}}_i'\beta + \beta'\tilde{\mathbf{D}}_i\tilde{\mathbf{D}}_i'\beta)\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}] \\ &= E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}],\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{Z}_i\mathbf{D}_i'I_i \xrightarrow{p} E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i'I_i^{ML}].$$

The conclusion then follows. □

Step C.6.1.6. $(\hat{\sigma}^c)^{-1}(\hat{\beta}_1^c - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$.

Proof. By combining the results from Steps C.6.1.2–C.6.1.4 and by Lemma B.6,

$$\begin{aligned}&(A) \xrightarrow{p} 0, \\ &(B) \xrightarrow{d} \mathcal{N}(0, (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i'I_i^{ML}])^{-1}E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}](E[\tilde{\mathbf{D}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}])^{-1}),\end{aligned}$$

and therefore,

$$\sqrt{n}(\hat{\beta}^c - \tilde{\beta}_n) \xrightarrow{d} \mathcal{N}(0, (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i'I_i^{ML}])^{-1}E[\tilde{\epsilon}_i^2\tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}](E[\tilde{\mathbf{D}}_i\tilde{\mathbf{Z}}_i'I_i^{ML}])^{-1}).$$

The conclusion then follows from Steps C.6.1.1 and C.6.1.5. □

□

□

C.6.2 Consistency and Asymptotic Normality of $\hat{\beta}_1^s$ When $\Pr(ML(X_i) \in (0,1)) > 0$

Let $I_i^s = 1\{p^s(X_i;\delta_n) \in (0,1)\}$, $\mathbf{D}_i^s = (1, D_i, p^s(X_i;\delta_n))'$ and $\mathbf{Z}_i^s = (1, Z_i, p^s(X_i;\delta_n))'$. Let

$$\hat{\beta}^{c,s} = (\sum_{i=1}^{n}\mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s)^{-1}\sum_{i=1}^{n}\mathbf{Z}_i^sY_iI_i^s$$

and

$$\hat{\mathbf{\Sigma}}^{c,s} = (\sum_{i=1}^{n}\mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s)^{-1}(\sum_{i=1}^{n}(\hat{\epsilon}_i^s)^2\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s)(\sum_{i=1}^{n}\mathbf{D}_i^s(\mathbf{Z}_i^s)'I_i^s)^{-1},$$

where $\hat{\epsilon}_i^s = Y_i - (\mathbf{D}_i^s)'\hat{\beta}^{c,s}$. Here, we only show that $\hat{\beta}_1^{c,s} \xrightarrow{p} \beta_1$ if $S_n \to \infty$ and that $(\hat{\sigma}^s)^{-1}(\hat{\beta}_1^{c,s} - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$ if Assumption 5 holds when $\text{Var}(ML(X_i)|I_i^{ML} = 1) > 0$. For that, it suffices to show that

$$\hat{\beta}^{c,s} - \hat{\beta}^c = o_p(1)$$

if $S_n \to \infty$ and that

$$\sqrt{n}(\hat{\beta}^{c,s} - \hat{\beta}^c) = o_p(1),$$
$$n\hat{\mathbf{\Sigma}}^{c,s} \xrightarrow{p} (E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i^{ML}])^{-1} E[\tilde{\epsilon}_i^2 \tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i^{ML}](E[\tilde{\mathbf{D}}_i\tilde{\mathbf{Z}}_i' I_i^{ML}])^{-1}$$

if Assumption 5 holds. We have

$$\begin{aligned}
\hat{\beta}^{c,s} - \hat{\beta}^c &= (\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)' I_i^s)^{-1}\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s Y_i I_i^s - (\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i \\
&= (\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)' I_i^s)^{-1}(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s Y_i I_i^s - \frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i) \\
&\quad - (\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)' I_i^s)^{-1}(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)' I_i^s - \frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i' I_i)(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i.
\end{aligned}$$

By Lemma B.8, $\hat{\beta}^{c,s} - \hat{\beta}^c = o_p(1)$ if $S_n \to \infty$, and $\sqrt{n}(\hat{\beta}^{c,s} - \hat{\beta}^c) = o_p(1)$ under the boundedness imposed by Assumption 4 (c) if Assumption 5 holds.

By proceeding as in Step C.6.1.5 in Section C.6.1, we have

$$\frac{1}{n}\sum_{i=1}^n (\hat{\epsilon}_i^s)^2 \mathbf{Z}_i^s(\mathbf{Z}_i^s)' I_i^s = \frac{1}{n}\sum_{i=1}^n (\epsilon_i^s)^2 \mathbf{Z}_i^s(\mathbf{Z}_i^s)' I_i^s + o_p(1),$$

where $\epsilon_i^s = Y_i - (\mathbf{D}_i^s)'\beta$. Then, by Lemma B.8,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n (\hat{\epsilon}_i^s)^2 \mathbf{Z}_i^s(\mathbf{Z}_i^s)' I_i^s - \frac{1}{n}\sum_{i=1}^n \epsilon_i^2 \mathbf{Z}_i\mathbf{Z}_i' I_i \\
&= \frac{1}{n}\sum_{i=1}^n (Y_i^2 - 2Y_i(\mathbf{D}_i^s)'\beta + \beta'\mathbf{D}_i^s(\mathbf{D}_i^s)'\beta)\mathbf{Z}_i^s(\mathbf{Z}_i^s)' I_i^s - \frac{1}{n}\sum_{i=1}^n (Y_i^2 - 2Y_i\mathbf{D}_i'\beta + \beta'\mathbf{D}_i\mathbf{D}_i'\beta)\mathbf{Z}_i\mathbf{Z}_i' I_i + o_p(1) \\
&= o_p(1)
\end{aligned}$$

so that

$$\frac{1}{n}\sum_{i=1}^n (\hat{\epsilon}_i^s)^2 \mathbf{Z}_i^s(\mathbf{Z}_i^s)' I_i^s \xrightarrow{p} E[\tilde{\epsilon}_i^2 \tilde{\mathbf{Z}}_i\tilde{\mathbf{Z}}_i' I_i^{ML}].$$

Also, $\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)' I_i^s \xrightarrow{p} E[\tilde{\mathbf{Z}}_i\tilde{\mathbf{D}}_i' I_i^{ML}]$ by using Lemma B.8. The conclusion then follows. □

C.6.3 Consistency and Asymptotic Normality of $\hat{\beta}_1$ When $\Pr(ML(X_i) \in (0,1)) = 0$

Since $\Pr(ML(X_i) \in (0,1)) = 0$, $\mathbf{I}_n = 0$ with probability one. Hence,

$$\hat{\beta} = (\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i$$

with probability one. We use the notation and results provided in Appendix B. By Lemma B.5, under Assumption 3 (e), there exists $\mu > 0$ such that $d_{\Omega^*}^s$ is twice continuously differentiable on $N(\partial\Omega^*, \mu)$ and that

$$\int_{N(\partial\Omega^*,\delta)} g(x)dx = \int_{-\delta}^{\delta}\int_{\partial\Omega^*} g(u + \lambda\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\lambda)d\mathcal{H}^{p-1}(u)d\lambda$$

for every $\delta \in (0, \mu)$ and every function $g : \mathbb{R}^p \to \mathbb{R}$ that is integrable on $N(\partial\Omega^*, \delta)$.

Below we show that $\hat{\beta}_1 \xrightarrow{p} \beta_1$ if $n\delta_n \to \infty$ and $\delta_n \to 0$ and that $\hat{\sigma}^{-1}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} \mathcal{N}(0,1)$ if $n\delta_n^3 \to 0$ in addition. The proof proceeds in eight steps.

Step C.6.3.1. *There exist $\bar{\delta} > 0$ and a bounded function $r : \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times (-1,1) \times (0, \bar{\delta}) \to \mathbb{R}$ such that*

$$p^{ML}(u + \delta v \nu_{\Omega^*}(u); \delta) = k(v) + \delta r(u, v, \delta)$$

for every $(u, v, \delta) \in \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times (-1,1) \times (0, \bar{\delta})$, where

$$k(v) = \begin{cases} 1 - \frac{1}{2} I_{(1-v^2)}(\frac{p+1}{2}, \frac{1}{2}) & \text{for } v \in [0,1) \\ \frac{1}{2} I_{(1-v^2)}(\frac{p+1}{2}, \frac{1}{2}) & \text{for } v \in (-1,0). \end{cases}$$

Here $I_x(\alpha, \beta)$ is the regularized incomplete beta function (the cumulative distribution function of the beta distribution with shape parameters α and β).

Proof. By Assumption 3 (f) (ii), there exists $\bar{\delta} \in (0, \frac{\mu}{2})$ such that $ML(x) = 0$ for almost every $x \in N(\mathcal{X}, 3\bar{\delta}) \setminus \Omega^*$. By Taylor's theorem, for every $u \in \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta})$ and $a \in B(\mathbf{0}, 2\bar{\delta})$,

$$d^s_{\Omega^*}(u + a) = d^s_{\Omega^*}(u) + \nabla d^s_{\Omega^*}(u)' a + a' R(u, a) a,$$

where

$$R(u, a) = \int_0^1 (1 - t) D^2 d^s_{\Omega^*}(u + ta) dt.$$

Since $D^2 d^s_{\Omega^*}$ is continuous and $\mathrm{cl}(N(\partial\Omega^*, 2\bar{\delta}))$ is bounded and closed, $D^2 d^s_{\Omega^*}$ is bounded on $\mathrm{cl}(N(\partial\Omega^*, 2\bar{\delta}))$. Therefore, $R(\cdot, \cdot)$ is bounded on $\partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times B(\mathbf{0}, 2\bar{\delta})$. It also follows that

$$d^s_{\Omega^*}(u + a) = \nu_{\Omega^*}(u)' a + a' R(u, a) a,$$

since $d^s_{\Omega^*}(u) = 0$ and $\nabla d^s_{\Omega^*}(u) = \nu_{\Omega^*}(u)$ for every $u \in \partial\Omega^* \cap N(\mathcal{X}, 2\bar{\delta})$ by Lemma B.1. For $(u, v, \delta) \in \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times (-1,1) \times (0, \bar{\delta})$,

$$\begin{aligned}
& p^{ML}(u + \delta v \nu_{\Omega^*}(u); \delta) \\
&= \frac{\int_{B(\mathbf{0},1)} ML(u + \delta v \nu_{\Omega^*}(u) + \delta w) dw}{\int_{B(\mathbf{0},1)} dw} \\
&= \frac{\int_{B(\mathbf{0},1)} 1\{u + \delta v \nu_{\Omega^*}(u) + \delta w \in \Omega^*\} dw}{\mathrm{Vol}_p} \\
&= \frac{\int_{B(\mathbf{0},1)} 1\{d^s_{\Omega^*}(u + \delta(v \nu_{\Omega^*}(u) + w)) \geq 0)\} dw}{\mathrm{Vol}_p} \\
&= \frac{\int_{B(\mathbf{0},1)} 1\{\delta \nu_{\Omega^*}(u)'(v \nu_{\Omega^*}(u) + w) + \delta^2 (v \nu_{\Omega^*}(u) + w)' R(u, \delta(v \nu_{\Omega^*}(u) + w))(v \nu_{\Omega^*}(u) + w) \geq 0\} dw}{\mathrm{Vol}_p},
\end{aligned}$$

where Vol_p denotes the volume of the p-dimensional unit ball, and the second equality follows since $u + \delta v \nu_{\Omega^*}(u) + \delta w \in N(\mathcal{X}, 3\bar{\delta})$ and hence $ML(u + \delta v \nu_{\Omega^*}(u) + \delta w) = 0$ for almost every

$w \in B(\mathbf{0},1)$ such that $u + \delta v \nu_{\Omega^*}(u) + \delta w \notin \Omega^*$. Observe that

$$
\begin{aligned}
&1\{\delta \nu_{\Omega^*}(u)'(v\nu_{\Omega^*}(u)+w) + \delta^2 (v\nu_{\Omega^*}(u)+w)' R(u, \delta(v\nu_{\Omega^*}(u)+w))(v\nu_{\Omega^*}(u)+w) \geq 0\} \\
&= 1\{v + \nu_{\Omega^*}(u)\cdot w + \delta(v\nu_{\Omega^*}(u)+w)' R(u, \delta(v\nu_{\Omega^*}(u)+w))(v\nu_{\Omega^*}(u)+w) \geq 0\} \\
&= 1\{v + \nu_{\Omega^*}(u)\cdot w \geq 0\} \\
&\quad - \underbrace{1\{v + \nu_{\Omega^*}(u)\cdot w \geq 0, v + \nu_{\Omega^*}(u)\cdot w + \delta(v\nu_{\Omega^*}(u)+w)' R(u, \delta(v\nu_{\Omega^*}(u)+w))(v\nu_{\Omega^*}(u)+w) < 0\}}_{=a(u,v,w,\delta)} \\
&\quad + \underbrace{1\{v + \nu_{\Omega^*}(u)\cdot w < 0, v + \nu_{\Omega^*}(u)\cdot w + \delta(v\nu_{\Omega^*}(u)+w)' R(u, \delta(v\nu_{\Omega^*}(u)+w))(v\nu_{\Omega^*}(u)+w) \geq 0\}}_{=b(u,v,w,\delta)}.
\end{aligned}
$$

Note that the set $\{w \in B(\mathbf{0},1) : v + \nu(u)\cdot w \geq 0\}$ is a region of the p-dimensional unit ball cut off by the plane $\{w \in \mathbb{R}^p : v + \nu(u)\cdot w = 0\}$. The distance from the center of the unit ball to the plane is $|v|$. Using the formula for the volume of a hyperspherical cap (see e.g. Li (2011)), we have

$$
\int_{B(\mathbf{0},1)} 1\{v + \nu(u)\cdot w \geq 0\} dw = \begin{cases} \mathrm{Vol}_p - \frac{1}{2}\mathrm{Vol}_p I_{(2(1-v)-(1-v)^2)}(\frac{p+1}{2}, \frac{1}{2}) & \text{for } v \in [0,1) \\ \frac{1}{2}\mathrm{Vol}_p I_{(2(1+v)-(1+v)^2)}(\frac{p+1}{2}, \frac{1}{2}) & \text{for } v \in (-1,0). \end{cases}
$$

Therefore, for every $(u,v,\delta) \in \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times (-1,1) \times (0,\bar{\delta})$,

$$
p^{ML}(u + \delta v \nu_{\Omega^*}(u); \delta) = k(v) + \frac{\int_{B(\mathbf{0},1)} (-a(u,v,w,\delta) + b(u,v,w,\delta)) dw}{\mathrm{Vol}_p}.
$$

Now let $r(u,v,\delta) = \delta^{-1}(p^{ML}(u + \delta v \nu_{\Omega^*}(u); \delta) - k(v))$. Since $R(\cdot,\cdot)$ is bounded on $\partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times B(\mathbf{0}, 2\bar{\delta})$ and $\|\nu_{\Omega^*}(u)\| = 1$, there exists $\bar{r} > 0$ such that

$$
|(v\nu_{\Omega^*}(u)+w)' R(u, \delta(v\nu_{\Omega^*}(u)+w))(v\nu_{\Omega^*}(u)+w)| \leq \bar{r}
$$

for every $(u,v,w,\delta) \in \partial\Omega^* \cap N(\mathcal{X}, \bar{\delta}) \times (-1,1) \times B(\mathbf{0},1) \times (0,\bar{\delta})$. Therefore,

$$
0 \leq a(u,v,w,\delta) \leq 1\{0 \leq v + \nu_{\Omega^*}(u)\cdot w < \delta\bar{r}\}
$$

and

$$
0 \leq b(u,v,w,\delta) \leq 1\{-\delta\bar{r} \leq v + \nu_{\Omega^*}(u)\cdot w < 0\}.
$$

It then follows that

$$
\begin{aligned}
-\frac{\int_{B(\mathbf{0},1)} 1\{0 \leq v + \nu_{\Omega^*}(u)\cdot w < \delta\bar{r}\} dw}{\mathrm{Vol}_p} &\leq \frac{\int_{B(\mathbf{0},1)} (-a(u,v,w,\delta) + b(u,v,w,\delta)) dw}{\mathrm{Vol}_p} \\
&\leq \frac{\int_{B(\mathbf{0},1)} 1\{-\delta\bar{r} \leq v + \nu_{\Omega^*}(u)\cdot w < 0\} dw}{\mathrm{Vol}_p}.
\end{aligned}
$$

The set $\{w \in B(\mathbf{0},1) : 0 \leq v + \nu_{\Omega^*}(u)\cdot w < \delta\bar{r}\}$ is a region of the p-dimensional unit ball cut off by the two planes $\{w \in \mathbb{R}^p : v + \nu_{\Omega^*}(u)\cdot w = 0\}$ and $\{w \in \mathbb{R}^p : v + \nu_{\Omega^*}(u)\cdot w = \delta\bar{r}\}$. Its

Lebesgue measure is at most the volume of the $(p-1)$-dimensional unit ball times the distance between the two planes, so

$$-\delta \text{Vol}_{p-1}\bar{r} \leq -\int_{B(\mathbf{0},1)} 1\{0 \leq v + \nu_{\Omega^*}(u)\cdot w < \delta\bar{r}\}dw.$$

Likewise,

$$\int_{B(\mathbf{0},1)} 1\{-\delta\bar{r} \leq v + \nu_{\Omega^*}(u)\cdot w < 0\}dw \leq \delta \text{Vol}_{p-1}\bar{r}.$$

Therefore,

$$-\frac{\delta \text{Vol}_{p-1}\bar{r}}{\text{Vol}_p} \leq \frac{\int_{B(\mathbf{0},1)}(-a(u,v,w,\delta)+b(u,v,w,\delta))dw}{\text{Vol}_p} \leq \frac{\delta \text{Vol}_{p-1}\bar{r}}{\text{Vol}_p}.$$

It follows that

$$\begin{aligned} r(u,v,\delta) &= \delta^{-1}\frac{\int_{B(\mathbf{0},1)}(-a(u,v,w,\delta)+b(u,v,w,\delta))dw}{\text{Vol}_p} \\ &\in [-\frac{\text{Vol}_{p-1}\bar{r}}{\text{Vol}_p}, \frac{\text{Vol}_{p-1}\bar{r}}{\text{Vol}_p}], \end{aligned}$$

and hence r is bounded on $\partial\Omega^* \cap N(\mathcal{X},\bar{\delta}) \times (-1,1) \times (0,\bar{\delta})$. □

Step C.6.3.2. *For every $(u,v,\delta) \in \partial\Omega^* \cap N(\mathcal{X},\bar{\delta}) \times (-1,1) \times (0,\bar{\delta})$, $p^{ML}(u+\delta v \nu_{\Omega^*}(u);\delta) \in (0,1)$.*

Proof. Fix $(u,v,\delta) \in \partial\Omega^* \cap N(\mathcal{X},\bar{\delta}) \times (-1,1) \times (0,\bar{\delta})$. Suppose $v = 0$. By Step C.6.3.1, $p^{ML}(u) = \lim_{\delta'\to 0} p^{ML}(u;\delta') = k(0) = \frac{1}{2}$. This implies that there exists $\delta' \in (0,\delta)$ such that $p^{ML}(u;\delta') \in (0,1)$. It then follows that $0 < \mathcal{L}^p(B(u,\delta') \cap \Omega^*) \leq \mathcal{L}^p(B(x,\delta) \cap \Omega^*)$ and that $0 < \mathcal{L}^p(B(x,\delta') \setminus \Omega^*) \leq \mathcal{L}^p(B(x,\delta) \setminus \Omega^*)$. Therefore, $p^{ML}(u;\delta) = \frac{\mathcal{L}^p(B(u,\delta)\cap\Omega^*)}{\mathcal{L}^p(B(u,\delta))} \in (0,1)$.

Suppose $v \neq 0$ and let $\epsilon \in (0, \delta(1-|v|))$. Note that $B(u,\epsilon) \subset B(u+\delta v\nu_{\Omega^*}(u),\delta)$, since for any $x \in B(u,\epsilon)$, $\|u+\delta v\nu_{\Omega^*}(u) - x\| \leq \|\delta v \nu_{\Omega^*}(u)\| + \|u-x\| \leq \delta|v| + \epsilon < \delta$. Since $p^{ML}(u) = \frac{1}{2}$, there exists $\epsilon' \in (0,\epsilon)$ such that $p^{ML}(u;\epsilon') \in (0,1)$. It then follows that $0 < \mathcal{L}^p(B(u,\epsilon') \cap \Omega^*) \leq \mathcal{L}^p(B(u,\epsilon)\cap\Omega^*) \leq \mathcal{L}^p(B(u+\delta v\nu_{\Omega^*}(u),\delta)\cap\Omega^*)$ and that $0 < \mathcal{L}^p(B(x,\epsilon')\setminus\Omega^*) \leq \mathcal{L}^p(B(x,\epsilon)\setminus\Omega^*) \leq \mathcal{L}^p(B(u+\delta v\nu_{\Omega^*}(u),\delta)\setminus\Omega^*)$. Therefore, $p^{ML}(u+\delta v\nu_{\Omega^*}(u);\delta) = \frac{\mathcal{L}^p(B(u+\delta v\nu_{\Omega^*}(u),\delta)\cap\Omega^*)}{\mathcal{L}^p(B(u+\delta v\nu_{\Omega^*}(u),\delta))} \in (0,1)$. □

Step C.6.3.3. *Let $g : \mathbb{R}^p \to \mathbb{R}$ be a function that is bounded on $N(\partial\Omega^*,\delta') \cap N(\mathcal{X},\delta')$ for some $\delta' > 0$. Then, for $l \geq 0$, there exist $\tilde{\delta} > 0$ and constant $C > 0$ such that*

$$|\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}]| \leq C$$

for every $\delta \in (0,\tilde{\delta})$. If g is continuous on $N(\partial\Omega^*,\delta') \cap N(\mathcal{X},\delta')$ for some $\delta' > 0$, then

$$\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] = \int_{-1}^{1} k(v)^l dv \int_{\partial\Omega^*} g(x) f_X(x) d\mathcal{H}^{p-1}(x) + o(1)$$

$$\delta^{-1}E[Z_i p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] = \int_{0}^{1} k(v)^l dv \int_{\partial\Omega^*} g(x) f_X(x) d\mathcal{H}^{p-1}(x) + o(1)$$

for $l \geq 0$. Furthermore, if g is continuously differentiable and ∇g is bounded on $N(\partial\Omega^*, \delta') \cap N(\mathcal{X}, \delta')$ for some $\delta' > 0$, then

$$\delta^{-1} E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] = \int_{-1}^{1} k(v)^l dv \int_{\partial\Omega^*} g(x) f_X(x) d\mathcal{H}^{p-1}(x) + O(\delta)$$
$$\delta^{-1} E[Z_i p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] = \int_{0}^{1} k(v)^l dv \int_{\partial\Omega^*} g(x) f_X(x) d\mathcal{H}^{p-1}(x) + O(\delta)$$

for $l \geq 0$.

Proof. Let $\bar{\delta}$ be given in Step C.6.3.1. Under Assumption 3 (g), there exists $\tilde{\delta} \in (0, \bar{\delta})$ such that f_X is bounded, is continuously differentiable, and has bounded partial derivatives on $N(\partial\Omega^*, 2\tilde{\delta}) \cap N(\mathcal{X}, 2\tilde{\delta})$. Let $\tilde{\delta} \in (0, \bar{\delta})$ be such that both g and f_X are bounded on $N(\partial\Omega^*, 2\tilde{\delta}) \cap N(\mathcal{X}, 2\tilde{\delta})$. We first show that $p^{ML}(x;\delta) \in \{0,1\}$ for every $x \in \mathcal{X} \setminus N(\partial\Omega^*, \delta)$ for every $\delta \in (0, \tilde{\delta})$. Pick $x \in \mathcal{X} \setminus N(\partial\Omega^*, \delta)$ and $\delta \in (0, \tilde{\delta})$. Since $B(x,\delta) \cap \partial\Omega^* = \emptyset$, either $B(x,\delta) \subset \text{int}(\Omega^*)$ or $B(x,\delta) \subset \text{int}(\mathbb{R}^p \setminus \Omega^*)$. If $B(x,\delta) \subset \text{int}(\Omega^*)$, $p^{ML}(x;\delta) = 1$. If $B(x,\delta) \subset \text{int}(\mathbb{R}^p \setminus \Omega^*)$, $p^{ML}(x;\delta) = 0$, since $ML(x') = 0$ for almost every $x' \in B(x,\delta) \subset N(\mathcal{X}, 3\bar{\delta}) \setminus \Omega^*$ by the choice of $\bar{\delta}$. Therefore, $\{x \in \mathcal{X} : p^{ML}(x;\delta) \in (0,1)\} \subset N(\partial\Omega^*, \delta)$ for every $\delta \in (0, \tilde{\delta})$. By this and Lemma B.5, for $\delta \in (0, \tilde{\delta})$,

$$\begin{aligned}
&\delta^{-1} E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \\
&= \delta^{-1} \int p^{ML}(x;\delta)^l g(x) 1\{p^{ML}(x;\delta) \in (0,1)\} f_X(x) dx \\
&= \delta^{-1} \int_{N(\partial\Omega^*,\delta)} p^{ML}(x;\delta)^l g(x) 1\{p^{ML}(x;\delta) \in (0,1)\} f_X(x) dx \\
&= \delta^{-1} \int_{-\delta}^{\delta} \int_{\partial\Omega^*} p^{ML}(u + \lambda\nu_{\Omega^*}(u);\delta)^l g(u + \lambda\nu_{\Omega^*}(u)) 1\{p^{ML}(u + \lambda\nu_{\Omega^*}(u);\delta) \in (0,1)\} \\
&\qquad \times f_X(u + \lambda\nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*} \psi_{\Omega^*}(u,\lambda) d\mathcal{H}^{p-1}(u) d\lambda.
\end{aligned}$$

With change of variables $v = \frac{\lambda}{\delta}$, we have

$$\begin{aligned}
&\delta^{-1} E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \\
&= \int_{-1}^{1} \int_{\partial\Omega^*} p^{ML}(u + \delta v\nu_{\Omega^*}(u);\delta)^l 1\{p^{ML}(u + \delta v\nu_{\Omega^*}(u);\delta) \in (0,1)\} \\
&\qquad \times g(u + \delta v\nu_{\Omega^*}(u)) f_X(u + \delta v\nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*} \psi_{\Omega^*}(u,\delta v) d\mathcal{H}^{p-1}(u) dv.
\end{aligned}$$

For every $(u,v,\delta) \in \partial\Omega^* \setminus N(\mathcal{X}, \tilde{\delta}) \times (-1,1) \times (0, \tilde{\delta})$, $u + \delta v\nu_{\Omega^*}(u) \notin \mathcal{X}$, so

$$\begin{aligned}
&\delta^{-1} E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \\
&= \int_{-1}^{1} \int_{\partial\Omega^* \cap N(\mathcal{X},\tilde{\delta})} p^{ML}(u + \delta v\nu_{\Omega^*}(u);\delta)^l 1\{p^{ML}(u + \delta v\nu_{\Omega^*}(u);\delta) \in (0,1)\} \\
&\qquad \times g(u + \delta v\nu_{\Omega^*}(u)) f_X(u + \delta v\nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*} \psi_{\Omega^*}(u,\delta v) d\mathcal{H}^{p-1}(u) dv \\
&= \int_{-1}^{1} \int_{\partial\Omega^* \cap N(\mathcal{X},\tilde{\delta})} (k(v) + \delta r(u,v,\delta))^l g(u + \delta v\nu_{\Omega^*}(u)) f_X(u + \delta v\nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*} \psi_{\Omega^*}(u,\delta v) d\mathcal{H}^{p-1}(u) dv,
\end{aligned}$$

where the second equality follows from Steps C.6.3.1 and C.6.3.2. By Lemma B.5, $J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(\cdot,\cdot)$ is bounded on $\partial\Omega^* \times (-\tilde{\delta}, \tilde{\delta})$. Since r, g and f_X are also bounded, for some constant $C > 0$,

$$|\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}]| \leq C \int_{-1}^{1} \int_{\partial\Omega^* \cap N(\mathcal{X},\tilde{\delta})} d\mathcal{H}^{p-1}(u)dv,$$

which is finite by Assumption 3 (f) (i). Moreover, if g and f_X are continuous on $N(\partial\Omega^*, 2\tilde{\delta}) \cap N(\mathcal{X}, 2\tilde{\delta})$, by the Dominated Convergence Theorem,

$$\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \to \int_{-1}^{1} k(v)^l dv \int_{\partial\Omega^*} g(u) f_X(u) d\mathcal{H}^{p-1}(u),$$

where we use the fact from Lemma B.5 that $J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\lambda)$ is continuous in λ and $J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,0) = 1$.

Note that $ML(x) = 1$ for every $x \in \Omega^*$ and $ML(x) = 0$ for almost every $x \in N(\mathcal{X}, 2\tilde{\delta}) \setminus \Omega^*$. Also, for every $(u,v,\delta) \in \partial\Omega^* \cap N(\mathcal{X},\tilde{\delta}) \times (-1,1) \times (0,\tilde{\delta})$, $u + \delta v \nu_{\Omega^*}(u) \in \Omega^*$ if $v \in (0,1)$ and $u + \delta v \nu_{\Omega^*}(u) \in N(\mathcal{X}, 2\tilde{\delta}) \setminus \Omega^*$ if $v \in (-1,0]$. Therefore,

$$\begin{aligned}
&\delta^{-1}E[Z_i p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \\
&= \delta^{-1}E[ML(X_i) p^{ML}(X_i;\delta)^l g(X_i) 1\{p^{ML}(X_i;\delta) \in (0,1)\}] \\
&= \int_{-1}^{1} \int_{\partial\Omega^* \cap N(\mathcal{X},\tilde{\delta})} ML(u + \delta v \nu_{\Omega^*}(u))(k(v) + \delta r(u,v,\delta))^l g(u + \delta v \nu_{\Omega^*}(u)) \\
&\qquad\qquad \times f_X(u + \delta v \nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta v) d\mathcal{H}^{p-1}(u)dv \\
&= \int_{0}^{1} \int_{\partial\Omega^* \cap N(\mathcal{X},\tilde{\delta})} (k(v) + \delta r(u,v,\delta))^l g(u + \delta v \nu_{\Omega^*}(u)) f_X(u + \delta v \nu_{\Omega^*}(u)) J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta v) d\mathcal{H}^{p-1}(u)dv \\
&\to \int_{0}^{1} k(v)^l dv \int_{\partial\Omega^*} g(u) f_X(u) d\mathcal{H}^{p-1}(u).
\end{aligned}$$

Now suppose that g and f_X are continuously differentiable on $N(\partial\Omega^*, 2\tilde{\delta}) \cap N(\mathcal{X}, 2\tilde{\delta})$ and that ∇g and ∇f are bounded on $N(\partial\Omega^*, 2\tilde{\delta}) \cap N(\mathcal{X}, 2\tilde{\delta})$. Using the mean-value theorem, we obtain that, for any $(u,v,\delta) \in \partial\Omega^* \cap N(\mathcal{X},\tilde{\delta}) \times (-1,1) \times (0,\tilde{\delta})$,

$$\begin{aligned}
g(u + \delta v \nu_{\Omega^*}(u)) &= g(u) + \nabla g(y_g(u, \delta v \nu_{\Omega^*}(u)))' \delta v \nu_{\Omega^*}(u), \\
f_X(u + \delta v \nu_{\Omega^*}(u)) &= f_X(u) + \nabla f_X(y_f(u, \delta v \nu_{\Omega^*}(u)))' \delta v \nu_{\Omega^*}(u)
\end{aligned}$$

for some $y_g(u, \delta v \nu_{\Omega^*}(u))$ and $y_f(u, \delta v \nu_{\Omega^*}(u))$ that are on the line segment connecting u and $u + \delta v \nu_{\Omega^*}(u)$. In addition,

$$\begin{aligned}
J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta v) &= J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,0) + \frac{\partial J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u, y_J(u,\delta v))}{\partial\lambda}\delta v \\
&= 1 + \frac{\partial J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u, y_J(u,\delta v))}{\partial\lambda}\delta v
\end{aligned}$$

for some $y_J(u,\delta v)$ that is on the line segment connecting 0 and δv. By Lemma B.5, $\frac{\partial J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(\cdot,\cdot)}{\partial\lambda}$ is bounded on $\partial\Omega^*\times(-\tilde{\delta},\tilde{\delta})$. We then have

$$\begin{aligned}
&\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i)1\{p^{ML}(X_i;\delta)\in(0,1)\}]\\
=&\int_{-1}^{1}\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})}(k(v)+\delta r(u,v,\delta))^l(g(u)+\nabla g(y_g(u,\delta v\nu_{\Omega^*}(u)))'\delta v\nu_{\Omega^*}(u))\\
&\qquad\times(f_X(u)+\nabla f_X(y_f(u,\delta v\nu_{\Omega^*}(u)))'\delta v\nu_{\Omega^*}(u))(1+\tfrac{\partial J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,y_J(u,\delta v))}{\partial\lambda}\delta v)d\mathcal{H}^{p-1}(u)dv\\
=&\int_{-1}^{1}\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})}(k(v)^l g(u)f_X(u)+\delta h(u,v,\delta))d\mathcal{H}^{p-1}(u)dv\\
=&\int_{-1}^{1}k(v)^l dv\int_{\partial\Omega^*}g(u)f_X(u)d\mathcal{H}^{p-1}(u)+\delta\int_{-1}^{1}\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})}h(u,v,\delta)d\mathcal{H}^{p-1}(u)dv
\end{aligned}$$

for some function h bounded on $\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})\times(-1,1)\times(0,\tilde{\delta})$. It then follows that

$$\delta^{-1}E[p^{ML}(X_i;\delta)^l g(X_i)1\{p^{ML}(X_i;\delta)\in(0,1)\}]=\int_{-1}^{1}k(v)^l dv\int_{\partial\Omega^*}g(u)f_X(u)d\mathcal{H}^{p-1}(u)+O(\delta).$$

Also,

$$\begin{aligned}
&\delta^{-1}E[Z_i p^{ML}(X_i;\delta)^l g(X_i)1\{p^{ML}(X_i;\delta)\in(0,1)\}]\\
=&\int_{0}^{1}\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})}(k(v)+\delta r(u,v,\delta))^l g(u+\delta v\nu_{\Omega^*}(u))f_X(u+\delta v\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta v)d\mathcal{H}^{p-1}(u)dv\\
=&\int_{0}^{1}k(v)^l dv\int_{\partial\Omega^*}g(u)f_X(u)d\mathcal{H}^{p-1}(u)+O(\delta).
\end{aligned}$$

□

Step C.6.3.4. *Let* $S_{\mathbf{D}}=\lim_{\delta\to0}\delta^{-1}E[\mathbf{Z}_i\mathbf{D}_i'1\{p^{ML}(X_i;\delta)\in(0,1)\}]$ *and* $S_Y=\lim_{\delta\to0}\delta^{-1}E[\mathbf{Z}_iY_i1\{p^{ML}(X_i;\delta)\in(0,1)\}]$. *Then the second element of* $S_{\mathbf{D}}^{-1}S_Y$ *is* β_1.

Proof. Note that $D_i=Z_iD_i(1)+(1-Z_i)D_i(0)$ and $Y_i=Z_iY_{1i}+(1-Z_i)Y_{0i}$. By Step C.6.3.3,

$$\begin{aligned}
&S_{\mathbf{D}}\\
=&\begin{bmatrix}
2\bar{f}_X & \int_{\partial\Omega^*}E[D_i(1)+D_i(0)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x) & \int_{-1}^{1}k(v)dv\bar{f}_X\\
\bar{f}_X & \int_{\partial\Omega^*}E[D_i(1)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x) & \int_{0}^{1}k(v)dv\bar{f}_X\\
\int_{-1}^{1}k(v)dv\bar{f}_X & \int_{\partial\Omega^*}(\int_{0}^{1}k(v)dvE[D_i(1)|X_i=x]+\int_{-1}^{0}k(v)dvE[D_i(0)|X_i=x])f_X(x)d\mathcal{H}^{p-1}(x) & \int_{-1}^{1}k(v)^2dv\bar{f}_X
\end{bmatrix},
\end{aligned}$$

where $\bar{f}_X=\int_{\partial\Omega^*}f_X(x)d\mathcal{H}^{p-1}(x)$, and

$$S_Y=\begin{bmatrix}
\int_{\partial\Omega^*}E[Y_{1i}+Y_{0i}|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)\\
\int_{\partial\Omega^*}E[Y_{1i}|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)\\
\int_{\partial\Omega^*}(\int_{0}^{1}k(v)dvE[Y_{1i}|X_i=x]+\int_{-1}^{0}k(v)dvE[Y_{0i}|X_i=x])f_X(x)d\mathcal{H}^{p-1}(x)
\end{bmatrix}.$$

After a few lines of algebra, we have

$$\begin{aligned}
\det(S_{\mathbf{D}})=&\bar{f}_X^{-2}\int_{\partial\Omega^*}E[D_i(1)-D_i(0)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)\\
&\times(\int_{-1}^{0}(k(v)-\int_{-1}^{0}k(s)ds)^2dv+\int_{0}^{1}(k(v)-\int_{0}^{1}k(s)ds)^2dv),
\end{aligned}$$

which is nonzero under Assumption 3 (b) and (f) (i). After another few lines of algebra, we obtain that the second element of $S_{\mathbf{D}}^{-1}S_Y$ is

$$\frac{\int_{\partial\Omega^*} E[(D_i(1)-D_i(0))(Y_i(1)-Y_i(0))|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)}{\int_{\partial\Omega^*} E[D_i(1)-D_i(0)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)}.$$

On the other hand, by Step C.6.3.3,

$$\begin{aligned}
\beta_1 &= \lim_{\delta\to 0} E[\omega_i(\delta)(Y_i(1)-Y_i(0))] \\
&= \lim_{\delta\to 0} \frac{\delta^{-1}E[p^{ML}(X_i;\delta)(1-p^{ML}(X_i;\delta))(D_i(1)-D_i(0))(Y_i(1)-Y_i(0))1\{p^{ML}(X_i;\delta)\in(0,1)\}]}{\delta^{-1}E[p^{ML}(X_i;\delta)(1-p^{ML}(X_i;\delta))(D_i(1)-D_i(0))1\{p^{ML}(X_i;\delta)\in(0,1)\}]} \\
&= \frac{\int_{-1}^{1}k(v)(1-k(v))dv\int_{\partial\Omega^*}E[(D_i(1)-D_i(0))(Y_i(1)-Y_i(0))|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)}{\int_{-1}^{1}k(v)(1-k(v))dv\int_{\partial\Omega^*}E[D_i(1)-D_i(0)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)} \\
&= \frac{\int_{\partial\Omega^*}E[(D_i(1)-D_i(0))(Y_i(1)-Y_i(0))|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)}{\int_{\partial\Omega^*}E[D_i(1)-D_i(0)|X_i=x]f_X(x)d\mathcal{H}^{p-1}(x)}.
\end{aligned}$$

□

Step C.6.3.5. *If $n\delta_n\to\infty$ as $n\to\infty$, then $\hat{\beta}_1\xrightarrow{p}\beta_1$.*

Proof. It suffices to verify that the variance of each element of $\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i$ and $\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_iYI_i$ is $o(1)$. Here, we only verify that $\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^n p^{ML}(X_i;\delta_n)Y_iI_i)=o(1)$. Note that

$$E[Y_i^2|X_i]=E[Z_iY_{1i}^2+(1-Z_i)Y_{0i}^2|X_i]\le E[Y_{1i}^2+Y_{0i}^2|X_i].$$

Under Assumption 3 (g), there exists $\delta'>0$ such that $E[Y_{1i}^2+Y_{0i}^2|X_i]$ is continuous on $N(\partial\Omega^*,\delta')$. Since $\mathrm{cl}(N(\partial\Omega^*,\frac{1}{2}\delta'))$ is closed and bounded, $E[Y_{1i}^2+Y_{0i}^2|X_i]$ is bounded on $\mathrm{cl}(N(\partial\Omega^*,\frac{1}{2}\delta'))$. We have

$$\begin{aligned}
\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^n p^{ML}(X_i;\delta_n)Y_iI_i) &\le \frac{1}{n\delta_n}\delta_n^{-1}E[p^{ML}(X_i;\delta_n)^2Y_i^2I_i] \\
&= \frac{1}{n\delta_n}\delta_n^{-1}E[p^{ML}(X_i;\delta_n)^2E[Y_i^2|X_i]I_i] \\
&\le \frac{1}{n\delta_n}C
\end{aligned}$$

for some $C>0$, where the last inequality follows from Step C.6.3.3. The conclusion follows since $n\delta_n\to\infty$. □

Now let $\beta=(\beta_0,\beta_1,\beta_2)'=S_{\mathbf{D}}^{-1}S_Y$ and let $\epsilon_i=Y_i-\mathbf{D}_i'\beta$. We can write

$$\begin{aligned}
\sqrt{n\delta_n}(\hat{\beta}-\beta) &= (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)^{-1}\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^n \mathbf{Z}_i\epsilon_iI_i \\
&= (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)^{-1}\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^n \{(\mathbf{Z}_i\epsilon_iI_i-E[\mathbf{Z}_i\epsilon_iI_i])+E[\mathbf{Z}_i\epsilon_iI_i]\}.
\end{aligned}$$

Step C.6.3.6.

$$\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^{n}(\mathbf{Z}_i\epsilon_i I_i - E[\mathbf{Z}_i\epsilon_i I_i]) \xrightarrow{d} \mathcal{N}(0,\mathbf{V}),$$

where $\mathbf{V} = \lim_{n\to\infty}\delta_n^{-1}E[\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i I_i]$.

Proof. We use the triangular-array Lyapunov CLT and the Cramér-Wold device. Pick a nonzero $\lambda \in \mathbb{R}^p$, and let $V_{i,n} = \frac{1}{\sqrt{n\delta_n}}\lambda'(\mathbf{Z}_i\epsilon_i I_i - E[\mathbf{Z}_i\epsilon_i I_i])$. First,

$$\sum_{i=1}^{n}E[V_{i,n}^2] = \delta_n^{-1}\lambda'(E[\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i] - E[\mathbf{Z}_i\epsilon_i I_i]E[\mathbf{Z}_i'\epsilon_i I_i])\lambda.$$

By Step C.6.3.3,

$$E[\mathbf{Z}_i\epsilon_i I_i] = E[\mathbf{Z}_i(Y_i - \mathbf{D}_i'\beta)I_i] = O(\delta_n),$$

so

$$\delta_n^{-1}E[\mathbf{Z}_i\epsilon_i I_i]E[\mathbf{Z}_i'\epsilon_i I_i] = o(1).$$

We have

$$\begin{aligned}
E[\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i] &= E[(Y_i - \beta_0 - \beta_1 D_i - \beta_2 p^{ML}(X_i;\delta_n))^2\mathbf{Z}_i\mathbf{Z}_i' I_i] \\
&= E[Z_i(Y_{1i} - \beta_0 - \beta_1 D_i(1) - \beta_2 p^{ML}(X_i;\delta_n))^2\mathbf{Z}_i\mathbf{Z}_i' I_i] \\
&\quad + E[(1-Z_i)(Y_{0i} - \beta_0 - \beta_1 D_i(0) - \beta_2 p^{ML}(X_i;\delta_n))^2\mathbf{Z}_i\mathbf{Z}_i' I_i].
\end{aligned}$$

Since $E[Y_{1i}|X_i]$, $E[Y_{0i}|X_i]$, $E[D_i(1)|X_i]$, $E[D_i(0)|X_i]$, $E[Y_{1i}^2|X_i]$, $E[Y_{0i}^2|X_i]$, $E[Y_{1i}D_i(1)|X_i]$ and $E[Y_{0i}D_i(0)|X_i]$ are continuous on $N(\partial\Omega^*,\delta')$ for some $\delta' > 0$ under Assumption 3 (g), $\lim_{n\to\infty}\delta_n^{-1}E[\epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i]$ exists and finite. Therefore,

$$\sum_{i=1}^{n}E[V_{i,n}^2] \to \lambda'\mathbf{V}\lambda < 0.$$

We next verify the Lyapunov condition: for some $t > 0$,

$$\sum_{i=1}^{n}E[|V_{i,n}|^{2+t}] \to 0.$$

We have

$$\begin{aligned}
\sum_{i=1}^{n}E[|V_{i,n}|^4] &= \frac{1}{n\delta_n}\delta_n^{-1}E[|\lambda'(\mathbf{Z}_i\epsilon_i I_i - E[\mathbf{Z}_i\epsilon_i I_i])|^4] \\
&\le \frac{1}{n\delta_n}2^{3c}\delta_n^{-1}\{E[|\lambda'\mathbf{Z}_i\epsilon_i I_i|^4] + |\lambda'E[\mathbf{Z}_i\epsilon_i I_i]|^4\}
\end{aligned}$$

by the c_r-inequality. Repeating using the c_r-inequality gives

$$\begin{aligned}
\delta_n^{-1}E[|\lambda'\mathbf{Z}_i\epsilon_i I_i|^4] &= \delta_n^{-1}E[|\lambda'\mathbf{Z}_i(Y_i - \beta_0 - \beta_1 D_i - \beta_2 p^{ML}(X_i;\delta_n))|^4 I_i] \\
&\le 2^{3c}\delta_n^{-1}E[(|\lambda'\mathbf{Z}_i|^4)(|Y_i|^4 + |\beta_0|^4 + |\beta_1|^4 D_i + |\beta_2|^4 p^{ML}(X_i;\delta_n)^4)I_i] \\
&\le 2^{3c}(\lambda_1+\lambda_2+\lambda_3)^4\delta_n^{-1}E[(Y_i^4 + \beta_0^4 + \beta_1^4 + \beta_2^4)I_i] \\
&= 2^{3c}O(1)
\end{aligned}$$

for some finite constant c, where the last equality holds by Step C.6.3.3 under Assumption 3 (g). Moreover,

$$\begin{aligned}\delta_n^{-1}|\lambda' E[\mathbf{Z}_i\epsilon_i I_i]|^4 &= \delta_n^3|\lambda'\delta_n^{-1}E[\mathbf{Z}_i\epsilon_i I_i]|^4 \\ &= \delta_n^3 O(1) \\ &= o(1).\end{aligned}$$

Therefore, when $n\delta_n \to \infty$,

$$\sum_{i=1}^n E[|V_{i,n}|^4] \to 0,$$

and the conclusion follows from the Lyapunov CLT and the Cramér-Wold device. □

Step C.6.3.7. $n\delta_n\hat{\mathbf{\Sigma}} \xrightarrow{p} S_{\mathbf{D}}^{-1}\mathbf{V}(S_{\mathbf{D}}')^{-1}$.

Proof. We have

$$\begin{aligned}\frac{1}{n\delta_n}\sum_{i=1}^n \hat{\epsilon}_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i =& \frac{1}{n\delta_n}\sum_{i=1}^n (Y_i - \mathbf{D}_i'\hat{\beta})^2\mathbf{Z}_i\mathbf{Z}_i' I_i \\ =& \frac{1}{n\delta_n}\sum_{i=1}^n (\epsilon_i - \mathbf{D}_i'(\hat{\beta}-\beta))^2\mathbf{Z}_i\mathbf{Z}_i' I_i \\ =& \frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i \\ & - \frac{2}{n\delta_n}\sum_{i=1}^n (Y_i - \mathbf{D}_i'\beta)((\hat{\beta}_0-\beta_0) + D_i(\hat{\beta}_1-\beta_1) + p^{ML}(X_i;\delta_n)(\hat{\beta}_2-\beta_2))\mathbf{Z}_i\mathbf{Z}_i' I_i \\ & + \frac{1}{n\delta_n}\sum_{i=1}^n ((\hat{\beta}_0-\beta_0) + D_i(\hat{\beta}_1-\beta_1) + p^{ML}(X_i;\delta_n)(\hat{\beta}_2-\beta_2))^2\mathbf{Z}_i\mathbf{Z}_i' I_i \\ =& \frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i + o_p(1)O_p(1),\end{aligned}$$

where the last equality follows from the result that $\hat{\beta} - \beta = o_p(1)$ and from application of Step C.6.3.3 as in Steps C.6.3.5 and C.6.3.6. To show $\frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i \xrightarrow{p} \mathbf{V}$, it suffices to verify that the variance of each element of $\frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i$ is $o(1)$. We only verify that $\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2 p^{ML}(X_i;\delta_n)^2 I_i) = o(1)$. Using the c_r-inequality, we have that for some constant

c,

$$\begin{aligned}
\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^{n}\epsilon_i^2 p^{ML}(X_i;\delta_n)^2 I_i) &\leq \frac{1}{n\delta_n}\delta_n^{-1}E[\epsilon_i^4 I_i] \\
&= \frac{1}{n\delta_n}\delta_n^{-1}E[(Y_i-\beta_0-\beta_1 D_i-\beta_2 p^{ML}(X_i))^4 I_i] \\
&\leq \frac{1}{n\delta_n}2^{3c}\delta_n^{-1}E[(Y_i^4+\beta_0^4+\beta_1^4 D_i+\beta_2^4 p^{ML}(X_i)^4)I_i] \\
&\leq \frac{1}{n\delta_n}2^{3c}\delta_n^{-1}E[(Y_i^4+\beta_0^4+\beta_1^4+\beta_2^4)I_i] \\
&= \frac{1}{n\delta_n}2^{3c}O(1) \\
&= o(1),
\end{aligned}$$

where the second last equality holds by Step C.6.3.3 under Assumption 3 (g). Therefore,

$$\frac{1}{n\delta_n}\sum_{i=1}^{n}\hat{\epsilon}_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i \xrightarrow{p} \mathbf{V}.$$

It follows that

$$n\delta_n\hat{\mathbf{\Sigma}} = (\frac{1}{n\delta_n}\sum_{i=1}^{n}\mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}(\frac{1}{n\delta_n}\sum_{i=1}^{n}\hat{\epsilon}_i^2\mathbf{Z}_i\mathbf{Z}_i' I_i)(\frac{1}{n\delta_n}\sum_{i=1}^{n}\mathbf{D}_i\mathbf{Z}_i' I_i)^{-1} \xrightarrow{p} S_{\mathbf{D}}^{-1}\mathbf{V}(S_{\mathbf{D}}')^{-1}.$$

□

Step C.6.3.8. $\hat{\sigma}^{-1}(\hat{\beta}_1-\beta_1)\xrightarrow{d}\mathcal{N}(0,1)$.

Proof. Let $\beta_n = S_{\mathbf{D}}^{-1}\delta_n^{-1}E[\mathbf{Z}_i Y_i I_i]$. We then have

$$\begin{aligned}
\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^{n}E[\mathbf{Z}_i\epsilon_i I_i] &= \sqrt{n\delta_n}\delta_n^{-1}E[\mathbf{Z}_i(Y_i-\mathbf{D}'\beta)I_i] \\
&= \sqrt{n\delta_n}\delta_n^{-1}E[\mathbf{Z}_i(Y_i-\mathbf{D}_i'\beta_n+\mathbf{D}_i'(\beta_n-\beta))I_i] \\
&= \sqrt{n\delta_n}\delta_n^{-1}\{E[\mathbf{Z}_i Y_i I_i]-E[\mathbf{Z}_i\mathbf{D}_i' I_i]\beta_n+E[\mathbf{Z}_i\mathbf{D}_i' I_i](\beta_n-\beta)\} \\
&= \sqrt{n\delta_n}\{(S_{\mathbf{D}}-\delta_n^{-1}E[\mathbf{Z}_i\mathbf{D}_i' I_i])S_{\mathbf{D}}^{-1}\delta_n^{-1}E[\mathbf{Z}_i Y_i I_i] \\
&\qquad + \delta_n^{-1}E[\mathbf{Z}_i\mathbf{D}_i' I_i]S_{\mathbf{D}}^{-1}(\delta_n^{-1}E[\mathbf{Z}_i Y_i I_i]-S_Y)\} \\
&= \sqrt{n\delta_n}(O(\delta_n)O(1)+O(1)O(\delta_n)) \\
&= O(\sqrt{n\delta_n}\delta_n),
\end{aligned}$$

where we use Step C.6.3.3 for the second last equality. Thus, when $n\delta_n^3\to 0$,

$$\begin{aligned}
\sqrt{n\delta_n}(\hat{\beta}-\beta) &= (\frac{1}{n\delta_n}\sum_{i=1}^{n}\mathbf{Z}_i\mathbf{D}_i' I_i)^{-1}\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^{n}\{(\mathbf{Z}_i\epsilon_i I_i-E[\mathbf{Z}_i\epsilon_i I_i])+E[\mathbf{Z}_i\epsilon_i I_i]\} \\
&\xrightarrow{d}\mathcal{N}(0,S_{\mathbf{D}}^{-1}\mathbf{V}(S_{\mathbf{D}}')^{-1}).
\end{aligned}$$

The conclusion then follows from Step C.6.3.7. □

□

C.6.4 Consistency and Asymptotic Normality of $\hat{\beta}_1^s$ When $\Pr(ML(X_i) \in (0,1)) = 0$

Let $I_i^s = 1\{p^s(X_i;\delta_n) \in (0,1)\}$, $\mathbf{D}_i^s = (1, D_i, p^s(X_i;\delta_n))'$ and $\mathbf{Z}_i^s = (1, Z_i, p^s(X_i;\delta_n))'$. $\hat{\beta}^s$ and $\hat{\mathbf{\Sigma}}^s$ are given by

$$\hat{\beta}^s = (\sum_{i=1}^n \mathbf{Z}_i^s (\mathbf{D}_i^s)' I_i^s)^{-1} \sum_{i=1}^n \mathbf{Z}_i^s Y_i I_i^s.$$

and

$$\hat{\mathbf{\Sigma}}^s = (\sum_{i=1}^n \mathbf{Z}_i^s (\mathbf{D}_i^s)' I_i^s)^{-1} (\sum_{i=1}^n (\hat{\epsilon}_i^s)^2 \mathbf{Z}_i^s (\mathbf{Z}_i^s)' I_i^s) (\sum_{i=1}^n \mathbf{D}_i^s (\mathbf{Z}_i^s)' I_i^s)^{-1},$$

where $\hat{\epsilon}_i^s = Y_i - (\mathbf{D}_i^s)'\hat{\beta}^s$. It is sufficient to show that

$$\hat{\beta}^s - \hat{\beta} = o_p(1),$$

if $S_n \to \infty$ and that

$$\sqrt{n\delta_n}(\hat{\beta}^s - \hat{\beta}) = o_p(1),$$
$$n\delta_n \hat{\mathbf{\Sigma}}^s \xrightarrow{p} S_{\mathbf{D}}^{-1} \mathbf{V} (S_{\mathbf{D}}')^{-1}$$

if Assumption 5 holds.

Step C.6.4.1. *Let $\{V_i\}_{i=1}^\infty$ be i.i.d. random variables. If $E[V_i|X_i]$ and $E[V_i^2|X_i]$ are bounded on $N(\partial\Omega^*, \delta') \cap N(\mathcal{X}, \delta')$ for some $\delta' > 0$, and $S_n \to \infty$, then*

$$\frac{1}{n\delta_n} \sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I_i^s - \frac{1}{n\delta_n} \sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_i = o_p(1)$$

for $l = 0, 1, 2, 3, 4$. If, in addition, Assumption 5 holds, then

$$\frac{1}{\sqrt{n\delta_n}} \sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I_i^s - \frac{1}{\sqrt{n\delta_n}} \sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_i = o_p(1)$$

for $l = 0, 1, 2$.

Proof. We have

$$\begin{aligned}
&\frac{1}{n\delta_n} \sum_{i=1}^n V_i p^s(X_i;\delta_n)^l I_i^s - \frac{1}{n\delta_n} \sum_{i=1}^n V_i p^{ML}(X_i;\delta_n)^l I_i \\
=&\frac{1}{n\delta_n} \sum_{i=1}^n V_i p^s(X_i;\delta_n)^l (I_i^s - I_i) + \frac{1}{n\delta_n} \sum_{i=1}^n V_i (p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l) I_i.
\end{aligned}$$

We first consider $\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i$. By using the argument in the proof of Step C.6.3.3 in Section C.6.3, we have

$$
\begin{aligned}
&|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i]| \\
&= \delta_n^{-1}|E[E[V_i|X_i]E[p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l|X_i]I_i]| \\
&\le \delta_n^{-1}E[|E[V_i|X_i]||E[p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l|X_i]|I_i] \\
&= \int_{-1}^1\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})} |E[V_i|X_i = u+\delta_n v\nu_{\Omega^*}(u)]||E[p^s(u+\delta_n v\nu_{\Omega^*}(u);\delta_n)^l - p^{ML}(u+\delta_n v\nu_{\Omega^*}(u);\delta_n)^l]| \\
&\qquad\qquad \times f_X(u+\delta_n v\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta_n v)d\mathcal{H}^{p-1}(u)dv,
\end{aligned}
$$

where the choice of $\tilde{\delta}$ is as in the proof of Step C.6.3.3. By Lemma B.7, for $l = 0, 1, 2$,

$$
\begin{aligned}
&|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i]| \\
&\le \frac{1}{S_n}\int_{-1}^1\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})} |E[V_i|X_i = u+\delta_n v\nu_{\Omega^*}(u)]|f_X(u+\delta_n v\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta_n v)d\mathcal{H}^{p-1}(u)dv \\
&= O(S_n^{-1}).
\end{aligned}
$$

Also, by Lemma B.7,

$$
\begin{aligned}
&|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^3 - p^{ML}(X_i;\delta_n)^3)I_i]| \\
&= |\delta_n^{-1}E[V_i(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n)^2 + p^s(X_i;\delta_n)p^{ML}(X_i;\delta_n) + p^{ML}(X_i;\delta_n)^2)I_i]| \\
&\le \delta_n^{-1}E[|E[V_i|X_i]||E[(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n)^2 + p^s(X_i;\delta_n)p^{ML}(X_i;\delta_n) + p^{ML}(X_i;\delta_n)^2)|X_i]|I_i] \\
&\le 3\delta_n^{-1}E[|E[V_i|X_i]|E[|p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n)||X_i]I_i] \\
&= \int_{-1}^1\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})} |E[V_i|X_i = u+\delta_n v\nu_{\Omega^*}(u)]|E[|p^s(u+\delta_n v\nu_{\Omega^*}(u);\delta_n) - p^{ML}(u+\delta_n v\nu_{\Omega^*}(u);\delta_n)|] \\
&\qquad\qquad \times f_X(u+\delta_n v\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u,\delta_n v)d\mathcal{H}^{p-1}(u)dv \\
&\le (\frac{1}{S_n\epsilon^2} + \epsilon)O(1)
\end{aligned}
$$

for every $\epsilon > 0$. We can make the right-hand side arbitrarily close to zero by taking sufficiently small $\epsilon > 0$ and sufficiently large S_n, which implies that $|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^3 - p^{ML}(X_i;\delta_n)^3)I_i]| = o(1)$ if $S_n \to \infty$. Likewise,

$$
\begin{aligned}
&|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^4 - p^{ML}(X_i;\delta_n)^4)I_i]| \\
&= |\delta_n^{-1}E[V_i(p^s(X_i;\delta_n)^2 + p^{ML}(X_i;\delta_n)^2)(p^s(X_i;\delta_n) + p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))I_i]| \\
&\le \delta_n^{-1}E[|E[V_i|X_i]||E[(p^s(X_i;\delta_n)^2 + p^{ML}(X_i;\delta_n)^2)(p^s(X_i;\delta_n) + p^{ML}(X_i;\delta_n))(p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n))|X_i]|I_i] \\
&\le 8\delta_n^{-1}E[|E[V_i|X_i]|E[|p^s(X_i;\delta_n) - p^{ML}(X_i;\delta_n)||X_i]I_i] \\
&= o(1).
\end{aligned}
$$

As for variance, for $l = 0, 1, 2$,

$$\begin{aligned}\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i) &\le \frac{1}{n\delta_n}\delta_n^{-1}E[V_i^2(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2 I_i] \\ &\le \frac{1}{n\delta_n}\delta_n^{-1}E[E[V_i^2|X_i]E[(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2|X_i]I_i] \\ &\le \frac{4}{n\delta_n S_n}\delta_n^{-1}E[E[V_i^2|X_i]I_i] \\ &= O((n\delta_n S_n)^{-1}),\end{aligned}$$

and for $l = 3, 4$,

$$\begin{aligned}\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i) &\le \frac{1}{n\delta_n}\delta_n^{-1}E[V_i^2(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)^2 I_i] \\ &\le \frac{1}{n\delta_n}\delta_n^{-1}E[V_i^2 I_i] \\ &= o(1).\end{aligned}$$

Therefore, $\frac{1}{n\delta_n}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i = o_p(1)$ if $S_n \to \infty$ for $l = 0, 1, 2, 3, 4$, and $\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^n V_i(p^s(X_i;\delta_n)^l - p^{ML}(X_i;\delta_n)^l)I_i = o_p(1)$ if $n^{-1/2}S_n \to \infty$ for $l = 0, 1, 2$.

We next show that $\frac{1}{n\delta_n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l(I_i^s - I_i) = o_p(1)$ if $S_n \to \infty$ for $l \ge 0$. We have

$$\begin{aligned}|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)]| &= \delta_n^{-1}|E[V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)]| \\ &\le \delta_n^{-1}E[|E[V_i|X_i]||E[p^s(X_i;\delta_n)^l(I_i^s - I_i)|X_i]|] \\ &= \delta_n^{-1}E[|E[V_i|X_i]|E[|I_i^s - I_i||X_i]].\end{aligned}$$

Since $I_i^s - I_i \le 0$ with strict inequality only if $I_i = 1$,

$$E[|I_i^s - I_i||X_i] = -E[I_i^s - I_i|X_i]I_i = (1 - E[I_i^s|X_i])I_i = \Pr(p^s(X_i;\delta_n) \in \{0,1\}|X_i)I_i.$$

We then have

$$\begin{aligned}&|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)]| \\ &\le \delta_n^{-1}E[|E[V_i|X_i]|\Pr(p^s(X_i;\delta_n) \in \{0,1\}|X_i)I_i] \\ &\le \delta_n^{-1}E[|E[V_i|X_i]|((1 - p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n})I_i] \\ &\le \int_{-1}^1\int_{\partial\Omega^*\cap N(\mathcal{X},\tilde{\delta})} |E[V_i|X_i = u + \delta_n v\nu_{\Omega^*}(u)]|\{(1 - p^{ML}(u + \delta_n v\nu_{\Omega^*}(u);\delta_n))^{S_n} \\ &\qquad + p^{ML}(u + \delta_n v\nu_{\Omega^*}(u);\delta_n)^{S_n}\}f_X(u + \delta_n v\nu_{\Omega^*}(u))J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}(u, \delta_n v)d\mathcal{H}^{p-1}(u)dv,\end{aligned}$$

where the second inequality follows from Lemma B.7. Note that for every $(u, v) \in \partial\Omega^* \cap N(\mathcal{X}, \tilde{\delta}) \times (-1, 1)$, $\lim_{\delta\to 0} p^{ML}(u + \delta_n v\nu_{\Omega^*}(u);\delta_n) = k(v) \in (0, 1)$ by Step C.6.3.1 in Section C.6.3. Since $E[V_i|X_i]$, f_X and $J_{p-1}^{\partial\Omega^*}\psi_{\Omega^*}$ are bounded, by the Bounded Convergence Theorem,

$$|E[\frac{1}{n\delta_n}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)]| = o(1)$$

if $S_n \to \infty$.

As for variance,

$$\begin{aligned}\mathrm{Var}(\frac{1}{n\delta_n}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)) &\le \frac{1}{n\delta_n}\delta_n^{-1}E[V_i^2 p^s(X_i;\delta_n)^{2l}(I_i^s - I_i)^2] \\ &\le \frac{1}{n\delta_n}\delta_n^{-1}E[V_i^2|I_i^s - I_i|] \\ &= \frac{1}{n\delta_n}\delta_n^{-1}E[E[V_i^2|X_i]E[|I_i^s - I_i||X_i]] \\ &= o(1).\end{aligned}$$

Lastly, we show that, for $l \ge 0$, $\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I_i^s - I_i) = o_p(1)$ if Assumption 5 holds. Let $\eta_n = \gamma\frac{\log n}{S_n}$, where γ is the one satisfying Assumption 5. We have

$$\begin{aligned}&|E[\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^{n} V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)]| \\ &\le \sqrt{n\delta_n^{-1}}E[|E[V_i|X_i]|((1-p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n})I_i] \\ &= \sqrt{n\delta_n^{-1}}E[|E[V_i|X_i]|((1-p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n}))1\{p^{ML}(X_i;\delta_n) \in (0,\eta_n)\cup(1-\eta_n,1)\}] \\ &\quad + \sqrt{n\delta_n^{-1}}E[|E[V_i|X_i]|((1-p^{ML}(X_i;\delta_n))^{S_n} + p^{ML}(X_i;\delta_n)^{S_n}))1\{p^{ML}(X_i;\delta_n) \in (\eta_n,1-\eta_n)\}] \\ &\le (\sup_{x\in N(\partial\Omega^*,2\tilde{\delta})\cap N(\mathcal{X},2\tilde{\delta})}|E[V_i|X_i = x]|)(\sqrt{n\delta_n^{-1}}\Pr(p^{ML}(X_i;\delta_n) \in (0,\eta_n)\cup(1-\eta_n,1)) \\ &\qquad + 2\sqrt{n\delta_n}(1-\eta_n)^{S_n}\delta_n^{-1}E[1\{p^{ML}(X_i;\delta_n) \in (\eta_n,1-\eta_n)\}]).\end{aligned}$$

By Assumption 5, $\sqrt{n\delta_n^{-1}}\Pr(p^{ML}(X_i;\delta_n) \in (0,\eta_n)\cup(1-\eta_n,1)) = o(1)$. For the second term,

$$\begin{aligned}2\sqrt{n\delta_n}(1-\eta_n)^{S_n}\delta_n^{-1}E[1\{p^{ML}(X_i;\delta_n) \in (\eta_n,1-\eta_n)\}] &\le 2\sqrt{n\delta_n}(1-\eta_n)^{S_n}\delta_n^{-1}E[I_i] \\ &= 2\sqrt{n\delta_n}(1-\eta_n)^{S_n}O(1).\end{aligned}$$

Observe that $\eta_n = \gamma\frac{\log n}{S_n} = \gamma\frac{\log n}{n^{1/2}}\frac{1}{n^{-1/2}S_n} \to 0$, since $n^{-1/2}S_n \to \infty$ and $\frac{\log n}{n^{1/2}} \to 0$. Using the fact that $e^t \ge 1+t$ for every $t \in \mathbb{R}$, we have

$$\begin{aligned}\sqrt{n\delta_n}(1-\eta_n)^{S_n} &\le \sqrt{n\delta_n}(e^{-\eta_n})^{S_n} \\ &= \sqrt{n\delta_n}e^{-\eta_n S_n} \\ &= \sqrt{n\delta_n}e^{-\gamma\log n} \\ &= \sqrt{n\delta_n}n^{-\gamma} \\ &= n^{1/2-\gamma}\delta_n^{1/2} \\ &\to 0,\end{aligned}$$

since $\gamma > 1/2$. As for variance,

$$\begin{aligned}\mathrm{Var}(\frac{1}{\sqrt{n\delta_n}}\sum_{i=1}^n V_i p^s(X_i;\delta_n)^l(I_i^s - I_i)) &\leq \delta_n^{-1}E[V_i^2 p^s(X_i;\delta_n)^{2l}(I_i^s - I_i)^2] \\ &\leq \delta_n^{-1}E[E[V_i^2|X_i]E[|I_i^s - I_i||X_i]I_i] \\ &= o(1).\end{aligned}$$

□

We have

$$\begin{aligned}&\hat{\beta}^s - \hat{\beta} \\ &= (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s)^{-1}\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s Y_i I_i^s - (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)^{-1}\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i \\ &= (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s)^{-1}(\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s Y_i I_i^s - \frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i) \\ &\quad - (\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s)^{-1}(\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s - \frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)(\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i\mathbf{D}_i'I_i)^{-1}\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i Y_i I_i.\end{aligned}$$

By Step C.6.4.1, $\hat{\beta}^s - \hat{\beta} = o_p(1)$ if $S_n \to \infty$, and $\sqrt{n\delta_n}(\hat{\beta}^s - \hat{\beta}) = o_p(1)$ if Assumption 5 holds.

By proceeding as in Step C.6.3.7 in Section C.6.3, we have

$$\frac{1}{n\delta_n}\sum_{i=1}^n(\hat{\epsilon}_i^s)^2\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s = \frac{1}{n\delta_n}\sum_{i=1}^n(\epsilon_i^s)^2\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s + o_p(1),$$

where $\epsilon_i^s = Y_i - (\mathbf{D}_i^s)'\beta$. Then, by Step C.6.4.1,

$$\begin{aligned}&\frac{1}{n\delta_n}\sum_{i=1}^n(\hat{\epsilon}_i^s)^2\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s - \frac{1}{n\delta_n}\sum_{i=1}^n \epsilon_i^2\mathbf{Z}_i\mathbf{Z}_i'I_i \\ &= \frac{1}{n\delta_n}\sum_{i=1}^n(Y_i^2 - 2Y_i(\mathbf{D}_i^s)'\beta + \beta'\mathbf{D}_i^s(\mathbf{D}_i^s)'\beta)\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s - \frac{1}{n\delta_n}\sum_{i=1}^n(Y_i^2 - 2Y_i\mathbf{D}_i'\beta + \beta'\mathbf{D}_i\mathbf{D}_i'\beta)\mathbf{Z}_i\mathbf{Z}_i'I_i + o_p(1) \\ &= o_p(1)\end{aligned}$$

so that

$$\frac{1}{n\delta_n}\sum_{i=1}^n(\hat{\epsilon}_i^s)^2\mathbf{Z}_i^s(\mathbf{Z}_i^s)'I_i^s \xrightarrow{p} \mathbf{V}.$$

Also, $\frac{1}{n\delta_n}\sum_{i=1}^n \mathbf{Z}_i^s(\mathbf{D}_i^s)'I_i^s \xrightarrow{p} S_{\mathbf{D}}$ by using Step C.6.4.1. The conclusion then follows. □

□

C.7 Proof of Proposition A.2

We can prove Part (a) using the same argument in the proof of Proposition 1 (a). For Part (b), suppose to the contrary that there exists $x_d \in \mathcal{X}_d^A$ such that $\mathcal{L}^{p_c}(\{x_c \in \mathcal{X}_c^A(x_d) : p^{ML}(x_d, x_c) \in \{0,1\}\}) > 0$. Without loss of generality, assume $\mathcal{L}^{p_c}(\{x_c \in \mathcal{X}_c^A(x_d) : p^{ML}(x_d, x_c) = 1\}) > 0$. The proof proceeds in five steps.

Step C.7.1. $\mathcal{L}^{p_c}(\mathcal{X}_c^A(x_d) \cap \mathcal{X}_{c,1}(x_d)) > 0$.

Step C.7.2. $\mathcal{X}_c^A(x_d) \cap \text{int}(\mathcal{X}_{c,1}(x_d)) \neq \emptyset$.

Step C.7.3. $p^{ML}(x_d, x_c) = 1$ *for any* $x_c \in \text{int}(\mathcal{X}_{c,1}(x_d))$.

Step C.7.4. *For every* $x_c^* \in \mathcal{X}_c^A(x_d) \cap \text{int}(\mathcal{X}_{c,1}(x_d))$*, there exists* $\delta > 0$ *such that* $B(x_c^*, \delta) \subset \mathcal{X}_c^A(x_d) \cap \text{int}(\mathcal{X}_{c,1}(x_d))$.

Step C.7.5. $E[Y_{1i} - Y_{0i} | X_i \in A]$ *is not identified.*

Following the argument in the proof of Proposition 1 (b), we can prove Steps C.7.1–C.7.3. Once Step C.7.4 is established, we prove Step C.7.5 by following the proof of Step C.1.4 in Proposition 1 (b) with $B(x_c^*, \delta)$ and $B(x_c^*, \epsilon)$ in place of $B(x^*, \delta)$ and $B(x^*, \epsilon)$, respectively, using the fact that $\Pr(X_{ci} \in B(x_c^*, \epsilon) | X_{di} = x_d) > 0$ by the definition of support. Here, we provide the proof of Step C.7.4.

Proof of Step C.7.4. Pick an $x_c^* \in \mathcal{X}_c^A(x_d) \cap \text{int}(\mathcal{X}_{c,1})$. Then, $x^* = (x_d, x_c^*) \in A$. Since A is open relative to $\mathcal{X}$, there exists an open set $U \in \mathbb{R}^p$ such that $A = U \cap \mathcal{X}$. This implies that for any sufficiently small $\delta > 0$, $B(x^*, \delta) \cap \mathcal{X} \subset U \cap \mathcal{X} = A$. It then follows that $\{x_c \in \mathbb{R}^{p_c} : (x_d, x_c) \in B(x^*, \delta) \cap \mathcal{X}\} \subset \{x_c \in \mathbb{R}^{p_c} : (x_d, x_c) \in A\}$, equivalently, $B(x_c^*, \delta) \cap \mathcal{X}_c(x_d) \subset \mathcal{X}_c^A(x_d)$. By choosing a sufficiently small $\delta > 0$ so that $B(x_c^*, \delta) \subset \text{int}(\mathcal{X}_{c,1}(x_d)) \subset \mathcal{X}_c(x_d)$, we have $B(x_c^*, \delta) \subset \mathcal{X}_c^A(x_d) \cap \text{int}(\mathcal{X}_{c,1}(x_d))$. □

□

C.8 Proof of Theorem A.1

The proof is analogous to the proof of Theorem 1. The only difference is that, when we prove the convergence of expectations, we show the convergence of the expectations conditional on X_{di}, and then take the expectations over X_{di}. □

D Machine Learning Simulation: Details

Parameter Choice. For the variance-covariance matrix $\mathbf{\Sigma}$ of X_i, we first create a 100×100 symmetric matrix $\mathbf{V}$ such that the diagonal elements are one, $\mathbf{V}_{ij}$ is nonzero and equal to $\mathbf{V}_{ji}$ for $(i, j) \in \{2, 3, 4, 5, 6\} \times \{35, 66, 78\}$, and everything else is zero. We draw values from $\text{Unif}(-0.5, 0.5)$ independently for the nonzero off-diagonal elements of $\mathbf{V}$. We then create matrix $\mathbf{\Sigma} = \mathbf{V} \times \mathbf{V}$, which is a positive semidefinite matrix.

For α_0 and α_1, we first draw $\tilde{\alpha}_{0j}$, $j = 51, ..., 100$, from $\text{Unif}(-100, 100)$ independently across j, and draw $\tilde{\alpha}_{1j}$, $j = 1, ..., 100$, from $\text{Unif}(-150, 200)$ independently across j. We then set $\tilde{\alpha}_{0j} = \tilde{\alpha}_{1j}$ for $j = 1, ..., 50$, and calculate α_0 and α_1 by normalizing $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ so that $\text{Var}(X_i' \alpha_0) = \text{Var}(X_i' \alpha_1) = 1$.

Training of Prediction Model. We first randomly split the sample $\{(\tilde{Y}_i, \tilde{X}_i, \tilde{D}_i, \tilde{Z}_i)\}_{i=1}^{\tilde{n}}$ into train (80%) and test datasets (20%). We use random forests on the training sample to obtain

the prediction model μ_z and validate its performance on the test sample. The trained algorithm has an accuracy of 97% on the test data.

E Empirical Policy Application: Details

E.1 Hospital Cost Data

We use publicly available Healthcare Cost Report Information System (HCRIS) data,[40] to project[41] safety net eligibility and funding amounts for all hospitals in the dataset. This data set contains information on various hospital characteristics including utilization, number of employees, medicare cost data and financial statement data.

The data is available from financial year 1996 to 2019. As the coverage is higher for 2018 (compared to 2019), we utilize the data corresponding to the 2018 financial year. Hospitals are uniquely identified in a financial year by their CMS (Center for Medicaid and Medicare Services) Certification Number. We have data for 4,705 providers for the 2018 financial year. We focus on 4,648 acute care and critical access hospitals that are either located in one of the 50 states or Washington DC.

Disproportionate patient percentage

Disproportionate patient percentage is equal to the percentage of Medicare inpatient days attributable to patients eligible for both Medicare Part A and Supplemental Security Income (SSI) summed with the percentage of total inpatient days attributable to patients eligible for Medicaid but not Medicare Part A.[42] In the data, this variable is missing for 1560 hospitals. We impute the disproportionate patient percentage to 0 when it is missing.

Uncompensated care per bed

Cost of uncompensated care refers to the care provided by the hospital for which no compensation was received from the patient or the insurer. It is the sum of a hospital's bad debt and the financial assistance it provides.[43] The cost of uncompensated care is missing for 86 hospitals, which we impute to 0. We divide the cost of uncompensated care by the number of beds in the hospital to obtain the cost per bed. The data on bed count is missing for 15 hospitals, which we drop from the analysis, leaving us with 4,633 hospitals in 2,473 counties.

[40] We use the RAND cleaned version of this dataset, which can be accessed `https://www.hospitaldatasets.org/`

[41] We use the methodology detailed in the CARE ACT website to project funding based on 2018 financial year cost reports.

[42] For the precise definition, see `https://www.cms.gov/Medicare/Medicare-Fee-for-Service-Payment/AcuteInpatientPPS/dsh`.

[43] The precise definition can be found at `https://www.aha.org/fact-sheets/2020-01-06-fact-sheet-uncompensated-hospital-care-cost`.

Profit Margin

Hospital profit margins are indicative of the financial health of the hospitals. We calculate profit margins as the ratio of net income to total revenue where total revenue is the sum of net patient revenue and total other income. After the calculation, profit margins are missing for 92 hospitals, which we impute to 0.

Funding

We calculate the projected funding using the formula on the CARES ACT website. Hospitals that do not qualify on any of the three dimensions are not given any funding. Each eligible hospital is assigned an individual facility score, which is calculated as the product of disproportionate patient percentage and number of beds in that hospital. We calculate cumulative facility score as the sum of all individual facility scores in the dataset. Each hospital receives a share of $10 billion, where the share is determined by the ratio of individual facility score of that hospital to the cumulative facility score. The amount of funding received by hospitals is bounded below at $5 million and capped above at $50 million.

E.2 Hospital Utilization Data

We use the publicly available COVID-19 Reported Patient Impact and Hospital Capacity by Facility dataset for our outcome variables. This provides facility level data on hospital utilization aggregated on a weekly basis, from July 31st onwards. These reports are derived from two main sources – (1) HHS TeleTracking and (2) reporting provided directly to HHS Protect by state/territorial health departments on behalf of health care facilities.[44]

The hospitals are uniquely identified for a given collection week (which goes from Friday to Thursday) by their CMS Certification number. All hospitals that are registered with CMS by June 1st 2020 are included in the population. We merge the hospital cost report data with the utilization data using the CMS certification number. According to the terms and conditions of the CARES Health Care Act, the recipients may use the relief funds only to "prevent, prepare for, and respond to coronavirus" and for "health care related expenses or lost revenues that are attributable to coronavirus". Therefore, for our analysis we focus on 4 outcomes that were directly affected by COVID-19, for the week spanning July 31st to August 6th 2020. The outcome measures are described below.

1. Total reports of patients currently hospitalized in an adult inpatient bed who have laboratory-confirmed or suspected COVID-19, including those in observation beds reported during the 7-day period.

2. Total reports of patients currently hospitalized in an adult inpatient bed who have laboratory-confirmed COVID-19 or influenza, including those in observation beds. Including patients who have both laboratory-confirmed COVID-19 and laboratory confirmed influenza during the 7-day period.

[44] Source: `https://healthdata.gov/Hospital/COVID-19-Reported-Patient-Impact-and-Hospital-Capa/anag-cw7u`.

3. Total reports of patients currently hospitalized in a designated adult ICU bed who have suspected or laboratory-confirmed COVID-19.

4. Total reports of patients currently hospitalized in a designated adult ICU bed who have laboratory-confirmed COVID-19 or influenza, including patients who have both laboratory-confirmed COVID-19 and laboratory-confirmed influenza.

In the dataset, when the values of the 7 day sum are reported to be <4, they are replaced with -999,999. We recode these values to missing.

E.3 Computing Quasi Propensity Score

As the three determinants of safety net eligibility are continuous variables, we can think of this setting as a multi-dimensional regression discontinuity design and a suitable setting to apply our method. In this setting, the X_i are disproportionate patient percentage, uncompensated care per bed and profit margin. Funding eligibility (Z_i) is determined algorithmically using these 3 dimensions. D_i is the amount of funding received by hospital i, which depends on both safety net eligibility status Z_i, number of beds in the hospital, and disproportionate patient percentage. Before calculating QPS, we normalize each characteristic of X_i to have mean 0 and variance 1. For each hospital and every $\delta \in \{0.01, 0.025, 0.05, 0.075, 0.1, 0.25, 0.5\}$, we draw 1000 times from a δ-ball around the normalized covariate space and calculate QPS by averaging funding eligibility Z_i over these draws.