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Mark Armstrong

Jidong Zhou
Yale University

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By

Mark Armstrong and Jidong Zhou

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YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

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Consumer Information and the Limits to Competition*

Mark Armstrong
Department of Economics
Oxford University

Jidong Zhou
School of Management
Yale University

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Abstract

This paper studies competition between firms when consumers observe a private signal of their preferences over products. Within the class of signal structures which induce pure-strategy pricing equilibria, we derive signal structures which are optimal for firms and those which are optimal for consumers. The firm-optimal policy amplifies underlying product differentiation, thereby relaxing competition, while ensuring consumers purchase their preferred product, thereby maximizing total welfare. The consumer-optimal policy dampens differentiation, which intensifies competition, but induces some consumers to buy their less-preferred product. Our analysis sheds light on the limits to competition when the information possessed by consumers can be designed flexibly.

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JEL classification: D43, D83, L13

1 Introduction

In many markets firms supply differentiated products and compete in prices. Consumers might not always possess all information about product attributes when they

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make their purchase decision, and market performance crucially depends on how much information consumers possess about products. If little such information is available, consumers cannot meaningfully compare products and so will regard them as close substitutes. This induces firms to compete fiercely, so price is low, but consumers often end up purchasing a mismatched product. When more product information is available, the quality of the consumer-product match improves, but this also endows firms with greater market power and so raises price. This broad trade-off between match quality and price is well known, but there has been little systematic investigation of the interaction between the amount of product information consumers have and the intensity of competition. For example, is more information for consumers always associated with increased market power of firms? Can "biased" information provision which favors one firm over others help relax price competition? In which direction is the trade-off between match quality and market price optimally resolved for consumers? More generally, what kind of information provision is optimal for firm profit and what kind is optimal for consumer welfare? This paper aims to answer these questions.

These questions are also important when a third-party can design the information environment that consumers face. For instance, consumers often gather information and make purchases via a platform (e.g., Amazon or Expedia) which hosts competing firms. The platform can control many aspects of the information environment consumers face, such as how detailed is the product information it displays, whether to post customer reviews or its own reviews, whether to offer personalized recommendations, how flexibly consumers can filter and compare products, and so on. How should the platform design its information environment if it aims to maximize industry profit, or consumer surplus, or a weighted sum of them?[1] When some products (e.g., insurance plans) are difficult for consumers to compare, regulators sometimes intervene by specifying how firms should disclose product information, aiming to enhance consumers' ability to select their preferred product. But is a more transparent information environment always better for consumers, given the potentially adverse effect on price competition of information improvement?[2]

[1]What a platform aims to maximize depends partly on which sides of the market it can levy fees, which in turn depends on the relative intensity of platform competition across the two sides of the market. We do not explicitly model strategic platforms in this paper.

[2]See, e.g., https://go.cms.gov/2U2lZJe for information transparency regulation in the US health insurance market. Sometimes the regulator may even require firms to offer "standardized" products to facilitate comparison. This involves a similar trade-off to information regulation: more standardized products are close substitutes and so facilitate competition on price, but prevent some consumers with particular preferences from obtaining a product tailored to those preferences. See, e.g., Ericson and Starc (2016) for a discussion and empirical analysis of this issue in the context of health insurance

Formally, as described in section 2, we study an *ex ante* symmetric duopoly market where two firms each costlessly supply a single variety of a product and compete in prices. Consumers are initially uncertain about their preferences for the varieties, but before purchase they receive a private signal of these preferences. Consumers then update their beliefs about their preferences and make their choice given the pair of prices offered by firms. The information environment or *signal structure*, which governs the mapping between consumers' true preferences and the signals they receive, is publicly known to both firms and consumers. We wish to understand how the signal structure affects competition and welfare. In particular, we explore the limits to competition in this market: which signal structures induce the highest profit for firms and which generate the highest surplus for consumers?[3]

Solving our problem at the most general level is challenging. Consumer preferences are generally two-dimensional in our setup, and current understanding of information design in such cases is limited.[4] For this reason, we first focus on the case when the outside option for consumers is irrelevant, and so only the relative valuation between the two products—a scalar variable—matters for consumer decisions. Even with scalar heterogeneity, under some signal structures the only equilibria in the pricing game between firms involve mixed strategies, which can be hard to deal with. For this reason, we mostly focus on signal structures which induce a pure strategy equilibrium in the pricing game. (Towards the end of the paper, we report what progress can be made when the outside option might bind or when mixed strategies are allowed.)

Our analysis starts in section 3.1 with a distinct pricing problem, which does not concern information design *per se*. We describe the set of relative valuation distributions (if any) which support a given pair of prices as equilibrium prices. It turns out that a price pair can be implemented if and only if the valuation distribution lies between two *bounds*, which are determined by each firm's no-deviation condition in equilibrium. Using these bounds we characterize the range of possible pure strategy equilibrium

exchange.

[3]Another interpretation of the consumer-optimal problem is that sometimes consumers can commit to how much product information to acquire, or use, before firms make their pricing decisions. For instance, a consumer could strategically delegate her purchase decision to an agent (e.g., a purchasing alliance) who commits to focus more on price than other product characteristics in order to stimulate price competition among suppliers.

[4]It is well known that the posterior (expected) consumer valuation distribution induced by any signal structure is a mean-preserving contraction of the underlying prior distribution. However, unlike the scalar case, a multidimensional mean-preserving contraction has no simple characterization as in Rothschild and Stiglitz (1970). See, for example, section 7.2 of Dworczak and Martini (2019) for discussion of this point.

prices when any valuation distribution is allowed.

In our information design problem, however, only some of the posterior valuation distributions can be induced by signal structures. Using the bounds approach, we show in section 3.2 for both firms and consumers that any asymmetric signal structure which treats the two firms differently is dominated by a symmetric signal structure. Surprisingly, this is true even for an individual firm, i.e., the firm which is favored by an asymmetric signal will prefer some symmetric signal structure which treats it and its rival the same. Intuitively, when a firm is treated unfavorably with asymmetric signals, it has an incentive to set a low price. Our result shows that this force can be sufficiently strong so that even the favored firm suffers from this fierce competition. This result suggests that firms have congruent interests when it comes to the design of consumer information, and "biased recommendations" are not the best approach to improve profit within this framework.

The bounds are used in sections 3.3 and 3.4 to characterize the firm-optimal or the consumer-optimal information policy (which must be symmetric). The firm-optimal signal structure amplifies "perceived" product differentiation in order to relax competition, and does so by reducing the likelihood that consumers *ex post* are near-indifferent between products. Compared to full information disclosure, the firm-optimal information policy can significantly improve industry profit. This suggests that providing consumers with more product information need not raise firms' market power. The firm-optimal policy usually enables consumers to buy their preferred product for sure, in which case total welfare is also maximized.

The consumer-optimal signal structure by contrast dampens "perceived" product differentiation in order to stimulate competition, and does so by increasing the number of consumers who are near-indifferent between products. In the consumer-optimal policy, a consumer with strong preferences can buy her preferred product for sure, but a less choosy consumer receives less precise information and may end up with the inferior product. Thus, in contrast to the firm-optimal policy, the consumer-optimal policy does not maximize total welfare. The consumer-optimal policy usually induces rather low market prices, suggesting that for information design with only two firms the price competition effect tends to outweigh the match quality effect for consumers.

In section 3.5 we characterize the welfare frontier when the information environment consumers face is flexible. Any profit between zero and the firm-optimal profit level is feasible. For a given profit level, the efficient frontier is achievable if the profit is no less than the profit under full disclosure; otherwise, mismatch is needed to induce fierce competition, in which case the efficient frontier is not achievable. This welfare frontier can be used to solve the Ramsey problem, i.e., the optimal information policy which maximizes a weighted sum of industry profit and consumer surplus.

In section 4 we discuss several extensions to this analysis. First, the outside option becomes relevant for the analysis if valuations for the products are relatively dispersed *ex ante*. Optimal information design in such a case is a hard problem in general. However, if valuations are sufficiently dispersed, we show that the "rank" signal structure, which informs consumers which product they prefer but nothing else, allows firms to obtain *first-best* profit, in which case the outside option binds for all consumers. The rank structure eliminates consumers who are near-indifferent between products, and with sufficiently dispersed preferences, it is an equilibrium for firms to charge consumers their (expected) valuation of the preferred product. Second, we explore whether considering the wider class of signals which allow mixed strategy pricing equilibria can improve outcomes. We show that allowing symmetric mixed strategy equilibria could at best improve consumer surplus only slightly. Third, we consider markets with more than two firms. There we show that the "top product" signal structure which informs consumers of their most preferred product but nothing else can sometimes enable firms to achieve the first-best outcome, and even if it does not achieve first-best profit, the same signal structure bounds industry profit away from zero regardless of the number of firms. When there are many firms, the "top two" signal structure which informs consumers of their best *two* products (but without ranking them) can approximately achieve the first-best outcome for consumers: it induces marginal-cost pricing, and the sacrifice of match quality when consumers buy one of the best two products is negligible with many firms. With many firms full information disclosure can also approximately achieve the first-best for consumers, but it is dominated by the top-two structure for a finite number of firms.

As with many other optimal policies in economics, the optimal information policies derived in this paper might be "too complicated" to implement in practice. Nevertheless, these optimal policies provide useful guidance for which kinds of informations help boost profits and which help consumers. To relax competition and improve profit, we need to reduce the number of price-sensitive consumers who are near-indifferent between (the best two) products. A simple information policy to do so is the rank or the top-product signal structure. To improve consumer welfare, we want to induce a fierce competition but at the same time not cause too much product mismatch. A simple information policy which can serve this purpose is the top-two signal structure.

Related literature. One strand of the relevant literature considers a monopolist's incentive to provide information to enable consumers to discover their valuation for its product. An early paper on this topic is Lewis and Sappington (1994). They study a monopoly market and show, within the class of "truth or noise" signal structures, how it is optimal for the firm either to disclose no information or all information. Johnson and Myatt (2006) derive a similar result for a more general class of information structures which induce rotations of the demand curve. Anderson and Renault (2006) argue that partial information disclosure before consumers search can be optimal for a monopolist if consumers need to pay a search cost to buy the product (in which case they learn their valuation automatically). Importantly, they allow for general signal structures as in the later Bayesian persuasion literature and show that firm-optimal information disclosure takes the coarse form whereby a consumer is informed merely whether her valuation lies above a threshold.

Roesler and Szentes (2017) study the signal structure which is best for consumers (rather than the firm) in a monopoly model. They show that the optimal signal structure can be found within the class of posterior distributions which induce unit-elastic demand functions. (These unit-elastic demand functions play a similar role to the posterior distribution bounds in our analysis.) They show that partial rather than complete learning is optimal for consumers, and that the optimal information structure induces *ex ante* efficient trade and maximizes total welfare.[5] In their setup, where trade is always efficient, the firm-optimal signal structure is to disclose no information at all, in which case the firm can extract all surplus by charging a price equal to the expected valuation. With competition, however, this is no longer true since without any information consumers regard the firms' products as perfect substitutes and firms earn zero profit. (Indeed in our duopoly model we show that disclosing no information is nearly optimal for *consumers*, rather than firms.) Therefore, the firm-oriented problem is more interesting and challenging in our setting with competition than it is with monopoly. With competition, the consumer-optimal policy also exhibits some significant differences with the monopoly case in Roesler and Szentes. For example, it usually causes product mismatch so that the allocation is sometimes inefficient, the induced residual demand for each firm is unit-elastic for upward but not downward price deviations, and the consumer-optimal signal structure is not the least profitable policy for firms.

Our paper concerns information design in an oligopoly setting. Most of the previous research on this topic studies the "decentralized" disclosure policies of individual firms. For example, Ivanov (2013) studies competitive disclosure when each firm decides how

[5] Condorelli and Szentes (2020) study the related problem of how to choose the demand curve to maximize consumer surplus, given the monopolist chooses its price optimally in response. (The consumer accurately observes her realized valuation in this model.) Choi, Kim, and Pease (2019) extend Roesler and Szentes (2017) to the set-up of Anderson and Renault (2006), and derive the consumer-optimal policy in the context of a search good.

much information about its own product to release and what price to charge. He focuses on information structures which rotate demand as in Johnson and Myatt (2006), and shows that full disclosure is the only symmetric equilibrium when the number of firms goes to infinity.[6] Hwang, Kim, and Boleslavsky (2020) show that the same result holds if general signal structures are allowed (and more generally are able to show that increasing the number of firms induces each firm to reveal more information). Intuitively, with many firms, a consumer's valuation for the best rival product (if other firms fully disclose their information) is high. To compete for the consumer, a firm discloses all information as that is the policy which maximizes the posterior probability she has a high valuation.[7]

Instead of studying equilibrium disclosure by individual firms, though, we focus on a "centralized" design problem, such as when a platform mediates the information flow from products to consumers. This enables us to discuss signals which reflect *relative* valuations across products, e.g., which rank the products for consumers. This more general signal structure introduces a number of additional features. In our framework, for instance, full information disclosure is not the firm-optimal design even with many firms, and the rank policy sometimes yields first-best profit for firms (which can never be achieved with a decentralized design). In addition, when decentralized signals are not fully revealing, there are welfare losses since consumers sometimes choose a less preferred product, while with a centralized structure it is possible to have a coarse signal structure (e.g., the rank signal) which maintains efficiency. Finally, Hwang *et al.* (2020) show that the equilibrium price (and hence profit) often falls relative to the full-information price when individual firms choose their disclosure policy non-cooperatively, while by construction the centralized firm-optimal policy must boost profit.

Other papers have also studied "centralized" aspects of the design of consumer information. Anderson and Renault (2009) study comparative advertising in a duopoly model where each firm unilaterally chooses between fully disclosing its own product information, fully disclosing information about both products, or disclosing nothing. Among other results, they make the point that disclosing more information improves match quality but also softens price competition. Jullien and Pavan (2019) make a similar point when they study how platforms' information management affects their com-

[6] Bar-Isaac, Caruana, and Cuñat (2012) study competitive product design within a sequential search market. They consider designs which rotate demand, and show that a reduction in the search cost induces more firms to choose niche product design (which can be interpreted as full information disclosure in the context of information design when consumers have a common prior).

[7] A similar result appears in other recent works which study competitive disclosure but without price competition, such as Board and Lu (2018).

petition in two-sided markets. Both papers consider specific classes of signal structures. When general signal structures are allowed, as we show in this paper, the monotonic relationship between the amount of information and the degree of price competition fails. Dogan and Hu (2019) study the consumer-optimal disclosure policy in a sequential search framework with many firms. Consumers receive a signal of their valuation for a particular product only when they visit its seller. The information policy is chosen by a third party, and the disclosure is only about each individual product (and so there is no disclosure about relative valuations across products). Because the reservation value in this search framework is static, their problem is similar to the monopoly problem studied by Roesler and Szentes (2017). Moscarini and Ottaviani (2001) study a duopoly model of price competition similar to ours, where the consumer receives a private signal of her relative valuation for the two products. A major difference, however, is that they assume the relative valuation is binary and the signal is binary as well and is further symmetric across states. In that simple setting, they can allow firms to be *ex ante* asymmetric, but the pricing equilibrium often involves complicated mixed strategies.[8] Moscarini and Ottaviani (2001) also point out that both firm profit and consumer welfare can vary non-monotonically with the amount of information consumers have, but they do not explicitly study the optimal signal structure for firms or consumers.

More broadly, our paper belongs to the recent literature on Bayesian persuasion and information design. See Kamenica and Gentzkow (2011) for a pioneering paper in this literature, and Bergemann and Morris (2019) and Kamenica (2019) for recent surveys. Among its many applications, its method and insights have been used to revisit classic problems within Industrial Organization. For instance, Bergemann, Brooks, and Morris (2015) study third-degree price discrimination by a monopolist. In contrast to the consumer-side design problem in Roesler and Szentes (2017) and our model, their paper considers a firm-side design problem where signals are sent to the firm about consumer preferences, and consumers accurately know their valuation from the start. A given signal structure corresponds to a particular partition of consumers. If all ways to partition consumers are possible, the paper shows that any combination of profit (above the no-discrimination benchmark) and consumer surplus which sum to no more than maximum total welfare can be implemented. Elliot, Galeotti, and Koh (2020) extend Bergemann *et al.* to the competition case with product differentiation, and derive conditions under which market segmentation through information can earn firms

[8] When we study the "top product" signal structure in section 4.3, the resulting posterior valuation distribution is binary. There we construct the mixed strategy pricing equilibrium similarly as in Moscarini and Ottaviani (2001) but with an arbitrary number of firms. Beyond the binary case, it is generally intractable to study mixed strategy pricing equilibrium with product differentiation.

the first-best profit. Li (2020) solves the consumer-optimal problem in the competition case by applying the segmentation idea in Bergemann *et al.* to each submarket where there is a dominant firm which is preferred by all consumers there.[9]

2 The model

A risk-neutral consumer wishes to buy a single unit of a differentiated product which is costlessly supplied by two risk-neutral firms, 1 and 2. The consumer's valuation for the unit from firm $i = 1, 2$ is denoted $v_i \geq 0$. The consumer is initially uncertain about her valuations and holds a prior belief about the joint distribution of (v_1, v_2). Throughout the paper we assume that firms are symmetric *ex ante*, in the sense that the prior distribution for (v_1, v_2) is symmetric between v_1 and v_2. We assume that the support of the prior distribution lies inside the square $[V, V+1]^2$. Here, $V \geq 0$ represents the "basic utility" from either product, while 1 is the normalized range of product valuations. Let $\mu \equiv \mathbb{E}[v_i]$ denote the expected valuation of either product (or a random product).

The consumer has an outside option which is sure to have payoff zero. Until section 4.1, however, we assume this outside option is irrelevant and the consumer always purchases one of the two products. (As shown in Lemma 3 in the appendix, this is the case if the basic utility V is sufficiently high and we restrict attention to pure strategy pricing equilibrium.) In this case the consumer cares only about the difference in her valuations, $x \equiv v_1 - v_2$, and so consumer heterogeneity becomes one-dimensional. Let $F(x)$ be the prior distribution for x which is symmetric around 0 and has support in $[-1, 1]$.

Before purchase the consumer observes a private signal of x. The signal is generated according to a signal structure $\{\sigma(s|x), S\}$, where S is the signal space and $\sigma(s|x)$ specifies the distribution of signal s when the true preference parameter is x. We assume the signal structure is common knowledge to the consumer and to both firms, and determined before firms choose prices. This assumption is plausible when the signal structure is publicly chosen by some third-party such as a platform or a regulator as a relatively long-term decision. After observing signal s, the consumer updates her belief

[9] Similar problems are also studied in the Varian (1980) setup where firms supply a homogeneous product but each firm has some captive consumers who know only one firm. See, e.g., Armstrong and Vickers (2019), Bergemann, Brooks, and Morris (2020), and Shi and Zhang (2020). In many circumstances, market segmentation also depends on how much preference information consumers are willing to share with firms. See, e.g., Ali, Lewis, and Vasserman (2019) and Ichihashi (2020) for relevant works which consider general information structures.

about her relative preference x to, say, $F_s(x)$. Given the consumer is risk neutral, only the expected x given s, $\mathbb{E}_{F_s}[x]$, matters for her choice. Let $G(x)$ be the distribution of this expected x, and we refer to it as the consumer's *posterior* distribution. It is jointly determined by the prior distribution for x and the signal structure. Even though some signal structures may be hard to implement in practical terms, this approach is a general way to model the consumer's information environment and enables us to explore the limits of competition when information is fully flexible.[10]

Firms know the consumer's prior and the signal structure, and hence know the posterior distribution $G(x)$, but they do not observe her private signal, and so they each choose a single price regardless of the signal received by the consumer. We assume that prices are always accurately observed by the consumer. If p_i is firm i's price, then the consumer prefers to buy from firm 1 if her expected x after seeing a signal satisfies $x > p_1 - p_2$, prefers to buy from firm 2 if $x < p_1 - p_2$ (and is indifferent when $x = p_1 - p_2$). Firms set prices simultaneously to maximize their expected profit, and we use Bertrand-Nash equilibrium as the solution concept for the pricing game.

We aim to investigate how the signal structure affects competition and the ability of the consumer to buy her preferred product. In particular, we search for those signal structures which maximize industry profit and those which maximize consumer surplus. As discussed in the introduction, for tractability we focus on those signal structures which induce a *pure strategy* pricing equilibrium. (We will discuss signal structures which might induce mixed strategy equilibria in section 4.2.)

2.1 Examples of signals

It is useful initially to consider some simple signal structures. To illustrate these, assume here that the prior distribution is uniform, so $F(x) = \frac{1}{2}(1 + x)$.

Full information disclosure: Here the signal perfectly reveals the true preference x, e.g., where $s \equiv x$, and so the posterior and prior distributions for x coincide. The equilibrium price is then 1, which is also industry profit (given full consumer participation). The consumer always buys her preferred product, which maximizes total welfare, but the

[10] One interpretation of the signal structure in our setup is that an information designer (e.g., a platform) might know a consumer's preferences (e.g., due to its past interactions with the consumer) and can send the consumer a signal contingent on those preferences (e.g., personalized recommendations), but it commits to the signal structure before seeing the consumer's preferences. The more standard, also less informationally demanding, interpretation from the Bayesian persuasion literature is that the signal structure is an information experimentation device, and even consumers with the same preferences may learn different noisy signals from the same information environment.

market price is relatively high. The consumer's expected surplus is $(\mu + \frac{1}{4}) - 1 = \mu - \frac{3}{4}$, where μ is the expected match utility of a random product, and $\frac{1}{4} = \mathbb{E}_F[\max\{x, 0\}]$ is the match efficiency improvement under full information disclosure relative to random purchase.

No information disclosure: In this case the signal is completely uninformative (i.e., the distribution of s does not depend on x) and the posterior G is degenerate at $x = 0$. In particular, the consumer views the two products as perfect substitutes. Both the equilibrium price and industry profit are 0, and consumer surplus is simply μ since the consumer buys a product randomly in equilibrium. This consumer surplus is higher than that under full information disclosure, and so the disutility she incurs from buying a random product is outweighed by the low equilibrium price she pays. (As we will see in section 4.3, when there are more than two firms, the top-two signal structure which informs consumers of their best two products but without ranking them can achieve the same pricing outcome but without causing too much product mismatch.)

"Truth or rank" signal: Here the signal accurately informs the consumer of her x with probability $\theta < 1$ and otherwise only informs her whether $x > 0$ or $x < 0$ (i.e., which product she prefers). (Suppose the rank signals are distinguishable from the truth signals.) Then the posterior distribution G has two mass points at $-\frac{1}{2}$ and $\frac{1}{2}$ (with mass $\frac{1-\theta}{2}$ at each) and is otherwise the same as the prior but with a reduced density $\frac{\theta}{2}$. In this case, if θ is sufficiently close to 1 (so the two mass points do not carry too much weight), a symmetric equilibrium price exists and is equal to $\frac{1}{\theta} > 1$, and so firms earn more than they did under full disclosure.[11] (Consumer surplus, however, is lower than with full disclosure because in either case the consumer always buys her preferred product.)

The first two of these examples illustrate the consumer's trade-off between match quality and the intensity of competition, and how the latter effect can dominate. The first and third examples suggest that one way to soften competition and enhance industry profit is to reduce the chance that the consumer is indifferent between the two products, and more information for consumers is not always associated with more market power of firms. The three examples together imply that full information disclosure

[11] This is the case if $\theta \geq 3 - \sqrt{5} \approx 0.76$. If θ is below the threshold, the incentive for each firm to undercut and steal the rival's consumers on the mass points becomes so strong that there is no pure strategy equilibrium. When $\theta = 0$, we have the pure rank signal structure which has independent interest and which will be discussed in section 4.

is optimal neither for firms nor for the consumer. As we will see in section 4.3, these insights carry over to the case when there are more than two firms.

All these signal structures are symmetric in the sense that they induce a symmetric posterior G. More generally, as shown in this final example, a signal structure can induce an asymmetric posterior and make *ex ante* symmetric products appear asymmetric.

"Truth or biased recommendation" signal: Modify the previous example so that when the consumer is not provided with full information she learns whether $x > b$ or $x < b$ for some $b \in (0, 1)$. (An interpretation is that firm 1's product is recommended if $x > b$ and otherwise firm 2's product is recommended.) The posterior now has a mass point with mass $\frac{1}{2}(1-\theta)(b+1)$ at $\frac{1}{2}(b-1)$ and a mass point with mass $\frac{1}{2}(1-\theta)(1-b)$ at $\frac{1}{2}(b+1)$. One can check that given $b \in (0, 1)$, if θ is sufficiently close to 1 there is an asymmetric pure strategy equilibrium with prices $p_1 = \frac{1}{\theta}(1-\Delta)$ and $p_2 = \frac{1}{\theta}(1+\Delta)$, where $\Delta \equiv \frac{b}{3}(1-\theta) > 0$. Compared to the symmetric case with $b = 0$, the favored firm 2 raises its price while firm 1 lowers its price. Firm 2's equilibrium market share rises to $\frac{1}{2}(1+\Delta)$ and firm 1's falls to $\frac{1}{2}(1-\Delta)$, and so the biased recommendation helps firm 2 but harms firm 1. Industry profit is $\frac{1}{\theta}(1+\Delta^2)$, which is even greater than in the symmetric case. (The consumer then must suffer from this biased recommendation as it harms match efficiency and total welfare.) This final example suggests that the use of asymmetric signals might be a useful tool to improve industry profit. However, as we will see in section 3.2, this turns out not to be possible once we consider general signal structures.

2.2 Preliminaries

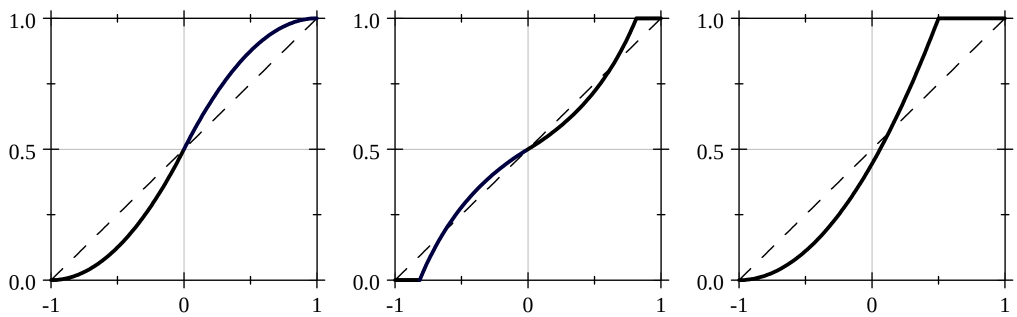
To study more general signal structures we use the following well-known results to simplify the problem. For a given prior F, the only restriction on the posterior G imposed by Bayesian consistency is that it is a *mean-preserving contraction (MPC)* of F, i.e.,

$$\int_{-1}^{x} G(\tilde{x})d\tilde{x} \leq \int_{-1}^{x} F(\tilde{x})d\tilde{x} \text{ for } x \in [-1, 1], \text{ with equality at } x = 1\ . \tag{1}$$

Moreover, any G which is an MPC of F can be generated by means of some signal structure.[12] Therefore, instead of analyzing the signal structure directly, we can work with the posterior distribution G and look for a posterior which is optimal for firms or

[12] See, for example, Blackwell (1953), Rothschild and Stiglitz (1970), Gentzkow and Kamenica (2016), and Roesler and Szentes (2017).

consumers subject only to the MPC constraint (1). Figure 1 depicts some examples of MPCs of the uniform prior (where the dashed line represents the uniform prior CDF and the bold curve shows the posterior CDF).



(a) (b) (c)

Figure 1: Some examples of MPCs

The following two observations will be useful: First, when G is symmetric, the mean-preserving requirement is satisfied automatically, and so a simpler condition

$$\int_{-1}^{x} G(\tilde{x})d\tilde{x} \leq \int_{-1}^{x} F(\tilde{x})d\tilde{x} \text{ for } x \in [-1,0] \tag{2}$$

is sufficient for G to be an MPC of F. Second, if a symmetric G further crosses F at most once and from below in the negative range $x \in (-1,0)$, as on Figures 1a and 1b, then

$$\int_{-1}^{0} G(x)dx \leq \int_{-1}^{0} F(x)dx \tag{3}$$

is sufficient for G to be an MPC of F.

We also need a measure of the efficiency of product choice corresponding to a posterior G. If the consumer always chooses her preferred product (i.e., chooses product 1 if the expected x is positive and product 2 otherwise, as would be the case if the two firms offered the same price), total surplus is

$$\mu + \mathbb{E}_G[\max\{x,0\}] = \mu + \int_{-1}^{0} G(x)dx \ , \tag{4}$$

where the equality follows after integration by parts and uses the fact that G has a zero mean (which implies $\int_{-1}^{1} G(x)dx = 1$). Since μ is the expected value of a random product, the integral term reflects the match efficiency improvement under posterior G

relative to no information disclosure. (Note that this observation does not need G to be symmetric.)

Define

$$\delta \equiv \mathbb{E}_F[\max\{x,0\}] = \int_{-1}^{0} F(x)dx \ . \tag{5}$$

This parameter δ captures the underlying extent of product differentiation in the market. Since G is an MPC of F, condition (1) at $x = 0$ implies that (4) cannot exceed the maximum total surplus $\mu + \delta$. In other words, match efficiency cannot increase when the consumer observes a noisy signal of her preferences, as is intuitive. When the inequality (3) is strict—as in Figures 1a and 1c—mismatch occurs with the posterior G. There is no product mismatch when there is equality in (3). This is the case under full disclosure, and more generally when the consumer is fully informed about whether $x > 0$ or $x < 0$, even though she may not be fully informed about the magnitude of x.

3 Optimal signal structures

We derive the firm and consumer optimal signal structures in two steps. First, we ignore the MPC constraint (1) and derive the constraints on G needed to implement a target pair of prices (p_1, p_2) in pure strategy equilibrium. Second, we search for the prices, together with their supporting G, which are optimal for firms or the consumer, subject to the constraint that G is an MPC of F.

3.1 Bounds on posteriors

In this section we derive bounds on the distributions $G(x)$ which implement a given pair of prices (p_1, p_2) in pure strategy equilibrium. The analysis here is separate from the information design problem, and may have some independent interest more generally for oligopoly theory.

Define

$$L_{p_1,p_2}(x) \equiv \max\left\{0, 1 - \frac{\pi_1}{\max\{0, p_2 + x\}}\right\} \tag{6}$$

and

$$U_{p_1,p_2}(x) \equiv \min\left\{1, \frac{\pi_2}{\max\{0, p_1 - x\}}\right\} \ , \tag{7}$$

where

$$\pi_i = \frac{p_i^2}{p_1 + p_2} \ . \tag{8}$$

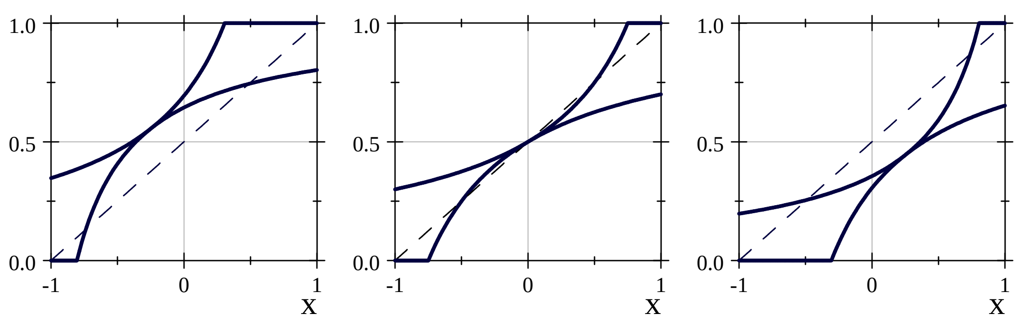
(When both prices are zero, we stipulate that $\pi_i = 0$ in (8).) The following lemma shows that these functions are the required bounds on G.

Lemma 1 *(i) A distribution G implements (p_1, p_2) as equilibrium prices if and only if*

$$L_{p_1,p_2}(x) \leq G(x) \leq U_{p_1,p_2}(x) \tag{9}$$

for all $x \in [-1, 1]$, where the two bounds are defined in (6) and (7) and π_i in (8) is firm i's equilibrium profit; furthermore $p_1 = p_2 > 0$ if and only if $G(0) = \frac{1}{2}$.
(ii) (p_1, p_2) can be equilibrium prices under some G if and only if $p_j - 1 \leq \pi_i$ for $i \neq j$ (which implies $|p_i - p_j| \leq 1$ and $p_i \leq 2$).

The two bounds L_{p_1,p_2} and U_{p_1,p_2} arise from each firm's no-deviation requirement for equilibrium. They are illustrated in Figure 2 below. (The dashed line in each plot indicates the CDF corresponding to the uniform prior and can be ignored for the current discussion.)



(a) $p_1 < p_2$ (b) $p_1 = p_2$ (c) $p_1 > p_2$

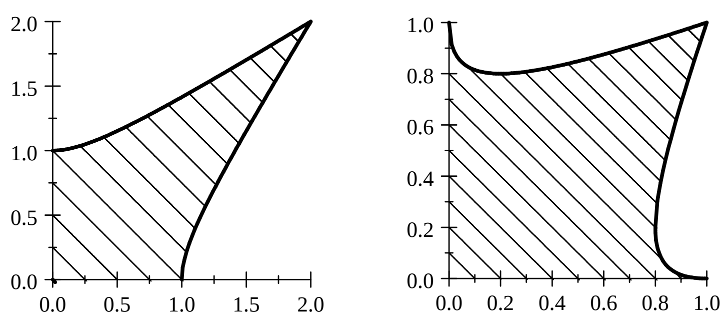
Figure 2: Bounds on G to implement pure strategy equilibrium prices

The lower bound L_{p_1,p_2} increases with x and begins to be positive at $x = \pi_1 - p_2$ (which exceeds -1 given $p_2 - 1 \leq \pi_1$), and it is concave whenever it is positive. The upper bound U_{p_1,p_2} also increases with x and reaches 1 at $x = p_1 - \pi_2$ (which is below 1 given $p_1 - 1 \leq \pi_2$), and it is convex whenever it is less than 1. The two bounds coincide and have the same slope at $x = p_1 - p_2$.

Part (i) of Lemma 1 says that for a distribution G to induce a pair of equilibrium prices (p_1, p_2), we only need it to be between the two associated bounds, and in particular G can be irregular (e.g., the distribution can have atoms). But G has to be smooth at $x = p_1 - p_2$ given the two bounds are tangent to each other at that point. (This is intuitive because if G has an atom at $x = p_1 - p_2$, a firm obtains a discrete jump in demand if it slightly undercuts its rival, in which case (p_1, p_2) cannot be an equilibrium.) In Figure 2a, we have $G(0) > \frac{1}{2}$ for any G above the lower bound, i.e., it is more likely that the consumer prefers firm 2's product, and that is why firm 2 sets a higher price; the opposite is true in Figure 2c.

From the equilibrium profit expression (8), firm i's equilibrium market share is $p_i/(p_1+p_2)$. Hence, both equilibrium profits and market shares are determined entirely by equilibrium prices and do not depend separately on the shape of G, and the firm with the higher equilibrium price necessarily has the higher market share and the higher profit. Part (i) also shows that if the distribution is symmetric and induces a pure strategy equilibrium, the equilibrium prices must be symmetric. (Notice, however, that an asymmetric distribution can also induce symmetric equilibrium prices if $G(0) = \frac{1}{2}$.)

Using the bounds result in part (i), part (ii) of Lemma 1 characterizes the set of possible pure strategy equilibrium prices when G can be any distribution on $[-1, 1]$. This range is depicted as the shaded area in Figure 3a. In particular, the highest possible pure strategy equilibrium price for either firm is 2. Using part (ii), one can also derive all possible pairs of profits (π_1, π_2) as described in Figure 3b. It is clear that the best outcome for each firm is the symmetric outcome with $p_1 = p_2 = 2$ in which case each earns profit 1. This suggests that firms have congruent interests when it comes to jointly choosing the valuation distribution (e.g., via joint product design). No firm will be happy with an asymmetric distribution even if it is favored and earns more than its rival. As we will see in next section, a similar result holds when we consider information design in which case not all (posterior) valuation distributions are available due to the MPC constraint.



(a) prices (p_1, p_2) (b) profits (π_1, π_2)

Figure 3: Set of possible pure strategy equilibrium outcomes

The symmetric case $p_1 = p_2 = p$, shown on Figure 2b, is important in our subsequent analysis. With symmetric prices the bounds simplify to

$$L_p(x) = \max\left\{0, 1 - \frac{p}{2\max\{0, p+x\}}\right\} \tag{10}$$

and

$$U_p(x) = \min\left\{1, \frac{p}{2\max\{0, p-x\}}\right\} . \tag{11}$$

As indicated on Figure 2b above, these bounds are mirror images of each other, in the sense that $L_p(x) \equiv 1 - U_p(-x)$. Therefore, if a symmetric G lies between the bounds in the negative range $x \in [-1, 0]$, it will lie between the bounds over the whole range $[-1, 1]$. A useful property of the two bounds (10) and (11) is that they rotate clockwise about the point $(0, \frac{1}{2})$ as p increases, and in particular both of them increase with p for $x < 0$. Intuitively, to induce a higher price we need fewer price-sensitive consumers around $x = 0$, which requires the bounds to be flatter.

In the following, we often consider the case when the market has a pure strategy pricing equilibrium in the benchmark with full disclosure. Lemma 1 implies that this is the case if and only if the (symmetric) prior distribution F lies between the bounds L_p and U_p for some p. Since the two bounds are tangent to each other at $x = 0$, F must be differentiable at that point and let $f(0)$ be the corresponding density. Then the symmetric equilibrium price under full information is $p_F = 1/(2f(0))$.

Condition 1 *F lies between the bounds L_{p_F} and U_{p_F} with $p_F = 1/(2f(0))$.*

The oligopoly literature often invokes one of the following stronger conditions to ensure the existence of a pure strategy equilibrium:

Condition 2 *(i) $1 - F$ is log-concave on $[-1, 0]$, and (ii) F is log-concave on $[-1, 0]$.*

Condition 3 *F has density f which is log-concave on $[-1, 1]$.*

It is well known (see, e.g., Bagnoli and Bergstrom (2005)) that Condition 3 implies that both $1 - F$ and F are log-concave on $[-1, 1]$, and hence Condition 2. Using the observation that U_p and $1 - L_p$ are log-convex whenever they are less than one, one can also see that Condition 2 implies Condition 1.[13] We will use some of these conditions in parts of the following analysis.

[13]Notice that at $x = 0$, both U_{p_F} and F equal $\frac{1}{2}$ and have the same slope $f(0)$. Then, if F is log-concave on $[-1, 0]$, we have $F \leq U_{p_F}$ over that range given the upper bound is log-convex. Similarly, when $1 - F$ is log-concave on $[-1, 0]$, using the fact that $1 - L_p$ is log-convex whenever L_p is positive, we can also see that $F \geq L_{p_F}$ over that range. Therefore, Condition 2 implies that F lies between L_{p_F} and U_{p_F}.

3.2 The suboptimality of asymmetric posteriors

In the example with biased recommendations in section 2.1, we saw that relative to certain symmetric signals, asymmetric signals which favor one firm over the other can improve industry profit and the favored firm's profit. We show using Lemma 1 that this insight does not continue to apply when general signal structures are allowed.

Lemma 2 *For each firm or the consumer, an asymmetric posterior which induces a pure strategy equilibrium is dominated by a symmetric posterior which also induces a pure strategy equilibrium. (The dominance is strict if the asymmetric posterior induces asymmetric prices.)*

The result concerning firms states that even if a firm earns more than its rival under an asymmetric posterior, it will nevertheless prefer some symmetric posterior. Since a symmetric posterior induces a symmetric equilibrium, both firms then are better off and so industry profit improves. This implies that similar to the case in the previous section where there was no information constraint, firms have congruent interests when it comes to the design of consumer information. Intuitively, when a firm is treated unfavorably under an asymmetric signal structure, it has an incentive to set a low price. Lemma 2 shows that this force can be sufficiently strong so that even the favored firm suffers from the resulting fierce competition. This suggests regardless of whether a designer (e.g., a platform) wants to maximize industry profit or one firm's profit (e.g., the profit of the platform's own product), "biased recommendations" achieved through asymmetric signals may not be the best approach.

For the consumer, asymmetric signal structures result in mismatch (due to both biased information and price dispersion). This negative effect on the consumer turns out to outweigh the potential benefit from lower prices in an asymmetric market.

Using Lemma 2, in the following we restrict our attention to symmetric posteriors when we derive the firm or consumer optimal solution.

3.3 Firm-optimal policy

Since we can restrict attention to symmetric posteriors and symmetric prices, and since there is full consumer participation, maximizing industry profit corresponds to maximizing the symmetric equilibrium price p. We therefore wish to find the highest p such that there exists a G lying between L_p in (10) and U_p in (11) which is an MPC of the prior F. Notice that, whenever a symmetric G is a solution to this problem which induces price p, a symmetric posterior which takes the form L_p for negative x (and so takes the form U_p for positive x) must be a solution as well.[14] Therefore, we can restrict attention to symmetric posteriors which take the form L_p for negative x.

We illustrate our approach by considering the uniform-prior example with $F(x) = \frac{1}{2}(1+x)$. As on Figure 2b, given that L_p is concave whenever it is positive, it crosses the linear prior at most once and from below in the range of negative x. Then according to (3) a symmetric G which is equal to L_p for negative x is an MPC of the prior if and only if $\int_{-1}^{0} L_p(x)dx \le \int_{-1}^{0} F(x)dx = \frac{1}{4}$. Since L_p increases with p for negative x, the optimal price p^* must make this inequality bind, so that

$$\int_{-1}^{0} L_p(x)dx = \int_{-\frac{p}{2}}^{0} \left(1 - \frac{p}{2(p+x)}\right) dx = \tfrac{1}{2}(1-\log 2)p = \tfrac{1}{4}$$

and so

$$p^* = \frac{1}{2(1-\log 2)} \approx 1.63 \ .$$

This optimal price is about 63% higher than the full-information price $p_F = 1$ in this example. The optimal symmetric posterior equals L_{p^*} for negative x, and the way we derive the optimal price also implies that this is the unique optimal solution.[15] Notice also that (3) holds with equality when $G = L_{p^*}$, and so the firm-optimal posterior leads to perfect matching and so maximizes total welfare. (The consumer does not possess full information, but she has enough information always to buy her preferred product.)

This argument continues to hold provided that for each p, L_p crosses F at most once and from below in the range of negative x. This is true, for instance, when F is convex for negative x (or the density f is weakly single-peaked at $x = 0$). More generally, we have the following result:

Proposition 1 *Under Condition 2(i), the firm-optimal signal structure induces the symmetric equilibrium price*

$$p^* = \frac{2\delta}{1-\log 2} \ , \tag{12}$$

where δ is defined in (5), and it is uniquely implemented by the symmetric posterior which is equal to L_{p^*} in the range of negative x, where L_p is defined in (10). With the firm-optimal signal structure there is no mismatch and total welfare is also maximized.

[14] Formally, we have $\int_{-1}^{x} L_p(\tilde{x})d\tilde{x} \le \int_{-1}^{x} G(\tilde{x})d\tilde{x} \le \int_{-1}^{x} F(\tilde{x})d\tilde{x}$ for $x \in [-1, 0]$, where the first inequality is because L_p is the lower bound of G and the second is because G is an MPC of F. Therefore, according to (2) a symmetric posterior which equals L_p for negative x must be an MPC of F.

[15] If there is another posterior which induces the same price p^* but differs from L_{p^*} on a positive measure of $x < 0$, then it must violate the MPC constraint.

The firm-optimal posterior distribution depends on the prior only through δ. (A prior with a higher δ leads to a higher optimal price and so a flatter posterior.) Figure 4 illustrates the optimal posterior with a uniform prior (where the dashed lines are the prior density and CDF, respectively). In particular, the density is U-shaped. Intuitively, to soften price competition we reduce the number of consumers around $x = 0$, as these consumers are the most price sensitive, and push consumers towards the two extremes insofar as this is feasible given the pure-strategy and MPC constraints.[16]

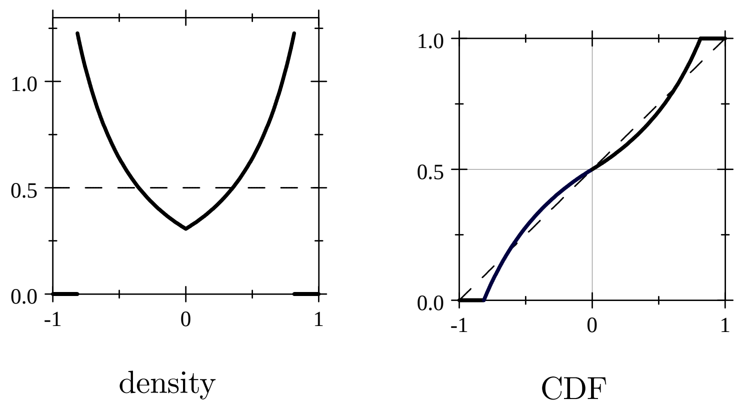


Figure 4: Firm-optimal posterior

The uniform example shows how the firm-optimal policy can significantly improve profit compared to the simple benchmark of full disclosure, and so giving more information to consumers does not always lead to greater market power for firms. More generally, with a log-concave prior density firms can achieve profit at least 63% higher than with full disclosure. The following result provides more details.

Corollary 1 *Under Condition 3, the firm-optimal price in (12) satisfies* $\eta p_F \leq p^* \leq 2\eta p_F$ *where* $\eta = \frac{1}{2(1-\log 2)} \approx 1.63$.

This result also shows that the advantage of considering general signal structures, relative to some frequently-used signal structures, can be significant. For instance, the often-used "truth-or-noise" structure (whereby the signal s is equal to the true x with some probability and otherwise the signal is a random realization of x) induces a degree of mismatch so that consumers become more concentrated around $x = 0$, and so

[16]With this posterior distribution, when a firm, say, firm 2 deviates to a price lower than p^*, its residual demand takes the form of the upper bound and so is unit-elastic (i.e., its profit is unchanged), since the upper bound was derived from firm 2's no deviation condition. While if it deviates to a price higher than p^*, its demand takes the form of the lower bound and so its profit strictly falls.

it cannot enhance profit relative to the full-information policy. The same is true more generally when the distribution for x is "rotated" about $x = 0$ as studied by Johnson and Myatt (2006). Therefore, the use of unrestricted signal structures, which allow consumers to buy their preferred product, enables firms to do at least 63% better than they could with those more restricted signals.

Discussion: For more general priors than those covered by Proposition 1, the firm-optimal price is the highest p such that

$$\int_{-1}^{\tilde{x}} L_p(x)dx \leq \int_{-1}^{\tilde{x}} F(x)dx \text{ for } \tilde{x} \in [-1, 0] \ .$$

In general, however, L_p and F can cross multiple times in the range of negative x, in which case it becomes somewhat harder to solve for the optimal price. Moreover, (3) might hold strictly and so there could be welfare loss associated with the firm-optimal signal structure. Figure 5 illustrates both points, where the prior shown as the dashed curve is initially convex and then concave in the range of negative x. The highest price such that L_p is an MPC of the prior is shown as the solid curve, where the integrals of the two curves up to the crossing point a are equal (so with any higher price the MPC constraint would be violated). Here, since L_{p^*} lies below the prior for x above a, (3) holds strictly and there is some mismatch at the optimum.

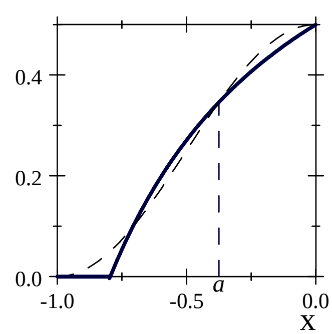


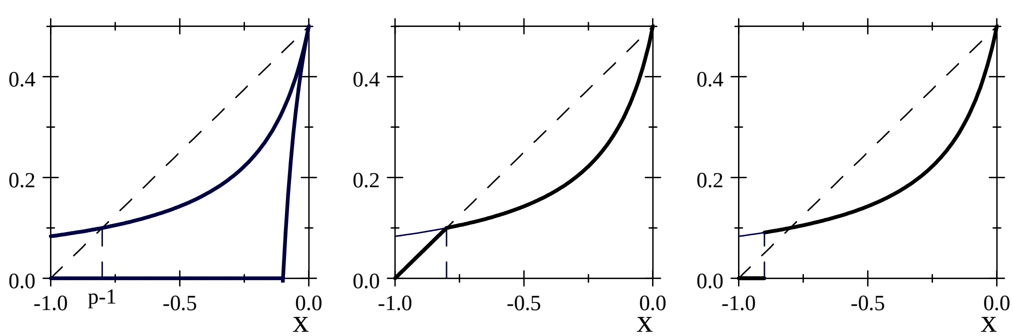
Figure 5: Firm-optimal posterior with a less regular prior

Notice also that if the prior distribution is so dispersed that the MPC constraint never binds (e.g., when the prior distribution is binary at $x = -1$ and 1), our bounds analysis in section 3.1 implies that the highest possible (pure strategy) profit 2 is achievable. Beyond the information design problem, this observation is relevant, for example, when firms can coordinate on their product design (which determines G) to maximize their joint profits.

3.4 Consumer-optimal policy

We turn next to the optimal information policy for the consumer. Unlike firms, the consumer does not care solely about the induced price but also about the reliability of the product match. When a posterior G induces a symmetric equilibrium with price p, the consumer always buys her *ex post* preferred product, so from (4) we know that her expected surplus is $\mu - p + \int_{-1}^{0} G(x)dx$. To maximize this, we first find the highest possible G to maximize match efficiency $\int_{-1}^{0} G(x)dx$ for a given price p, subject to the bounds condition $L_p \leq G \leq U_p$ and the MPC constraint, and then identify the optimal price.

We illustrate our approach by considering the uniform-prior example again. For the consumer to do better than with full information disclosure (in which case the equilibrium price is $p_F = 1$), the induced price must be below 1 to counteract any potential product mismatch. Then the associated bounds must be steep enough at $x = 0$ so that they lie below the prior CDF for x close to zero. This is illustrated in Figure 6a below, where the two bold curves are the two bounds in the range of negative x. Since we want G to be as high as possible, it is now the *upper* bound U_p which will constrain G, rather than the lower bound which was relevant for the firm-optimal policy. Since the upper bound is convex, for any price $p < 1$ the upper bound cuts the prior CDF at $x_p \equiv p - 1 \in [-1, 0)$ and from above, again as illustrated in Figure 6a.



(a) (b) (c)

Figure 6: Consumer-optimal G for given price p

Therefore, for a given price $p < 1$, a consumer-optimal (symmetric) G is simply

$$G(x) = \min\{F(x), U_p(x)\} \tag{13}$$

for negative x, as shown as the bold curve in Figure 6b. First, this G is clearly between the two bounds associated with price p and is also an MPC of the prior.[17] Second, for $x \in [x_p, 0]$, this G already equals the upper bound; for $x < x_p$, this G equals F and so $\int_{-1}^{x_p} G(x)dx$ already reaches its maximum given the MPC constraint.

Notice, however, for $x < x_p$, the consumer-optimal G can take other forms provided that it is between the two bounds and satisfies $\int_{-1}^{x_p} G(x)dx = \int_{-1}^{x_p} F(x)dx$. Figure 6c depicts one alternative solution. It is also clear that for any consumer-optimal G, the strict inequality of (3) must hold because G is below F for $x \in [x_p, 0]$. Therefore, in contrast to the firm-optimal solution, there must be welfare losses at the consumer optimum and the consumer sometimes buys her less preferred product.

Expression (13) implies that the maximum consumer surplus for a given price $p < 1$ is

$$\mu + \int_{-1}^{0} \min\{F(x), U_p(x)\}dx - p \ . \tag{14}$$

One can check that the derivative of (14) with respect to p is $\frac{1}{2}(p - \log p - 3)$ when F is uniform, which decreases with p in the range $[0, 1]$. Then the optimal price is $p^{**} \approx 0.05$ which is the root of $p - \log p = 3$ in the range $[0, 1]$, and a consumer-optimal symmetric posterior is given by expression (13) with $p = p^{**}$ for negative x.

The above argument continues to hold if for each price $p < p_F$, the upper bound U_p crosses F once and from above in the range of negative x. This is true, for example, when F is concave. More generally, we have the following result:

Proposition 2 *Under Conditions 1 and 2(ii), a consumer-optimal signal structure induces the symmetric equilibrium price*

$$p^{**} = \frac{-\gamma}{1-\gamma} F^{-1}(\tfrac{1}{2}\gamma) \ , \tag{15}$$

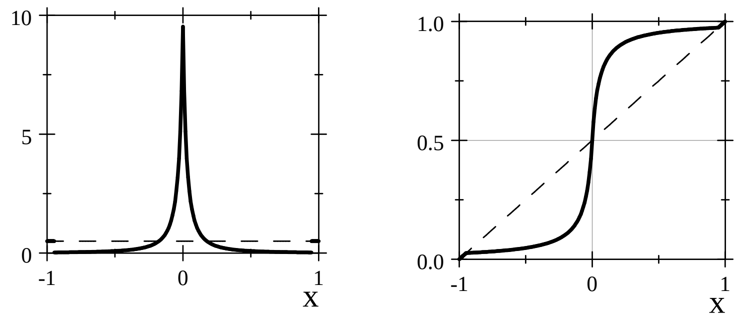
where $\gamma \approx 0.05$ *is the root of* $\gamma - \log \gamma = 3$ *in the range* $[0, 1]$, *and it can be implemented by a symmetric posterior which is equal to* $\min\{F(x), U_{p^{**}}(x)\}$ *in the range of negative* x, *where* U_p *is defined in (11). With the consumer-optimal signal structure there is mismatch and total welfare is not maximized.*

(Here Condition 1 is needed to ensure that the full-information benchmark has a pure strategy equilibrium price p_F, so the argument that the consumer-optimal price must be below p_F continues to hold.)

Figure 7 depicts the consumer-optimal posterior distribution when the prior is uniform (and when the posterior takes the particular form in Figure 6b). The number

[17] Note that at the full information price p_F we have $L_{p_F} \leq F$, and so $L_p < F$ for negative x when $p < p_F$ given the lower bound increases in price for negative x. Hence, for any price $p < p_F$, (13) must be above L_p.

of price-sensitive consumers near $x = 0$ is amplified compared with the prior distribution, and this forces firms to reduce their price in equilibrium. Those consumers near $x = 0$ do not have strong preferences about which product they buy, and so there is only limited welfare loss due to product mismatch. Those consumers with very strong preferences, however, are sure to buy their preferred product and at a low price.[18]



density

CDF

Figure 7: Consumer-optimal posterior

The uniform example has indicated that the consumer-optimal price is rather low. This remains true more generally.

Corollary 2 *Under Condition 3, the consumer-optimal price in (15) satisfies* $\gamma p_F \leq p^{**} \leq \min\{\gamma, \frac{1}{2}p_F\}$ *where* $\gamma \approx 0.05$ *is defined in Proposition 2.*

Therefore, the insight from the previous no-disclosure example in section 2.1 is generally true that for the consumer the price effect appears more important than the match quality effect. Of course, when there are more firms, the match quality effect will become more important, and we will discuss this issue in section 4.3.

Discussion: For more general prior distributions, we can consider the family of posteriors illustrated by Figure 6c above. Given Lemma 2, it can be shown that the consumer-optimal policy can be implemented by a symmetric posterior which is equal to $G_p^m(x) = U_p(x)$ for $x \in [m, 0]$ and $= 0$ for $x < m$, where $m \in [-1, 0]$ is a constant. (This is in the spirit of Roesler and Szentes (2017).) The consumer problem is then to choose (p, m) to maximize $\mu + \int_{-1}^{0} G_p^m(x)dx - p$, subject to G_p^m being an MPC of F. It is a well-defined optimization problem, but it does not have a general analytical solution if the upper bound U_p crosses F multiple times in the range of negative x.

[18]In contrast to footnote 16, the consumer-optimal posterior distribution implies that when a firm unilaterally increases its price its residual demand is unit-elastic.

Notice also that the consumer-optimal policy is often approximated by the solution to a simpler "relaxed" problem, which is to choose a distribution G, without considering the MPC constraint, in order to maximize equilibrium consumer surplus. Then the consumer-optimal G for a given p is simply U_p, and so the problem is to choose the price p to maximize $\mu + \int_{-1}^{0} U_p(x)dx - p$. This yields the optimal price $p \approx 0.055$ which is also rather small. This approximated solution is relevant if the prior is sufficiently dispersed such that the MPC constraint never binds, or if we consider a joint product design problem.

3.5 The welfare limits

Having discussed the signal structures which maximize profit and which maximize consumer surplus, we now explore the welfare frontier in the space of profit and consumer surplus. Let p^* be the firm-optimal price or industry profit. (This need not take the form (12) if the assumption in Proposition 1 does not hold.) We know that it is implemented by a symmetric posterior which equals L_{p^*} for negative x. Then any price or industry profit $p < p^*$ is feasible, since it can be implemented by a symmetric posterior which equals L_p for negative x. (This is because $L_p < L_{p^*}$ and so L_p is an MPC of the prior whenever L_{p^*} is.)

To derive the welfare frontier, we need to find the maximum possible consumer surplus for each given industry profit $p \leq p^*$. Lemma 2 shows that for the consumer any asymmetric posterior is dominated by some symmetric one (which may induce a different industry profit). What we need here is a stronger result that for the consumer any asymmetric posterior is dominated by a symmetric posterior which induces the *same* industry profit. This turns out to be true when the prior satisfies the assumptions used for Proposition 2. Once we rule out asymmetric posteriors, the analysis for the consumer-optimal problem immediately yields the maximum possible consumer surplus for a given price p.

Proposition 3 *Any profit between* 0 *and* p^* *is feasible, and for a given profit* $p \in [0, p^*]$, *if Conditions 1 and 2(ii) hold, the highest possible consumer surplus is* $\mu+\delta-p$ *if* $p \geq p_F$ *and is (14), which is concave in* p, *if* $p < p_F$.

The bold curve in Figure 8 illustrates the welfare frontier in the uniform example, where the two dots indicate the firm-optimal and the consumer-optimal outcomes respectively. The downward sloping dashed line is the efficient frontier with maximum total welfare $\mu + \delta$ (where for convenience we set $\mu = 2$ and $\delta = \frac{1}{4}$). Here, the full-information price is $p_F = 1$, and for prices above p_F the feasible frontier coincides with the efficient frontier. For lower prices, the feasible frontier lies strictly inside the efficient frontier and it is concave. This figure remains qualitatively the same for other prior distributions.[19]

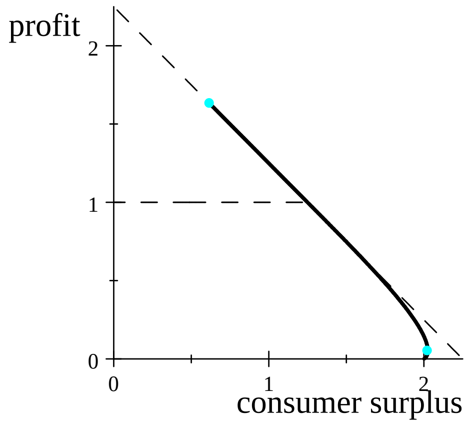


Figure 8: Welfare frontier

This discussion enables us to solve the Ramsey problem of maximizing a weighted sum of profit and consumer surplus. If the weight on profit is higher than that on consumer surplus, it is clear from Figure 8 that the solution simply coincides with the firm-optimal policy. While, as is more usual, if the weight on consumer surplus is higher the solution lies on the concave part of the outer frontier, where the optimal price is below p_F but above the consumer-optimal price p^{**} and is lower when the weight on consumer surplus is greater. This also implies that full information disclosure is optimal only in the knife edge case when we want to maximize the unweighted total welfare.

4 Extensions and discussions

We have so far focused on the situation with two firms, where the outside option never binds for the consumer, and we restricted attention to signal structures which induce a pure strategy pricing equilibrium. The problem without these restrictions is in general hard to solve. In this section, we report the progress we can make when we relax these restrictions. (The omitted proofs and details in this section are relegated to the online appendix.)

[19] For a given profit $p < p^*$, the lowest value of consumer surplus is usually generated by an asymmetric posterior. This problem appears hard to solve due to the MPC constraint, unless we focus on symmetric posteriors (in which case we can show that the minimum consumer surplus for a given price p is $\mu - \frac{1}{2}(1 + \log 2)p$).

4.1 Relevant outside option

In our model, the consumer's valuations for the two varieties are (v_1, v_2) and they lie inside $[V, V+1]^2$. When the outside option sometimes binds for the consumer, the scalar parameter $x = v_1 - v_2$ is no longer sufficient to determine consumer choice and in general we would need to deal with two-dimensional consumer heterogeneity captured by the pair of valuations (v_1, v_2). As we discussed in the introduction, current understanding of information design in such cases is limited. In particular, a mean-preserving contraction has no simple characterization when consumer heterogeneity is multidimensional, which prevents us solving the general two-dimensional problem. Progress can be made, however, in the two cases when first-best outcomes are feasible and when consumer heterogeneity is one-dimensional.

First-best outcome is feasible: If a signal structure maximizes total surplus and allocates all of that surplus to firms in equilibrium, then it must be optimal for firms. This is feasible under certain conditions. Let $\mu_H = \mathbb{E}[\max\{v_1, v_2\}]$ be the expected valuation of the consumer's preferred product according to the prior, and let $\mu_L = \mathbb{E}[\min\{v_1, v_2\}]$ be the expected valuation of her less preferred product. (These are related by $\mu_L + \mu_H = 2\mu$.) Note that μ_H is the maximum possible welfare in the market.

Proposition 4 *If*

$$\mu_H \geq 2\mu_L \ , \tag{16}$$

the rank signal structure (which only informs the consumer which product she prefers) implements an equilibrium in which firms earn the first-best profit.

This result is straightforward to understand. With the rank signal structure, the posterior distribution is binary: with probability half the consumer's expected valuations for the two products are (μ_H, μ_L), and with remaining probability her expected valuations are (μ_L, μ_H). Then $p_1 = p_2 = \mu_H$ is the equilibrium given condition (16), and so firms earn the first-best profit. (In equilibrium a firm earns $\frac{1}{2}\mu_H$, and if it unilaterally deviates to μ_L (or slightly below), it will have the whole demand and so earn a deviation profit μ_L. Condition (16) ensures that such a deviation is unprofitable.)

Intuitively, (16) is more likely to be satisfied when the valuations are more dispersed. For example, if (v_1, v_2) are uniformly distributed on the square $[0,1]^2$, then $\mu_H = \frac{2}{3}$ and $\mu_L = \frac{1}{3}$ and so (16) is (just) satisfied. More generally, as shown in the online appendix, (16) holds if v_1 and v_2 are independently distributed with a density which weakly decreases over the support $[0,1]$.[20]

[20] If instead of duopoly the two products were jointly supplied by a multiproduct monopolist, the

One can also consider whether there exists a signal structure which enables the consumer to achieve her first-best outcome. To achieve that outcome, we need $p_1 = p_2 = 0$ in equilibrium, which in turn requires that the consumer regards the products as perfect substitutes. In that case, she is unable to choose the preferred product more than half the time. Consequently, it is impossible for the consumer to obtain the first-best surplus except in the trivial case in which the products are perfect substitutes.

One-dimensional consumer heterogeneity: Notice that given $\mu_L \geq V$ and $\mu_H \leq V + 1$, condition (16) requires $V \leq 1$, i.e., the basic utility is small relative to the range of valuations. In other words, firms earn the first-best profit with the rank signal structure when consumer valuations are sufficiently dispersed, in which case the participation constraint is binding. By contrast, sections 2 and 3 studied the situation where valuations were sufficiently concentrated ($V > 3$ as shown in Lemma 3 in the appendix), in which case the participation constraint was irrelevant. One way to bridge the gap between these two situations is to consider a special case where consumer heterogeneity is actually one-dimensional.

Suppose a consumer values product 1 at $v_1 = 1 + \frac{1}{2}x$ and product 2 at $v_2 = 1 - \frac{1}{2}x$, where $x = v_1 - v_2 \in [-1, 1]$ indicates her relative preference for product 1 and each consumer's average valuation $\frac{1}{2}(v_1 + v_2) \equiv 1$ is constant. Let F be the symmetric prior distribution for x, and let $\delta = \int_{-1}^{0} F(x)dx$ measure the dispersion of the prior distribution. The consumer will buy a product only if its net surplus is positive. In the online appendix, we solve the optimal symmetric signal structure in this one-dimensional setup when the prior density is log-concave, and show how the optimal policy varies with δ. For example, when δ is sufficiently low, the firm-optimal policy is the same as in section 3; when δ is sufficiently high, the firm-optimal policy is the rank signal structure and it earns the first-best profit for firms; in between the optimal policy is a mixture of these two policies.

4.2 Allowing mixed pricing strategies

It is hard to deal systematically with signal structures which induce mixed strategy pricing equilibrium, when the bounds approach in section 3 does not apply. Instead, in this section we derive an upper bound for consumer surplus across all symmetric signal structures which induce a symmetric (pure or mixed strategy) equilibrium, and show that this upper bound is close to the maximum consumer surplus available with pure

rank signal structure with associated prices $p_1 = p_2 = \mu_H$ allows the firm to fully extract surplus regardless of whether (16) holds. See Ichihashi (2020) for a related observation.

strategies.

Consider the model introduced in section 2 with a zero outside option. The following proposition reports the main result.

Proposition 5 *Suppose $V \geq 1$ and $v_1 - v_2$ has a log-concave density. Then the maximum consumer surplus available using only pure strategies attains at least 98.4% of the maximum consumer surplus available across all symmetric signal structures which induce a symmetric pure or mixed strategy equilibrium.*

This result demonstrates that it is not possible that consumers can do significantly better if the class of (symmetric) signal structures is broadened to permit mixed pricing strategies in equilibrium. (Note, however, we have not found an example where the use of mixed strategies improves consumer surplus at all.) Intuitively, mixed strategy pricing usually does not intensify price competition and the resulting price dispersion further causes product mismatch, in which case it does not benefit consumers.

Ideally one would also like to obtain a tight upper bound on the profit obtained using either pure or mixed strategies, and see how closely the optimal profit under the pure strategy restriction approaches such an upper bound. This appears to be a harder problem, though, and we leave it for future work.

4.3 More than two firms

We now explore the case with an arbitrary number of firms. Consider $n \geq 2$ symmetric firms, each producing a variety of a differentiated product. Let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}_+^n$ denote the consumer's valuations for the n varieties, and let $F(\mathbf{v})$ be the symmetric joint prior distribution. Before her purchase, the consumer receives a private signal of $\mathbf{v}$ which is generated by a public signal structure $\{\sigma(\mathbf{s}|\mathbf{v}), \mathbf{S}\}$. Let $G(\mathbf{v})$ be the joint posterior distribution of the expected valuations after receiving the signal. As before, G is an MPC of F (in the sense that they have the same mean and $\int \phi dG \leq \int \phi dF$ for any convex function ϕ whenever the integrals exist), and any G which is an MPC of F can be generated by some signal structure. (See, e.g., Blackwell (1953), and Strassen (1965).) It would be interesting to know how the number of firms affects optimal information design, but this general problem is hard to solve. A full analysis would require consideration of multi-dimensional consumer heterogeneity even in situations where the outside option could be ignored. This invalidates both the bounds approach and the Rothschild-Stiglitz characterization of the MPC constraint.

Here we consider two special signal structures and explore the insights from them. A "top product" signal structure informs the consumer which product is her best match but nothing else. In the duopoly case this is just the rank signal structure. A "top two" signal structure informs the consumer which two products are her best matches but nothing else. In the duopoly case this corresponds to no information disclosure. Recall $\mu \equiv \mathbb{E}[v_i]$, and let $\mu_{n:n} \equiv \mathbb{E}[\max\{v_i\}]$ denote the expected valuation for the top product.

Proposition 6 *(i) The "top product" signal structure achieves the first-best profit for firms if*

$$\frac{\mu_{n:n}}{\mu} \geq \frac{n^2}{2n-1} \; ; \tag{17}$$

otherwise it leads to a mixed strategy pricing equilibrium in which the industry profit is bounded away from zero even if $n \to \infty$.
(ii) Suppose the outside option is irrelevant and $\{v_i\}$ are i.i.d. and have a common density function which is log-concave. Both full information disclosure and the "top two" signal structure asymptotically achieve the first-best outcome for the consumer when $n \to \infty$, but the latter performs strictly better for any finite n.

Similar to the first-best result for duopoly in section 4.1, the "top product" signal structure induces an equilibrium where each firm sets price $p = \mu_{n:n}$ if $\frac{1}{n}\mu_{n:n} \geq \mu_{<n:n}$, where $\mu_{<n:n} \equiv \mathbb{E}[v_j | v_j \neq \max\{v_i\}]$ is the expected valuation for a non-top product. This condition can be written as (17) by using the identity $\frac{1}{n}\mu_{n:n} + (1 - \frac{1}{n})\mu_{<n:n} = \mu$. (17) is usually harder to be satisfied when there are more firms in the market, and it must fail for sufficiently large n if v_i has a finite upper bound.[21]

When (17) does not hold, the "top product" signal structure induces a mixed strategy pricing equilibrium. We show in the proof that the equilibrium industry profit converges to $\lim_{n\to\infty} \mu_{n:n} - \mu_{<n:n} > 0$ when there are many firms. This contrasts with the literature discussed in the introduction, where firms disclose only information about their own product, where firms disclose all information and price goes to zero when there are many firms.

The consumer faces a trade off between price and match quality, and with just two firms we saw that this trade off was typically resolved by revealing little product information to consumers. With many firms, however, this trade-off usually vanishes. For example, with full information disclosure, the consumer can choose her preferred product and under the stated condition for part (ii) she also pays a zero price as $n \to \infty$, and so in the limit this policy achieves the first best for the consumer. The same outcome

[21]For example, for the standard uniform distribution (17) holds only for $n = 2$; however, for a distribution with a heavy tail it can hold for a large n (e.g., for a Pareto distribution $F(x) = 1 - x^{-\alpha}$ with $\alpha > 1$, for any given n it holds if α is sufficiently close to 1).

can also be achieved by the "top two" signal structure: the consumer regards the best two products as perfect substitutes and so competition always drives price to zero for any $n \geq 2$ (which also implies that this signal structure outperforms no disclosure for the consumer); on the other hand, the mismatch from buying one of the best two products becomes negligible as $n \to \infty$ if the valuation distribution has a thin tail (which is the case if the density is log-concave). With a finite number of firms, however, the usual trade-off persists: full disclosure leads to perfect matching but also a positive mark-up; the "top two" signal structure induces the lowest possible price but results in some mismatch. Under the stated condition, we show that the price effect dominates and so the consumer always prefers the "top two" signal structure. This simple signal structure becomes even more appealing to the consumer than full disclosure if the need to choose among many products involves high information processing costs for the consumer.

5 Conclusion

This paper has studied the limits to competition when product information possessed by consumers can be designed flexibly. Among signal structures which induce pure strategy pricing equilibrium, we derived the optimal policy for firms and for consumers. The firm-optimal signal structure amplifies perceived product differentiation by reducing the number of consumers who regard the products as close substitutes. The firm-optimal signal structure typically enables consumers to buy their preferred product, and so it maximizes total welfare too. In particular, the rank or the top-product signal structure which only informs consumers of which product is their best match can sometimes be optimal for firms. The consumer-optimal policy, in contrast, dampens perceived product differentiation by increasing the number of marginal consumers and so implements a low price. This low price can only be achieved by inducing a degree of product mismatch, however, and so the policy does not maximize total welfare. The top-two signal structure which informs consumers of their best *two* products (but without ranking them) can be close to be optimal for consumers.

Besides the extensions discussed in section 4, there are other interesting extensions to explore. One would be to consider situations where firms were asymmetric *ex ante*, including the case of vertical differentiation where one firm was known to provide a higher match utility than its rival. One way to model this asymmetry is to consider an asymmetric prior distribution F, and then one could investigate whether the optimal information policy maintains, amplifies or reduces this prior asymmetry, and whether firms continue to have aligned interests over the design of consumer information. The bounds analysis which is independent of the information problem still applies in this more general problem, but the main challenge stems from dealing with the MPC constraint.

Another possible extension would be to allow consumers to be heterogeneous *ex ante*. For instance, a consumer's valuation v_i for product i might be decomposed as $v_i = a_i + b_i$, where consumers know the vector $(a_1, ...)$ from the start, from other information sources, and there is scope to manipulate information only about the vector $(b_1, ...)$. If there was enough heterogeneity in $(a_1, ...)$, then one might be able to rule out mixed pricing strategies in equilibrium, rather than assuming them away as we mostly did in this paper.

Finally, it would be valuable to embed this analysis within a framework in which the "information designer" is modelled explicitly as an economic agent. Platforms typically compete with each other to provide intermediation services. If a profit-maximizing platform chooses what product information to reveal to consumers, and also chooses its fees to each side of the market, then the relative competitive intensity among platforms on the two sides of the market and the platform's equilibrium fee structure will presumably affect whether its information policy is focussed more on delivering firm profit or consumer surplus.

Technical Appendix

A sufficient condition for the outside option to be irrelevant. Recall that in our model the consumer's valuations for the two varieties are (v_1, v_2) and they lie inside $[V, V+1]^2$. The consumer is initially uncertain about her valuations. She receives a signal from a signal structure $\{\sigma(s|v_1, v_2), S\}$ before purchase, and then estimates her valuation for each product and makes a purchase decision. When we focus on signal structures which induce a pure strategy pricing equilibrium, the following lemma reports a sufficient condition for the zero outside option to be irrelevant.

Lemma 3 *If $V > 3$, then under any signal structure which induces a pure strategy equilibrium, equilibrium prices are below V and all consumers obtain positive surplus when they buy from either firm.*

Proof. With any signal structure the posterior valuation for a product lies in the range $[V, V+1]$. Suppose a signal structure induces firms $i = 1, 2$ to offer respective prices p_1 and p_2 and to obtain profits π_1 and π_2. It is clear that neither p_1 nor p_2 can exceed $V+1$.[22] Firm j will serve all consumers if it deviates to a low price p (or slightly below it) such that $V-p \geq V+1-p_i$, i.e., if $p \leq p_i - 1$. (Given $p_i \leq V+1$, the inequality $V-p \geq V+1-p_i$ ensures that $V-p \geq 0$ so that all consumers prefer to buy from firm j than to buy nothing.) Therefore, we must have

$$p_i - 1 \leq \pi_j \ . \tag{18}$$

If firms are labelled so $p_1 \geq p_2$, then (18) implies $p_1 - 1 \leq \pi_2 \leq p_2$ so that the price difference $p_1 - p_2$ cannot exceed 1. Adding the pair of inequalities (18) implies that

$$p_1 + p_2 - 2 \leq \pi_1 + \pi_2 \leq p_1 \ ,$$

where the second inequality follows since industry profit cannot exceed the maximum price p_1, and so $p_2 \leq 2$. Since $p_1 - p_2 \leq 1$ it follows that $p_1 \leq 3$, and so when $3 < V$ we must have $\max\{p_1, p_2\} < V$. ■

The proof shows that equilibrium prices in a pure strategy equilibrium can never exceed 3 and so must be below V under the stated condition. The outside option is then irrelevant in the sense that if at most one firm deviates from equilibrium all consumers continue to participate. The condition of $V > 3$ essentially requires that valuations have a concentrated distribution in the sense that the range of valuations is small relative to the basic utility of the product.

Proof of Lemma 1. (i) Let π_i, $i = 1, 2$, denote firm i's equilibrium profit when (p_1, p_2) are equilibrium prices. (They are now unspecified constants.) As already shown in Lemma 3, given the valuation difference between the two products is at most 1, in any pure strategy equilibrium we must have $p_i - 1 \leq \pi_j$ for $i \neq j$ to ensure that no firm has a unilateral incentive to use a low price to capture all the demand. We can therefore focus on prices and profits which satisfy this condition.

We first show the "if and only if" condition on G for (p_1, p_2) to be the associated equilibrium prices. (p_1, p_2) are the equilibrium prices under G if and only if no firm has a unilateral incentive to deviate from their equilibrium price. If firm 2 deviates to price $p_2' \neq p_2$, the consumer buys from firm 2 if $x \leq p_1 - p_2'$. (Thus we suppose that if G has a mass point at $x = p_1 - p_2'$, firm 2 serves all consumers at that mass point. This is

[22] If firm j did choose price $p_j > V+1$ in equilibrium, then no consumer will buy from it, and firm i acts as a monopolist and its optimal price must be $p_i > 0$. Then firm j can earn a strictly positive profit by deviating to a price slightly below p_i, as under any signal structure there must be a positive probability that the consumer weakly prefer product j over product i given the two products are symmetric *ex ante*.

the natural tie-breaking rule given that the firm can achieve this outcome by charging a price slightly below p_2'.) Therefore, firm 2 has no incentive to deviate if and only if

$$p_2' G(p_1 - p_2') \le \pi_2$$

holds for all p_2'. By changing variables from p_2' to $x = p_1 - p_2'$, we can write this no-deviation requirement as $(p_1 - x)G(x) \le \pi_2$, or $G(x) \le U_{p_1,p_2}(x)$ for any $x \in [-1, 1]$. (It is unprofitable for firm 2 to set a negative price p_2', and so there are restrictions on G only in the range where $p_2' = p_1 - x > 0$, which is why there is $\max\{0, \cdot\}$ in the denominator of (7). In addition, a CDF cannot exceed 1 which is why there is $\min\{1, \cdot\}$ in (7).)

Likewise, firm 1 has no incentive to deviate if and only if

$$p_1'(1 - G((p_1' - p_2)^-)) \le \pi_1$$

for all p'. (As with firm 2, if G has an atom at $x = p_1' - p_2$ the natural tie-breaking assumption is that firm 1 serves all consumers at x. The deviation demand is written as $1 - G((p_1' - p_2)^-)$ as a CDF is defined to be right-continuous.) By changing variables from p_1' to $x = p_1' - p_2$, this constraint can be written as $(p_2 + x)(1 - G(x^-)) \le \pi_1$, or $G(x^-) \ge L_{p_1,p_2}(x)$ for any $x \in [-1, 1]$. Given the lower bound is a continuous function, this condition is actually equivalent to $G(x) \ge L_{p_1,p_2}(x)$ for any $x \in [-1, 1]$ and $0 \ge L_{p_1,p_2}(-1)$.[23] The latter condition holds automatically given $p_2 - 1 \le \pi_1$ and so can be ignored.

It remains to show that $U_{p_1,p_2} \ge L_{p_1,p_2}$ for any $x \in [-1, 1]$ (so that a G between the two bounds exists) if and only if π_i is defined as in (8). Suppose first both prices are positive. Notice that L_{p_1,p_2} is increasing and becomes positive at $x = \pi_1 - p_2$ (which is no less than -1 given $p_2 - 1 \le \pi_1$), and is concave whenever it is positive. U_{p_1,p_2} is increasing and reaches 1 at $x = p_1 - \pi_2$ (which is no greater than 1 given $p_1 - 1 \le \pi_2$), and is convex whenever it is below 1. Let

$$q_i \equiv \frac{\pi_i}{p_i}$$

denote firm i's market share, and we must have $q_1 + q_2 = 1$ under full consumer participation. One can then check that the two bounds coincide and equal firm 2's market share q_2 at $x = p_1 - p_2$. These properties of the two bounds imply that $U_{p_1,p_2} \ge L_{p_1,p_2}$ for any $x \in [-1, 1]$ if and only if the two bounds have the same slope at $x = p_1 - p_2$,

[23] If G has a mass point at $x = -1$, the former constraint alone does not rule out the possibility of $L_{p_1,p_2}(-1) > 0$ (which, however, is not compatible with $G(x^-) \ge L_{p_1,p_2}(x)$ at $x = -1$).

which requires $\pi_2/p_2^2 = \pi_1/p_1^2$. Together with $\pi_i = p_i q_i$ and $q_1+q_2 = 1$, this then implies $q_i = p_i/(p_1 + p_2)$ and $\pi_i = p_i^2/(p_1 + p_2)$ as defined in (8).

Suppose now at least one price is zero. If $p_1 > p_2 = 0$, one can check that the only possibility for $U_{p_1,0} \geq L_{p_1,0}$ for all $x \in [-1, 1]$ is when $\pi_1 = p_1$ and $\pi_2 = 0$ (if we stipulate $0/0 = \infty$), in which case $U_{p_1,0} = L_{p_1,0} = 0$ for $x < p_1$ and $U_{p_1,0} = 1 > L_{p_1,0}$ for $x \geq p_1$. (The case of $p_2 > p_1 = 0$ is similar.) If $p_1 = p_2 = 0$, they can be sustained as equilibrium prices only if the consumer regards the two products as perfect substitutes, i.e., if G is degenerate at $x \equiv 0$. One can check that both $L_{0,0}$ and $U_{0,0}$ equal the step distribution function with all the mass at $x = 0$ (if we stipulate $0/0 = \infty$). Hence, our result holds trivially.

We now prove the symmetric-price result in (i) when prices are positive. Notice that $U_{p_1,p_2}(0) = \frac{\pi_2}{p_1}$ and $L_{p_1,p_2}(0) = 1 - \frac{\pi_1}{p_2}$. The "only if" part is straightforward. If $p_1 = p_2$, then (8) implies that $U_{p_1,p_2}(0) = L_{p_1,p_2}(0) = \frac{1}{2}$, and so $G(0) = \frac{1}{2}$. To prove the "if" part, suppose $G(0) = \frac{1}{2}$. If (p_1, p_2) are the equilibrium prices under G, the bounds condition (9) at $x = 0$ requires

$$1 - \frac{\pi_1}{p_2} \leq \frac{1}{2} \leq \frac{\pi_2}{p_1} \ ,$$

from which we have $\frac{p_2}{p_1} \leq 2q_1$ and $\frac{p_1}{p_2} \leq 2q_2$, where q_i is firm i's market share defined before. This, together with $q_1 + q_2 = 1$, implies $\frac{p_1}{p_2} + \frac{p_2}{p_1} \leq 2$. This can hold only if $p_1 = p_2$.

(ii) If (p_1, p_2) are the equilibrium prices under some distribution G, we must have $p_j - 1 \leq \pi_i$ as pointed out before, and part (i) implies that firm i's equilibrium profit π_i must take the form in (8). To prove the "if" part, first consider positive prices. Notice that when $p_j - 1 \leq \pi_i$, we have $U_{p_1,p_2}(1) = 1$ and $L_{p_1,p_2}(-1) = 0$. Meanwhile, as shown in the proof of part (i) we have $U_{p_1,p_2} \geq L_{p_1,p_2}$ for any x when π_i takes the form in (8). Hence, there always exists a distribution G which is between the two bounds. Then the desired result follows from part (i). If $p_1 = p_2 = 0$, they satisfy $p_j - 1 \leq \pi_i$ and can be supported as equilibrium prices by a degenerate distribution at $x \equiv 0$. If $p_1 > p_2 = 0$, then $p_j - 1 \leq \pi_i$ requires $p_1 \leq 1$. In this case, the two prices can be sustained in equilibrium by, for example, a degenerate distribution at $x \equiv p_1$. (The case of $p_2 > p_1 = 0$ is similar.)

Finally, we show $|p_i - p_j| \leq 1$ and $p_i \leq 2$ in any pure strategy equilibrium. From $p_i - 1 \leq \pi_j \leq p_j$, it is clear that the price difference cannot exceed 1. To prove the second result, suppose $p_1 \geq p_2$ without loss of generality. Then $p_1 - 1 \leq \pi_2 \leq \frac{1}{2}p_2 \leq \frac{1}{2}p_1$, where the second inequality is because in equilibrium firm i's market share is $q_i = p_i/(p_1 + p_2)$ and so the firm with a higher price has a higher market share. Then $p_1 \leq 2$ follows immediately. ■

Proof of Lemma 2. (i) First consider the firm problem. Suppose that an asymmetric posterior G induces an equilibrium with prices (p_1, p_2). Without loss of generality, let us consider the case with $p_1 \le p_2$ (and so $\pi_1 \le \pi_2$). (Figure 2a illustrates the two bounds on G in this case. Recall that even an asymmetric posterior will induce symmetric equilibrium prices $p_1 = p_2$ if $G(0) = \frac{1}{2}$.) In this case, we must have $\pi_1 + \pi_2 \le p_2$. Since G is an MPC of the prior, its lower bound L_{p_1,p_2} must satisfy

$$\int_{-1}^{\tilde{x}} L_{p_1,p_2}(x)dx \le \int_{-1}^{\tilde{x}} F(x)dx \tag{19}$$

for any $\tilde{x} \in [-1, 1]$. Now consider a symmetric CDF $\hat{G}(x)$ which equals $L_{2\pi_2}(x)$ for negative x, where $L_{2\pi_2}$ is the lower bound of the posteriors which induce a symmetric equilibrium with price $p = 2\pi_2$. We show below that $\hat{G}$ is an MPC of the prior, and so it is a legitimate posterior and induces a symmetric equilibrium in which each firm earns π_2. That is, $\hat{G}$ outperforms G for each firm.

To prove $\hat{G}$ is an MPC of the prior, given (19) it suffices to show $L_{2\pi_2} \le L_{p_1,p_2}$ for negative x. $L_{2\pi_2}$ is positive and equal to $1 - \frac{\pi_2}{2\pi_2+x}$ when $x > -\pi_2$; $L_{p_1,p_2}(x)$ is positive and equal to $1 - \frac{\pi_1}{p_2+x}$ when $x > \pi_1 - p_2$. Given $\pi_1 + \pi_2 \le p_2$, the latter is positive in a larger range of x, and at the same time for $x > -\pi_2$ we have

$$\frac{\pi_1}{p_2 + x} \le \frac{\pi_1}{\pi_1 + \pi_2 + x} \le \frac{\pi_2}{2\pi_2 + x} ,$$

where the second inequality is from $\pi_1 \le \pi_2$. Therefore, we must have $L_{2\pi_2} \le L_{p_1,p_2}$. (If $p_1 < p_2$, then $\pi_1 < \pi_2$ and $\pi_1 + \pi_2 < p_2$, and so $L_{2\pi_2} < L_{p_1,p_2}(x)$ whenever the latter is positive. The same argument then works for a symmetric price p which is slightly above $2\pi_2$. As a result, $\hat{G}$ strictly outperforms G for each firm.)

(ii) We then consider the consumer problem. Suppose that an asymmetric posterior G induces an equilibrium with prices (p_1, p_2) and industry profit π. It is now more convenient to focus on the case of $p_1 \ge p_2$. (Figure 2c illustrates the two bounds on G in this case.) Let $\Delta = p_1 - p_2 \ge 0$, and also notice that we must have $\pi_2 \le \frac{1}{2}\pi \le \pi_1$ and $\pi \le p_1$. Total match efficiency in this equilibrium is

$$\mu + \int_{-1}^{0} G(x)dx - \int_{0}^{\Delta} xdG(x) , \tag{20}$$

where the last term is the efficiency loss from the mismatch caused by the price dispersion.[24]

[24] When the consumer has an expected $x \in (0, \Delta)$, she should choose firm 1 (and get utility v_1) from the social planner's perspective, but she actually chooses firm 2 (and gets utility v_2 instead). This leads to an efficiency loss $v_1 - v_2 = x$.

If $\int_{-1}^{0} G(x)dx < \pi$, consumer surplus then must be less than μ, and so G will be dominated by no information disclosure (which induces a symmetric but degenerated posterior).[25]

From now on we assume

$$\int_{-1}^{0} G(x)dx \geq \pi \ . \tag{21}$$

We aim to show that there exists a symmetric posterior which induces a symmetric equilibrium with the same industry profit π but (weakly) higher total efficiency. (If $p_1 > p_2$, the following argument implies that a strict improvement exists.)

Consider a symmetric CDF

$$\hat{G}(x) = \begin{cases} 0 & \text{if } x < m \\ U_\pi(x) & \text{if } m \leq x \leq 0 \end{cases} ,$$

where $U_\pi(x)$ is the upper bound of posteriors which induce a symmetric equilibrium price $p = \pi$, and $m \in (-1, 0)$ is a constant such that

$$\int_{-1}^{0} G(x)dx = \int_{-1}^{0} \hat{G}(x)dx \ . \tag{22}$$

Notice that given $\pi_2 \leq \frac{1}{2}\pi$ and $p_1 \geq \pi$, we have

$$G(x) \leq U_{p_1,p_2}(x) = \frac{\pi_2}{p_1 - x} \leq U_\pi(x) = \frac{\pi/2}{\pi - x} \tag{23}$$

whenever U_{p_1,p_2} is less than 1 (which must be the case for a negative x). Meanwhile, given $U_\pi(x) \leq \frac{1}{2}$ for negative x and (21), we have

$$\int_{-\pi/2}^{0} U_\pi(x)dx \leq \frac{\pi}{4} < \int_{-1}^{0} G(x)dx \ ,$$

where the second inequality is implied by (21). These two observations imply that (22) has a unique solution $m \in (-1, -\frac{\pi}{2})$, and $\hat{G}$ crosses G at most once and from below. The latter observation and (22) imply that

$$\int_{-1}^{\tilde{x}} \hat{G}(x)dx \leq \int_{-1}^{\tilde{x}} G(x)dx \leq \int_{-1}^{\tilde{x}} F(x)dx$$

for $\tilde{x} \in [-1, 0]$, where the second inequality is because G is an MPC of F. Therefore, the symmetric $\hat{G}$ is an MPC of F and so is a legitimate posterior.

[25] In this case, the dominating distribution does not generate the same profit as the original distribution. This is why this Lemma 2 cannot be directly used to prove the welfare frontier result in Proposition 3 later.

Meanwhile, given $m < -\frac{\pi}{2}$ and $-\frac{\pi}{2}$ is the x at which the lower bound $L_\pi(x)$ becomes positive, $\hat{G}$ must be also between L_π and U_π for negative x. Hence, $\hat{G}$ induces a symmetric equilibrium with industry profit π. It also yields total welfare $\mu + \int_{-1}^{0} \hat{G}(x)dx$, which is (weakly) greater than (20) under G given (22). Therefore, a symmetric posterior $\hat{G}$ with m defined in (22) is at least the same good as G for the consumer. ■

Proof of Proposition 1. Notice that whenever L_p is positive, $1 - L_p = \frac{p}{2(p+x)}$ is log-convex. Therefore, when $1 - F$ is log-concave in the range of negative x, the single-crossing property as in the uniform example continues to hold. Then the same argument as in that example implies that the firm-optimal symmetric price must solve

$$\int_{-1}^{0} L_p(x)dx = \int_{-1}^{0} F(x)dx = \delta \ ,$$

from which we derive p^* in (12). This equality immediately implies that the symmetric posterior G which equals L_{p^*} for negative x is the unique optimal posterior, and it also leads to no mismatch and so maximizes total surplus. ■

Proof of Corollary 1. When f is log-concave, the full-information price is well defined and is equal to $p_F = 1/(2f(0))$. When f is log-concave, it must be weakly increasing and so F must be weakly convex in the range of negative x. Then we have

$$\delta = \int_{-1}^{0} F(x)dx \geq \int_{\frac{-1}{2f(0)}}^{0} (\tfrac{1}{2} + xf(0))dx = \frac{1}{8f(0)} = \tfrac{1}{4}p_F \ , \tag{24}$$

where the inequality follows since F lies above its tangent at $x = 0$. Using this result and (12), we have $p^* \geq \frac{1}{2(1-\log 2)}p_F$.

On the other hand, when f is log-concave, F is log-concave and so $\frac{F(x)}{f(x)}$ is increasing. Then we have

$$\delta = \int_{-1}^{0} F(x)dx = \int_{-1}^{0} \frac{F(x)}{f(x)}dF(x) \leq \frac{F(0)^2}{f(0)} = \tfrac{1}{2}p_F \ . \tag{25}$$

Using this result and (12), we have $p^* \leq \frac{1}{1-\log 2}p_F$. ■

Proof of Proposition 2. Following the same argument as in the main text, we know that the consumer-optimal symmetric price must be below the full-information price p_F. Notice that $U_p = \frac{p}{2(p-x)}$ is log-convex when it is less than one. Hence, when prior CDF F is log-concave in the range $[-1, 0]$, the upper bound U_p crosses F once and from above in that range for any $p < p_F$. Then the same argument used for the uniform example shows that an optimal G given $p < p_F$ is given by (13). This immediately implies that under the consumer-optimal solution there is mismatch and total welfare is not maximized.

With G in (13), consumer surplus is (14) and its derivative with respect to p is

$$\int_{x_p}^{0} \frac{\partial U_p(x)}{\partial p} dx - 1 = \frac{1}{2}\left(\frac{p}{p - x_p} - \log\frac{p}{p - x_p} - 3\right) , \tag{26}$$

where x_p is the intercept point of $F(x)$ and $U_p(x)$. Since $F(x_p) \equiv U_p(x_p)$, it follows that $\frac{p}{p-x_p} = 2F(x_p)$, and so (26) equals $\frac{1}{2}(2F(x_p) - \log(2F(x_p)) - 3)$. This decreases in p since the intercept point x_p increases with p (which is because the upper bound crosses F from above). Therefore, the optimal intercept point x^* satisfies $2F(x^*) = \gamma$, or $x^* = F^{-1}(\frac{1}{2}\gamma)$. The optimal price p^{**} then satisfies $\frac{p^{**}}{p^{**}-x^*} = 2F(x^*) = \gamma$, from which we obtain $p^{**} = \frac{-\gamma}{1-\gamma}x^*$ and so (15). ■

Proof of Corollary 2. When f is log-concave, F is convex on $[-1, 0]$, and so it is below the linear line $\frac{1}{2}(1 + x)$. This implies $F^{-1}(\frac{1}{2}\gamma) \geq \gamma - 1$. Using this result and (15), we have $p^{**} \leq \gamma$. Also, p^{**} should never exceed δ. (This is because δ is the maximum match efficiency improvement relative to a random match. Given the consumer-optimal policy is better than no-information disclosure (in which case consumers buy a random product at a zero price), the consumer-optimal price should not exceed δ.) When f is log-concave, we have known from (25) that $\delta \leq \frac{1}{2}p_F$, and so $p^{**} \leq \frac{1}{2}p_F$.

On the other hand, when F is convex on $[-1, 0]$, we have $\frac{\gamma}{2} = F(x^*) \geq \frac{1}{2} + f(0)x^* = \frac{1}{2}(1 + \frac{x^*}{p_F})$, where the first equality is from the proof of Proposition 2 and the final equality is from $p_F = \frac{1}{2f(0)}$. This implies $\frac{-x^*}{p_F} \geq 1 - \gamma$. Since $p^{**} = \frac{-\gamma}{1-\gamma}x^*$, we deduce $p^{**} \geq \gamma p_F$. ■

Proof of Proposition 3. Under Conditions 1 and 2(ii), the full-information price p_F is well defined and F is log-concave on $[-1, 0]$. When $p \geq p_F$, it is possible to find a supporting posterior G between the bounds which is an MPC of F such that (3) holds with equality (i.e. the full efficiency is achieved). This is because given $p < p^*$ the lower bound L_p must have (3) hold with strict inequality, while given $p \geq p_F$ the log-convex upper bound U_p must be above the log-concave F in the range of $x < 0$. One way to construct such a G is: $G(x) = L_p(x)$ for $x < m$ and $G(x) = U_p(x)$ for $m \leq x \leq 0$, where m is chosen to make (3) bind. Since there is no mismatch with such a posterior, this achieves the maximum consumer surplus $\mu + \delta - p$ for a given p.

To prove the result for $p < p_F$, we first need the following result to rule out asymmetric posteriors:

Lemma 4 *When F is log-concave on $[-1, 0]$, if an asymmetric posterior induces a pure strategy equilibrium with industry profit less than p_F, there exists a symmetric posterior* *which induces the same profit but weakly higher total welfare (and so weakly higher consumer surplus).*

Proof. Suppose that a posterior G induces an asymmetric equilibrium with industry profit $\pi < p_F$. Without loss of generality, suppose the equilibrium prices are $p_1 \geq p_2$, and so the bounds for G are like in Figure 2c. Let $\Delta = p_1 - p_2 \geq 0$, and notice in this case we have $\pi_2 \leq \frac{1}{2}\pi \leq \pi_1$ and $\pi \leq p_1$. As explained in the proof of Lemma 1, total welfare in this case is (20).

Now consider a symmetric CDF $\tilde{G}(x) = \min\{F(x), U_\pi(x)\}$ for negative x, where U_π is the upper bound for posteriors which induce a symmetric equilibrium with price $p = \pi$. Given $\pi < p_F$, the log-convex U_π must cross the log-concave F once and from above in the range of negative x. Notice that $\tilde{G}$ must be an MPC of the prior and so is a legitimate posterior. It is also between the bounds U_π and L_π since $L_\pi < L_{p_F} \leq F$. Hence, $\tilde{G}$ induces a symmetric equilibrium with profit π, the same as under G, and total welfare $\mu + \int_{-1}^{0} \tilde{G}(x)dx$. If this total welfare is higher than that under G, then $\tilde{G}$ leads to higher consumer surplus.

Notice that U_{p_1,p_2}, the upper bound of G, is log-convex, and so it crosses F once at, say, $\hat{x} \in (-1, 0)$. Then

$$\begin{aligned}
\int_{-1}^{\Delta} G(x)dx - \Delta G(\Delta) &\leq \int_{-1}^{0} G(x)dx \\
&\leq \int_{-1}^{\hat{x}} F(x)dx + \int_{\hat{x}}^{0} U_{p_1,p_2}(x)dx \\
&= \int_{-1}^{0} \min\{F(x), U_{p_1,p_2}(x)\}dx \\
&\leq \int_{-1}^{0} \min\{F(x), U_\pi(x)\}dx \\
&= \int_{-1}^{0} \tilde{G}(x)dx \ .
\end{aligned}$$

Here the second inequality used that G is an MPC of F (which implies $\int_{-1}^{\hat{x}} G(x)dx \leq \int_{-1}^{\hat{x}} F(x)dx$), and the last inequality used that given $\pi_2 \leq \frac{1}{2}\pi$ and $\pi \leq p_1$, we have

$$U_{p_1,p_2}(x) = \frac{\pi_2}{p_1 - x} < U_\pi(x) = \frac{\pi/2}{\pi - x}$$

whenever the former is less than 1 (which must be the case for a negative x). This proves the desired result.[26] ■

[26] Note that this argument does not always extend to the case when U_{p_1,p_2} crosses F multiple times, and that is why we need F to be log-concave.

Now we can focus on symmetric posteriors when we solve for the maximum consumer surplus given industry profit $p < p_F$. Then the same analysis as in the consumer-optimal problem in section 3.4 applies, and the maximum consumer surplus, for a given $p < p_F$, equals (14). In this case total welfare is not maximized, and as shown in the proof of Proposition 2, the derivative of (14) with respect to p decreases in p. Hence, the welfare frontier is concave in p, and it reaches the optimum at the positive consumer-optimal price in (15). ■

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Online Appendix
[Not For Publication]

In this online appendix, we report the omitted proofs and details for the various extensions discussed in section **4.**

A.1 Section 4.1 on relevant outside option

We first report a sufficient condition for (16), the condition for firms to obtain the first-best profit.

Claim 1 *Suppose that v_1 and v_2 are independently distributed with density which weakly decreases over the support $[0,1]$. Then condition (16) is satisfied and the rank signal structure generates first-best profit.*

Proof. Using $\mu_H + \mu_L = 2\mu$, we rewrite (16) as $3\mu_H \geq 4\mu$. Let $H(v_i)$ be the CDF for each valuation, with weakly decreasing density $h(v_i)$. Then the CDF for the variable $\max\{v_1, v_2\}$ is $H^2(v)$, and

$$\begin{aligned} 3\mu_H - 4\mu &= \int_0^1 \{3[1 - H^2(v)] - 4[1 - H(v)]\}dv \\ &= \int_0^1 [1 - H(v)][3H(v) - 1]dv \\ &= \int_0^1 \frac{(1-z)(3z-1)}{h(H^{-1}(z))}dz \ , \end{aligned}$$

where $H^{-1}(\cdot)$ is the inverse function to $H(\cdot)$. Here, the first equality is from integration by parts and the assumption that the lower bound of v_i is zero, and the final equality follows by changing variables from v to $z = H(v)$. Noting that the above integrand is negative for $z < \frac{1}{3}$ and positive for $z > \frac{1}{3}$, and that $h(H^{-1}(z))$ weakly decreases with z, it follows that

$$3\mu_H - 4\mu \geq \frac{1}{h(H^{-1}(\frac{1}{3}))} \int_0^1 [(1-z)(3z-1)]dz = 0$$

as claimed. ■

In the following, we report the details of the one-dimensional model introduced in section *4.1*. Suppose the consumer values product 1 at $v_1 = 1 + \frac{1}{2}x$ and product 2 at $v_2 = 1 - \frac{1}{2}x$, where $x = v_1 - v_2 \in [-1, 1]$ indicates her relative preference for product 1 and the consumer's average valuation $\frac{1}{2}(v_1 + v_2) \equiv 1$ is constant. Let F be the symmetric prior distribution for x, and let $\delta = \int_{-1}^{0} F(x)dx$ measure the dispersion of the prior distribution. The information environment is the same as introduced in section 2. The consumer will buy a product only if its expected net surplus is positive. We restrict our attention to symmetric signal structures which induce a pure strategy pricing equilibrium.

Extended bounds analysis. We first extend the posterior bounds analysis to this new setup when the market is fully covered in equilibrium. (This analysis will be used to prove the full-coverage result in Claim 2 below.) Consider a symmetric posterior $G(x)$ which induces a symmetric equilibrium price p. Suppose firm 2 deviates to p'. A type-x consumer will buy from firm 2 if and only if $1 - \frac{x}{2} - p' \geq \max\{0, 1 + \frac{x}{2} - p\}$, i.e., if $x \leq \min\{2(1-p'), p - p'\}$. Hence, the no-deviation condition for firm 2 is:

$$p'G(\min\{2(1-p'), p-p'\}) \leq \tfrac{1}{2}p \tag{27}$$

holds for any p' and with equality at $p' = p$. To implement a price $p \leq \frac{1}{2}$ (which is the lowest valuation for a product), the extensive margin $2(1-p')$ does not matter and so the upper bound of G is (11) as before. To implement a higher price $p > \frac{1}{2}$, we need to deal with the extensive margin explicitly.[27]

For convenience, define

$$U_p^M(x) = \min\left\{1, \frac{p}{\max\{0, 2-x\}}\right\} \ ; \ \ U_p(x) = \min\left\{1, \frac{p}{2\max\{0, p-x\}}\right\} \ . \tag{28}$$

Here, $U_p(x)$ is the same upper bound (11) as before, and $U_p^M(x)$ is the upper bound when the outside option binds. One can check that U_p^M and U_p intersect only once, and let $\tilde{x}_p$ denote the solution to $U_p^M(x) = U_p(x)$. Then $U_p^M > U_p$ if and only if $x < \tilde{x}_p$. (Note that $\tilde{x}_p \leq 1$ given p never exceeds $\frac{3}{2}$, the highest valuation for a product.) Using this notation, condition (27) can be written as

$$G(x) \leq \max\{U_p^M(x), U_p(x)\}$$

and

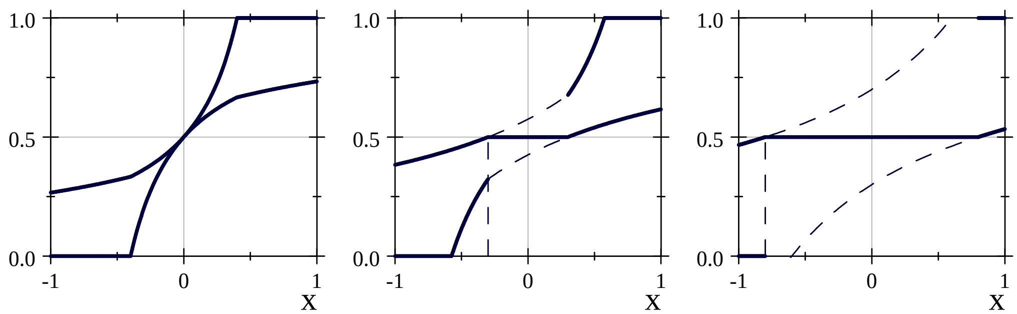
$$G(\min\{-\tilde{x}_p, 0\}) = \tfrac{1}{2} \ . \tag{29}$$

Using firm 1's no-deviation condition, we can derive the lower bound of G. But given G is symmetric, the lower bound is simply the mirror image of the upper bound.

The qualitative form of the bounds depend on the size of p as shown in Figure A1 below. For $p \leq \frac{1}{2}$, the bounds are the same as in section 3. For price $\frac{1}{2} < p \leq 1$, we

[27] Note that if $p > 1$ then a consumer with posterior $x \approx 0$ will not participate. However, the signal could induce a gap in the posterior distribution around $x = 0$, in which case it is possible to have full coverage with a price $p > 1$.

have $\tilde{x}_p \in (-1, 0]$ so the upper bound takes the form of U_p^M for $x < \tilde{x}_p$ as illustrated in Figure A1-a. The upper bound passes through the point $(0, \frac{1}{2})$, and the bounds conditions automatically imply (29) which is now $G(0) = \frac{1}{2}$. In particular, the lower bound in the range $x \in [-1, 0]$ is the same as in section 3. For a price $1 < p < \frac{4}{3}$, the bounds are shown in Figure A1-b. We have $\tilde{x}_p \in (0, \frac{p}{2})$ where $\frac{p}{2}$ is the value of x where U_p reaches 1. The crucial difference is that now (29) implies $G(-\tilde{x}_p) = \frac{1}{2}$. This requires $G(x) = \frac{1}{2}$ for any $x \in [-\tilde{x}_p, \tilde{x}_p]$, and so in this middle range there are no consumers and the upper bound and the lower bound coincide. Finally, for price $p \geq \frac{4}{3}$, we have $\tilde{x}_p > \frac{p}{2}$ and the middle range is so large that the bounds are as shown in Figure A1-c. In particular, the lower bound for negative x is a step function with discontinuity at $-\tilde{x}_p$.



(a) $\frac{1}{2} < p \leq 1$ (b) $1 < p < \frac{4}{3}$ (c) $p \geq \frac{4}{3}$

Figure A1: Bounds on G to implement price $p > \frac{1}{2}$ with binding outside option

Full market coverage. We now show that the optimal signal structure must induce full market coverage in equilibrium.

Claim 2 *A firm-optimal or consumer-optimal symmetric signal structure induces an equilibrium with full market coverage.*

Proof. It is easy to prove that the market is fully covered in the consumer-optimal solution. A consumer-optimal signal structure must be weakly better for the consumer than no information disclosure, where consumers buy a random product at price zero and so consumer surplus is 1. Since the match efficiency improvement relative to random match is at most $\delta = \int_{-1}^{0} F(x)dx \leq \frac{1}{2}$, firms cannot earn more than δ in the consumer-optimal solution. Suppose in contrast to the claim that the market is fully covered that some consumers do not buy, in which case the price must exceed 1 and consumers around $x = 0$ are excluded. Then a feasible unilateral deviation is to offer price at 1, in which case at least half of the consumers will buy from the deviating firm. Hence, each firm's equilibrium profit must be greater than $\frac{1}{2}$, which is a contradiction.

The argument for the firm-optimal policy is less straightforward. To prove that the market is fully covered in the firm-optimal solution, it suffices to show that for any signal structure which induces a partial-coverage equilibrium, there exists another signal structure which induces a full-coverage equilibrium with a strictly higher industry profit.

Consider a symmetric posterior distribution G which is an MPC of F and which induces an equilibrium where each firm charges $p > 1$ and only a fraction $\alpha < 1$ of consumers buy. (If $p \leq 1$ all consumers would buy in equilibrium.) In this case, industry profit is αp. Notice that $\tilde{x}_p \equiv 2(p-1) > 0$ solves $1 + \frac{x}{2} = p$, so consumers with $x \geq \tilde{x}_p$ buy from firm 1 and those with $x \leq -\tilde{x}_p$ buy from firm 2. Other consumers in the range of $(-\tilde{x}_p, \tilde{x}_p)$ are excluded from the market. Industry profit in this equilibrium must be no less than one, i.e., $\alpha p \geq 1$, since each firm could attract half the consumers by offering price 1.

Suppose firm 1 offers the equilibrium price p but firm 2 deviates to p'. A type-x consumer will buy from firm 2 if and only if $1 - \frac{x}{2} - p' \geq \max\{0, 1 + \frac{x}{2} - p\}$. This requires $x \leq \min\{2(1-p'), p - p'\}$. The no-deviation condition for the partial-coverage equilibrium is then $p'G(\min\{2(1-p'), p-p'\}) \leq \frac{1}{2}\alpha p$ for any p', with equality at $p' = p$. Changing variables yields

$$G(x) \leq \alpha \max\{U_p^M(x), U_p(x)\} \text{ and } G(-\tilde{x}_p) = \frac{\alpha}{2} ,$$

where U_p^M and U_p are given in (28). Here, $U_p^M > U_p$ if and only if $x < \tilde{x}_p$. The upper bound passes through the point $(-\tilde{x}_p, \frac{\alpha}{2})$. For our purpose, we only need the lower bound which is the mirror image of the upper bound:

$$L_{\alpha,p}(x) = \begin{cases} 1 - \alpha U_p(-x) & \text{if } x < -\tilde{x}_p \\ \max\{\frac{\alpha}{2}, 1 - \alpha U_p^M(-x)\} & \text{if } -\tilde{x}_p \leq x < 0 \end{cases} .$$

In the following, we will use

$$L_{\alpha,p}^-(x) = \begin{cases} 1 - \alpha U_p(-x) & \text{if } x < -\tilde{x}_p \\ \frac{\alpha}{2} & \text{if } -\tilde{x}_p \leq x < 0 \end{cases} \tag{31}$$

which is weakly lower than $L_{\alpha,p}(x)$.

Let $\hat{p} = \alpha p \geq 1$ and construct a new symmetric posterior which is equal to

$$L_{1,\hat{p}}(x) = \begin{cases} 1 - U_{\hat{p}}(-x) & \text{if } x < -\tilde{x}_{\hat{p}} \\ \frac{1}{2} & \text{if } -\tilde{x}_{\hat{p}} \leq x < 0 \end{cases} \tag{32}$$

in the range of negative x. Note that this is the lower bound of posteriors which support a full-coverage equilibrium with price $\hat{p} \geq 1$. In the following, we show that $L_{1,\hat{p}}$ is a strict MPC of G in the sense of $\int_{-1}^{u} L_{1,\hat{p}}(x)dx < \int_{-1}^{u} G(x)dx$ for any $u \in (-1, 0]$. (Then a similar posterior associated with a price slightly above $\hat{p}$ must be an MPC of G.) Since $L_{\alpha,p}^{-} \leq G$, it suffices to show $L_{1,\hat{p}}$ is a strict MPC of $L_{\alpha,p}^{-}$. One can check that $L_{1,\hat{p}}$ crosses $L_{\alpha,p}^{-}$ only once and from below in the range of negative x. Therefore, it suffices to show

$$\int_{-1}^{0} L_{1,\hat{p}}(x)dx < \int_{-1}^{0} L_{\alpha,p}^{-}(x)dx \ .$$

Using (31) and (32), one can rewrite this condition as

$$1 - \tfrac{1}{2}\alpha p \times \left(1 + \log\left(\frac{4}{\alpha p} - 2\right)\right) < (2-\alpha)(1 - \tfrac{1}{2}p) - \tfrac{1}{2}\alpha p \times \log\left(\frac{4}{\alpha p} - \frac{2}{\alpha}\right) \ ,$$

which further simplifies to

$$\alpha p \times \log \frac{2-p}{2-\alpha p} < 2(\alpha - 1)(p-1) \ .$$

Given $\log x \leq x - 1$, a sufficient condition for the above inequality is

$$\frac{\alpha p}{2 - \alpha p} > \frac{2(p-1)}{p} \ .$$

Since $\alpha p \geq 1$, we have $\frac{\alpha p}{2-\alpha p} \geq 1$. Therefore, the above condition holds if $1 > \frac{2(p-1)}{p}$ or $p < 2$. This must be true given p should never exceed the highest possible valuation $\frac{3}{2}$. This completes the proof. ■

Firm-optimal solution. We now report the firm-optimal symmetric solution.

Claim 3 *When $1 - F$ is log-concave, the firm-optimal solution involves no mismatch, and is as follows:*
(i) when $\delta \leq \frac{1}{2}(1 - \log 2) \approx 0.153$, the firm-optimal price is p^* in (12), which satisfies $p^* \leq 1$, and is uniquely implemented by L_{p^*};
(ii) when $\frac{1}{2}(1 - \log 2) < \delta < \frac{1}{3}$, the firm-optimal price $p^* \in (1, \frac{4}{3})$ solves $\frac{p}{2}[1 + \log(\frac{4}{p} - 2)] = 1 - \delta$ and is uniquely implemented by a modified posterior lower bound;
(iii) when $\delta \geq \frac{1}{3}$ (which is equivalent to (16)), the firm-optimal price is $p^* = \mu_H = 1 + \delta$ which earns firms the first-best profit and is implemented by the rank signal structure.

Proof. The lower bound of the posterior for $x \in [-1, 0]$ across the three cases depicted in Figure A1 can be defined as

$$\tilde{L}_p(x) = \begin{cases} L_p(x) & \text{if } x < \min\{0, -\tilde{x}_p\} \\ \frac{1}{2} & \text{if } \min\{0, -\tilde{x}_p\} \leq x < 0 \end{cases} \ ,$$

and it increases with p. Given $1-F$ is log-concave and $1-L_p$ is log-convex, $\tilde{L}_p$ crosses F at most once and from below in the range of negative x. This implies that the optimal posterior must take the form of the lower bound, and the optimal price p^* solves

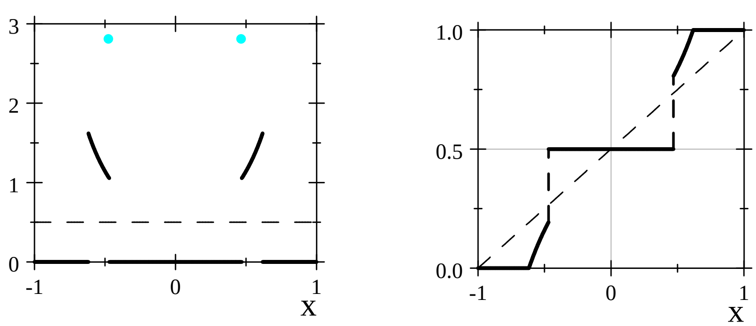
$$\int_{-1}^{0} \tilde{L}_p(x)dx = \delta = \int_{-1}^{0} F(x)dx \ . \tag{33}$$

(This implies there is no mismatch under the firm-optimal signal structure.) We then have: (i) if $\delta \leq \int_{-1}^{0} \tilde{L}_1(x)dx = \frac{1}{2}(1-\log 2)$, (33) has a unique solution $p^* = \frac{2\delta}{1-\log 2} \leq 1$ and $\tilde{L}_{p^*}$ takes the form in Figure A1-a; (ii) if $\frac{1}{2}(1-\log 2) < \delta < \int_{-1}^{0} \tilde{L}_{4/3}(x)dx = \frac{1}{3}$, (33) has a unique solution $p^* \in (1, \frac{4}{3})$ which solves

$$\int_{-1}^{0} \tilde{L}_p(x)dx = 1 - \tfrac{p}{2}[1 + \log(\tfrac{4}{p} - 2)] = \delta \ ,$$

and $\tilde{L}_{p^*}$ takes the form in Figure A1-b; (iii) if $\delta \geq \frac{1}{3}$, which implies (16), (33) has a unique solution $p^* = 1 + \delta$ and $\tilde{L}_{p^*}$ takes the form in Figure A1-c. ■

Intuitively, if the prior distribution is sufficiently concentrated, the firm-optimal price must be low so that the outside option is irrelevant and the solution is the same as in section 3. In contrast, if the prior distribution is sufficiently dispersed that (16) holds the first-best outcome is achievable. In between, the optimal solution is a mixture of these two cases, and it changes smoothly with δ. In all three cases, there is no product mismatch and so total welfare is maximized as well.



density CDF

Figure A2: Firm-optimal G when outside option binds

To illustrate, consider the uniform example with $\delta = \frac{1}{4}$. Case (ii) in Claim 3 applies and the optimal G is as described in Figure A2. The distribution has two symmetric mass points (represented as the dots on the density figure) and no consumers located between them. When δ becomes larger, the optimal distribution has more weight on the two mass points, and as δ approaches $\frac{1}{3}$ it converges to a binary distribution which is implemented by the rank structure and earns firms the first-best profit.

Consumer-optimal solution. The consumer-optimal policy is not affected by the presence of the outside option. As pointed out in the proof of Claim 2, the consumer-optimal price is no greater than δ. Given $\delta = \int_{-1}^{0} F(x)dx \leq \frac{1}{2}$, the consumer-optimal price is no greater than the minimum valuation for a product in our setup. Therefore, the outside option is irrelevant for the consumer-optimal problem.

A.2 Section 4.2 on mixed strategies

Here we report the proof of Proposition *5*. Let F denote the symmetric prior distribution of $x = v_1 - v_2$ and G denote a symmetric posterior distribution of (expected) x. Consumer surplus under G is no greater than $\mu + \int_{-1}^{0} G(x)dx$ (which is the surplus when each consumer buys their preferred product) minus industry profit in a symmetric equilibrium with posterior G. We first derive a lower bound on that industry profit:

Claim 4 *Suppose $V \geq 1$ and let G be a symmetric distribution for $x = v_1 - v_2$. Then in any symmetric equilibrium (with pure or mixed strategies) industry profit is no lower than*

$$\max_{x\in[-1,1]} : \frac{-2xG(x)}{1 - G(x)} \ . \tag{34}$$

Proof. Note that (34) is zero if and only if the distribution G is degenerate at $x = 0$, in which case equilibrium profit is also zero and the result holds. Suppose now that (34) is positive, and slightly abusing the notation we denote its value by $p > 0$. Since x which solves (34) must be negative, we have $p \leq 2$. Suppose in contrast to the statement that there is an equilibrium where each firm obtains profit π^* strictly below $p/2$. Firm 1, say, will never choose a price below π^* in this equilibrium (as then it obtains lower profit even if it serves all consumers). Since a firm's profit increases with its rival's price, firm 2's profit π^* is then at least equal to the maximum profit it can obtain if firm 1 chooses price π^*. Given $\pi^* < p/2 \leq 1 \leq V$, if firm 1 chooses price π^* the outside option is not relevant for consumers, regardless of the price chosen by firm 2. Hence firm 2's profit π^* satisfies

$$\pi^* \geq \max_{p'} : p' \times \Pr\{v_2 - p' \geq v_1 - \pi^*\} = \max_{p'} : p'G(\pi^* - p') = \max_{x\in[-1,1]} : (\pi^* - x)G(x) \ ,$$

where the final equality follows after changing to the variable $x = \pi^* - p'$. Thus for any $x \in [-1, 1]$ we have $(1 - G(x))\pi^* \geq -xG(x)$, in which case π^* is at least equal to $p/2$. As this contradicts our assumption, the result is proved. ■

Slightly abusing notation let p denote (34) for a given G. (The proof of Lemma 4 shows that $p \leq 2$.) Then an upper bound on consumer surplus with posterior G is $\mu + \int_{-1}^{0} G(x)dx - p$. By construction, for any $x \in [-1, 0)$ we have $G(x)/(1-G(x)) \leq \frac{p}{-2x}$, or $G(x) \leq \frac{p}{p-2x}$. (If the lower bound p in Lemma 4 is attained, then G should equal $\frac{p}{p-2x}$ for some $x < 0$.) Since G cannot exceed $\frac{1}{2}$ for $x \in [-1, 0]$, it follows that G in the negative range lies below the upper bound

$$G(x) \leq \hat{U}_p(x) \equiv \min\left\{\frac{1}{2}, \frac{p}{p-2x}\right\} . \tag{35}$$

Here, the upper bound $\hat{U}_p$ increases with p and x, and reaches $\frac{1}{2}$ at $x = -\frac{1}{2}p \geq -1$.

Given the prior distribution F, let τ_p denote the maximum match efficiency when the lower bound on industry profit is p, i.e., $\tau_p =: \max_G \int_{-1}^{0} G(x)dx$ subject to (i) G lying below the upper bound $\hat{U}_p(x)$ in (35) (and touching it at some $x < 0$) and (ii) G being a symmetric MPC of F. Then an upper bound on consumer surplus is $\mu + \max_p(\tau_p - p)$. Notice that for a given p, the optimization problem in defining τ_p is similar to the problem of finding the best symmetric G for consumers under the pure strategy restriction in section 3.4, except that here is the relevant upper bound for G is $\hat{U}_p$ instead of U_p in (11) which for negative x equals $p/(2p - 2x)$ and lies below $\hat{U}_p$. However, for small p, which is usually the relevant case for the consumer-optimal solution, the two bounds are very close and for this reason the use of mixed strategies cannot significantly benefit consumers.

The remaining task is to calculate τ_p, and this can be done in a manner similar to the way we found the consumer-optimal policy for a given price with pure strategies. If the prior has a log-concave density, then F is log-concave on $[-1, 0]$, while the upper bound $\hat{U}_p$ is log-convex in the range $[-1, -\frac{1}{2}p]$. For relatively small p, which will be the relevant case, the upper bound $\hat{U}_p$ therefore crosses the prior F twice.[28] See Figure A3 for an illustration when the prior is uniform. Let $\hat{x}_p$ denote the smaller of the two crossing points given p (i.e. the smaller solution to $\frac{p}{p-2x} = F(x)$). As in section 3.4, two necessary conditions for a feasible G are that it satisfies the MPC constraint at the intercept point $\hat{x}_p$, and that G lies below $\hat{U}_p$ for $x \in [\hat{x}_p, 0]$. The bold curve on Figure A3 is then a convenient candidate for the optimal G.

[28] To see that with the p which maximizes $\tau_p - p$ the upper bound $\hat{U}_p$ crosses the prior F, we argue as follows. For any p we must have $\tau_p \leq \delta$ since G is an MPC of F. Let $\tilde{p}$ denote the price such that $\hat{U}_{\tilde{p}}$ just touches F. Then setting $G \equiv F$ solves the stated problem for $\tau_{\tilde{p}}$, in which case $\tau_{\tilde{p}} = \delta$. For $p > \tilde{p}$, when the upper bound $\hat{U}_p$ lies strictly above F, we must have $\tau_p - p \leq \delta - p < \tau_{\tilde{p}} - \tilde{p}$. As claimed, then, the p which maximizes $\tau_p - p$ is no greater than $\tilde{p}$ and so the upper bound crosses F.

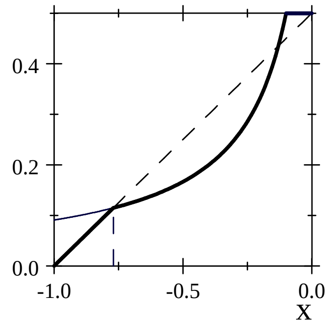


Figure A3: Consumer-optimal way to reach the profit lower bound p

We have not shown that the candidate G illustrated as the bold curve in Figure A3 is an MPC of F, as $\hat{U}_p$ is above F for x close to zero. Therefore, the resulting τ_p based on this G is an upper bound on the feasible match efficiency when the MPC constraint is fully considered.

As with expression (14), an upper bound on consumer surplus given p is therefore

$$\mu + \int_{-1}^{\hat{x}_p} F(x)dx + \int_{\hat{x}_p}^{0} \hat{U}_p(x)dx - p \; . \tag{36}$$

The derivative of this expression with respect to p is

$$\int_{\hat{x}_p}^{-\frac{1}{2}p} \frac{\partial \hat{U}_p(x)}{\partial p} dx - 1 = \frac{1}{2}\left(\frac{p}{p-2\hat{x}_p} - \log\frac{2p}{p-2\hat{x}_p} - \frac{5}{2}\right) \; .$$

It equals $\frac{1}{2}(F(\hat{x}_p) - \log(2F(\hat{x}_p)) - \frac{5}{2})$ by using $\frac{p}{p-2\hat{x}_p} = F(\hat{x}_p)$. Note that $\hat{x}_p$ increases with p given that $\hat{U}_p$ crosses F from above at the smaller of the two crossing points. This derivative therefore decreases with p, and so the point $\hat{x}^*$ which maximizes the upper bound (36) satisfies $F(\hat{x}^*) = \hat{\gamma}$, where $\hat{\gamma} \approx 0.043$ is the root of $\gamma - \log(2\gamma) = \frac{5}{2}$. Evaluating the upper bound (36) at this crossing point $\hat{x}^*$ shows the maximum consumer surplus upper bound to be

$$\mu + \int_{-1}^{\hat{x}^*} F(x)dx - \hat{x}^* F(\hat{x}^*) = \mu - \int_{-1}^{\hat{x}^*} x dF(x) \; . \tag{37}$$

On the other hand, following the proof of Proposition 2, the optimal consumer surplus under the pure strategy restriction is

$$\begin{aligned} \mu + \int_{-1}^{x^*} F(x)dx + \int_{x^*}^{0} U_{p^{**}}(x)dx - p^{**} &= \mu + \int_{-1}^{x^*} F(x)dx - p^{**}\left(1 + \tfrac{1}{2}\log\frac{p^{**}}{p^{**}-x^*}\right) \\ &= \mu + \int_{-1}^{x^*} F(x)dx - x^* F(x^*) \\ &= \mu - \int_{-1}^{x^*} x dF(x) \; . \end{aligned} \tag{38}$$

(Here, the second equality used $\frac{p^{**}}{p^{**}-x^*} = 2F(x^*) = \gamma$, the definition of γ, and (15), while the last equality follows by integration by parts.) This is the same expression as (37) but using $x^* < \hat{x}^*$.

Finally, we show that consumer surplus with pure strategies in (38) comes close to reaching the upper bound in (37). The ratio of (38) to (37) is

$$\frac{\mu - \int_{-1}^{x^*} x dF(x)}{\mu - \int_{-1}^{\hat{x}^*} x dF(x)} = \frac{\mu - x^* F(x^*) + \int_{-1}^{x^*} F(x)dx}{\mu - \hat{x}^* F(\hat{x}^*) + \int_{-1}^{\hat{x}^*} F(x)dx} > \frac{\mu - x^* F(x^*) + \int_{-1}^{x^*} F(x)dx}{\mu - x^* F(\hat{x}^*) + \int_{-1}^{x^*} F(x)dx}$$

$$> \frac{\mu - x^* F(x^*)}{\mu - x^* F(\hat{x}^*)} > \frac{\mu + F(x^*)}{\mu + F(\hat{x}^*)} \geq \frac{1 + F(x^*)}{1 + F(\hat{x}^*)} = \frac{1 + \frac{1}{2}\gamma}{1 + \hat{\gamma}} \approx 0.984 \; .$$

Here, the first inequality uses $\int_{x^*}^{\hat{x}^*} F(x)dx < (\hat{x}^* - x^*)F(\hat{x}^*)$, the third inequality uses $-1 < x^* < 0$, and the final inequality uses the fact that $V \geq 1$ implies $\mu \geq 1$. Thus, when the prior has log-concave density the maximum consumer surplus attainable with pure strategies attains at least 98.4% of the consumer surplus which could be available when mixed pricing strategies were permitted. This completes the proof of Proposition 5.

A.3 Section 4.3 on more firms

We first report the proof of Proposition 6. (i) The first-best result is explained in the main text. Here we report a mixed-strategy pricing equilibrium under the top-product signal structure when condition (17) does not hold, and show that the industry profit in this equilibrium is bounded away from zero even if $n \to \infty$. (It is straightforward but lengthy to verify the equilibrium. All the details are available upon request.[29])

For notational simplicity, define $v = \mu_{n:n}$ and $\delta = \mu_{n:n} - \mu_{<n:n}$, where $\mu_{n:n}$ is the expected valuation for the top product and $\mu_{<n:n}$ is the expected valuation for a non-top product. Under the top-product signal structure, with probability $\frac{1}{n}$ a product is regarded as the top one and so evaluated at v by the consumer, and with probability $1 - \frac{1}{n}$ it is regarded as a non-top one and so evaluated at $v - \delta$. In the following, we focus on the case with $\delta < (1 - \frac{1}{n})v$, the opposite of condition (17) for the best-first result. Then the pricing game with binary valuations has no pure strategy equilibrium. Let $a_n > 0$ be the solution to

$$\frac{1+a}{2+a} = \left(\frac{1}{(n-1)a}\right)^{n-1} \; . \tag{39}$$

[29] It is an extension of the duopoly analysis in Moscarini and Ottaviani (2001) when consumers have binary valuations and receive binary signals.

(Note that the left-hand side increases in a from $\frac{1}{2}$ to 1 and the right-hand side decreases in a from ∞ to 0. Hence, (39) has a unique solution, and it is also easy to see $\lim_{n\to\infty} a_n = 0$.)

When $\delta \leq \frac{v}{2+a_n}$, it is an equilibrium that each firm independently draws a price from the following distribution:

$$\Gamma(p) = \begin{cases} 1 - \left(\frac{n\pi}{p+\delta}\right)^{\frac{1}{n-1}} & \text{if } p \in [l, l+\delta) \\ \frac{n}{n-1}\left(1 - \frac{\pi}{p-\delta}\right) & \text{if } p \in [l+\delta, l+2\delta] \end{cases} ,$$

where $l = a_n \delta$ and $\pi = \frac{l+\delta}{n}$ is each firm's equilibrium profit. (The condition $\delta \leq \frac{v}{2+a_n}$ ensures $l + 2\delta \leq v$.) As $n \to \infty$, the price distribution does not degenerate to 0, and industry profit $n\pi$ converges to $\lim_{n\to\infty} \delta > 0$.

When $\frac{v}{2+a_n} < \delta < (1 - \frac{1}{n})v$, it is an equilibrium that each firm independently draws a price from the following distribution:

$$\Gamma(p) = \begin{cases} 1 - \left(\frac{n\pi}{p+\delta}\right)^{\frac{1}{n-1}} & \text{if } p \in [l, v-\delta) \\ 1 - \left(\frac{n\pi}{v}\right)^{\frac{1}{n-1}} & \text{if } p \in [v-\delta, l+\delta) \\ \frac{n}{n-1}\left(1 - \frac{\pi}{p-\delta}\right) & \text{if } p \in [l+\delta, v) \\ 1 & \text{if } p \geq v \end{cases} ,$$

where l uniquely solves

$$\frac{\delta}{(n-1)l} = \left(\frac{l+\delta}{v}\right)^{\frac{1}{n-1}}$$

and $\pi = \frac{l+\delta}{n}$ is each firm's equilibrium profit. (The condition for this case ensures $l < v - \delta < l + \delta < v$.) This distribution has a gap in the interval $[v - \delta, l + \delta)$ and a mass point on the top. As $n \to \infty$, $l = 0$ and the price distribution does not degenerate to 0, and industry profit $n\pi$ converges to $\lim_{n\to\infty} \delta > 0$ as in the previous case.

(ii) When the outside option is irrelevant and $\{v_i\}$ are i.i.d., the case under full information disclosure is the Perloff and Salop (1985) model.[30] Slightly abusing the notation, let $F(v)$ and $f(v)$ be each v_i's CDF and density function respectively. When $f(v)$ is log-concave, it is known that the equilibrium price in Perloff-Salop is

$$p_F = \frac{1/n}{\int_0^\infty f(v) dF(v)^{n-1}} , \tag{40}$$

and it decreases in n and goes to 0 as $n \to \infty$. Therefore, full disclosure achieves the first-best outcome for the consumer when there are many firms.

[30] Perloff, J., and S. Salop (1985): "Equilibrium with Product Differentiation," *Review of Economic Studies*, 52(1), 107–120.

Under the top-two signal structure, as explained in the main text, firms charge a zero price in equilibrium. Hence, consumer surplus is $\frac{1}{2}(\mu_{n:n} + \mu_{n-1:n})$, where $\mu_{n-1:n}$ is the expected valuation for the second best product. When f is log-concave, we must have $\lim_{n\to\infty}(\mu_{n:n} - \mu_{n-1:n}) = 0$. Therefore, the top-two signal structure also achieves the first-best outcome for the consumer when there are many firms.

Now consider the case with a finite n. The consumer prefers the top-two signal structure if $\frac{1}{2}(\mu_{n:n} + \mu_{n-1:n}) > \mu_{n:n} - p_F$, or equivalently if $\mu_{n:n} - \mu_{n-1:n} < 2p_F$. The CDF of $\max\{v_i\}$ is $F_n(v) \equiv F(v)^n$ and the CDF of the second-best match is $F_{n-1}(v) \equiv F(v)^n + n(1 - F(v))F(v)^{n-1}$. Then

$$\begin{aligned}
\mu_{n:n} - \mu_{n-1:n} &= \int_0^\infty [1 - F_n(v)]dv - \int_0^\infty [1 - F_{n-1}(v)]dv \\
&= n\int_0^\infty (1 - F(v))F(v)^{n-1}dv \\
&= n(\mu_{n:n} - \mu_{n-1:n-1}) \ .
\end{aligned}$$

That is, the expected gap between the best match and the second-best match equals n times the expected difference between the best match among n options and the best match among $n-1$ options. Therefore, it suffices to show $\mu_{n:n} - \mu_{n-1:n-1} < \frac{2}{n}p_F$. In fact, a stronger version of this inequality holds:

$$\mu_{n:n} - \mu_{n-1:n-1} \le \frac{p_F}{n} \ . \tag{41}$$

This means that the match efficiency improvement from having one more firm in the Perloff-Salop model is less than the new entrant's equilibrium profit. This is already proved in Anderson et al. (1995) when f is log-concave.[31] (They use this observation to prove in their Theorem 1 that the Perloff-Salop model with free entry leads to excessive entry.)

Here we report a simpler proof developed in Tan and Zhou (2020).[32] As in Anderson et al. (1995), using the quantile density function $l(t) \equiv f(F^{-1}(t))$ one can rewrite (41) as

$$(n-1)n^2 \int_0^1 \frac{(1-t)t^{n-1}}{l(t)}dt \int_0^1 t^{n-2}l(t)dt < 1 \ . \tag{42}$$

When f is log-concave, the quantile density function $l(t)$ must be concave. Lemma 3 in Tan and Zhou (2020) shows that (42) holds for any concave $l(t)$ if and only if it

[31] Anderson, S., A. de Palma, and Y. Nesterov (1995): "Oligopolistic Competition and the Optimal Provision of Products," *Econometrica*, 63(6), 1281–1301.

[32] Tan, G., and J. Zhou (2020): "Consumer Heterogeneity and Inefficiency in Oligopoly Markets," mimeo.

holds for any triangle-shaped function $l(t) = \frac{t}{\alpha}$ for $t \in [0, \alpha]$ and $\frac{1-t}{1-\alpha}$ for $t \in [\alpha, 1]$, where $\alpha \in (0, 1)$ is a constant. (This is basically because any concave function can be represented as a mixture of triangle-shaped functions.) Substituting the triangle-shaped function into (42) yields

$$\left(1 + \frac{\alpha^n}{(n-1)(1-\alpha)}\right)(1 - \alpha^{n-1}) < 1 \Leftrightarrow \alpha \frac{1-\alpha^{n-1}}{1-\alpha} < n - 1 ,$$

which must be true.

Finally, we report a sufficient condition for the outside option to be irrelevant in the case with n firms if we focus on signal structures which induce a pure strategy pricing equilibrium. Given the following result, the outside option is irrelevant in the sense that if at most one firm deviates from equilibrium all consumers continue to participate.

Claim 5 *If $V > 3$, then under any signal structure which induces a pure strategy pricing equilibrium, there are at least two firms which offer equilibrium prices below V and the consumer can obtain a positive surplus from buying at least from these firms.*

Proof. Without loss of generality consider a pure strategy equilibrium with prices $p_1 \geq p_2 \geq \cdots \geq p_n$. Let π_i be firm i's equilibrium profit.

Case 1: $p_{n-1} = p_n = 0$. In this case our result holds trivially.

Case 2: $p_{n-1} > p_n = 0$, i.e., one firm sets a zero price and all other firms set strictly positive prices. This is impossible in equilibrium as firm n can then always raise its price slightly and at the same time still has a positive demand. (This is because under any signal structure, given products are *ex ante* symmetric, there must be a positive probability that the consumer weakly prefers firm n's product over others.)

Case 3: $p_n > 0$, i.e., all firms set strictly positive prices. In this case, all firms must earn strictly positive profits in equilibrium. Suppose in contrast a firm earns a zero profit due to a zero demand. It can then deviate and lower its price to obtain a positive profit, for example, by setting a strictly positive price which is below both p_n and the consumer's highest possible (expected) valuation for its product. This implies that in equilibrium we must have $p_i \leq V + 1$ and $p_1 - p_n \leq 1$. (Recall that $V + 1$ is the valuation upper bound, and if $p_1 - p_n > 1$, the consumer would never buy from firm 1.)

Now consider the no-deviation conditions for firms $n-1$ and n. First, in equilibrium we must have $p_{n-1} - 1 \leq \pi_n$, since firm n can win the whole market by setting a price slightly below $p_{n-1} - 1$. (Note that given $p_{n-1} \leq V + 1$, the consumer prefers to buy product n at price $p_{n-1} - 1$ than to buy nothing.) Second, we also have $p_n - 1 \leq \pi_{n-1}$ for a similar reason. Adding these two inequalities together implies

$$p_{n-1} + p_n - 2 \leq \pi_n + \pi_{n-1} \leq p_{n-1} ,$$

where the second inequality follows since the total profit of firms $n-1$ and n cannot exceed the maximum price p_{n-1}. Then we deduce $p_n \leq 2$. Since $p_1 - p_n \leq 1$, it follows $p_1 \leq 3$, and so when $3 < V$, all prices must be less than V. ■