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By

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Mixed Bundling in Oligopoly Markets*

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Abstract

This paper proposes a framework for studying competitive mixed bundling with an arbitrary number of firms. We examine both a firm's incentive to introduce mixed bundling and equilibrium tariffs when all firms adopt the mixed-bundling strategy. In the duopoly case, relative to separate sales, mixed bundling has ambiguous impacts on prices, profit and consumer surplus; with many firms, however, mixed bundling typically lowers all prices, harms firms and benefits consumers.

Keywords: bundling, multiproduct pricing, price competition, oligopoly
JEL classification: D43, L13, L15

1 Introduction

There are many circumstances where consumers are offered a package of products at a discounted price relative to the sum of the component prices. This selling strategy is called "mixed bundling." Examples include software suites, TV-internet-phone bundles, home and auto insurance bundles, package tours, value meals, lawn care and landscaping packages, gas and electricity in some regions, and so on. (In the extreme form of "pure bundling," all component products are sold in a package only and no individual products are available for purchase.)

The possible rationales for bundling and the impact of bundling on market performance are classic economic questions that have received wide attention. Early research on bundling focuses on the monopoly case. It is pointed out that, aside from some obvious reasons such as the cost savings in production and transactions,

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bundling can be a strategy to price discriminate and extract more surplus from consumers (e.g., Stigler (1968), Adams and Yellen (1976), and McAfee, McMillan, and Whinston (1989)).[1] Bundling can also be used by a multiproduct firm to exclude smaller rivals that only supply a subset of the products (e.g., Whinston (1990), Choi and Stefanadis (2001), Carlton and Waldman (2002), and Nalebuff (2004)). This is the usual antitrust concern about bundling.

In many examples of bundling, however, the market structure is relatively stable and several competing multiproduct firms operate there. In that case, bundled discounting is usually not intended to exclude rivals from the market but is simply a business strategy to attract consumers to buy more products from the same firm. This paper studies mixed bundling in such a competitive environment. Given the prevalence of bundled discounting among competing firms, there is already substantial research on this phenomenon, such as Matutes and Regibeau (1992), Anderson and Leruth (1993), Reisinger (2004), Thanassoulis (2007, 11), and Armstrong and Vickers (2010). Nevertheless, all the existing papers focus on the duopoly case. Little is understood about how the degree of competition in terms of the number of firms might affect firms' incentives to adopt the mixed-bundling strategy and the impact of mixed bundling on market performance. This is the first paper that studies competitive mixed bundling with an arbitrary number of firms. It makes the following contributions.

First, we offer a random-utility framework for studying competitive mixed bundling. To have product differentiation,[2] the existing works usually use a two-dimensional Hotelling model where consumers with different preferences for products are distributed on a square. In a multiproduct environment, it is not convenient to extend this spatial approach of product differentiation to the case with more than two firms. In this paper, we instead adopt a multiproduct version of the random utility framework in Perloff and Salop (1985). This framework can easily accommodate any number of firms; in the duopoly case, it can be converted into a two-dimensional Hotelling model such that we can compare our results with those in the existing literature.[3]

Second, we extend the existing insights on a firm's incentive to use the mixed-bundling strategy (e.g., from McAfee, McMillan, and Whinston (1989), and Chen

[1] See, for example, also Long (1984), Schmalensee (1984), Fang and Norman (2006), and Chen and Riordan (2013).

[2] Introducing product differentiation is necessary for studying competitive bundling if firms have similar cost conditions. Otherwise, prices would settle at marginal costs and there would be no meaningful scope for offering a bundling discount.

[3] Anderson and Leruth (1993) also use a random-utility framework in their duopoly model of mixed bundling, but they focus on the logit setting where the utility shock follows the extreme value distribution. Another important difference is that in our model the utility shock is at the level of individual products, while in Anderson and Leruth (1993) the utility shock is at the bundle level (so that a consumer might like both products individually but dislike the package of the two products).

and Riordan (2013)) to a general oligopoly model. The oligopoly problem can actually be formulated as a monopoly problem but with a random outside option that depends on the equilibrium prices. The particular structure of the outside option in a symmetric competition environment enables us to obtain some clean results. For instance, for any continuous joint valuation distribution, each firm has a unilateral incentive to introduce mixed bundling when there are only two firms,[4] or when there are many firms and a certain tail-behavior condition is satisfied.

Third, we explain why the problem of mixed bundling is much harder once we go beyond duopoly. The main challenge, when there are more than two firms, is how to calculate a firm's demand. From a firm, a consumer can buy both products, one product only, or nothing. In the third option, the consumer can buy both products from a single rival firm to take advantage of its bundling discount or mix and match across all rival firms to assemble a better bundle. Which is better depends on whether the best matched products among the rival firms are from the same firm or two different firms and also depends on the magnitude of the bundling discount. We develop an approach to calculate the demand and characterize the necessary conditions for a mixed-bundling equilibrium (if it exists). However, the equilibrium conditions are hard to deal with in general, and further analytical progress is made only in the duopoly case and the case with many firms. In particular, in the latter case, the equilibrium prices typically have a simple approximation: the bundling discount is approximately equal to half of the single-product markup. When the production cost is zero, the approximate pricing scheme features "50% off for the second product."

Finally, and perhaps most importantly, we show that the impacts of mixed bundling on market prices, profit and consumer welfare can qualitatively depend on the number of firms in the market.[5] The example often highlighted in the existing research is when consumers are uniformly distributed on the Hotelling square. In that case, compared to separate sales, bundling reduces all prices, harms firms and benefits consumers. This leads to the usual perception that mixed bundling is pro-competitive. In this paper, we first argue that this insight is incomplete. There are many other duopoly examples with different valuation distributions where bundling raises single-product prices or even all prices relative to separate sales so that it is possible for bundling to benefit firms and harm consumers or even harm all players. Therefore, in duopoly the impacts of mixed bundling are in general ambiguous, and we should be cautious about the policy implications drawn from some convenient examples such as the widely used Hotelling model with uniformly distributed con-

[4] A similar duopoly result is also derived by Armstrong and Vickers (2010) in their Hotelling setup.

[5] With the assumption of full market coverage, bundling must always harm total welfare as it causes too much one-stop shopping and so sub-optimal match between consumers and products.

sumers. However, with many firms, we show that mixed bundling has less ambiguous impacts, and it usually makes all products cheaper, and therefore harms firms and benefits consumers.

The intuition behind these results is as follows. The bundling discount creates a new competition boundary on which consumers are indifferent between buying both products from a firm and buying both from its rivals. For these marginal consumers, if the firm makes one of its single products slightly cheaper (but keeps the price of the other single product and the bundling discount unchanged), they will switch to buying both products from it. This renders the price reduction "doubly profitable," which is a force for bundling to intensify price competition and which is emphasized in the literature. However, the bundling discount also shifts the position of all the marginal consumers who will respond to a firm's price reduction and so potentially changes their density. This effect is subtler and depends on the shape of the consumer valuation distribution. For example, in the Hotelling model with a uniform distribution, this second effect does not exist because the position of those marginal consumers does not affect their density. This is why in that case all prices go down in the regime of mixed bundling. For other distributions, it can be well the case that the bundling discount decreases the density of marginal consumers, which goes against and sometimes even dominates the "double profit" effect. This is the source of the potentially ambiguous impacts of mixed bundling. However, if the bundling discount is small, we show that the second effect is of second order relative to the "double profit" effect, and so mixed bundling intensifies competition. In other words, a small bundling discount is generally pro-competitive. When there are many firms, competition leaves little scope for offering a bundling discount and the discount is indeed small in equilibrium. This explains why mixed bundling has unambiguous impacts when there are many firms.

Among the existing papers on competitive mixed bundling, Armstrong and Vickers (2010) is the most general study so far if there are only two firms and each consumer needs to buy all products.[6] They consider a general symmetric consumer distribution on the Hotelling square, allow for the existence of an exogenous shopping cost and also consider elastic demand and general nonlinear pricing schedules. Our paper is more general in terms of considering more than two firms, but otherwise, it focuses on the simple case with unit demand and without an exogenous shopping cost. Some of our analysis (e.g., the local-deviation argument used in various places) parallels theirs.

There are also many works on competitive pure bundling. See, for example, Matutes and Regibeau (1988), Economides (1989), Kim and Choi (2015), Zhou

[6] Thanassoulis (2007, 11) study the case when some "small" consumers only need one product and highlight the possible distributional effect of bundling on different types of consumers. This consumer heterogeneity is absent in both Armstrong and Vickers (2010) and this paper.

(2017), Hurkens, Jeon, and Menicucci (2019), and Shuai, Yang, and Zhang (2019).[7] The random-utility framework used in this paper follows Zhou (2017). Pure bundling is easier to deal with, so more analytical progress has been made there in a general oligopoly model. It is shown that compared to separate sales, pure bundling tends to relax price competition when the number of firms is above a threshold. A somewhat opposite result is derived in this paper for mixed bundling. This contrast highlights an important difference between mixed and pure bundling: with mixed bundling, the bundling discount is endogenous and becomes small when there are many firms, while pure bundling is like mixed bundling with a fixed and sufficiently large discount regardless of the number of firms. As discussed above, the pro-competitiveness of mixed bundling when there are many firms relies on the discount being small in equilibrium.

Bundling has also been studied in other competitive environments such as auctions. See, for instance, Zhou (2017) for a discussion on how pure bundling among competing firms is related to pure bundling in multi-object auctions as studied in Palfrey (1983) and Chakraborty (1999) (and also to information disclosure in single-object auctions as studied in Board (2009) and Ganuza and Penalva (2010)). From a mechanism-design perspective, Armstrong (2000), Avery and Hendershott (2000), and Jehiel, Meyer-ter-Vehn, and Moldovanu (2007) study the possibility of mixed bundling being a feature of revenue-maximizing design in multi-object auctions. The first two papers mainly focus on the case with binary valuations; in a setup with continuous valuations, Jehiel, Meyer-ter-Vehn, and Moldovanu (2007) show that introducing a discount for the bidder who receives the whole package of objects improves the revenue relative to separate sales and pure bundling. They also calculate the optimal discount in a two-bidder example. This strand of research, however, is very different from studying the equilibrium mixed-bundling tariff among competing firms.

The rest of the paper is organized as follows. Section 2 introduces the model and studies the benchmark case of separate sales. Section 3 examines a firm's individual incentive to introduce a bundling discount. Section 4 characterizes the demand and the equilibrium pricing schedule when all firms use the mixed-bundling strategy and also derives the general formulas of how mixed bundling affects industry profit and consumer surplus compared to separate sales. Section 5 deals with two special cases with two or many firms. Section 6 discusses issues such as multi-stop shopping cost and bundling premium, and Section 7 concludes. All omitted proofs are presented in the appendix.

[7] See Section 7 in Stole (2007) and Section 4 in Armstrong (2016) for surveys of the literature on competitive bundling.

2 The model and the benchmark

Consider a market where each consumer needs to buy two products 1 and 2. The measure of consumers is normalized to one. There are $n \geq 2$ firms, each supplying both products. The unit production cost of each product is normalized to zero, so we can regard prices as markups. Each product is horizontally differentiated across firms (e.g., each firm produces a different variety of the product), but there is no product compatibility issue and consumers can freely mix and match. We adopt a multiproduct version of the random utility framework in Perloff and Salop (1985) to model product differentiation. Let $\mathbf{X}^k \equiv (X_1^k, X_2^k)$, $k = 1, \cdots, n$, denote the random match utilities of firm k's two products for a consumer, and they are privately observed by the consumer. We assume that $\mathbf{X}^k$ is i.i.d. across consumers (e.g., consumers have idiosyncratic tastes for the products from different firms), and is also i.i.d. across firms (so firms are *ex ante* symmetric). Suppose $\mathbf{X}^k$ is distributed according to a common joint cumulative distribution function (cdf) $F(x_1, x_2)$. F has a full-dimensional support $S \subset \mathbb{R}^2$ and a bounded and differentiable probability density function (pdf) $f(x_1, x_2)$. Let $F_i(x)$ and $f_i(x)$, $i = 1, 2$, be the marginal cdf and pdf of X_i^k, and let $[\underline{x}_i, \overline{x}_i]$ be its support (where $\underline{x}_i = -\infty$ and $\overline{x}_i = \infty$ are allowed). We generally allow correlation in a consumer's match utilities for the two products supplied by the same firm; but for some results we consider the special "i.i.d." case with $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ and $F_1 = F_2$, i.e., the case when the two products in each firm are symmetric and have independent match utilities.

We consider a discrete-choice framework where the incremental utility from consuming more than one variety of a product is zero and so a consumer only wants to buy one variety of each product.[8] We also assume that a consumer has unit demand for her preferred variety of each product. If a consumer consumes two products with match utilities (x_1, x_2) (which can be purchased from different firms) and makes a total payment T, she obtains surplus $(x_1 + x_2) - T$.[9]

If a firm sells its two products separately, it chooses a price vector $(\hat{p}_1, \hat{p}_2)$. Let $\hat{P} \equiv \hat{p}_1 + \hat{p}_2$ be the associated bundle price. If a firm adopts the mixed-bundling strategy, it chooses a pair of single-product prices (p_1, p_2) together with a bundling discount $\delta > 0$.[10] Let $P \equiv p_1 + p_2 - \delta$ be the associated bundle price. In either regime

[8]This assumption is standard in the literature on competitive bundling, though it is not always without loss of generality. For example, reading another article on the same subject in a different newspaper, or reading another chapter on the same topic in a different textbook, sometimes improves utility. There are works on consumer demand which extend the usual discrete choice model by allowing consumers to consume multiple versions of a product (see, e.g., Gentzkow (2007)).

[9]Most of bundling papers assume such an additive utility function, and this is also compatible with perfect complements under the assumption of full market coverage. There is some research which studies bundling of substitutes or complements (see, e.g., Long (1984), Armstrong (2013), and Haghpanah and Hartline (2019)).

[10]We assume that consumer purchase cannot be monitored and so it is impossible to implement

the timing is that firms choose their prices simultaneously, and then consumers make their choices after observing all the match utilities and prices. As often assumed in the literature on competitive bundling, the market is fully covered (i.e., all consumers buy both products). This will be the case if consumers do not have outside options, or if they have a sufficiently high basic valuation for each product on top of the above match utilities.

For convenience, we introduce a few pieces of notation. Denote by

$$Y_i^{-k} \equiv \max_{k' \neq k} X_i^{k'}$$

the match utility of firm k's best rival product i, and by

$$Z_i^k \equiv X_i^k - Y_i^{-k}$$

the match utility of firm k's product i relative to its best rival product. When firms sell their products separately and charge the same prices, a consumer will buy firm k's product i if and only if $Z_i^k > 0$. Note that X_i^k and Y_i^{-k} are independent of each other given X_i^k is i.i.d. across firms. Since firms are symmetric, we suppress the superscripts k and $-k$ thereafter.

Let $G(y_1, y_2) \equiv F(y_1, y_2)^{n-1}$ be the joint cdf of (Y_1, Y_2), $g(y_1, y_2)$ be the associated joint pdf, and $G_i(y) \equiv F_i(y)^{n-1}$ be the marginal cdf of Y_i. The joint cdf of (Z_1, Z_2) is

$$H(z_1, z_2) \equiv \int_S F(y_1 + z_1, y_2 + z_2) dG(y_1, y_2) \ ,$$

and let $h(z_1, z_2)$ be the associated joint pdf. (Whenever there is no confusion we ignore the integral region S thereafter.) The marginal cdf of Z_i is $H_i(z_i) \equiv \int F_i(y + z_i) dG_i(y)$ with support $[\underline{x}_i - \overline{x}_i, \overline{x}_i - \underline{x}_i]$, and let $h_i(z_i)$ be the associated marginal pdf. In particular, due to firm symmetry, we have

$$H_i(0) = 1 - \frac{1}{n} \quad \text{and} \quad h_i(0) = \int f_i(y) dF_i(y)^{n-1} \ . \tag{1}$$

Here $H_i(0)$ is the chance that a firm's product i is worse than its best rival product, and $h_i(0)$ is the density of consumers who are indifferent between this firm's product i and its best rival product.

Separate-sales benchmark. We first report the equilibrium in the benchmark regime of separate sales. Since firms compete on each product separately, the market for each product is an independent Perloff-Salop model where only the marginal distribution of that product's match utility matters. Consider the market for product i, and let $\hat{p}_i$ be the (symmetric) equilibrium price.[11] Suppose a firm deviates to

a pricing strategy with a bundling premium $\delta < 0$. See Section 6.2 for a further discussion.

[11] In the duopoly case, Perloff and Salop (1985) have shown that the pricing game has no asymmetric equilibrium. Beyond duopoly Caplin and Nalebuff (1991) show that there is no asymmetric equilibrium in the logit model. More recently, Quint (2014) proves a general result (see Lemma 1 there) which implies that our pricing game of separate sales has no asymmetric equilibrium if f_i is log-concave.

price $\hat{p}_i'$, while other firms stick to the equilibrium price $\hat{p}_i$. Then the demand for the deviating firm's product i is

$$q_i(\hat{p}_i') = \Pr[X_i - \hat{p}_i' > Y_i - \hat{p}_i] = 1 - H_i(\hat{p}_i' - \hat{p}_i) \ .$$

In equilibrium the demand is $q_i(\hat{p}_i) = \frac{1}{n}$ due to firm symmetry, and this is also easy to see by using (1).

The deviating firm's profit from product i is $\hat{p}_i' q_i(\hat{p}_i')$, and for $\hat{p}_i$ to be the equilibrium price the profit should be maximized at $\hat{p}_i' = \hat{p}_i$. From the first-order condition we derive

$$\hat{p}_i = \frac{1}{nh_i(0)} \ , \tag{2}$$

where $h_i(0)$ is defined in (1). Henceforth, we assume that this first-order condition is also sufficient for defining the equilibrium price. This is the case, for example, when f_i is log-concave (see Caplin and Nalebuff (1991)).[12] In the example of uniform distribution with $F_i(x) = x$, we have $h_i(0) = 1$ and so $\hat{p}_i = 1/n$. In the example of extreme value distribution with $F_i(x) = e^{-e^{-x}}$ (which generates the logit model), we have $h_i(0) = (n-1)/n^2$ and so $\hat{p}_i = n/(n-1)$. Generally, $\hat{p}_i$ decreases in n if f_i is log-concave (see, e.g., Anderson, de Palma, and Nesterov (1995), and Zhou (2017)), and $\lim_{n\to\infty} \hat{p}_i = 0$ if and only if $\lim_{x\to\overline{x}_i} \frac{f_i(x)}{1-F_i(x)} = \infty$ (see Zhou (2017)). The latter must be true if f_i is strictly positive on a bounded support.

3 Incentive to use mixed bundling

We first examine, starting from separate sales, whether a firm has a unilateral incentive to introduce the mixed-bundling strategy (so that separate sales cannot be an equilibrium outcome). We need another two pieces of notation: the cdf of Z_i, conditional on $Z_j = z_j$ where $j \neq i$, is

$$H_i(z_i|z_j) \equiv \int_{-\infty}^{z_i} h_i(\zeta_i|z_j) d\zeta_i \ ,$$

where $h_i(\zeta_i|z_j) \equiv h(\zeta_i, z_j)/h_j(z_j)$ is the conditional pdf of Z_i.

Suppose a firm unilaterally deviates from separate sales and introduces a *small* bundling discount $\delta > 0$ (but keeps its single-product prices the same as in the separate-sales equilibrium). Figure 1 below depicts how this small deviation affects consumer demand in the space of (z_1, z_2), where Ω_i, $i = 1, 2$, indicates consumers who buy only product i from the firm in question and Ω_b indicates consumers who buy both products from it. (This local-deviation approach follows McAfee, McMillan, and Whinston (1989) who study bundling incentive in the monopoly case. Figure 1 below is similar to their Figure III.)

[12] Many often used distributions such as uniform, normal, logistic, and extreme value have a log-concave density. Caplin and Nalebuff (1991) provide a weaker sufficient condition which requires f_i to be $-\frac{1}{n+1}$-concave for a given n.

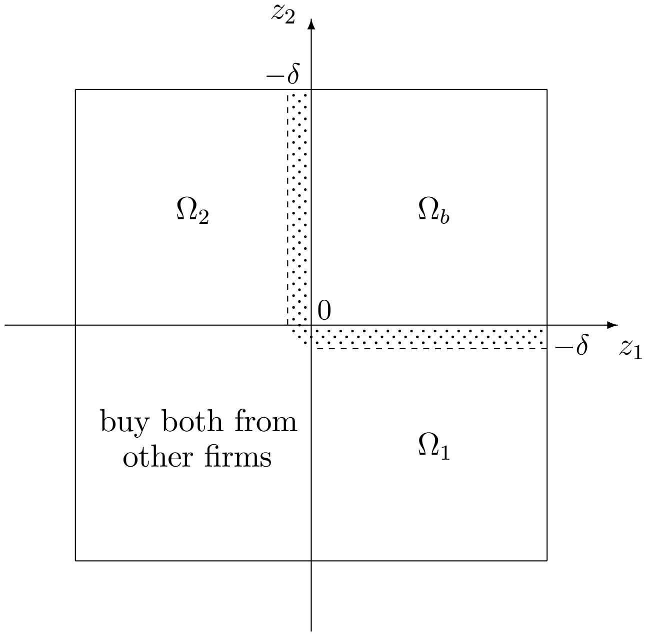

Figure 1: The impact of a small bundling discount on demand

The negative effect of the deviation is that the deviating firm earns δ less from the consumers who buy both products from it. In the regime of separate sales, the measure of those consumers is

$$\Omega_b = 1 - H_1(0) - H_2(0) + H(0,0) = \frac{2}{n} - 1 + H(0,0) \ ,$$

where we have used $H_i(0) = 1 - \frac{1}{n}$. So the (first-order) loss from the small deviation is $\delta\Omega_b$.

The positive effect of the deviation is that more consumers buy both products from the deviating firm, i.e., the region Ω_b expands as indicated on the graph. Those consumers on the two shaded rectangle areas switch from buying only one product to buying both products from the deviating firm, and those on the small shaded triangle area switch from buying nothing to buying both products from the deviating firm.

Notice that given a bounded and continuous joint density, the small triangle area is a second-order effect when δ is small, so only the two rectangle areas matter. The measure of consumers on the vertical rectangle area is $\delta \int_0^\infty h(0, z_2)dz_2$, and the deviating firm now makes an extra profit $\hat{p}_1 - \delta$ from each of them. This yields a (first-order) gain

$$\delta\hat{p}_1 \int_0^\infty h(0, z_2)dz_2 = \frac{\delta}{nh_1(0)} \times h_1(0) \int_0^\infty h_2(z_2|0)dz_2 = \frac{\delta}{n}[1 - H_2(0|0)] \ , \tag{3}$$

where the first equality used the equilibrium condition (2) in the regime of separate sales. Similarly, the measure of consumers on the horizontal rectangle area is $\delta \int_0^\infty h(z_1, 0)dz_1$, and the deviating firm now makes an extra profit $\hat{p}_2 - \delta$ from each of them. This yields another (first-order) gain

$$\delta\hat{p}_2 \int_0^\infty h(z_1, 0)dz_1 = \frac{\delta}{n}[1 - H_1(0|0)] \ . \tag{4}$$

Introducing a small bundling discount is therefore profitable if the sum of these two gains

$$\frac{\delta}{n}[2 - H_1(0|0) - H_2(0|0)] \tag{5}$$

is greater than the loss $\delta\Omega_b$, i.e., if

$$n[1 - H(0,0)] > H_1(0|0) + H_2(0|0) \ . \tag{6}$$

The following result reports simple primitive conditions for (6) to hold:

Proposition 1 *Starting from separate sales with prices defined in (2), each firm has a strict unilateral incentive to introduce mixed bundling if (6) holds.*
(i) For a given distribution F, (6) holds if $n = 2$, or if n is sufficiently large and $\lim_{z_i \to \overline{x}_i - \underline{x}_i} \frac{h(z_1,z_2)}{h_1(z_1)h_2(z_2)} \neq 0$.
(ii) For a given n, (6) holds if X_1 and X_2 are independent, negatively dependent (in the sense that $\Pr(X_i > a|X_j > b)$ is decreasing in b for any a), or limitedly positively dependent (in the sense that $\Pr(X_i > a|X_j > b) \geq \Pr(X_i > a)$ for any a and b, and $\frac{d}{dt}H_i(0|H_j^{-1}(t)) > -1$ for $t \in [1 - \frac{1}{n}, 1]$).

The duopoly and the independence result are relatively easy to understand. Both can be seen from the condition for $\hat{p}_i$ to be the equilibrium price in the benchmark regime of separate sales. With separate sales, if a firm unilaterally lowers its price, say, $\hat{p}_1$ by a small δ, the (first-order) loss is $\delta(\Omega_1 + \Omega_b)$ where $\Omega_1 + \Omega_b$ is the number of consumers who buy product 1 from the firm (and is actually equal to $\frac{1}{n}$ in a symmetric equilibrium), and the (first-order) gain is $\delta\hat{p}_1 \int_{-\infty}^{\infty} h(0, z_2)dz_2$ where $\delta \int_{-\infty}^{\infty} h(0, z_2)dz_2$ is the measure of consumers who switch to buying product 1 from the firm in question, i.e., the vertical shaded area on Figure 1 extended to the lower bound of z_2. These two terms must be equal in equilibrium. Since $h(0, z_2)$ is symmetric around $z_2 = 0$ in the duopoly case, this implies that the first gain in (3) equals $\frac{1}{2}\delta(\Omega_1 + \Omega_b)$. A similar result holds for the second gain (4) when a firm lowers its price $\hat{p}_2$ slightly. Therefore, the sum of the two gains equals $\delta(\Omega_1 + \Omega_b)$ and it is clearly greater than the loss $\delta\Omega_b$ caused by a small bundling discount.

In the independence case, the loss $\delta\Omega_b$ actually *equals* either of the two gains. We can interpret $\delta\Omega_b = \delta \int_0^{\infty} \int_0^{\infty} h(z_1, z_2)dz_1 dz_2$ as the loss when a firm was able to lower $\hat{p}_1$ by a small δ only among the consumers who have $z_2 > 0$, and $\delta\hat{p}_1 \int_0^{\infty} h(0, z_2)dz_2$ in (3) as the associated gain. When Z_1 and Z_2 are independent, these two effects cancel out each other if and only if two similar effects are also equal when the condition $z_2 > 0$ is removed, i.e., if $\delta \int_{-\infty}^{\infty} \int_0^{\infty} h(z_1, z_2)dz_1 dz_2 = \delta\hat{p}_1 \int_{-\infty}^{\infty} h(0, z_2)dz_2$, where the first term is equal to $\delta(\Omega_1 + \Omega_b)$. As explained above, this is just the equilibrium condition for $\hat{p}_1$. Therefore, the sum of the two gains must exceed the loss. (This explanation is the same as in the monopoly case in McAfee, McMillan, and Whinston (1989).)

The result with many firms has a similar intuition as in the independence case. In the proof we show that the loss $\delta\Omega_b$ is approximately equal to one of the gains when n is large. Intuitively, when there are many firms, the measure of consumers in the region of Ω_b is close to zero since a firm's product is almost surely dominated by the best rival product. Then the potential correlation between Z_1 and Z_2 in that region plays a rather limited role.

In general, given other firms are selling their products separately, a firm's problem of whether to introduce mixed bundling is essentially a monopoly problem where a consumer's net valuation for its product i is $X_i - (Y_i - \hat{p}_i)$. Here $Y_i - \hat{p}_i$ is regarded as a random outside option. Then our incentive results in part (ii) of Proposition 1 are closely related to the existing works on the profitability of mixed bundling in a monopoly setting. For example, McAfee, McMillan, and Whinston (1989) have shown a general sufficient condition for mixed bundling to be profitable and the condition must hold when valuations are independent across products. Using a copula approach, Chen and Riordan (2013) have further identified simple primitive conditions when valuations are dependent. (Our proof for the cases with dependent valuations closely follows their approach.) However, the additional structure in our symmetric oligopoly setting leads to the result for $n = 2$ (which has also been derived by Armstrong and Vickers (2010) in their Hotelling model), and the result for a large n (which is new in the literature).

4 Mixed-bundling equilibrium

In this section, we characterize a mixed-bundling pricing equilibrium (if it exists) and derive the general formula of the impacts of mixed bundling on profit and consumer surplus relative to separate sales. We will consider some special cases in next section where more analytical progress can be made.

Consider a symmetric mixed-bundling equilibrium (p_1, p_2, δ), where p_i is the price of single product i and δ is the bundling discount. We focus on the equilibrium with $\delta \le \min\{p_1, p_2\}$, in which case $P = p_1 + p_2 - \delta \ge \max\{p_1, p_2\}$ and so the bundle is no cheaper than any single product.[13]

4.1 Demand

We first need to investigate firms' demand in the mixed-bundling regime. Suppose that a firm unilaterally deviates to a pricing schedule (p'_1, p'_2, δ') with $\delta' \le$

[13] We are not claiming that it is impossible to have an equilibrium with $P < \max\{p_1, p_2\}$. (In that case, at least one single product is never sold alone as long as consumers have free disposal of either single product from the bundle.) Such a possible equilibrium, however, involves firms playing weakly dominated strategies: a firm can always earn the same profit by reducing the price of the more expensive single product so that $P = \max\{p_1, p_2\}$.

$\min\{p_1', p_2'\}$,[14] while other firms stick to the equilibrium pricing schedule. Then for a consumer who values this firm's products at (x_1, x_2) and the best products from other firms at (y_1, y_2), she has the following four purchase options:

(a) buy both products from the deviating firm, in which case her surplus is $x_1 + x_2 - (p_1' + p_2' - \delta')$;

(b) buy product 1 from the deviating firm but product 2 elsewhere, in which case her surplus is $x_1 + y_2 - p_1' - p_2$;

(c) buy product 2 from the deviating firm but product 1 elsewhere, in which case her surplus is $y_1 + x_2 - p_1 - p_2'$;

(d) buy both products from other firms, in which case her surplus is $\Theta - (p_1 + p_2 - \delta)$, where Θ is a random variable conditional on (y_1, y_2) as defined in (7) below.

When the consumer buys only one product, say, product i from some other firm, she will buy the one with the highest match utility y_i. When she buys both products from other firms, however, she does not always buy the two with the highest match utilities (y_1, y_2) if $n \geq 3$. This is because she may choose to buy the two products from a single firm due to the bundling discount but (y_1, y_2) are not realized at that firm. For this reason, (y_1, y_2) is not a sufficient statistic for the match utilities from other firms. This is the main source of the complication in studying competitive bundling when we go beyond the duopoly case. To derive Θ, we discuss two cases:

First, if y_1 and y_2 are realized at the same firm, how to buy in option (d) is simple: the consumer will just buy both products from that firm, and so $\Theta = y_1 + y_2$. Conditional on $Y_1 = y_1$ and $Y_2 = y_2$, this event occurs with probability

$$\lambda(y_1, y_2) \equiv \frac{(n-1)f(y_1, y_2)F(y_1, y_2)^{n-2}}{g(y_1, y_2)} ,$$

where the numerator is the probability in the density sense that $Y_1 = y_1$ and $Y_2 = y_2$ are realized in the same firm among $n - 1$ ones, and the denominator is the joint pdf of (Y_1, Y_2), i.e., the probability that $Y_1 = y_1$ and $Y_2 = y_2$ in the density sense.[15] Notice that (i) when $n = 2$, $\lambda(y_1, y_2) = 1$; (ii) when the two products at each firm have independent match utilities, $\lambda(y_1, y_2)$ simplifies to $\frac{1}{n-1}$ as expected.

Second, with the rest of the probability $1 - \lambda(y_1, y_2)$, y_1 and y_2 are realized at two different firms. Then the consumer faces the trade-off between consuming better-matched products by two-stop shopping, in which case she gets surplus $y_1 + y_2 - (p_1 + p_2)$, or enjoying the bundling discount by one-stop shopping, in which case she gets surplus $Y(y_1, y_2) - (p_1 + p_2 - \delta)$, where $Y(y_1, y_2)$ denotes the match utility of the best bundle among $n - 1$ firms *conditional* on $Y_1 = y_1$ and $Y_2 = y_2$ being realized at different firms. Hence, in this second case, $\Theta = \max\{Y(y_1, y_2), y_1 + y_2 - \delta\}$. (The conditional distribution of $Y(y_1, y_2)$ is important but complicated for demand calculation. We characterize it in Lemma 2 in the appendix.)

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[14] As explained in footnote 13, other pricing schedules are weakly dominated.

[15] More explicitly, $g(y_1, y_2) = (n-1)F(y_1, y_2)^{n-3}[f(y_1, y_2)F(y_1, y_2) + (n-2)\frac{\partial F}{\partial y_1}\frac{\partial F}{\partial y_2}]$.

In sum, conditional on $Y_1 = y_1$ and $Y_2 = y_2$, we have

$$\Theta = \begin{cases} y_1 + y_2 & \text{with probability } \lambda(y_1, y_2) \\ \max\{Y(y_1, y_2), y_1 + y_2 - \delta\} & \text{with probability } 1 - \lambda(y_1, y_2) \end{cases} . \tag{7}$$

When $n = 2$, y_1 and y_2 must be from the same firm and so $\Theta = y_1 + y_2$ for sure. Then the problem can be converted into an often used two-dimensional Hotelling model by using two "location" random variables $Z_1 = X_1 - Y_1$ and $Z_2 = X_2 - Y_2$.

Given (y_1, y_2, θ) where θ is a realization of Θ, Figure 2 below describes how a consumer chooses among the four purchase options in the space of (x_1, x_2).

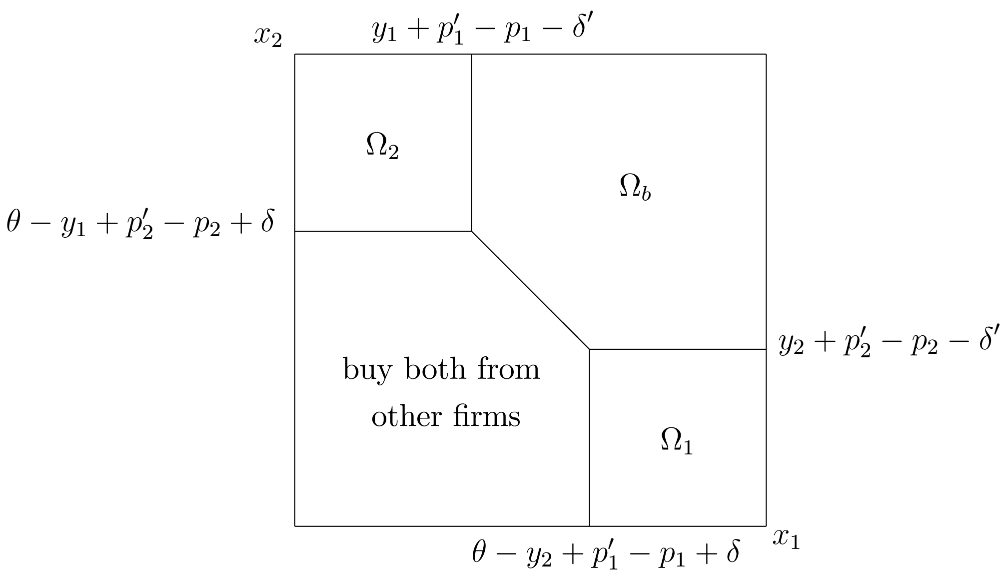

Figure 2: The pattern of consumer choice conditional on (y_1, y_2, θ)

As before, Ω_i, $i = 1, 2$, indicates the region where the consumer buys only product i from the deviating firm, and Ω_b indicates the region where the consumer buys both products from it. Then integrating the area of Ω_i over (y_1, y_2, θ) yields the demand for the deviating firm's single product i, and integrating the area of Ω_b over (y_1, y_2, θ) yields the demand for its bundle.

From Figure 2, we can see that the equilibrium demand for a firm's single product 1 is

$$\Omega_1(\delta) \equiv \mathbb{E}[\int_{\underline{x}_2}^{y_2-\delta} \int_{\theta-y_2+\delta}^{\overline{x}_1} f(x_1, x_2) dx_1 dx_2] , \tag{8}$$

and the equilibrium demand for a firm's single product 2 is

$$\Omega_2(\delta) \equiv \mathbb{E}[\int_{\theta-y_1+\delta}^{\overline{x}_2} \int_{\underline{x}_1}^{y_1-\delta} f(x_1, x_2) dx_1 dx_2] . \tag{9}$$

(All the expectations in this paper are taken over (y_1, y_2, θ).) Given full market coverage, the equilibrium demand depends only on the bundling discount δ but not on any single-product price. Let $\Omega_b(\delta)$ be the equilibrium demand for a firm's bundle. Then we must have

$$\Omega_i(\delta) + \Omega_b(\delta) = \frac{1}{n} \, . \tag{10}$$

This is because, with full market coverage, all consumers buy product i, so $1/n$ of them should buy it from a particular firm (via either single product purchase or bundle purchase). This also implies that $\Omega_1(\delta) = \Omega_2(\delta)$, even when the two products are asymmetric.

4.2 Equilibrium prices

We now characterize the necessary conditions for (p_1, p_2, δ) to be an equilibrium pricing schedule. Using Figure 2, one can write down a firm's deviation profit function and then derive the first-order conditions. To better understand the economics behind the first-order conditions, here we adopt the following graphic approach by considering a few local deviations. (This local-deviation argument is in the spirit of the analysis in McAfee, McMillan, and Whinston (1989) and is also used in Armstrong and Vickers (2010). Readers who want to skip the details can jump to Proposition 2 directly.)

First, suppose a firm unilaterally raises its bundling discount to $\delta' = \delta + \varepsilon$, where $\varepsilon > 0$ is small, while keeps its single-product prices unchanged. Figure 3a below describes, conditional on (y_1, y_2, θ), how this small deviation affects consumer choices: Ω_b expands because now more consumers buy both products from the deviating firm.

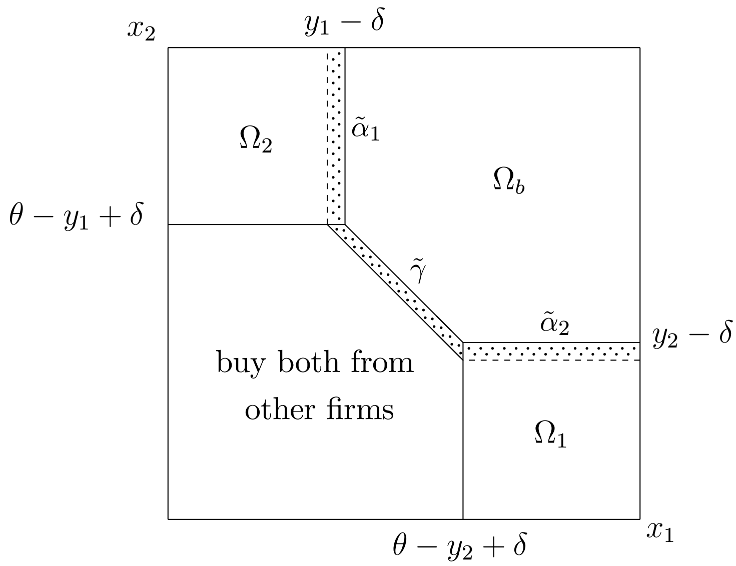

Figure 3a: Price deviation and consumer choice I

The marginal consumers who adjust their purchase are distributed on the shaded areas. Here

$$\tilde{\alpha}_1 = \int_{\theta-y_1+\delta}^{\overline{x}_2} f(y_1-\delta, x_2)dx_2 \text{ and } \tilde{\alpha}_2 = \int_{\theta-y_2+\delta}^{\overline{x}_1} f(x_1, y_2-\delta)dx_1$$

are the densities of marginal consumers along the vertical and horizontal line segments on the graph, respectively, and

$$\tilde{\gamma} = \int_{y_1-\delta}^{\theta-y_2+\delta} f(x_1, \theta-x_1)dx_1$$

is the density of marginal consumers along the diagonal line segment. For the marginal consumers on the vertical shaded area (which has a measure of $\varepsilon\tilde{\alpha}_1$), they switch from buying only product 2 to buying both products from the deviating firm, and so the firm makes $p_1-\delta-\varepsilon$ extra profit from each of them. Similarly, the deviating firm makes $p_2-\delta-\varepsilon$ extra profit from each of the marginal consumers on the horizontal shaded area (which has a measure of $\varepsilon\tilde{\alpha}_2$). For those marginal consumers on the diagonal shaded area (which has a measure of $\varepsilon\tilde{\gamma}$), they switch from buying both products from other firms to buying both from the deviating firm. So the deviating firm makes $p_1+p_2-\delta-\varepsilon$ extra profit from each of them. The only negative effect of the deviation is that those consumers on Ω_b who were already purchasing both products at the deviating firm now each pay ε less. The sum of all these effects integrated over (y_1, y_2, θ) should be equal to zero in equilibrium. After all the second-order effects being discarded, this yields the following first-order condition:

$$\alpha_1(p_1-\delta)+\alpha_2(p_2-\delta)+\gamma(p_1+p_2-\delta)=\Omega_b(\delta) \ , \tag{11}$$

where

$$\alpha_i \equiv \mathbb{E}[\tilde{\alpha}_i], \quad \gamma \equiv \mathbb{E}[\tilde{\gamma}] \ , \tag{12}$$

and $\Omega_b(\delta)$ is defined in (10).

Second, suppose a firm unilaterally raises its stand-alone price p_1 to $p_1' = p_1+\varepsilon$ and its bundling discount to $\delta' = \delta+\varepsilon$ (such that its bundle price remains unchanged). Figure 3b below describes how this small deviation affects consumer choices: Ω_1 shrinks because now fewer consumers buy a single product 1 from the deviating firm.

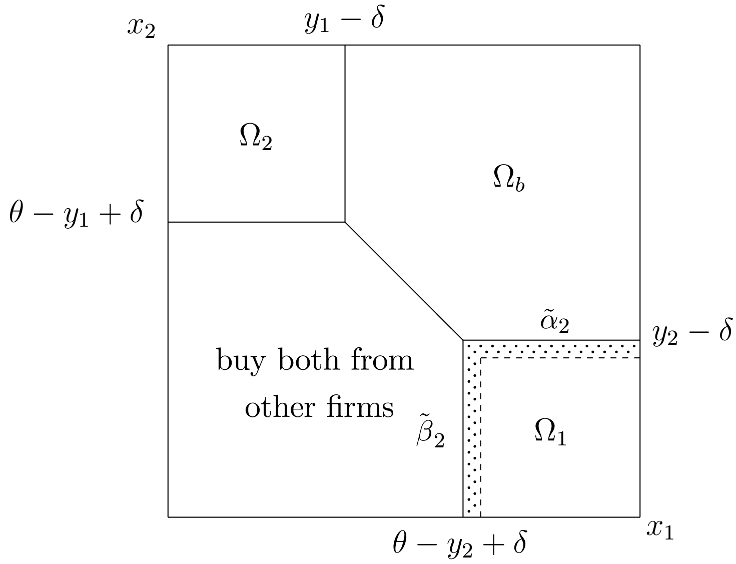

Figure 3b: Price deviation and consumer choice II

Here

$$\tilde{\beta}_2 = \int_{\underline{x}_2}^{y_2-\delta} f(\theta - y_2 + \delta, x_2) dx_2$$

is the density of marginal consumers along the vertical line segment on the graph. For those marginal consumers on the horizontal shaded area (which has a measure of $\varepsilon\tilde{\alpha}_2$), they switch from buying only product 1 to buying both products from the deviating firm. So the firm makes $p_2 - \delta$ extra profit from each of them. For those marginal consumers on the vertical shaded area (which has a measure of $\varepsilon\tilde{\beta}_2$), they switch from buying product 1 to buying nothing from the deviating firm. So the firm loses p_1 from each of them. The direct revenue effect of this deviation is that the firm earns ε more from each consumer on Ω_1. The sum of these effects integrated over (y_1, y_2, θ) should be equal to zero in equilibrium. This yields another first-order condition:

$$\alpha_2(p_2 - \delta) + \Omega_1(\delta) = \beta_2 p_1 \ , \tag{13}$$

where

$$\beta_2 \equiv \mathbb{E}[\tilde{\beta}_2] \ , \tag{14}$$

and $\Omega_1(\delta)$ is defined in (8).

Third, suppose a firm slightly raises its stand-alone price p_2 to $p_2' = p_2 + \varepsilon$ and its bundling discount to $\delta' = \delta + \varepsilon$ (such that its bundle price remains unchanged). (If the two products are symmetric, there is no need to consider this third deviation.) Then Ω_2 shrinks as described in Figure 3c below.

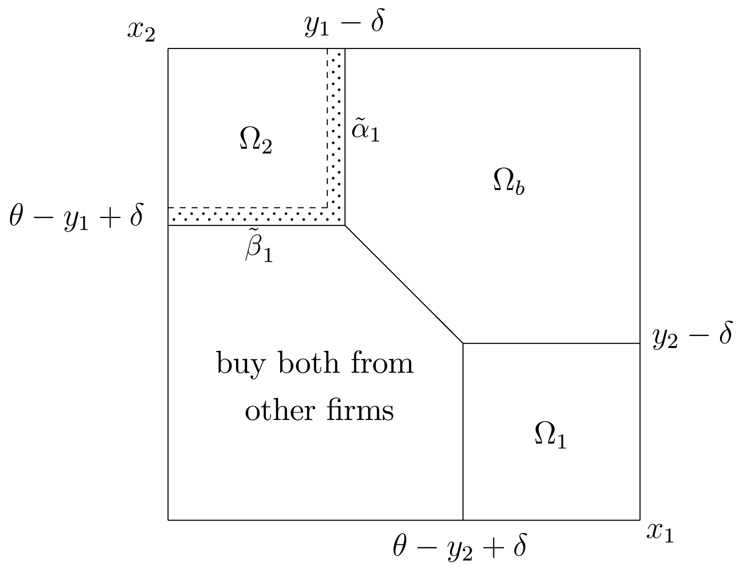

Figure 3c: Price deviation and consumer choice III

Here

$$\tilde{\beta}_1 = \int_{\underline{x}_1}^{y_1-\delta} f(x_1, \theta - y_1 + \delta) dx_1$$

is the density of marginal consumers along the horizontal line segment on the graph. A similar argument as before yields the third first-order condition:

$$\alpha_1(p_1 - \delta) + \Omega_2(\delta) = \beta_1 p_2 \ , \tag{15}$$

where

$$\beta_1 \equiv \mathbb{E}[\tilde{\beta}_1] \ , \tag{16}$$

and $\Omega_2(\delta)$ is defined in (8).

The following result rewrites the above three first-order conditions:[16]

Proposition 2 *If a symmetric mixed-bundling equilibrium with $\delta \leq \min\{p_1, p_2\}$ exists, the single-product prices p_1 and p_2 and the bundling discount δ must satisfy*

$$(\alpha_1 + \beta_2 + \gamma)p_1 + \gamma p_2 - (\alpha_1 + \gamma)\delta = \frac{1}{n} \ , \tag{17}$$

$$(\alpha_2 + \beta_1 + \gamma)p_2 + \gamma p_1 - (\alpha_2 + \gamma)\delta = \frac{1}{n} \ , \tag{18}$$

and

$$(\beta_2 - \alpha_1)p_1 + (\beta_1 - \alpha_2)p_2 + (\alpha_1 + \alpha_2)\delta = 2\Omega_1(\delta) \ , \tag{19}$$

where α_i, β_i and γ are defined in (12), (14) and (16) as functions of δ only.

[16] Notice that (17) is derived from the first and the second first-order condition (11) and (13) by using (10), and (18) is derived from the first and the third one (11) and (15) by using (10). Adding the second first-order condition (13) to the third one (15) and using $\Omega_1(\delta) = \Omega_2(\delta)$ yield (19).

One can check that the first two conditions (17) and (18) are actually the first-order conditions from considering a deviation of raising a single-product price but keeping the bundling discount unchanged. We refer to these two conditions as *single-product price equations.* Since they are linear in p_1 and p_2, from them one can solve p_1 and p_2 as functions of δ. Substituting them into the third condition, which is referred to as *the discount equation*, yields an equation of δ. These equations are more complicated than they appear because all α_i, β_i and γ are functions of δ. In general little can be said on how mixed bundling affects market prices relative to separate sales. More progress will be made in a few special cases studied in Section 5.

Discussion: equilibrium existence. To prove the existence of the above symmetric equilibrium, we need to show that (i) the system of necessary conditions (17)-(19) has a solution with $\delta \leq \min\{p_1, p_2\}$, and (ii) the necessary conditions are also sufficient for defining the equilibrium prices. Unfortunately, both issues are hard to investigate in general. For the first one, we will prove it in the i.i.d. case when $n = 2$ under a log-concavity condition or when n is sufficiently large. For the second one, no analytical progress has been made in general even in the duopoly case.[17] This is an unsolved problem in the literature on mixed bundling.[18]

4.3 Impact of mixed bundling

Given the assumption of full market coverage, the impact of mixed bundling on total welfare is straightforward. Total welfare is solely determined by the match quality between consumers and products. Since the bundling discount induces consumers to one-stop shop too often, mixed bundling must lower match quality and so total welfare relative to separate sales. In the following, we examine the impacts of mixed bundling on industry profit and consumer surplus.

Let $\pi(p_1, p_2, \delta)$ denote the equilibrium industry profit. Then

$$\pi(p_1, p_2, \delta) = p_1 + p_2 - n\delta\Omega_b(\delta) \ .$$

Every consumer buys both products, but those who buy both from the same firm

[17] See, for example, a discussion of this issue in footnote 19 in Armstrong and Vickers (2010). They claim that in the Hotelling setup with uniformly distributed consumers the first-order conditions are also sufficient for defining the equilibrium. We can further verify that in the i.i.d. duopoly case, if the valuation distribution F is uniform or exponential, each firm's profit function is locally concave at the equilibrium prices.

[18] In our pricing game, each firm's profit function is continuous since the consumer match utility distribution is assumed to be continuous. If firms do not choose an infinite price (e.g., due to consumers' budget constraints), their pricing strategy space is compact. Then it is well-known that our pricing game must have a mixed-strategy equilibrium. Moreover, given our game is symmetric, according to Becker and Damianov (2006), there must exist a symmetric mixed-strategy equilibrium.

pay δ less. Thus, relative to separate sales the impact of mixed bundling on industry profit is

$$\begin{aligned}\Delta\pi \equiv \pi(p_1,p_2,\delta) - \pi(\hat{p}_1,\hat{p}_2,0) &= (p_1-\hat{p}_1)+(p_2-\hat{p}_2) - n\delta\Omega_b(\delta) \\ &= P-\hat{P}+n\delta\Omega_1(\delta)\ , \end{aligned} \tag{20}$$

where we used $\Omega_1+\Omega_b=1/n$ in the second equality. (Recall that $\hat{p}_i$ and $\hat{P}=\hat{p}_1+\hat{p}_2$ are respectively the stand-alone price and the bundle price in the regime of separate sales.) Two simple cases are: if bundling lowers both stand-alone prices (and so the bundle price as well), it must harm industry profit; if bundling raises the bundle price, it must enhance industry profit. (When the two products are asymmetric, a more expensive bundle does not necessarily require each single product be more expensive, but the sum of the two stand-alone prices must increase.)

Let $v(\tilde{p}_1,\tilde{p}_2,\tilde{\delta})$ denote the consumer surplus when all firms charge stand-alone prices $(\tilde{p}_1,\tilde{p}_2)$ and offer a bundling discount $\tilde{\delta}$. Given full market coverage, an envelope argument implies that $v_i(\tilde{p}_1,\tilde{p}_2,\tilde{\delta})=-1$, $i=1,2$, and $v_3(\tilde{p}_1,\tilde{p}_2,\tilde{\delta})=n\Omega_b(\tilde{\delta})$, where the subscripts indicate partial derivatives. This is because raising $\tilde{p}_i$ by a small ε will make every consumer pay ε more, and raising the discount $\tilde{\delta}$ by ε will save ε for every consumer who buy both products from the same firm.[19] Then relative to separate sales, the impact of mixed bundling on consumer surplus is

$$\begin{aligned}\Delta v &\equiv v(p_1,p_2,\delta)-v(\hat{p}_1,\hat{p}_2,0) \\ &= \int_{\hat{p}_1}^{p_1} v_1(\tilde{p}_1,p_2,\delta)d\tilde{p}_1 + \int_{\hat{p}_2}^{p_2} v_2(\hat{p}_1,\tilde{p}_2,\delta)d\tilde{p}_2 + \int_0^{\delta} v_3(\hat{p}_1,\hat{p}_2,\tilde{\delta})d\tilde{\delta} \\ &= (\hat{p}_1-p_1)+(\hat{p}_2-p_2)+n\int_0^{\delta}\Omega_b(\tilde{\delta})d\tilde{\delta} \\ &= \hat{P}-P-n\int_0^{\delta}\Omega_1(\tilde{\delta})d\tilde{\delta}\ , \end{aligned} \tag{21}$$

where we used $\Omega_1+\Omega_b=1/n$ in the last equality. This formula implies that if bundling makes each single product cheaper, it must improve consumer welfare even if the bundling discount causes some product mismatch. This is simply because of a revealed-preference argument: consumers can at least buy the same products as they would buy in the case of separate sales but now at lower prices. In contrast, if bundling makes the bundle more expensive, it must harm consumers. This is because in this case we already know that industry profit must go up but bundling always reduces total welfare.[20]

[19] More rigorously, raising the discount slightly will also increase the number of consumers who choose to one-stop shop, but the affected consumers are those who were initially almost indifferent between one-stop shopping and two-stop shopping. Therefore, the impact from these marginal consumers is of second order when ε is small.

[20] From (20) and (21), we have $\Delta\pi+\Delta v = n[\delta\Omega_1(\delta)-\int_0^{\delta}\Omega_1(\tilde{\delta})d\tilde{\delta}]$, a formula of how mixed bundling impacts total welfare relative to separate sales. Consistent with the claim made before, this impact must be negative as $\Omega_1(\tilde{\delta})$ decreases in $\tilde{\delta}$.

In sum, if bundling makes each single product cheaper, it must harm firms and benefit consumers; if bundling makes the bundle more expensive, it must help firms and harm consumers; the less clear case is when bundling makes single products more expensive but the bundle cheaper.

5 Special cases

To make more progress, in this section we focus on the i.i.d. case where the two products at each firm are symmetric and have independent match utilities. Slightly abusing the notation, let $F(x)$ and $f(x)$ be the common cdf and pdf of X_i, and let

$$H(z) = \int F(y+z)dF(y)^{n-1} \text{ and } h(z) = \int f(y+z)dF(y)^{n-1}$$

be respectively the common cdf and pdf of $Z_i = X_i - Y_i$. (When $n = 2$, $h(z)$ is symmetric around zero.) Let p be the common single-product price, and let $\alpha = \alpha_i$ and $\beta = \beta_i$. Then the single-product price equations (17) and (18) simplify to

$$p = \frac{1/n + (\alpha+\gamma)\delta}{\alpha+\beta+2\gamma} \ , \tag{22}$$

and the discount equation (19) simplifies to

$$(\beta - \alpha)\, p + \alpha\delta = \Omega_1(\delta) \ . \tag{23}$$

The difficulty in making analytical progress is from the complication of α, β, γ, and $\Omega_1(\delta)$. However, they are simple in the duopoly case, and they also have simple (first-order) Taylor approximations if δ is small, which, as we show below, is usually the case when the number of firms is large.

5.1 Revisit the duopoly case

We first revisit the duopoly case. With $\Theta = y_1 + y_2$, our random utility model can be converted into a two-dimensional Hotelling model with two "location" variables $Z_i = X_i - Y_i$, $i = 1, 2$. This case has been extensively studied in the literature (see, e.g., Armstrong and Vickers (2010) for a general treatment). Here we report some results that have not been noticed before.

Using the symmetry of h, one can check that[21]

$$\alpha = \beta = h(\delta)[1 - H(\delta)] \ ; \ \ \gamma = 2\int_0^\delta h(t)^2 d \ ; \ \ \Omega_1(\delta) = [1 - H(\delta)]^2 \ .$$

[21] It may not be obvious to derive the expression for γ from (12). Using $\theta = y_1 + y_2$ in the duopoly case, we have $\gamma = \mathbb{E}[\int_{y_1-\delta}^{y_1+\delta} f(x_1)f(y_1+y_2-x_1)dx_1] = \mathbb{E}[\int_{-\delta}^{\delta} f(y_1-t)f(y_2+t)dt]$, where the second equality is from changing the variable from x_1 to $t = y_1 - x_1$. Then the definition of h implies $\gamma = \int_{-\delta}^{\delta} h(-t)h(t)dt = 2\int_0^\delta h(t)^2 dt$, where the second equality is from the symmetry of h in the duopoly case. All these expressions for α, β, γ and Ω_1 can also be seen from Figure 5 below.

Thus, the single-product price equation (22) further simplifies to

$$p = \frac{\delta}{2} + \frac{1}{4(\alpha + \gamma)} , \tag{24}$$

and the discount equation (23) simplifies to

$$\delta = \frac{1 - H(\delta)}{h(\delta)} . \tag{25}$$

The most often studied example in the literature is when H is a uniform distribution.[22] Suppose H is uniform on $[-1, 1]$. Then it is easy to check that in the regime of separate sales, each product is sold at price $\hat{p} = 1$; in the regime of mixed bundling, the single-product price drops to $p = \frac{11}{12}$, the bundling discount is $\delta = 0.5$, and so the bundle price is $P = \frac{4}{3}$. Both the stand-alone products and the bundle become *cheaper* under mixed bundling. Therefore, in this example, mixed bundling intensifies price competition and results in a prisoner's dilemma outcome for firms ($\Delta\pi \approx -0.6$), but it benefits consumers ($\Delta v \approx 0.52$). This observation is what the existing literature highlights.

In the following, we argue that the impacts of mixed bundling in the duopoly case are actually sensitive to the underlying valuation distribution. The following table reports the comparison between the two regimes in a few other examples (where the distributions are F instead of H):

	$\hat{p}$	p	δ	P	$\Delta\pi$	Δv	$\Delta(\pi + v)$
Uniform	0.5	0.57	1/3	0.81	−0.16	0.10	−0.06
Normal	1.77	1.85	1.06	2.63	−1.10	0.93	−0.17
Exponential	1	1.5	1	2	0.07	−0.22	−0.15
Pareto	5/8	1.48	1.37	1.58	0.37	−0.52	−0.15

Table 1: Impact of mixed bundling in duopoly

Prices. In the example of uniform distribution with $F(x) = x$ or standard normal distribution, compared to separate sales, mixed bundling makes each single product more expensive but the bundle cheaper; in the example of standard exponential distribution or Pareto distribution with $F(x) = 1 - \frac{1}{x^2}$ on $[1, \infty)$, it (weakly) increases all prices. More generally, we have the following results:

Proposition 3 *In the i.i.d. duopoly case, suppose that $1 - H(z)$ is log-concave (i.e., $\frac{d}{dz}\frac{1-H(z)}{h(z)} \leq 0$) in $z > 0$ (which is true if f is log-concave). Then*
(i) the system of (25) and (24) has a unique solution with $\delta \in (0, p)$;
(ii) relative to separate sales, mixed bundling lowers the bundle price, and it raises the single-product price if $h(z)$ is decreasing and $\frac{d}{dz}\frac{1-H(z)}{h(z)} \geq -\frac{1}{2}$ for $z > 0$.

[22] This is the standard case in the Hotelling model, but in our random utility setup it is possible only if we consider correlated match utilities across firms since $X - Y$ cannot have a uniform distribution when X and Y are independent of each other.

Recall that our demand analysis is predicated on $\delta \leq p$, i.e., the bundle is no cheaper than each single product. This is true at least under the log-concavity condition as shown in result (i), but can also be true even beyond this log-concavity case as suggested by the Pareto example. Result (ii) shows that under the log-concavity condition the bundle becomes cheaper in the mixed-bundling regime, but each single product becomes more expensive if $\frac{1-H(z)}{h(z)}$ does not decrease too fast (which requires the tail of the density $h(z)$ decrease fast enough). (It appears harder to find a simple general condition for each single product to become cheaper.) The proof of result (ii) also reveals that once $1-H$ becomes log-convex, even the bundle will become more expensive in the mixed-bundling regime. $1-H$ is log-concave in the first two examples and log-convex in the last example, and the exponential example is the edge case where $1-H$ is log-linear and so the bundle price remains unchanged.[23]

Profit and consumer surplus. Armstrong and Vickers (2010) have derived a sufficient condition in their Proposition 4 for mixed bundling to harm firms and benefit consumers in the duopoly case. With our notation, the condition is $\frac{d}{dz}\frac{1-H(z)}{h(z)} \leq -\frac{1}{4}$ for $z \geq 0$. When this condition is not satisfied, the welfare impacts of mixed bundling can be reversed. For instance, in the exponential or Pareto example mixed bundling helps firms but harms consumers. It is also possible that *both* firms and consumers suffer from mixed bundling. Consider a generalized Pareto distribution with $F(x) = 1-(1-ax)^{\frac{1}{a}}$, where $a \in [0,1]$ and the support is $[0, \frac{1}{a}]$. It has a log-concave density, and it becomes the exponential distribution when $a = 0$ and the uniform distribution when $a = 1$. Figure 4 below depicts how the impacts of mixed bundling in this example vary with a.

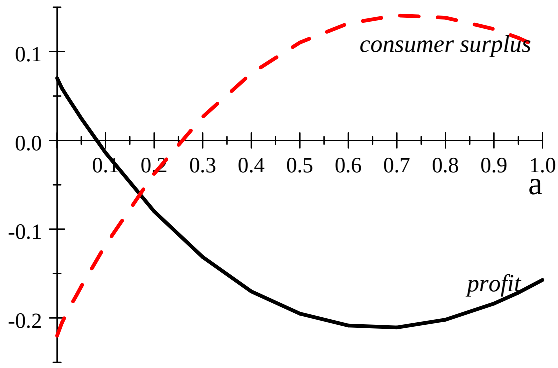

Figure 4: The example of generalized Pareto distribution

When a is sufficiently large, mixed bundling harms firms and benefits consumers as in the uniform example; when a is sufficiently small, the opposite is true as in the exponential example; in between mixed bundling harms both firms and consumers.

[23] When F is expotential, H is a double exponential distribution and so $1-H$ is log-linear in $z > 0$.

Intuition. Why can the underlying valuation distribution qualitatively matter for the impacts of mixed bundling? This can be seen from how the bundling discount affects firms' single-product pricing incentives. Suppose firm 1 raises p_1 by a small $\varepsilon > 0$ but keeps p_2 and δ unchanged. Figure 5 below depicts how this small deviation affects the demand in the (z_1, z_2) space.

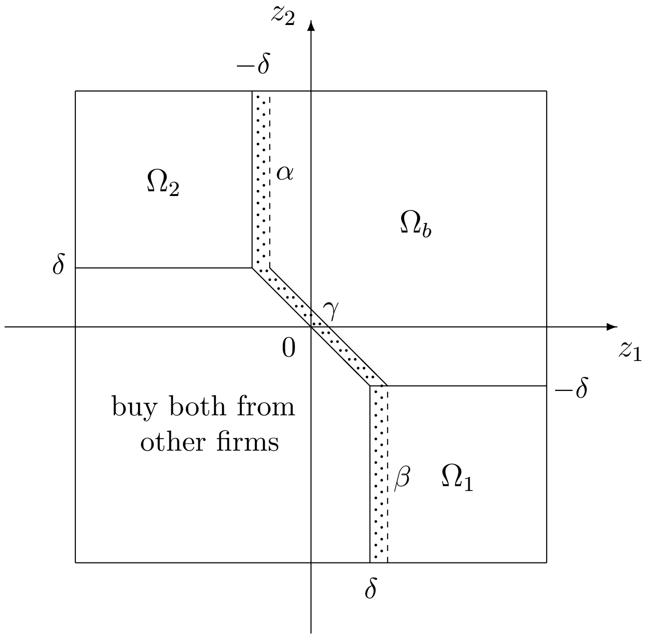

Figure 5: Price deviation and consumer choice in duopoly

Roughly, the marginal consumers with a measure of $\varepsilon(\alpha+\beta+\gamma)$ stop buying firm 1's product 1, causing a loss $\varepsilon p_1(\alpha+\beta+\gamma)$.[24] Among these marginal consumers, those on the diagonal shaded area γ actually stop buying the whole bundle, which causes an extra loss $\varepsilon(p_2-\delta)\gamma$. As emphasized in the literature, this "double loss" effect, or "double profit" effect if we consider a price reduction, is the source for bundling to intensify price competition. However, notice that bundling also changes the position of marginal consumers and so potentially their density: in separate sales with $\delta = 0$, $\alpha+\beta+\gamma = h(0)$; in mixed bundling with $\delta > 0$, $\alpha+\beta+\gamma$ becomes smaller if $h(z)$ is single-peaked at $z = 0$. This is a force to relax price competition. When h is uniform, this second force does not exist, so as we have seen mixed bundling lowers all prices; but if h decreases fast enough on both sides of $z = 0$, this second force can play a dominant role such that bundling raises all prices as we have seen in the exponential or Pareto example. Therefore, when h has heavier tails (in which case the tails usually decrease faster), it is more likely that mixed bundling helps relax price competition.

In sum, in the duopoly case the impacts of mixed bundling on prices, profit and consumer welfare are ambiguous in general. We should be cautious about the

[24] More precisely, those marginal consumers on the shaded area α only pay $p_1 - \delta$ less when they switch from buying the bundle to buying product 2 only, so the loss should be $\varepsilon[p_1(\alpha+\beta+\gamma)-\alpha\delta]$.

policy implications drawn from some convenient examples such as the widely used Hotelling model with uniformly distributed consumers.

5.2 The case with many firms

Another case where we can make analytical progress is when n is large. In this case, we show that mixed bundling has less ambiguous impacts, and under a mild condition it intensifies price competition, harms firms and benefits consumers.

We first report a useful approximation result based on Taylor expansion when δ is small:

Lemma 1 *For any given $n \geq 2$, if δ is close to zero, we have the following approximations in the i.i.d. case:*

$$\begin{aligned} \alpha &\approx \frac{h(0)}{n} - \left(\frac{h(0)^2}{n-1} + \frac{h'(0)}{n} \right) \delta \ , \\ \beta &\approx \left(1 - \frac{1}{n}\right) h(0) - \left(h(0)^2 - \frac{h'(0)}{n} \right) \delta \ , \\ \gamma &\approx \frac{n}{n-1} h(0)^2 \delta \ , \\ \Omega_1(\delta) &\approx \frac{1}{n}\left(1 - \frac{1}{n}\right) - \frac{2}{n} h(0)\delta \ , \end{aligned} \tag{26}$$

where $h(0) = \int f(x) dF(x)^{n-1}$ and $h'(0) = \int f'(x) dF(x)^{n-1}$.

These approximations are much simpler than the original expressions and give us some idea of how these objects vary with δ and n. One observation that is useful for our analysis below is $\alpha + \beta + \gamma \approx h(0)$.

When n is large, we show in the appendix that the system of (22) and (23) usually has a solution with δ close to zero. Then Lemma 1 can be used to approximate the equilibrium mixed-bundling prices.

Proposition 4 *Suppose $\lim_{n\to\infty} \hat{p} = 0$, where $\hat{p} = \frac{1}{nh(0)}$ is the separate-sales price in (2), and $\frac{|h'(0)|}{h(0)}$ is uniformly bounded for any n (which is true if $\frac{|f'(x)|}{f(x)}$ is uniformly bounded). When n is large, the system of (22) and (23) has a solution with $\delta \in (0, p)$ and it can be approximated as*

$$p \approx \frac{1}{nh(0)} \frac{1 + h(0)\delta}{1 + \frac{n}{n-1} h(0)\delta}; \quad \delta \approx \frac{1}{\frac{2h'(0)}{h(0)} + \frac{2n^2-3n+2}{n^2-n} nh(0)} \ . \tag{27}$$

Both the single-product price and the bundle price are lower than in the regime of separate sales.

Note that both required conditions are satisfied if $f(x)$ is strictly positive on a bounded support and $|f'(x)| < \infty$.[25] This proposition implies that when there are many firms in the market, mixed bundling tends to be pro-competitive relative to separate sales (though the impact is small as the outcome is close to perfect competition in either case). As we discussed in the duopoly case, the bundling discount δ makes the competition boundary γ doubly profitable when a firm lowers a single-product price, which is a force to intensify price competition; at the same time δ also shifts the position of marginal consumers along the boundaries α, β and γ and so may affect the density of them, which can be a force in the opposite direction. When δ is small, however, from the approximations in (26), we see that $\alpha + \beta + \gamma \approx h(0)$, i.e., the small discount has no first-order effect on the density of marginal consumers. As a result, only the former "double profit" effect matters. This implies that mixed bundling is pro-competitive if δ is small (which is usually the case when n is large). This discussion also suggests that even with a small number of firms, if the bundling discount is capped at a low level, price competition is fiercer in the regime of mixed bundling than in the regime of separate sales.

For a large n, we can further simplify the approximations in (27) to $p \approx \hat{p}$ and $\delta \approx \frac{1}{2}\hat{p}$. That is, the single-product price is approximately equal to the price in the regime of separate sales and the bundling discount is approximately half of the single-product price. The mixed-bundling scheme in this limit case can thus be interpreted as "50% off for the second product."[26]

When Proposition 4 holds, mixed bundling reduces all prices relative to separate sales, so it must harm firms and benefit consumers according to Section 4.3. Together with the duopoly case, this suggests that the impacts of mixed bundling can qualitatively depend on the number of firms in the market.

6 Discussion

6.1 Multi-stop shopping cost as a bundling discount

A situation similar to mixed bundling is when firms use linear pricing strategies but consumers face an exogenous multi-stop shopping cost $\delta > 0$. That is, if a consumer

[25] Among the examples we studied in the duopoly case, the normal, exponential, and Pareto distribution have an unbounded support, and the generalized Pareto distribution with $a \in (0, 1)$ has $f(\overline{x}) = 0$ and $\lim_{x \to \overline{x}} f'(x)/f(x) = \infty$. But if we consider a properly truncated version of those distributions, these issues disappear and meanwhile the results in those examples remain qualitatively unchanged. For instance, for a truncated exponential distribution with support $[0, 5]$, mixed bundling improves profit by about 0.05 and reduces consumer surplus by about 0.19 in the duopoly case.

[26] This interpretation works only when the production cost is zero. If there is a positive production cost c for each product, we have $\delta \approx (\hat{p} - c)/2$, i.e., the bundling discount is approximately equal to half of the single product's markup.

buys two products from two different firms, she needs to pay an extra cost δ, which reflects, for instance, the extra travelling cost or the transaction cost of paying an additional bill. If the shopping cost is not prohibitively high and some consumers still multi-stop shop, it then affects consumer purchase behavior exactly the same as the bundling discount. Therefore, the method developed in this paper can be used to investigate the impact of multi-stop shopping cost on competition and market performance. We show that the number of firms can play an important role in determining whether the shopping cost harms or improves consumer welfare.

Following a similar analysis as in Section 4.2, one can derive the first-order conditions for (p_1, p_2) to be the equilibrium prices:

$$(\alpha_1 + \beta_2 + \gamma)p_1 + \gamma p_2 = \frac{1}{n} \ ; \quad (\alpha_2 + \beta_1 + \gamma)p_2 + \gamma p_1 = \frac{1}{n} \ .$$

(These two conditions differ from the single-product price equations (17) and (18) because in this shopping-cost case δ does not directly affect firm profit.) In the i.i.d. case, we have

$$p = \frac{1}{n(\alpha + \beta + 2\gamma)} \ .$$

In general it is unclear how $\alpha + \beta + 2\gamma$ varies with the shopping cost δ. However, if the shopping cost δ is small, we can invoke the approximations in Lemma 1 and show that

$$p \approx \frac{1}{nh(0)} \frac{1}{1 + \frac{n}{n-1}h(0)\delta} < \hat{p} \ . \tag{28}$$

That is, introducing a small shopping cost will induce a fiercer competition among firms. The underlying reason is the same as in the case with a small bundling discount explained before.[27]

Although a small shopping friction lowers market prices, it also adversely affects the match quality between consumers and products. As a result, its impact on consumers is less clear.

Proposition 5 *Suppose δ is an exogenous multi-stop shopping cost and firms compete in linear prices. In the i.i.d. case, if δ is small, it intensifies price competition for any n, but it benefits consumers if and only if $n \leq 3$.*

Intuitively, the price-reduction effect of shopping costs is more significant when there are fewer competitors, while the match quality effect is larger when there are more firms. That is why introducing a small shopping cost benefits consumers in equilibrium only when there are relatively few firms. Notice that when n is large,

[27] This argument, however, can fail when δ is larger. For example, when δ is sufficiently large, the situation will be as if all firms use a pure-bundling strategy. According to Zhou (2017), we know that pure bundling often induces higher market prices than in separate sales when n is above a threshold. Hence, in general whether the presence of shopping cost δ intensifies or softens price competition depends both on the magnitude of δ and the number of firms.

the impact on consumers is opposite to what we see in the mixed-bundling case, but this is simply because δ is a shopping cost here but a beneficial discount there.

6.2 Bundling premium

So far we have ruled out the possibility of bundling premium (in which case the bundle price exceeds the sum of component prices). Bundling premium, even if it is desirable for firms, is usually hard to implement as it requires firms be able to monitor consumer purchase behavior; otherwise no consumers would buy the bundle at an additional cost. It might be becoming technically more feasible in the online market (e.g., for some digital goods) where firms have better monitoring technologies.

Starting from the separate-sales equilibrium, the analysis of the incentive to introduce a small bundling premium is very similar to Section 3. The demand pattern when a firm unilaterally charges a small bundling premium $\tau > 0$ is depicted on Figure 6 where Ω_b shrinks and both Ω_1 and Ω_2 expand.

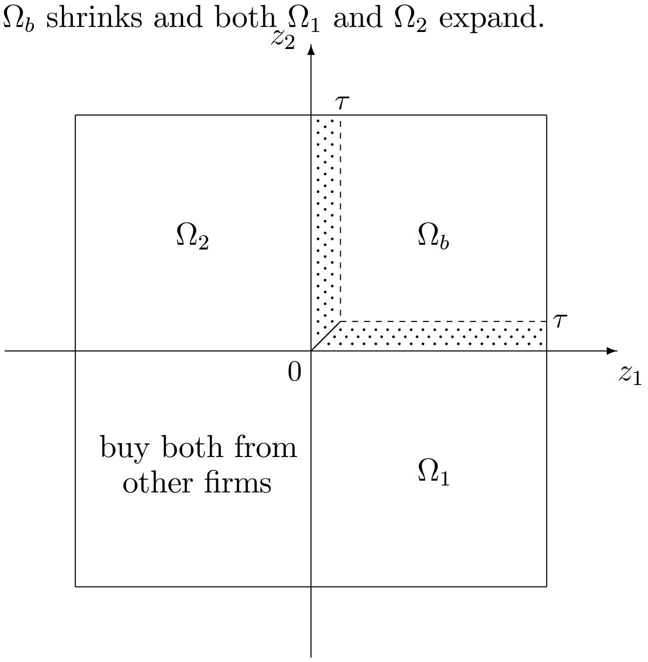

Figure 6: The impact of a small bundling premium on demand

Compared to Figure 1, the competition boundaries differ: Ω_1 and Ω_2 are now connected by a boundary (the short diagonal segment), while Ω_b and the region of buying both products from other firms become disconnected. This, however, does not affect the first-order analysis of a small deviation. One can show that introducing a small bundling premium is profitable if condition (6) is reversed. Therefore, if bundling premium is also feasible, a unilateral deviation to mixed bundling is generically profitable. (McAfee, McMillan, and Whinston (1989) made a similar observation in the monopoly case.)

The necessary conditions for an equilibrium with a bundling premium (if it exists) can be derived similarly as in the case of bundling discount. Instead of going through all the analysis again, here we briefly consider the simple i.i.d. duopoly case. If an equilibrium with a single-product price p and a bundling premium $\tau > 0$ exists, then one can check that (p, τ) must satisfy

$$2p = \frac{H(\tau)}{h(\tau)} \frac{2 - H(\tau)}{1 - H(\tau)} \ ; \ 2(p + \tau) = \frac{1 - H(\tau)}{h(\tau)} \ .$$

They imply an equation of τ:

$$\frac{1 - H(\tau)}{h(\tau)} - \frac{H(\tau)}{h(\tau)} \frac{2 - H(\tau)}{1 - H(\tau)} = 2\tau \ .$$

This equation, however, has no solution of $\tau > 0$ if h is log-concave (which is the case if f is log-concave). When $\tau = 0$, the left-hand side equals $-\frac{1}{h(0)}$, less than the right-hand side; meanwhile, the left-hand side decreases in τ if h is log-concave while the right-hand side increases in τ.[28] Therefore, under the log-concavity condition, this i.i.d. duopoly case has no equilibrium with a bundling premium. Of course, in a more general case (e.g., when the valuations for the two products are correlated), it is possible that bundling premium arises in equilibrium and it can be investigated similarly as in Section 4.

7 Conclusion

This paper has studied competitive mixed bundling in an oligopoly market by using a random-utility framework. It explains the source of difficulty in studying competitive mixed bundling beyond the duopoly case, and develops a method to calculate the demand and characterize the necessary conditions for equilibrium tariffs. Analytical progress on the impacts of bundling on prices, profit and consumer welfare is only made in the duopoly case (where we derive some new results compared to the existing literature) and the case with many firms (where a simple approximation of the equilibrium tariff is offered). We show that the impact of bundling in the duopoly case is sensitive to the underlying consumer valuation distribution, while in the case with many firms, the impact is less ambiguous and bundling tends to lower market prices, harm firms and benefit consumers. This suggests that the number of firms can qualitatively matter for the assessment of the impact of mixed bundling.

As in most theoretical studies on bundling, we have focused on the case with two products only. This is not meant to be realistic. In many examples of bundling, firms sell more than two products. The problem when there are more than two products is that the pricing strategy space will become much more complicated since firms can set a distinct price for each subset of its products. This is hard to deal with even in the duopoly case, as discussed in the appendix of Armstrong and Vickers (2010).

[28] When h is log-concave, both $1 - H$ and H are log-concave, and so $\frac{1-H}{h}$ is decreasing and $\frac{H}{h}$ is increasing; meanwhile, $\frac{2-H}{1-H}$ is clearly increasing.

One possible way to proceed is to consider simple pricing policies such as two-part tariffs or bundle-size pricing schemes as in Chu, Leslie, and Sorensen (2011). Another feature which is not captured in this paper and many other bundling papers is that some consumers may only need a subset of the products. This possible demand heterogeneity has been studied in Thanassoulis (2007, 11) and it can generate some interesting distributional effect of bundling on different types of consumers.

We have also focused on the case where each firm makes their mixed-bundling decisions independently. There are examples where rival firms coordinate on cross-firm joint-purchase discounts (e.g., a discounted city pass that covers various separately owned museums). See, e.g., Gans and King (2006), Armstrong (2013), and Jeitschko, Jung, and Kim (2017) for research on this type of cross-firm bundling. These papers consider single-product firms, but in principle, cross-firm joint-purchase discounts can also be offered by competing multiproduct firms.

Appendix

Proof of Proposition 1: *The duopoly result.* When $n = 2$, we have $Y_i = X_i$. Then conditional on $Z_i = 0$ (i.e., $X_i^1 = X_i^2$), X_j^1 and X_j^2 should share the same conditional distribution. This implies $H_1(0|0) = H_2(0|0) = 1/2$. Meanwhile, $H(0, 0)$ is always strictly less than $H_i(0) = 1/2$. Then

$$2[1 - H(0, 0)] > 2[1 - H_i(0)] = 1 = H_1(0|0) + H_2(0|0) \ .$$

The independence result. When the two products have independent valuations, $H_i(0|0) = H_i(0)$ and $H(0, 0) = H_1(0)H_2(0)$. Given $H_i(0) = 1 - \frac{1}{n}$, it is ready to check that the gain (5) is twice the loss $\delta\Omega_b$ for any n.

To prove the other sufficient conditions for (6), we use the copula approach introduced in Chen and Riordan (2013). (A classic reference on copula is Nelson, 2006.) Let $C(t_1, t_2)$ be the copula associated with the joint cdf H such that $H(z_1, z_2) = C(H_1(z_1), H_2(z_2))$. According to the Sklar's Theorem, such a copula exists uniquely for a given joint cdf if its marginal distributions are continuous. Therefore, a joint cdf can be represented by its marginal cdf's and a copula. A copula itself is a joint cdf on $[0, 1]^2$ with uniform marginal distributions, and it captures the dependence structure of the original distribution. Let $C_i(t_1, t_2)$ be the partial derivative with respect to t_i. Let $d(t) \equiv C(t, t)$ be the diagonal section of C, and it is increasing and uniformly continuous on $[0, 1]$. The following properties on copula are useful:

(a) $C(t_1, 0) = C(0, t_2) = 0$;

(b) $C(t_1, 1) = t_1$ and $C(1, t_2) = t_2$;

(c) $C_i(t_1, t_2)$ is the conditional distribution of t_{-i} given t_i;

(d) $\max\{0, 2t-1\} \le d(t) \le t$.

We first claim that (6) is equivalent to

$$1 - d(t) > (1-t)d'(t) \text{ at } t = 1 - \frac{1}{n} . \tag{29}$$

The definition of copula and $H_i(0) = 1 - \frac{1}{n}$ imply that

$$H(0,0) = C(H_1(0), H_2(0)) = d(1 - \frac{1}{n}) .$$

Using the fact

$$h(z_1, z_2) = C_{12}(H_1(z_1), H_2(z_2))h_1(z_1)h_2(z_2) \tag{30}$$

and property (a), one can check that $H_1(0|0) = C_2(t,t)$ and $H_2(0|0) = C_1(t,t)$ at $t = 1 - \frac{1}{n}$. Then (6) can be written as $n(1 - d(t)) > C_1(t,t) + C_2(t,t)$ at $t = 1 - \frac{1}{n}$ which is equivalent to (29).

The large-n result. Given $d(1) = 1$ (which can be seen from property (b)), (29) holds for a sufficiently large n if $d(t)$ is strictly convex at $t = 1$. Notice that $d''(1) = C_{11}(1,1) + 2C_{12}(1,1) + C_{22}(1,1) = 2C_{12}(1,1)$ since $C_{ii}(1,1) = 0$ (which is again from property (b)). Then $d''(1) > 0$ if and only if $C_{12}(1,1) > 0$, which is equivalent to the condition stated in the proposition according to (30).

To understand the intuition of this large-n result explained in the main text, we approximate the loss and gain from the small bundling discount using the copula. Note that $\Omega_b = \frac{2}{n} - 1 + H(0,0) = \frac{2}{n} - 1 + d(t)$ at $t = 1 - \frac{1}{n}$. When $t = 1 - \varepsilon$ with $\varepsilon \approx 0$, we have $d(t) \approx 1 - 2\varepsilon + \frac{1}{2}d''(1)\varepsilon^2$, where the approximation is from Taylor expansion, $d(1) = 1$ and $d'(1) = 2$ (both of which are from property (b)). Letting $\varepsilon = \frac{1}{n}$ yields $\Omega_b \approx \frac{1}{2n^2}d''(1) = \frac{1}{n^2}C_{12}(1,1)$ when n is large, where we have used the expression for $d''(1)$ derived before. On the other hand, $\hat{p}_1 \int_0^\infty h(0, z_2)dz_2 = \frac{1}{n}[1 - H_2(0|0)] = \frac{1}{n}[1 - C_1(t,t)]$ at $t = 1 - \frac{1}{n}$. When $t = 1 - \varepsilon$ with $\varepsilon \approx 0$, we have $C_1(t,t) \approx C_1(1,1) - \varepsilon(C_{11}(1,1) + C_{12}(1,1)) = C_1(1,1) - \varepsilon C_{12}(1,1)$. Letting $\varepsilon = \frac{1}{n}$ yields $\hat{p}_1 \int_0^\infty h(0, z_2)dz_2 \approx \frac{1}{n^2}C_{12}(1,1)$ when n is large. Therefore, the loss $\delta\Omega_b$ caused by a small discount δ is approximately equal to one of the gains, which is similar as in the independence case.

The negative-dependence result. Since $\Pr(X_i > a | X_j > b)$ decreases in b for any a, for any given realization of (Y_i, Y_j) we have $\Pr(X_i > a + Y_i | X_j > b + Y_j)$ decreases in b. Then $\Pr(Z_i > a | Z_j > b)$ decreases in b for any a. (This is called "right tail decreasing" in Nelson, 2006.) Corollary 5.2.6. in Nelson (2006) then implies that for any $t \in (0,1)$ we have

$$C_i(t,t) < \frac{t - C(t,t)}{1-t}, \; i = 1, 2 .$$

So $(1-t)d'(t) < 2(t - d(t))$. Then a sufficient condition for (29) is

$$1 - d(t) \ge 2(t - d(t)) \Leftrightarrow d(t) \ge 2t - 1 .$$

This is always true given property (d).

The positive-dependence result. As in the proof of Proposition 3 in Chen and Riordan (2013), (29) can be rewritten as

$$1-2t+d(t)+\int_t^1(1-\tilde{t})C_{11}(\tilde{t},t)d\tilde{t}+\int_t^1(1-\tilde{t})C_{22}(t,\tilde{t})d\tilde{t}>0 \text{ at } t=1-\frac{1}{n}\,. \quad (31)$$

(This can be verified by using integration by parts and property (b).) Given $\Pr(X_i > a|X_j > b) \geq \Pr(X_i > a)$ (which is called "positive quadrant dependence" in Nelson, 2006), we have $F(x_1,x_2) \geq F_1(x_1)F_2(x_2)$. This implies that $H(z_1,z_2) \geq H_1(z_1)H_2(z_2)$ and so $d(t) \geq t^2$ for any t. Also notice that $C_1(\tilde{t},t) = H_2(H_2^{-1}(t)|H_1^{-1}(\tilde{t})) = H_2(0|H_1^{-1}(\tilde{t}))$ at $t = 1-\frac{1}{n}$. Then our condition on the conditional distribution implies that $C_{11}(\tilde{t},t) > -1$ for $\tilde{t} \geq t = 1-\frac{1}{n}$. Similarly, $C_{22}(t,\tilde{t}) > -1$ for $\tilde{t} \geq t = 1-\frac{1}{n}$. Then the left-hand side of (31) is strictly greater than $(1-t)^2 - 2\int_t^1(1-\tilde{t})d\tilde{t} = 0$.

Omitted details in demand analysis in section 4.1: Recall that $Y(y_1,y_2)$ denotes the match utility of the best bundle among $n-1$ firms conditional on $Y_1 = y_1$ and $Y_2 = y_2$ being realized at different firms.

Lemma 2 *When $n \geq 3$, the cdf of $Y(y_1,y_2)$ is*

$$\begin{aligned} L(y|y_1,y_2) &= \frac{F_1(y-y_2|y_2)}{F_1(y_1|y_2)}\frac{F_2(y-y_1|y_1)}{F_2(y_2|y_1)} \\ &\quad \times\frac{1}{F(y_1,y_2)^{n-3}}\left(F(y_1,y-y_1)+\int_{y-y_1}^{y_2}\int_{\underline{x}_1}^{y-x_2}f(x_1,x_2)dx_1dx_2\right)^{n-3} \end{aligned} \quad (32)$$

for $y \in [\max\{y_1+\underline{x}_2, \underline{x}_1+y_2\}, y_1+y_2)$, where $F_i(y_i|y_j)$ is the conditional cdf of y_i.

Proof. For a given consumer, let $I(y_i)$, $i = 1,2$, be the identity of the firm where y_i is realized. The lower bound of $Y(y_1,y_2)$ is from the fact that the lowest possible match utility of the bundle from firm $I(y_i)$ is $y_i + \underline{x}_j$. We now calculate the conditional probability of $Y(y_1,y_2) < y$. This event occurs if and only if all of the following three conditions are satisfied: (i) $y_1 + X_2^{I(y_1)} < y$, (ii) $X_1^{I(y_2)} + y_2 < y$, and (iii) $X_1^k + X_2^k < y$ for all $k \neq I(y_1), I(y_2)$ among the $n-1$ competitors. Given y_1 and y_2, condition (i) holds with probability

$$\frac{F_2(y-y_1|y_1)}{F_2(y_2|y_1)}\,,$$

since the cdf of $X_2^{I(y_1)}$ conditional on y_1 and $X_2^{I(y_1)} < y_2$ is $F_2(x_2|y_1)/F_2(y_2|y_1)$. Similarly, condition (ii) holds with probability

$$\frac{F_1(y-y_2|y_2)}{F_1(y_1|y_2)}\,.$$

One can also check (with the help of a graph) that the probability that $X_1^k + X_2^k < y$ holds for a firm other than $I(y_1)$ and $I(y_2)$, is

$$\frac{1}{F(y_1, y_2)} \left(F(y_1, y - y_1) + \int_{y-y_1}^{y_2} \int_{\underline{x}_1}^{y-x_2} f(x_1, x_2) dx_1 dx_2 \right) .$$

(The term in the bracket is the unconditional probability that (X_1^k, X_2^k) lies in the region where $X_i^k < y_i$ and $X_1^k + X_2^k < y$.) Conditional on y_1 and y_2, these three events are independent of each other. Therefore, the conditional probability of $Y(y_1, y_2) < y$ is as stated in (32). ■

With this lemma, we can calculate the expectation of any function $\phi(Y_1, Y_2, \Theta)$ (if exists) as follows:

$$\begin{aligned} \mathbb{E}[\phi(Y_1, Y_2, \Theta)] &= \int_{(y_1, y_2)} [\lambda(y_1, y_2) \times \phi(y_1, y_2, y_1 + y_2) \qquad (33) \\ &\quad + (1 - \lambda(y_1, y_2)) \times \int_y \phi(y_1, y_2, \max\{y, y_1 + y_2 - \delta\}) dL(y|y_1, y_2)] dG(y_1, y_2) . \end{aligned}$$

Proof of Proposition 3: (i) When $1 - H(z)$ is log-concave, the right-hand side of (25) is decreasing, and so the equation has a unique solution $\delta > 0$. From (24) it is evident that $\delta < p$ if $\alpha + \gamma < \frac{1}{2\delta}$. This condition can be written as

$$h(\delta)(1 - H(\delta)) + 2 \int_0^\delta h(t)^2 dt < \frac{1}{2} \frac{h(\delta)}{1 - H(\delta)} \qquad (34)$$

by using (25) and the definitions of α and γ. At $\delta = 0$, the left-hand side is equal to $\frac{1}{2}h(0)$ and the right-hand side is equal to $h(0)$, and so (34) must hold. Meanwhile, the derivative of the left-hand side is $h'(\delta)(1 - H(\delta)) + h(\delta)^2$, and the derivative of the right-hand side is $\frac{1}{2(1-H(\delta))^2}[h'(\delta)(1 - H(\delta)) + h(\delta)]$. Given $1 - H(\delta)$ is log-concave and $1 - H(\delta) < \frac{1}{2}$ for $\delta > 0$, both derivatives are positive but the latter is at least twice the former, and so the result follows.

(ii) The bundle price in the regime of mixed bundling is $2p - \delta = \frac{1}{2(\alpha+\gamma)}$, and that in the regime of separate sales is $\frac{1}{h(0)}$. The former is smaller if and only if $\alpha + \gamma > \frac{1}{2}h(0)$ which equals

$$h(\delta)(1 - H(\delta)) + 2 \int_0^\delta h(t)^2 dt > \frac{1}{2}h(0) .$$

This is true as the equality holds at $\delta = 0$ and the left-hand side is increasing in δ if $1 - H(\delta)$ is log-concave. (Conversely, the opposite is true if $1 - H(\delta)$ is log-convex and so the left-hand side is decreasing.)

$p > \hat{p}$ if the right-hand side of (24) increases in δ given it is equal to $\hat{p}$ at $\delta = 0$. One can check this is true if $2(\alpha + \gamma)^2 > h'(\delta)(1 - H(\delta)) + h(\delta)^2$ for any $\delta > 0$. When h is decreasing in $\delta > 0$, we have $\gamma > 2h(\delta) \int_0^\delta h(t)dt = h(\delta)(2H(\delta) - 1)$, and so $\alpha + \gamma > h(\delta)H(\delta) > \frac{1}{2}h(\delta)$. Therefore, the desired condition holds if $h'(\delta)(1 - H(\delta)) + \frac{1}{2}h(\delta)^2 \leq 0$ for $\delta > 0$, or equivalently if $\frac{d}{d\delta}\frac{1-H(\delta)}{h(\delta)} \geq -\frac{1}{2}$ for $\delta > 0$.

Proof of Lemma 1: We first explain how to calculate $\mathbb{E}[\phi(Y_1, Y_2, \Theta)]$ defined in (33), where the expectation is taken over (Y_1, Y_2, Θ). Using (7) in the i.i.d. case, we have

$$\mathbb{E}[\phi(Y_1, Y_2, \Theta)] = \frac{1}{n-1}\int \phi(y_1, y_2, y_1 + y_2)dG$$

$$+\frac{n-2}{n-1}\int \left(L(y_1 + y_2 - \delta|y_1, y_2)\phi(y_1, y_2, y_1 + y_2 - \delta) + \int_{y_1+y_2-\delta}^{y_1+y_2} \phi(y_1, y_2, y)dL(y|y_1, y_2)\right) dG\,,$$

where $G(y_1, y_2) = F(y_1, y_2)^{n-1}$ and $L(y|y_1, y_2)$ is defined in (32). By integration by parts and using $L(y_1 + y_2|y_1, y_2) = 1$, we can simplify this to

$$\mathbb{E}[\phi(Y_1, Y_2, \Theta)] = \int \phi(y_1, y_2, y_1+y_2)dG - \frac{n-2}{n-1}\int \left(\int_{y_1+y_2-\delta}^{y_1+y_2} \frac{\partial}{\partial y}\phi(y_1, y_2, y)L(y|y_1, y_2)dy\right) dG\,.$$

Now let us derive the first-order approximation of α. (For our purpose, we do not need the higher-order approximations.) According to the formula above, we have

$$\alpha = \int f(y_1 - \delta)(1 - F(y_2 + \delta))dG + \frac{n-2}{n-1}\int \varphi(y_1, y_2, \delta)dG\,, \tag{35}$$

where

$$\varphi(y_1, y_2, \delta) = \int_{y_1+y_2-\delta}^{y_1+y_2} f(y_1 - \delta)f(y - y_1 + \delta)L(y|y_1, y_2)dy\,.$$

When $\delta \approx 0$, we have $f(y_1 - \delta) \approx f(y_1) - \delta f'(y_1)$, so

$$\int f(y_1 - \delta)dG \approx \int f(y_1)dF(y_1)^{n-1} - \delta\int f'(y_1)dF(y_1)^{n-1} = h(0) - h'(0)\delta\,.$$

We also have $1 - F(y_2 + \delta) \approx 1 - F(y_2) - \delta f(y_2)$, so

$$\int (1 - F(y_2 + \delta))dG \approx \int (1 - F(y_2))dF(y_2)^{n-1} - \delta\int f(y_2)dF(y_2)^{n-1} = \frac{1}{n} - h(0)\delta\,.$$

To approximate $\int \varphi(y_1, y_2, \delta)dG$, notice that $\varphi(y_1, y_2, 0) = 0$ and $\varphi_3(y_1, y_2, 0) = f(y_1)f(y_2)$ since $L(y|y_1, y_2)$ is independent of δ and $L(y_1 + y_2|y_1, y_2) = 1$. Hence,

$$\int \varphi(y_1, y_2, \delta)dG \approx \delta\int f(y_1)f(y_2)dG = \delta h(0)^2\,.$$

Substituting these approximations into (35) and discarding all higher order terms yields the approximation for α in (26). The other approximations can be derived similarly.

Proof of Proposition 4: We first show that when n is large, the system of (22) and (23) has a solution with a small δ under mild conditions.

Lemma 3 *Suppose* $\lim_{n\to\infty}\hat{p}=0$, *where* $\hat{p}=\frac{1}{nh(0)}$ *is the separate sales price in (2), and* $\lim_{n\to\infty}\frac{|h'(0)|}{n^{3/2}h(0)^2}<1$ *(which is true, e.g., when* $\frac{|f'(x)|}{f(x)}$ *is uniformly bounded). Then when* n *is sufficiently large, the system of (22) and (23) has a solution with* $\delta\in(0,\frac{1}{nh(0)})$.

Proof. Using (22), we rewrite (23) as an equation of δ:

$$\underbrace{(\beta-\alpha)\frac{1/n+(\alpha+\gamma)\delta}{\alpha+\beta+2\gamma}}_{\chi_L(\delta)}=\underbrace{\Omega_1(\delta)-\alpha\delta}_{\chi_R(\delta)}\ .$$

Denote the left-hand side by $\chi_L(\delta)$ and the right-hand side by $\chi_R(\delta)$.

We first show that $\chi_L(0)<\chi_R(0)$. At $\delta=0$, it is easy to verify that $\alpha=\frac{1}{n}h(0)$, $\beta=\left(1-\frac{1}{n}\right)h(0)$, $\gamma=0$ and $\Omega_1(0)=\frac{1}{n}\left(1-\frac{1}{n}\right)$. Then

$$\chi_L(0)=\frac{1}{n}(1-\frac{2}{n})<\chi_R(0)=\frac{1}{n}(1-\frac{1}{n})\ .$$

Next, we show that $\chi_L(\delta)>\chi_R(\delta)$ at $\delta=\frac{1}{nh(0)}$ when n is sufficiently large. The condition $\lim_{n\to\infty}\hat{p}=0$ implies that $\delta=\frac{1}{nh(0)}\approx 0$ when n is large. Replacing δ in (26) by $\frac{1}{nh(0)}$, we have

$$\alpha\approx\frac{h(0)}{n}-\left(\frac{h'(0)}{n}+\frac{h(0)^2}{n-1}\right)\frac{1}{nh(0)}=\left(\frac{1}{n}-\frac{1}{n(n-1)}\right)h(0)-\frac{h'(0)}{n^2h(0)}\ .$$

Similarly,

$$\beta\approx\left(1-\frac{1}{n}\right)h(0)+\left(\frac{h'(0)}{n}-h(0)^2\right)\frac{1}{nh(0)}=\left(1-\frac{2}{n}\right)h(0)+\frac{h'(0)}{n^2h(0)}\ ,$$

$$\gamma\approx\frac{nh(0)^2}{n-1}\frac{1}{nh(0)}=\frac{h(0)}{n-1}\ ,$$

$$\Omega_1(\delta)\approx\frac{1}{n}\left(1-\frac{1}{n}\right)-\frac{2h(0)}{n}\frac{1}{nh(0)}=\frac{1}{n}-\frac{3}{n^2}\ .$$

Notice that in each expression we just replaced δ by $\frac{1}{nh(0)}$ and *no* further approximations were made.

Notice that $\chi_L(\delta)>\chi_R(\delta)$ if and only if

$$\left[\frac{1}{n}+\delta(\alpha+\gamma)\right](\beta-\alpha)>\left[\Omega_1(\delta)-\delta\alpha\right](\alpha+\beta+2\gamma)\ . \tag{36}$$

Using the above approximations, we have

$$\alpha+\gamma\approx\frac{2h(0)}{n}-\frac{h'(0)}{n^2h(0)}\quad\text{and}\quad\beta-\alpha\approx\left(1-\frac{3}{n}+\frac{1}{n(n-1)}\right)h(0)+\frac{2h'(0)}{n^2h(0)}\ .$$

Then at $\delta=\frac{1}{nh(0)}$ the left-hand side of (36) equals

$$\begin{aligned}&\left[\frac{1}{n}+\frac{1}{nh(0)}\left(\frac{2h(0)}{n}-\frac{h'(0)}{n^2h(0)}\right)\right]\times\left[\left(1-\frac{3}{n}+\frac{1}{n(n-1)}\right)h(0)+\frac{2h'(0)}{n^2h(0)}\right]\\ =\ &\left[\frac{1}{n}+\frac{2}{n^2}-\frac{h'(0)}{n^3h(0)^2}\right]\times\left[1-\frac{3}{n}+\frac{1}{n(n-1)}+\frac{2h'(0)}{n^2h(0)^2}\right]h(0)\ .\end{aligned}\tag{37}$$

Using the approximations, at $\delta = \frac{1}{nh(0)}$ we also have

$$\Omega_1(\delta) - \delta\alpha \approx \frac{1}{n} - \frac{4}{n^2} + \frac{1}{n^2(n-1)} + \frac{h'(0)}{n^3 h(0)^2}$$

and

$$\alpha + \beta + 2\gamma \approx h(0) + \gamma \approx h(0) + \frac{h(0)}{n-1} = \frac{n}{n-1} h(0) \ ,$$

where we have used the fact that $\alpha + \beta + \gamma \approx h(0)$ when δ is small. Then the right-hand side of (36) equals

$$\left[\frac{1}{n} - \frac{4}{n^2} + \frac{1}{n^2(n-1)} + \frac{h'(0)}{n^3 h(0)^2} \right] \frac{n}{n-1} h(0) \ . \tag{38}$$

It is straightforward to show that (37) is greater than (38) if and only if

$$2 - \frac{1}{n(n-1)} + \frac{8-5n}{n^2} + \frac{6n-8}{n^3} \frac{h'(0)}{h(0)^2} - \frac{2(n-1)}{n^4} \frac{h'(0)^2}{h(0)^4} > 0 \ .$$

Treating the left-hand side as a quadratic function of $\frac{h'(0)}{nh(0)^2}$, one can show that this inequality holds if and only if $\frac{h'(0)}{nh(0)^2} \in \left(\frac{3n-4-n\sqrt{4n-5}}{2(n-1)}, \frac{3n-4+n\sqrt{4n-5}}{2(n-1)} \right)$. When n is large, this is equivalent to $\frac{|h'(0)|}{nh(0)^2} < \sqrt{n}$ or $\frac{|h'(0)|}{n^{3/2} h(0)^2} < 1$. Given $\lim_{n \to \infty} \frac{1}{nh(0)} = 0$, a simple sufficient condition is that $\frac{|h'(0)|}{h(0)}$ is uniformly bounded for any n, which is true if $\frac{|f'(x)|}{f(x)}$ is uniformly bounded for any x.[29] This completes the proof of the lemma. ■

Given the system (22) and (23) has a solution with a small δ when n is large, we can approximate each side of (23) around $\delta \approx 0$ by using (26) and discarding all higher order terms. Then it is straightforward to derive (27), from which it is evident that $p < \hat{p} = \frac{1}{nh(0)}$ and so bundling lowers all prices, and using the condition that $\frac{|h'(0)|}{h(0)}$ is uniformly bounded, one can also check $\delta < p$.

Proof of Proposition 5: Following a similar logic as in section 4.3, in this case we have $v_i(\tilde{p}_1, \tilde{p}_2, \tilde{\delta}) = -1$, $i = 1, 2$, and $v_3(\tilde{p}_1, \tilde{p}_2, \tilde{\delta}) = -[1 - n\Omega_b(\tilde{\delta})] = -n\Omega_1(\tilde{\delta})$. Raising the single-product prices has the same marginal effect as before, but now raising the shopping cost by ε will harm each multi-stop-shopping consumer by ε and the number of them is $1 - n\Omega_b(\tilde{\delta})$. (In the case of mixed bundling, a higher bundling discount benefits each one-stop-shopping consumer and so v_3 has the opposite sign.) Then

$$\Delta v \equiv v(p_1, p_2, \delta) - v(\hat{p}_1, \hat{p}_2, 0) = (\hat{p}_1 - p_1) + (\hat{p}_2 - p_2) - n \int_0^{\delta} \Omega_1(\tilde{\delta}) d\tilde{\delta} \ .$$

[29] Suppose $\frac{|f'(x)|}{f(x)} < M$ for a constant $M < \infty$. Then $-Mf(x) < f'(x) < Mf(x)$, and so $-M \int f(x) dF(x)^{n-1} < \int f'(x) dF(x)^{n-1} < M \int f(x) dF(x)^{n-1}$ for any n. That is, $-Mh(0) < h'(0) < Mh(0)$ for any n, and so $\frac{|h'(0)|}{h(0)}$ is uniformly bounded.

In the i.i.d. case with a small δ, we have

$$p \approx \frac{1}{nh(0)} \frac{1}{1 + \frac{n}{n-1}h(0)\delta} \approx \frac{1}{nh(0)} - \frac{\delta}{n-1} ,$$

and so

$$\Delta v \approx 2(\hat{p} - p) - n\delta\Omega_1(0) \approx \frac{2}{n-1}\delta - \frac{n-1}{n}\delta = \left(\frac{2}{n-1} - \frac{n-1}{n}\right)\delta ,$$

where we have used $\Omega_1(0) = \frac{1}{n}\left(1 - \frac{1}{n}\right)$. Therefore, when δ is small, Δv is positive for $n = 2, 3$ but negative for $n \geq 4$.

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