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Identification and Inference in First-Price Auctions with Risk Averse Bidders and Selective Entry*

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Abstract

We study identification and inference in first-price auctions with risk averse bidders and selective entry, building on a flexible entry and bidding framework we call the Affiliated Signal with Risk Aversion (AS-RA) model. Assuming that the econometrician observes either exogenous variation in the number of potential bidders (N) or a continuous instrument (z) shifting opportunity costs of entry, we provide a sharp characterization of the nonparametric restrictions implied by equilibrium bidding. Given variation in either competition or costs, this characterization implies that risk neutrality is nonparametrically testable in the sense that if bidders are strictly risk averse, then no risk neutral model can rationalize the data. In addition, if both instruments (discrete N and continuous z) are available, then the model primitives are nonparametrically point identified. We then explore inference based on these identification results, focusing on set inference and testing when primitives are set identified.

KEYWORDS: Auctions, entry, risk aversion, identification, set inference.

JEL CLASSIFICATIONS: D44, C57.

1 Introduction

Risk aversion and entry are both important considerations in real-world auction markets. While much empirical research has documented these factors individually, relatively little work has explored how they interact. This is at least in part because this interaction also raises a significant empirical challenge: selection into entry may undermine the exclusion

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restrictions necessary for identification of risk preferences. This paper provides a comprehensive analysis of identification in first price auctions with risk averse bidders and selective entry. We then explore inference based on these identification results, extending recent results of Chen, Christensen, and Tamer (2018) to construct confidence sets for identified sets when primitives are possibly set-identified.

Bidder risk attitutes are of fundamental importance in auction design—affecting, among other things, the revenue ranking between first-price and ascending auctions (Maskin and Riley (1984)), the structure of the optimal mechanism (Matthews (1987)), and whether the seller should disclose reserve prices (Li and Tan (2000)). Motivated by this fact, a substantial empirical literature has arisen on bidder risk preferences, finding evidence for risk aversion in a variety of real-world contexts, including in settings where bidders are firms. For instance, Baldwin (1995) and Athey and Levin (2001) find that bidding firms diversify risk across species in U.S. Forest Service timber auctions, Ackerberg, Hirano, and Shahriar (2017) show that bidder risk aversion rationalizes the use of buy-it-now options in eBay auctions, and Bajari and Hortacsu (2005) find that risk aversion explains bidder behavior in experiments. Meanwhile, using more structural approaches, Lu and Perrigne (2008) and Campo, Guerre, Perrigne, and Vuong (2011) find evidence for risk aversion in U.S. Forest Service timber auctions, while Kong (2019) finds that risk aversion can explain observed revenue differences between first-price and ascending auctions for oil and gas leases in New Mexico.[1]

A similarly substantial body of empirical research has also documented the prevalence of endogenous entry in real-world auction markets.[2] While this literature has evolved largely in parallel to the literature on risk aversion cited above, the conjunction between risk aversion and entry also raises important economic questions. For instance, Smith and Levin (1996) show that in environments with both risk aversion and entry, second-price auctions

[1]Findings of risk aversion in timber, oil and gas auctions are of particular interest as the players in both markets are firms. We view such findings as consistent with the hypothesis that, even within firms, all bidding is ultimately done by individuals. Hence, as usual in principle-agent models, the risk preferences of the bidding agents will typically be relevant even if one presumes that the firm itself is risk neutral.

[2]For instance, Hendricks, Pinkse, and Porter (2003) report that less than 25 percent of eligible bidders participate in U.S. Minerals Management Service "wildcat auctions" held from 1954 to 1970. Li and Zheng (2009) find that only about 28 percent of planholders in Texas Department of Transportation mowing contracts actually submit bids. Similar patterns have been reported for timber auctions (Athey, Levin, and Seira (2011), Li and Zhang (2010, 2015), Roberts and Sweeting (2013)), online auctions (Bajari and Hortacsu (2003)), highway procurement (Krasnokutskaya and Seim (2011), Bhattacharya, Roberts, and Sweeting (2014)) and corporate takeover markets (Gentry and Stroup (2019)) among others.

can yield higher revenue than first-price auctions, contradicting the usual revenue ranking (Maskin and Riley (1984)) which obtains with risk aversion alone.[3] Answers to many other policy questions—such as how the seller should regulate participation, or whether the seller should disclose the number of entrants—will similarly depend on the interaction between risk aversion and entry.

Econometrically, however, the interaction between risk aversion and entry also raises substantial challenges for identification and inference, particularly in settings where entry is potentially selective (Samuelson (1985), Ye (2007), Marmer, Shneyerov, and Xu (2013), Gentry and Li (2014), Roberts and Sweeting (2013)). Existing results on nonparametric point identification in auctions with risk averse bidders assume that the latent distribution of bidder valuations is invariant either to the seller's choice of auction format (Lu and Perrigne (2008)), or to the set of competitors faced (Guerre, Perrigne, and Vuong (2009)). But as shown by Li, Lu, and Zhao (2015), if risk averse bidders select into entry, the distribution of valuations among entrants will respond endogenously to both the auction format and the strength of competition faced. Hence both invariance assumptions typically fail in settings with selective entry, rendering identification of risk preferences correspondingly uncertain.

Motivated by these observations, we study identification and inference in first-price auctions with risk averse bidders and selective entry, building on a framework we call the Affiliated Signal with Risk Aversion (AS-RA) model. First proposed by Li, Lu, and Zhao (2015) (henceforth LLZ), this model considers a set of N symmetric potential bidders with wealth preferences described by a smooth concave Bernoulli utility function U, who compete in a first-price auction with entry. Potential bidders have independent private values, observe signals of their values prior to entry, and choose whether to incur a common-knowledge entry cost, with entrants learning their values and submitting bids. This framework nests many existing models as special cases, including the affiliated-signal (AS) models of Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2014) (henceforth GL); the mixed-strategy entry model of Levin and Smith (1994); and models with risk averse bidders but exogenous entry including Guerre, Perrigne, and Vuong (2009) (henceforth GPV), Campo, Guerre, Perrigne, and Vuong (2011) (henceforth CGPV) and Zincenko (2018). It thus represents a

[3] Less surprisingly, but also worth noting, this result also contrasts with revenue equivalence when bidders enter endogenously but are risk neutral (e.g. Levin and Smith (1994), Gentry, Li, and Lu (2017)).

natural focal point for researchers seeking to understand structural interactions between risk aversion and entry.

This paper makes two main contributions. First, we develop a suite of results on nonparametric and semiparametric identification of AS-RA model primitives based on variation either in the number of potential competitors N or in an instrument z influencing bidders' opportunity costs of entry. Assuming that neither U nor the *ex ante* distribution of bidders' private information depend on realizations of N and z, we provide a sharp characterization of the set of AS-RA primitives consistent with equilibrium bidding behavior (Theorem 1). This characterization implies that risk neutrality is nonparametrically testable even with only variation in N, in the strong sense that if bidders are strictly risk averse, then risk neutrality will be strictly outside the identified set. More generally, if only variation in N is available, primitives will be nonparametrically set identified, while if in addition a continuous instrument z inducing sufficiently rich entry variation is available, they will be point identified. These findings generalize prior results on nonparametric identification under either risk aversion with exogenous entry (GPV (2009)) or risk neutrality with AS entry (GL (2014)). We also show that the CRRA and CARA utility families imply semiparametric point identification of U, while a parametric signal-value copula yields conditional identification of AS-RA model primitives up to the unknown copula parameter.

Second, building on these identification results, we explore a new approach to inference within set identified auction models based on Chen, Christensen, and Tamer (2018) (henceforth CCT), who develop MCMC methods for inference on identified sets. CCT (2018)'s methods are ideally suited to our setting, since they accommodate both set identification of primitives and models in which the support of observables depends on parameters. The latter allows us to sidestep the well-known problem that in first price auctions the maximum bid is parameter-dependent, which violates usual regularity conditions for MLE inference as pointed out by Donald and Paarsch (1993). To operationalize inference based on CCT (2018), we parameterize bidder utility and distributions of values among entrants within flexible sieve-type families, re-interpreting nonparametric restrictions in Theorem 1 as constraints on model parameters. We then apply CCT (2018)'s results to develop a simple likelihood ratio test for risk neutrality, as well as confidence sets for identified sets of parameters. Finally, we evaluate the performance of these methods in a simulation study, focusing on a partially identified setting with variation in N only, with excellent practical results. To our knowledge, our paper is the first to apply CCT (2018) in an auction context, and we believe our implementation is also useful in other set-identified auction models.

The rest of this paper is organized as follows. Section 2 introduces the model. Section 3 analyzes identification. Section 4 explores set inference and Section 5 reports results from a Monte Carlo exercise. Finally, Section 6 concludes. Appendix A provides additional theoretical details, Appendix B collects technical proofs. Appendix C presents computation details for set inference.

2 The symmetric AS-RA model

We consider a population of independent first-price auctions, each involving allocation of a single indivisible good among $N\,(\geq 2)$ potential bidders via a first-price auction with entry. The number of potential bidders N varies on the set $\mathcal{N} \equiv \{N_1, ..., N_K\}$, where elements are ordered such that $N_1 < N_2 < ... < N_K$, and the subscript $k \in \mathcal{K} \equiv \{1, ..., K\}$ indexes levels of N. For each auction, the econometrician observes the number of potential bidders N, the number of bidders (entrants) n, and the vector of submitted bids $\mathbf{b}$, as well as an entry instrument z described below. We focus on a symmetric environment with independent private information, although our main identification insights extend to asymmetric bidders and unobserved auction heterogeneity as in GL (2014). All results extend immediately conditional on further auction-level covariates X, although for simplicity we suppress these in notation.

2.1 Model overview

We model entry and bidding as a two-stage game with the following timing. First, in Stage 1, each potential bidder i receives a private signal S_i of her (unknown) private value V_i, and all potential bidders simultaneously decide whether to undertake entry at an opportunity cost $c(z)$ described below. Next, in Stage 2, the n bidders who choose to enter in Stage 1 learn the realizations v_i of their private values V_i and submit bids. The Stage 2 mechanism is a standard first-price auction with non-binding reservation price $r = 0$, where the highest bidder wins and pays her bid.

Value-signal pairs (V_i, S_i) are drawn independently across bidders from a common joint distribution $F_{vs}(v, s)$, where higher pre-entry signals are good news in the sense that the distribution of V_i given S_i is stochastically increasing in S_i. We assume that V_i has a continuous marginal distribution F with support $[0, \bar{v}]$, where $\bar{v} \in (0, \infty)$. Without loss of generality, we normalize Stage 1 signals to standard uniform: $S_i \sim U[0, 1]$. By Sklar's theorem (see, e.g., Nelsen (1999)), we then have $F_{vs}(v, s) = C(F(v), s)$, where $C(a, s)$ is the unique bivariate copula describing dependence between V_i and S_i.

Conditional on choosing to enter, bidder i incurs an entry cost $c(z) > 0$, which potentially depends on an instrument z observable to both bidders and the econometrician. We follow Lu (2009) in modeling $c(z)$ as a pure opportunity cost of entry, with z interpreted as a factor affecting the value of opportunities foregone by entry. We assume that the support $\bar{\mathcal{Z}}$ of z is a closed subset of $\mathbb{R}$, although we allow $\bar{\mathcal{Z}}$ to be a singleton, discrete, or an interval.

Potential bidders are risk averse with risk preferences described by a symmetric, strictly monotone, weakly concave Bernoulli utility function $U(w)$, where w is post-auction wealth. Without loss of generality, we normalize U such that $U(0) = 0$ and $U(1) = 1$. For simplicity, we model bidders as having zero initial wealth and zero financial costs of entry. As described in Appendix A, however, these are in fact equivalent to normalizations in a more general setting with both nonzero initial wealth and financial (in addition to opportunity) costs of entry. In this more general case, following LLZ (2015), we interpret U as describing bidder i's utility of final wealth, normalized relative to the outcome that i enters the auction but does not win.

The number of potential competitors N, the entry cost $c(z)$, utility function U, ex ante value distribution F, and value-signal copula C are known to all potential bidders, with value-signal realization (v_i, s_i) being private information revealed to potential bidder i with timing described above. Although N is common knowledge prior to entry, the number of entrants (actual bidders) n is revealed to bidders only after the auction concludes.[4] In our view, this informational structure best reflects institutional practices typical in sealed-

[4]In circumstances where known n is considered a preferable assumption, one would condition bidding strategies on both N and n. This would substantially simplify identification: conditional on N, realizations of n would be effectively random, allowing for direct application of GPV (2009) identification arguments.

bid markets, where auctioneer announcements or industry experience convey knowledge of potential competition but bids are revealed only after the auction concludes.[5]

2.2 Structural assumptions

In what follows, we refer to (U, F, C, c) as the AS-RA model primitives, and (U, F, C) as the bid-stage primitives. We shall study identification of the bid-stage primitives of the AS-RA model based on variation in either N or z, assuming that both factors are excludable in the sense that true bid-stage primitives, subsequently denoted (U_0, F_0, C_0), are invariant to realizations of N and z.

Assumption 1. (U_0, F_0, C_0) and $c(\cdot)$ satisfy the following conditions:

1. For all $N \in \mathcal{N}$ and $z \in \bar{\mathcal{Z}}$, $Pr(V_i \leq v|N, z) = F_0(v)$ for any $v \in [0, \infty)$ and $Pr(F_0(V_i) \leq a, S_i \leq s|N, z) = C_0(a, s)$ for any $(a, s) \in [0, 1]^2$, and $U_0(\cdot)$ does not depend on N or z.
2. The entry cost function $c(z)$ is strictly increasing in z when $\bar{\mathcal{Z}}$ is not a singleton and continuous in z when $\bar{\mathcal{Z}}$ is an interval.

Exogenous variation in competition, either actual or potential, has been considered as a source of variation for testing and identification by many prior studies, including Haile, Hong, and Shum (2003), GPV (2009), and GL (2014) among others. Exogenous variation in an entry shifter z follows GL (2014) among others. While, for completeness, we analyze identification allowing for variation in both N and z, we expect that external instruments z which are excludable in the (strong) sense required may be challenging to find in practice.[6] For this reason, in both identification and inference, we will place particular emphasis on cases where only variation in N is available (or, equivalently, where $\bar{\mathcal{Z}}$ is a singleton). Importantly, however, our results extend immediately to settings with asymmetric bidders, in which case

[5]For example, in US highway procurement markets, the auctioneer will typically publish a list of planholders (potential entrants) on each contract prior to the letting date. But only a small fraction of planholders actually submit bids (Li and Zheng (2009)), and the set of bids received is only disclosed after the letting concludes. We view such auctions as naturally modeled by the assumption of known N but unknown n. Empirical support for the assumption of unknown n is provided by Kong (2019), who shows in the context of New Mexico oil and gas auctions that even when $n = 1$ the single bidder typically bids well above the reserve. This finding is difficult to rationalize when n is known, but follows immediately when n is unknown.

[6]In Assumption 1, we make implicit use of the fact that z shifts *opportunity*, rather than financial, costs of entry: if instead z shifted financial entry costs, then z would affect the normalization of $U(\cdot)$ and one could not assume that $U_0(x)$ is invariant to z. This interpretation is consistent with the structural AS-RA application of Kong (2017), in which z measures oil and gas auctions outside the specific region considered. As pointed out by a referee, however, one may also be concerned that opportunities to bid in other auctions could affect bidder wealth, in which case z would best be treated as a covariate rather than an instrument.

types of i's rivals are also natural candidates for instruments affecting bidder i's entry but excludable (in the sense of Assumption 1) with respect to i's primitives.

In addition to the key exclusion restrictions in Assumption 1, we assume that (U_0, F_0, C_0) belong to regularity classes defined as follows:

Assumption 2. $U_0 \in \mathcal{U}$, where $\mathcal{U}$ is the set of utility functions $U(\cdot)$ such that:

1. $U : [0, \infty) \to [0, \infty)$, $U(0) = 0$, and $U(1) = 1$.
2. $U(\cdot)$ is continuous on $[0, \infty)$ and admits three continuous derivatives on $(0, \infty)$, with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $(0, \infty)$.
3. Both $\lim_{x \downarrow 0} \frac{d}{dx}\left(\frac{U(x)}{U'(x)}\right)$ and $\lim_{x \downarrow 0} \frac{d^2}{dx^2}\left(\frac{U(x)}{U'(x)}\right)$ are finite.

Assumption 3. $F_0 \in \mathcal{F}$, where $\mathcal{F}$ is the set of probability distributions $F(\cdot)$ such that:

1. $F(\cdot)$ is supported on a compact interval $[0, \bar{v}]$, with $\bar{v} < \infty$.
2. $F(\cdot)$ is twice continuously differentiable with positive density on $[0, \bar{v}]$.

Assumption 4. $C_0 \in \mathcal{C}$, where $\mathcal{C}$ is the set of bivariate copula functions $C(a, s)$ such that, interpreted as a distribution over random variables (A, S) with uniform marginals:

1. $C(a, s)$ is continuous on $[0, 1] \times [0, 1]$.
2. For all $s \in [0, 1)$, the distribution of A given $S \geq s$ admits a continuous, bounded density with infimum support $\underline{a}(s)$ continuous in s, and for all points in its support except possibly the infimum $\underline{a}(s)$, this density is locally bounded away from zero, differentiable in a, and differentiable in s.
3. For all $a \in [0, 1]$, $C(a, s)$ is concave in s.

Assumptions 2 and 3 impose standard regularity conditions on U_0 and F_0, following GPV (2009) among others. Conditions 1 and 2 of Assumption 4 ensure that regularity conditions on F_0 pass through to selected distributions of V_i given $S_i \geq s$ arising in equilibrium, while nesting Samuelson (1985)'s model of perfectly selective entry within the class $\mathcal{C}$.[7] Finally, Condition 3 of Assumption 4 is equivalent to assuming that V_i is weakly increasing in S_i in the sense of first-order stochastic dominance; this can be seen most readily when $C_0(a, s)$ is differentiable, in which case $F(v|S_i = s) = \frac{\partial C_0(F_0(v), s)}{\partial s}$. We maintain Assumptions 1-4 throughout the analysis.

As in GPV (2009), rather than working with the utility function U_0 directly, it will frequently prove more convenient to use the following one-to-one transformations. Define $\lambda_0(x) \equiv \frac{U_0(x)}{U_0'(x)}$, and observe that in view of the normalizations above, we have $U_0(x) =$

[7] Formally, this model is nested by setting $C(a, s) = \min(a, s)$, in which case $C(a, s)$ does not admit a joint density but does satisfy the smoothness conditions in Assumption 4, which are sufficient for our results.

$\exp \int_1^x 1/\lambda_0(t)\,dt$. Furthermore, bearing in mind that $U_0(0) = 0$, $U_0' > 0$, and $U_0'' \leq 0$, we have $\lambda_0(0) = 0$ and $\lambda_0'(x) = 1 - \frac{U_0(x)}{U_0'(x)}\frac{U_0''(x)}{U_0'(x)} \geq 1$. It follows that $\lambda_0(\cdot)$ has a well-defined, monotone inverse $\lambda_0^{-1}(\cdot)$ satisfying $\lambda_0^{-1}(0) = 0$ and $\lambda_0^{-1,\prime} \leq 1$. We will work with U_0, λ_0, and λ_0^{-1} interchangeably, depending on context. Let Λ be the set of functions $\lambda(x)$ such that $\lambda(x) \equiv [U(x)/U'(x)]$ for some $U \in \mathcal{U}$, and let Λ^{-1} be the set of functions λ^{-1} which are inverses of some function $\lambda \in \Lambda$.

2.3 Equilibrium behavior

We focus on the unique symmetric, monotone equilibrium of the AS-RA model. Since the properties of this equilibrium have already been derived by LLZ (2015), we describe only its identification-relevant features here. We provide a complete derivation of the properties stated below, in a more general setting additionally accommodating nonzero initial wealth and financial costs of entry, in Appendix A.

For each competition level $k \in \mathcal{K}$ and each opportunity cost level $z \in \bar{\mathcal{Z}}$, equilibrium Stage 1 entry will involve a signal threshold $s_k(z) \in [0, 1]$ such that bidder j enters if and only if $S_j \geq s_k(z)$. The distribution of valuations among bidders choosing to enter conditional on observables (N_k, z) will therefore be described by the c.d.f.

$$F_k^0(v|z) \equiv F(v|S_j \geq s_k(z)) = \frac{F_0(v) - C_0(F_0(v), s_k(z))}{1 - s_k(z)}, \quad k = 1, ..., K. \tag{1}$$

In what follows, let $v_k(\alpha|z)$ be the quantile function of the post-entry value distribution $F_k^0(v|z)$, and $v_0(\alpha)$ be the quantile function of the ex ante value distribution $F_0(v)$. We also consider the case when $\bar{\mathcal{Z}}$ is a singleton, so that conditioning on z is trivial; in this case we will simply use $F_k^0(v)$ and $v_k(\alpha)$ to denote $F_k^0(v|z)$ and $v_k(\alpha|z)$ respectively.

Taking the entry threshold $s_k(z)$ as given, post-entry bidding at competition level $k \in \mathcal{K}$ will be described by a symmetric, monotone strategy $\beta_k(\cdot|z)$ such that entrant i drawing valuation v_i optimally submits bid $\beta_k(v_i|z)$. To characterize this equilibrium strategy, recall that (at the time of bidding) entrant i is uncertain whether any given potential rival j has entered. Hence, in equilibrium, entrant i submitting bid $\beta_k(y|z)$ expects to outbid any potential rival j in one of two events: either j does not enter (with probability $s_k(z)$), or j does enter (with probability $1 - s_k(z)$) but draws a valuation below y (with probability

$F_k^0(y|z)$). When all $N_k - 1$ potential rivals play equilibrium strategies, we may therefore write i's bidding problem as

$$\max_y U_0(v_i - \beta_k(y|z)) \cdot [s_k(z) + (1 - s_k(z))F_k^0(y|z)]^{N_k - 1}.$$

Taking a first-order condition with respect to y and enforcing the equilibrium condition $y = v_i$, we ultimately obtain the following differential equation characterizing the symmetric equilibrium post-entry bidding strategy $\beta_k(\cdot|z)$:

$$\beta_k'(v|z) = \lambda_0(v - \beta_k(v|z)) \frac{(N_k - 1)(1 - s_k(z))f_k^0(v|z)}{s_k(z) + (1 - s_k(z))F_k^0(v|z)}, \tag{2}$$

where $\beta_k'(v|z)$ and $f_k^0(v|z)$ are respectively the derivatives of $\beta_k(v|z)$ and $F_k^0(v|z)$ with respect to v. Combined with the boundary condition $\beta_k(v_k(0|z)|z) = 0$, the differential equation (2) uniquely determines the post-entry bidding strategy $\beta_k(\cdot|z)$.

Finally, consider the threshold $s_k(z)$ characterizing equilibrium entry at observables (N_k, z). In any equilibrium with nontrivial entry, this must be such that a bidder with signal $S_i = s_k(z)$ is just indifferent to entry against $N_k - 1$ rivals who play equilibrium strategies. Recalling that bidder i must forego the opportunity cost $c(z)$ to enter, this in turn implies the breakeven condition for nontrivial entry (i.e., $s_k(z) \in (0, 1)$):

$$\Pi(s_k(z), s_k(z); N_k) = U_0(c(z)), \tag{3}$$

where $\Pi(s_i, s_k(z); N_k)$ denotes the expected post-entry profit of a potential bidder with signal realization $S_i = s_i$ against $N_k - 1$ potential rivals who play equilibrium strategies. As shown in Appendix A, the breakeven condition (3) (see Appendix A for an equivalent expression (24)) will uniquely determine $s_k(z)$. Furthermore, for all $k \in \mathcal{K}$, if $s_k(z) \in (0, 1)$ then $s_k(z)$ is strictly increasing in both k and z. Finally, if $s_k(z) < 1$, then $s_l(z) < 1$ for all $l > k$.

2.4 Linking observables to unobservables

Let $\mathcal{Z} = \{z \in \bar{\mathcal{Z}} : s_1(z) \in [0, 1)\}$ denote the subset of realizations z such that equilibrium involves at least some bidding; in what follows, we focus on $z \in \mathcal{Z}$ without essential loss of generality.[8] For each $k \in \mathcal{K}$ and $z \in \mathcal{Z}$, let $G_k(b|z)$ be the equilibrium distribution of bids

[8]Whether entry occurs depends only on z, not on N, since $s_1(z) < 1$ implies $s_k(z) < 1$ for all $k \in \mathcal{K}$.

submitted at (N_k, z), $g_k(b|z)$ be the density of $G_k(b|z)$, and $b_k(\alpha|z)$ be the quantile function associated with $G_k(b|z)$. As usual, observing bids will (point-) identify $G_k(\cdot|z)$ for each $k \in \mathcal{K}$ and $z \in \mathcal{Z}$. Similarly, recalling $S_i \sim U[0,1]$, we may (point-) identify the equilibrium entry threshold $s_k(z)$ for each (k, z) from observed probabilities of entry:

$$s_k(z) = 1 - \frac{E[n|N_k, z]}{N_k}.$$

We next derive the key equilibrium inverse bidding function linking the directly identified objects $s_1(z), ..., s_K(z)$, $G_1(\cdot|z), ..., G_K(\cdot|z)$ to latent bid-stage primitives. Toward this end, following GPV (2009), we first apply the change of variables $b_i = \beta_k(v_i|z)$ to the first-order condition (2), then exploit strict monotonicity of b_i in v_i to re-express both bids and values in terms of their respective quantile functions $b_k(\alpha|z)$ and $v_k(\alpha|z)$. These transformations ultimately yield the following equilibrium quantile inverse bidding function, which is the basis for our subsequent analysis:

$$v_k(\alpha|z) = b_k(\alpha|z) + \lambda_0^{-1}(R_k(\alpha|z)), \quad k = 1, ..., K, \tag{4}$$

where the argument $R_k(\alpha|z)$ to the unknown function $\lambda_0^{-1}(\cdot)$ is defined as

$$R_k(\alpha|z) \equiv \frac{s_k(z) + (1 - s_k(z))\alpha}{(N_k - 1)(1 - s_k(z))g_k(b_k(\alpha|z)|z)}. \tag{5}$$

Properties of $\beta_k(\cdot|z)$ imply that $b_k(0|z) = 0$, that $b_k(\cdot|z)$ is differentiable on its domain, and that $R_k(\cdot|z)$ is continuous on $[0, 1]$ and differentiable on the same domain as $v_k(\cdot|z)$.[9] Furthermore, (point) identification of $s_k(z), G_k(\cdot|z)$ implies (point) identification of $b_k(\cdot|z)$ and $R_k(\cdot|z)$, and hence (point) identification of the right-hand side of (4) up to λ_0^{-1}.

3 Identified sets for bid-stage primitives

This section provides a sharp nonparametric characterization of restrictions on bid-stage primitives $(\lambda_0^{-1}, F_0, C_0)$ generated by the bid distributions $G_1(\cdot|z), ..., G_K(\cdot|z)$, taking entry thresholds $s_1(z)$, ..., $s_K(z)$ as given. Based on this characterization, we show that risk aversion is nonparametrically testable in the sense that, given variation in either N or z, risk neutrality is outside the identified set when bidders are strictly risk averse. We then

[9] I.e., either $[0, 1]$ if $\lim_{a \to 0} v_k'(a|z) < \infty$, or $(0, 1]$ if $\lim_{a \to 0} v_k'(a|z) = \infty$, when also $\lim_{a \to 0} R_k'(a|z) = \infty$.

derive implications for point identification based on variation in z and for identification with parametric utility, as well as for conditional identification with a parametric copula.

Our bid-stage focus is motivated by two findings in GL (2014). First, they show how to map restrictions on (F_0, C_0) implied by bidding behavior into identified sets for the entry cost function $c(z)$ via the breakeven condition (3), assuming that bidders are risk neutral so that λ_0^{-1} is the identity function. In Appendix A.3, we show that the sharp bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$ implies identified bounds on $c(z)$ through the breakeven condition (3). Second, in their risk-neutral context, GL (2014) find that entry-stage restrictions convey little additional information on bid-stage primitives. For this reason, we focus on restrictions on $(\lambda_0^{-1}, F_0, C_0)$ implied by equilibrium bidding, taking observed entry patterns as given. One could refine the identified sets below to incorporate entry-stage restrictions following Appendix A of GL (2014). But in view of GL (2014)'s findings, we do not pursue this exercise here.

3.1 Nonparametric bid-stage identifed set for $(\lambda_0^{-1}, F_0, C_0)$

We begin by analyzing the nonparametric bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$, denoted $\mathcal{I}$ and defined formally as follows:

Definition 1. The bid-stage identified set for $(\lambda_0^{-1}, F_0, C_0)$, denoted $\mathcal{I} \subset \Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$, is the set of all $(\lambda^{-1}, F, C) \in \Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$ which jointly satisfy equations (1) and (4) for all $k \in \mathcal{K}$ and all $z \in \mathcal{Z}$.

Equivalently, $\mathcal{I}$ is the subset of $\Lambda^{-1} \times \mathcal{F} \times \mathcal{C}$ such that, for all $k \in \mathcal{K}$ and $z \in \mathcal{Z}$, $G_k(\cdot|z)$ is the equilibrium bid distribution implied by each $(\lambda^{-1}, F, C) \in \mathcal{I}$ given $s_k(z)$.

We next provide a sharp characterization of $\mathcal{I}$, emphasizing restrictions on λ^{-1} generated by equilibrium bidding behavior. Toward this end, consider any candidate $\lambda^{-1} \in \Lambda^{-1}$. Under the hypothesis $\lambda^{-1} = \lambda_0^{-1}$, the quantile inverse bidding function (4) implies a unique set of candidates $\tilde{v}_1(\cdot|z; \lambda^{-1}), ..., \tilde{v}_K(\cdot|z; \lambda^{-1})$ for the unknown latent quantile functions $v_1(\cdot|z), ..., v_K(\cdot|z)$:

$$\tilde{v}_k(\alpha|z; \lambda^{-1}) \equiv b_k(\alpha|z) + \lambda^{-1}(R_k(\alpha|z)), \quad k = 1, ..., K. \tag{6}$$

By construction, these candidates $\tilde{v}_1(\cdot|z;\lambda^{-1}),...,\tilde{v}_K(\cdot|z;\lambda^{-1})$ are identified up to λ^{-1} and well-defined for any $\lambda^{-1} \in \Lambda^{-1}$. Furthermore, properties of $b_k(\cdot|z)$ and $R_k(\cdot|z)$ imply that, for all $\lambda^{-1} \in \Lambda^{-1}$, $\tilde{v}_k(\cdot|z;\lambda^{-1})$ is differentiable on the same domain as $v_k(\cdot|z)$.

Next observe that taking $\lambda_0^{-1} \in \Lambda^{-1}$ and entry behavior as given, primitives $(F_0, C_0) \in \mathcal{F} \times \mathcal{C}$ influence bidding behavior only through the latent quantile functions $v_1(\cdot|z),...,v_K(\cdot|z)$ (see (1)). To determine whether any candidate $\lambda^{-1} \in \Lambda^{-1}$ is consistent with bid-stage observables, it is therefore sufficient to determine whether there exists a structure $(F, C) \in \mathcal{F} \times \mathcal{C}$ consistent with the candidate quantile functions $\tilde{v}_1(\cdot|z;\lambda^{-1}),...,\tilde{v}_K(\cdot|z;\lambda^{-1})$ generated by λ^{-1} through (6). This turns out to reduce to a set of five restrictions on $\tilde{v}_1(\cdot|z;\lambda^{-1}),...,\tilde{v}_K(\cdot|z;\lambda^{-1})$, yielding the following sharp alternative characterization of the bid-stage identified set $\mathcal{I}$:

Theorem 1. *Let Λ_I^{-1} be the set of $\lambda^{-1} \in \Lambda^{-1}$ such that the candidate quantile functions $\tilde{v}_k(\cdot|z;\lambda^{-1})$ defined by (6) satisfy all of the following restrictions M, O, I, D and S:*

M *For all $k \in \mathcal{K}$, $z \in \mathcal{Z}$, and all $a \in (0,1]$, $\tilde{v}'_k(a|z;\lambda^{-1})$ is bounded away from zero.*

O *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $s_k(z) \leq s_l(z')$, $\tilde{v}_k(a|z;\lambda^{-1}) \leq \tilde{v}_l(a|z';\lambda^{-1})$ for all $a \in [0,1]$, with equality if $s_k(z) = s_l(z')$.*

I *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$, $\tilde{v}_k(1|z;\lambda^{-1}) = \tilde{v}_l(1|z';\lambda^{-1})$.*

D *For all $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $s_k(z) \leq s_l(z')$, and all $y, y' \in \mathbb{R}$ with $y' \geq y$,*

$$\big(1-s_k(z)\big)\left[\tilde{v}_k^{-1}(y'|z;\lambda^{-1})-\tilde{v}_k^{-1}(y|z;\lambda^{-1})\right] \geq \big(1-s_l(z')\big)\left[\tilde{v}_l^{-1}(y'|z';\lambda^{-1})-\tilde{v}_l^{-1}(y|z';\lambda^{-1})\right].$$

S *For all $k, l, m \in \mathcal{K}$ and $z, z', z'' \in \mathcal{Z}$ such that $s_k(z) < s_l(z') < s_m(z'')$,*

$$\frac{(1-s_k(z))\,\tilde{v}_k^{-1}(y|z;\lambda^{-1}) - (1-s_l(z'))\,\tilde{v}_l^{-1}(y|z';\lambda^{-1})}{s_l(z') - s_k(z)} \geq \frac{(1-s_l(z'))\tilde{v}_l^{-1}(y|z';\lambda^{-1}) - (1-s_m(z''))\tilde{v}_m^{-1}(y|z'';\lambda^{-1})}{s_m(z'') - s_l(z')}.$$

Then: for any $\lambda^{-1} \in \Lambda^{-1}$, there exists $(F, C) \in \mathcal{F} \times \mathcal{C}$ such that $(\lambda^{-1}, F, C) \in \mathcal{I}$ if and only if $\lambda^{-1} \in \Lambda_I^{-1}$. Moreover, in this case, $(\lambda^{-1}, F, C) \in \mathcal{I}$ for all $(F, C) \in \mathcal{F} \times \mathcal{C}$ such that for all $k \in \mathcal{K}$ and $z \in \mathcal{Z}$,

$$\tilde{v}_k^{-1}(y|z;\lambda^{-1}) = \frac{F(y) - C(F(y), s_k(z))}{1 - s_k(z)} \text{ for all } y \in [\tilde{v}_k(0|z;\lambda^{-1}), \tilde{v}_k(1|z;\lambda^{-1})].$$

Recalling that $\tilde{v}_1(\cdot|z;\lambda^{-1}),...,\tilde{v}_K(\cdot|z;\lambda^{-1})$ are identified up to λ^{-1}, Theorem 1 implies that we can express restrictions on λ^{-1} generated by equilibrium bidding solely in terms of directly identified objects. Moreover, for any $\lambda^{-1} \in \Lambda_I^{-1}$, only one set of selected distributions $F_1(\cdot|z),...,F_K(\cdot|z)$ can be consistent with equilibrium bidding, and the set of $(C, F) \in \mathcal{C} \times \mathcal{F}$ consistent with λ^{-1} contains all those which reproduce these. Restrictions M-S reflect properties of $F_1(\cdot|z),...,F_K(\cdot|z)$ implied by $(F, C) \in \mathcal{F} \times \mathcal{C}$. Restriction M (strict monotonicity) follows since each density $f_k(\cdot|z)$ is bounded. Restriction O (ordered quantile functions) reflects the fact that entrant values are stochastically increasing in $s_k(z)$. Restriction I (invariant top quantile) follows from stochastic ordering of V_i in S_i, together with the fact that for any $z \in \mathcal{Z}$, the set of entering types will include the potential bidder drawing the highest possible signal ($S_i = 1$). Restriction D (positive conditional densities) can be understood by noting that the c.d.f. of V_i given $S_i \in [s_k(z), s_l(z')]$ is proportional to $(1 - s_k(z))F_k(\cdot|z) - (1 - s_l(z'))F_l(\cdot|z')$. Finally, if V_i is stochastically increasing in S_i, then any conditional c.d.f. of the form $F(V_i|S_i \in [s, s'])$ must be decreasing in both s and s'. This in turn implies Restriction S (stochastically increasing conditional distributions).

Theorem 1 has three main implications.[10] First, and most important, risk aversion is nonparametrically testable within the AS-RA model, with or without a continuous instrument z. Second, if one observes a continuous instrument z which induces sufficient variation in entry, then λ_0^{-1} may be identified following GPV (2009). Finally, if U_0 belongs to a parametric family, then the identified set will typically reduce to a simpler structure involving point identification of U_0 as in CGPV (2011) and set identification of (F_0, C_0) as in GL (2014). We next develop each of these implications in turn.

3.2 Nonparametric testability of risk neutrality

Although our statement of Theorem 1 allows for variation in both N and z, we anticipate that valid external instruments z will in many cases be unavailable. We therefore view the special case of variation in N only as of particular practical importance. In this section, we show that risk neutrality is nonparametrically testable within the AS-RA model based on variation in N only, with or without variation in z.

Toward this end, temporarily suppose $\mathcal{Z}$ is a singleton, in which case we may omit the (now trivial) conditioning on z in notation. By Restriction I of Theorem 1, we have $\tilde{v}_k(1|\lambda_0^{-1}) = \tilde{v}_l(1|\lambda_0^{-1})$ for all $k, l \in \mathcal{K}$. Therefore by (6),

$$b_k(1) + \lambda_0^{-1}(R_k(1)) = b_l(1) + \lambda_0^{-1}(R_l(1)) \quad \forall k, l \in \mathcal{K}. \tag{7}$$

[10] Our sharp nonparametric identified set characterization contributes to a rapidly growing literature on partial identification in structure models; see, for example, Manski (2003), Manski and Tamer (2002), Haile and Tamer (2003), Fan and Wu (2010), Chesher and Rosen (2013), Molinari (2020) and the references therein.

Now consider any $k, l \in \mathcal{K}$ such that $R_l(1) < R_k(1)$ and define $\mathcal{R}_{kl} \equiv [R_l(1), R_k(1)]$. By the mean value theorem, over the interval $\mathcal{R}_{kl}$, we must have

$$\min_{x \in \mathcal{R}_{kl}} \lambda_0^{-1,\prime}(x) \leq \frac{\lambda_0^{-1}(R_k(1)) - \lambda_0^{-1}(R_l(1))}{R_k(1) - R_l(1)} \leq \max_{x \in \mathcal{R}_{kl}} \lambda_0^{-1,\prime}(x),$$

or equivalently substituting from (7),

$$\min_{x \in \mathcal{R}_{kl}} \lambda_0^{-1,\prime}(x) \leq -\frac{b_k(1) - b_l(1)}{R_k(1) - R_l(1)} \leq \max_{x \in \mathcal{R}_{kl}} \lambda_0^{-1,\prime}(x). \tag{8}$$

Recall that weak risk aversion ($U_0'' \leq 0$) implies $\lambda_0' \geq 1$ and hence $0 \leq \lambda_0^{-1,\prime} \leq 1$. Global risk neutrality ($U_0'' = 0$) corresponds to the special case $\lambda_0^{-1,\prime}(x) = 1$ for all x. We say that bidders are *strictly risk averse* at x if $U_0''(x) < 0$, or equivalently if $\lambda_0^{-1,\prime}(x) < 1$.

Now consider the implications of these facts for testing. If bidders are strictly risk averse for some $x \in \mathcal{R}_{kl}$, then we must have $\lambda^{-1,\prime}(x) < 1$ on an open subset of $\mathcal{R}_{kl}$, which since $\lambda^{-1,\prime} \leq 1$ and $b_k(1) - b_l(1) = \int_{R_k(1)}^{R_l(1)} \lambda^{-1,\prime}(x)\, dx$ implies

$$0 \leq -\frac{b_k(1) - b_l(1)}{R_k(1) - R_l(1)} < 1.$$

In contrast, if bidders are globally risk neutral, then $\inf_x \lambda_0^{-1,\prime}(x) = \sup_x \lambda_0^{-1,\prime}(x) = 1$ and therefore in view of (8) we must have

$$-\frac{b_k(1) - b_l(1)}{R_k(1) - R_l(1)} = 1.$$

Furthermore, since we cannot have $\lambda_0^{-1,\prime}(x) > 1$, it follows that *any* candidate $\lambda^{-1} \in \Lambda^{-1}$ rationalizing the data must satisfy $\lambda^{-1,\prime}(x) = 1$ for $x \in \mathcal{R}_{kl}$.

Taken together, these facts imply that risk neutrality is testable in the following strong sense, which for completeness we state reintroducing potential variation in z:

Corollary 1. *Let $\bar{R}(1) \equiv \sup_{k \in \mathcal{K}, z \in \mathcal{Z}} R_k(1|z)$ and $\underline{R}(1) \equiv \inf_{k \in \mathcal{K}, z \in \mathcal{Z}} R_k(1|z)$. Then the following statements hold:*

1. *If bidders are risk neutral, then any $\lambda^{-1} \in \Lambda_I^{-1}$ must satisfy $\lambda^{-1,\prime}(x) = 1$ for all $x \in [\underline{R}(1), \bar{R}(1)]$;*
2. *If, for any $x \in [\underline{R}(1), \bar{R}(1)]$, bidders are strictly risk averse at x, then no risk neutral model can rationalize bid-stage behavior.*

Note that Corollary 1 turns only on an invariant top quantile of values among entrants (Restriction I of Theorem 1). Restriction I is here a consequence of the assumption that valuations are stochastically increasing in signals, together with the fact that the set of entering types (if nonempty) will always include the potential bidder with the highest possible signal ($S_i = 1$). Importantly, however, a similar insight applies in any first-price auction where at least one quantile of values is invariant to either N or z. CGPV (2011) have considered parametric quantile restrictions, including quantile invariance, as a basis for estimation with parametric U_0. To our knowledge, however, the fact that quantile invariance also implies nonparametric testability of risk neutrality, including with only discrete instruments, has not previously been observed.

Remark. *In Section 4.2, we propose a likehood ratio test for risk neutrality, which uses all restrictions in Theorem 1. As suggested in prior versions of this paper, however, one could also develop testing procedures based directly on Corollary 1, which uses only Restriction I. For example, one could test restrictions of the form* $H_0 : b_l(1) - b_k(1) = R_k(1) - R_l(1)$ *against* $H_1 : b_l(1) - b_k(1) < R_k(1) - R_l(1)$*, suitably weighted across* $k, l \in \mathcal{K}$*, and adapting estimators in CGPV (2011) to conduct inference on* $\{b_k(1)\}_{k=1}^{K}$ *and* $\{R_k(1)\}_{k=1}^{K}$.

3.3 Identification with a continuous instrument z

As shown in Corollary 1, Theorem 1 yields substantive restrictions on primitives even based only on excludable variation in N. If a continuous entry instrument z is available, however, this can qualitatively sharpen identification in at least two respects.

First, if there exist $z, z' \in \mathcal{Z}$ such that for distinct $k, l \in \mathcal{K}$ we have $s_k(z) = s_l(z')$, then by Restriction O of Theorem 1 we must also have $v_k(\cdot|z) = v_l(\cdot|z')$. Following GPV (2009), we may therefore substitute from (4) to obtain the *compatibility condition*

$$b_k(a|z) + \lambda_0^{-1}(R_k(a|z)) = b_l(a|z') + \lambda_0^{-1}(R_l(a|z')), \quad \forall\, a \in [0,1]. \tag{9}$$

Further suppose that $R_k(0|z) = R_l(0|z') = 0$, which here holds if and only if the support of $V_i|S_i$ includes 0 at $S_i = s_k(z) = s_l(z')$. Results in GPV (2009) then establish that the compatibility condition (9) identifies λ_0^{-1} on its empirical domain:

Corollary 2. *Consider any distinct* $k, l \in \mathcal{K}$*. Suppose that there exist* $z, z' \in \mathcal{Z}$ *such that* $s_k(z) = s_l(z')$*, and that the support of* $V_i|S_i = s_k(z)$ *includes 0. Then* λ_0^{-1} *is identified on* $[0, \max_a \max\{R_k(a|z), R_l(a|z)\}]$ *and* $F_k^0(v|z)$ *is identified at* z.

Second, even when z induces insufficient entry variation to apply Corollary 2, access to a continuous instrument z allows us to extend the argument underlying Corollary 1 to obtain point identification of $\lambda_0^{-1,\prime}(r)$ for at least some r:[11]

[11] This fact was first noted, but not exploited, at the beginning of Section 5.3 in CGPV (2011).

Corollary 3. *For each $k \in \mathcal{K}$, let $\bar{R}_k(1) \equiv \sup_{z\in\mathcal{Z}} R_k(1|z)$ and $\underline{R}_k(1) \equiv \inf_{z\in\mathcal{Z}} R_k(1|z)$. Suppose there exists $z \in Int(\mathcal{Z})$ such that $s_k(z) \in (0,1)$. Then, for each $k \in \mathcal{K}$, $\bar{R}_k(1) > \underline{R}_k(1)$ and $\lambda_0^{-1,\prime}(r)$ is identified for all $r \in [\underline{R}_k(1), \bar{R}_k(1)]$.*

Even when $\lambda_0^{-1}(r)$ itself is not identified, identification of $\lambda_0^{-1,\prime}(r)$ may be of interest. For example, differentiating the definition $\lambda_0(x) \equiv U_0(x)/U_0'(x)$, one obtains

$$ARA(\lambda_0^{-1}(r)) = \frac{1}{r\lambda_0^{-1,\prime}(r)} - \frac{1}{r}, \tag{10}$$

where $ARA(r) \equiv -U''(r)/U'(r)$ is the Arrow-Pratt coefficient of absolute risk aversion. Thus, for example, if U_0 satisfies CARA, then the right-hand side of (10) must be constant, while if U_0 satisfies CRRA, then $ARA(\lambda_0^{-1}(r))/\lambda_0^{-1}(r)$ must be constant. In view of Corollary 3, both restrictions are testable in principle when z is continuous.

3.4 Point identification of λ_0^{-1} with parametric utility

In some applications, one may be willing to assume that U_0 belongs to a parametric family: i.e., that $\lambda_0^{-1} = \lambda^{-1}(\cdot;\gamma_0)$ for some $\gamma_0 \in \Gamma$, with Γ a compact subset of a finite dimensional Euclidean space. Theorem 1 will then often imply point identification of λ_0^{-1}, although potentially only set identification of other bid-stage primitives.

To see this, recall from Theorem 1 that $\gamma = \gamma_0$ implies that $\tilde{v}_l(1|z;\gamma)$ is constant for all k and z. Taking $\bar{v} \equiv \tilde{v}_k(1|z;\gamma)$ as an auxiliary parameter to be identified, we may equivalently express this restriction as

$$\bar{v} = b_k(1|z) + \lambda^{-1}(R_k(1|z);\gamma_0), \quad \forall\, k \in \mathcal{K},\, z \in \mathcal{Z}. \tag{11}$$

This parallels the system of estimating restrictions considered by CGPV (2011), here derived directly from AS entry. If the system (11) has a unique solution $(\gamma_0, \bar{v})$, then identification of λ_0^{-1} is immediate, with identification of $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ following through the quantile inverse bid function (6). Uniqueness will in general depend on both the parametric family considered and the scope of variation in (N, z). But for the CARA and CRRA utility families, the most widely employed single-parameter families, (11) can be shown to have a unique solution, leading to the following corollary:

Corollary 4. *Assume that U_0 belongs to either of the following parametric families:*

CRRA $U_0(x) = x^{1-\gamma_0}$*; $\gamma_0 \in [0, \bar{\gamma}]$ for some $\bar{\gamma} < 1$.*

CARA *$U_0(x) = x$ for $\gamma_0 = 0$; $U_0(x) \propto (1 - e^{-\gamma_0 x})$ for $\gamma_0 \in (0, \bar{\gamma}]$ with $\bar{\gamma} < \infty$.*

Further suppose that $R_k(1|z) \neq R_l(1|z')$ for some $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$. Then λ_0^{-1} and $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ are point identified.

Depending on the scope of variation in z, point identification of $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$ may not be sufficient to point-identify F_0 and C_0, hence model primitives as a whole may be only set-identified. But given point identification of λ_0^{-1} and $F_1^0(\cdot|z), ..., F_K^0(\cdot|z)$, we may construct identified sets for remaining bid-stage primitives F_0 and C_0, as well as the entry cost function $c(z)$, following GL (2014) as described in Appendix A.3.

3.5 Conditional identification with a parametric copula

In addition, or as an alternative, to parameterizing utility, one may be willing to assume that the value-signal copula C_0 belongs to a known parametric family: i.e., $C_0(a, s) = C(a, s; \theta_0)$, with θ_0 an element of a compact subset Θ of some Euclidean space.[12] In this case, (λ_0^{-1}, F_0) are typically identified up to the copula parameter θ_0, which can substantially simplify inference as we discuss in Section 4.4.

We develop this result focusing on variation in N, although it extends immediately to variation in z. Let Θ_I denote the bid-stage identified set for θ_0: i.e., the set of $\theta \in \Theta$ for which there exist some $(\lambda^{-1}, F) \in \Lambda^{-1} \times \mathcal{F}$ such that $(\lambda^{-1}, F, C(\cdot, \cdot; \theta)) \in \mathcal{I}$. For each $k \in \mathcal{K}$, define a quantile index function $h_k : [0, 1] \times \Theta \to [0, 1]$ as follows:

$$h_k(a; \theta) \equiv \frac{a - C(a, s_k; \theta)}{1 - s_k}. \tag{12}$$

Note that, interpreted as a function of a, each $h_k(\cdot; \theta)$ is identified up to the unknown copula parameter θ. Furthermore, at $\theta = \theta_0$, we have from (1) that for each $k \in \mathcal{K}$

$$F_k^0(y) = \frac{F_0(y) - C(F_0(y), s_k; \theta_0)}{1 - s_k} \equiv h_k(F_0(y); \theta_0). \tag{13}$$

Applying the change of variables $y = v_0(a)$ on both sides of (13) and inverting $F_k(\cdot)$ in the resulting expression, we obtain the identity $v_0(a) \equiv v_k(h_k(a; \theta_0))$, whose left-hand side crucially does not depend on k. Under the hypothesis $\theta = \theta_0$, we may thus transform the

[12] Parametric copula assumptions have also been proposed to correct for selection in other contexts, see e.g. Arellano and Bonhomme (2017).

quantile inverse first order conditions (4) into a system of compatibility conditions paralleling GPV (2009) by reindexing through $h_k(a;\theta)$:

$$b_k(h_k(a;\theta)) + \lambda_0^{-1}(R_k(h_k(a;\theta))) = b_l(h_l(a;\theta)) + \lambda_0^{-1}(R_l(h_l(a;\theta))) \text{ for all } k, l \in \mathcal{K}. \tag{14}$$

For $\theta \neq \theta_0$, equation (14) will *misspecify* the true equilibrium relationship, hence there need not exist $\lambda^{-1} \in \Lambda^{-1}$ satisfying (14). But if $\theta \in \Theta_I$, then there exists *at least one* candidate $\lambda^{-1} \in \Lambda^{-1}$ satisfying (14). In this case, under mild technical conditions on $C(a, s; \theta)$, one can extend arguments in GPV (2009) to recover this function from (14). In the following theorem, let $H(a, s; \theta) \equiv \frac{1-\partial C(a,s;\theta)/\partial a}{s+a-C(a,s;\theta)}$, and for any $\bar{r} > 0$, let $\Lambda^{-1}[0, \bar{r}]$ be the set of functions obtained by restricting elements of Λ^{-1} to the domain $[0, \bar{r}]$.

Theorem 2. *Consider any $\theta \in \Theta_I$ such that there exists at least one pair $k, l \in \mathcal{K}$ satisfying: (i) for all $a \in (0, 1)$, $\partial C(a, s_k; \theta)/\partial a < 1$ and $\partial C(a, s_l; \theta)/\partial a < 1$, and (ii) the set $A_{kl}(\theta) \equiv \{a \in [0, 1] : (N_k - 1)H(a, s_k; \theta) = (N_l - 1)H(a, s_l; \theta)\}$ is of Lebesgue measure zero. Then, for this θ, there exist unique $\lambda_\theta^{-1} \in \Lambda^{-1}[0, \max_{k\in\mathcal{K},a\in[0,1]} R_k(a)]$ and $F_\theta \in \mathcal{F}$ such that $(\lambda_\theta^{-1}, F_\theta, C(\cdot, \cdot; \theta)) \in \mathcal{I}$.*

Note that the conditions in Theorem 2 depend only on properties of $C(a, s; \theta)$, which can easily be checked numerically for any candidate $\theta \in \Theta$. Condition (i) implies that $R_k(0) = R_l(0) = 0$, so that 0 is in the equilibrium domain of λ_0^{-1}. Condition (ii) rules out pathologies which might lead bidding functions to be tangent on an open interval, in which case there could exist an interval of $a \in [0, 1]$ on which (14) would hold trivially for any $\lambda^{-1} \in \Lambda^{-1}$. So long as for each θ we can find at least one pair of competition levels such that both conditions hold—a very mild restriction in standard parametric families—Theorem 2 implies that identification essentially reduces to characterizing the identified set Θ_I for θ, which can simplify inference as we discuss in Section 4.4. Note also that at $\theta = \theta_0$, the *same* candidates $(\lambda_\theta^{-1}, F_\theta)$ must satisfy (14) for all $k, l \in \mathcal{K}$. This induces a continuum of identifying restrictions on θ_0 which must hold for all $\theta \in \Theta_I$.

4 Set inference on bid-stage primitives

We next consider inference based on the identification results above, focusing particularly on prospects for set inference when the model primitives is only partially identified. We consider an i.i.d. sample of L auctions where N varies exogenously on the set $\mathcal{N} = \{N_1, ..., N_K\}$ but no cost shifter z is available. For each auction $l = 1, ..., L$, we observe the number of potential bidders N_l, the entry decision $e_{il} \in \{0, 1\}$ for each potential bidder $i = 1, ..., N_l$, and the bid b_{il} submitted by each entrant. We parameterize bid-stage primitives flexibly in terms of Bernstein polynomial sieves as described below. We then apply results in CCT (2018) to develop tests for risk aversion and to construct confidence sets for bid-stage primitives using the restrictions in Theorem 1.

CCT (2018) propose three procedures to construct confidence sets for potentially set-identified parametric likelihood or moment (equality and inequality) based models. Their Procedures 1 and 2 are Monte Carlo simulation based confidence sets, and their Procedure 3 is a profile (quasi-) likelihood ratio based confidence sets for the identified set of a scalar parameter, using simple critical values from Chi-square distribution with one degree of freedom. Recall that the bid-stage primitives $(U_0, F_0, C_0) \in \mathcal{U} \times \mathcal{F} \times \mathcal{C}$ consist of smooth functions that also satisfy some shape restrictions and the additional restrictions imposed in Theorem 1. In what follows, we approximate the bid-stage primitives by flexible Bernstein polynomial sieves with large sieve dimensions so that the approximation error (or sieve bias) is of a smaller order and hence could be ignored in first order asymptotics (see, for example, Chen (2007)). We can then interpret bid-stage primitives as belonging to flexible parametric families so that CCT's results are applicable. We emphasize, however, that these families can be as flexible (over-parameterized) as desired, subject to standard concerns of computational cost and potential over-fitting. In particular, we do not require these flexible parameters to be point-identified, allowing researchers to target (insofar as possible) only restrictions implied directly by the AS-RA model.

4.1 Flexible parametric likelihood framework

Insofar as we aim to apply CCT (2018), we could in principle consider either flexible likelihood-based or optimal moment-type (optimal two-step GMM, continuous updating GMM, or generalized empirical likelihood) approaches to inference. We here explore the maximum likelihood approach, exploiting the fact that CCT (2018)'s results accommodate likelihood models with parameter dependent support (Appendix C of CCT). This is critical since, as pointed out by Donald and Paarsch (1993), the maximum predicted bid in a first-price auction will depend on the parameters to be estimated, leading to failure of standard regularity conditions for the asymptotic normality of MLE parameter estimation even in point-identified models. To our knowledge, this is the first exploration of CCT (2018)'s approach to set inference in first-price auctions. Our flexible parametric evaluation of the likelihood by substitution from first order conditions for equilibrium bidding is (we believe) also novel and should be applicable in other auction models.

Since we are primarily interested in bid-stage inference, we treat entry thresholds $\mathbf{s} \equiv (s_1, ..., s_K)$ as auxiliary parameters to be estimated. Let S denote the admissible set for $\mathbf{s}$: i.e., the set of $\mathbf{s} \in [0,1]^K$ such that $s_k \geq s_{k-1}$ for all $k = 2, ..., K$. Should one additionally wish to enforce entry constraints, one could start with bid-stage estimates obtained as below, then proceed to an efficient second step enforcing entry constraints.

We parameterize bid-stage primitives flexibly as follows. For computational reasons outlined in Appendix C, we parameterize in terms of the reciprocal $\bar{v}^{-1}$ rather than $\bar{v}$ directly; since $\bar{v}$ weakly exceeds the maximum observed bid, $\bar{v}^{-1}$ will belong to a known compact interval V^{-1}. For any nonnegative integers d, D with $D > 0$ and $d \leq D$, let $B_{d,D}(u)$ denote the dth Bernstein basis polynomial of degree D:

$$B_{d,D}(u) \equiv \binom{D}{d} u^d (1-u)^{D-d}, \quad u \in [0,1].$$

In inference, we target $\lambda_0(x) = U_0(x)/U_0'(x)$, rather its inverse $\lambda_0^{-1}(x)$. We parameterize $\lambda(x)$ as a shifted Bernstein polynomial of degree Q, with both range and domain scaled by the maximum valuation $\bar{v}$. Specifically, for $x \in [0, \bar{v}]$, we define $\lambda(x) = \bar{v}\tilde{\lambda}(\bar{v}^{-1}x)$, where $\tilde{\lambda}(u)$ is a shifted degree-Q Bernstein polynomial on $[0,1]$:

$$\tilde{\lambda}(u) \equiv u + \sum_{j=0}^{Q} \gamma_j B_{j,Q}(u), \quad u \in [0,1]. \tag{15}$$

We know $\lambda(0) = 0$, which implies $\gamma_0 = 0$, and that $\lambda'(x) \geq 1$, which we enforce by requiring $\gamma_j \geq \gamma_{j-1}$ for $j = 1, ..., Q$. We further assume $\gamma_Q \leq \bar{\gamma}$ for some constant $\bar{\gamma} < \infty$; a sufficient primitive condition for this is $RRA(x) < 1/(1-\bar{\gamma})$ for all $x \geq 0$, where $RRA(x) = -xU''(x)/U'(x)$ denotes the Arrow-Pratt coefficient of relative risk aversion. We take remaining coefficients $\gamma \equiv (\gamma_j)_{j=1}^{Q}$ as parameters to be estimated, belonging to a compact set Γ defined by the inequalities just described. Note that setting $Q = 1$ is equivalent to a CRRA model for U, in which case $\lambda(x) = x/(1 - RRA)$.

In principle, one could parameterize (F_0, C_0) using a flexible one-dimensional parameterization for F_0 and a flexible two-dimensional parameterization for C_0. Recall, however, that (F_0, C_0) enter only through the ex post distributions $F_1^0, ..., F_K^0$ (see (1)). In practice, therefore, we instead parameterize $F_1^0, ..., F_K^0$ directly, reinterpreting the conditions of Theorem 1 as constraints on the parameter space for $F_1^0, ..., F_K^0$. In our simulation exercise, which involves only discrete instruments, flexibly parameterizing $F_1^0, ..., F_K^0$ is easier for implementation, insofar as it directly targets restrictions implied by Theorem 1. When, however, a continuous entry instrument z is available, or when the cardinality of the set of entry instruments is otherwise, one may instead prefer to flexibly parameterize $(F_0, C_0) \in \mathcal{F} \times \mathcal{C}$ directly. Our proposed inference methods apply to either parameterization strategy, although we focus on flexibly parameterizing $F_1^0, ..., F_K^0$ in what follows.

Specifically, for each $k = 1, ..., K$, we first parameterize F_k as a Bernstein polynomial of degree P, scaled to the interval $[0, \bar{v}]$: i.e., we take $F_k(y) = \tilde{F}_k(\bar{v}^{-1}y)$, where

$$\tilde{F}_k(u) \equiv \sum_{j=0}^{P} \phi_{k,j} B_{j,P}(u), \quad u \in [0, 1]. \tag{16}$$

By definition, $F_k(0) = 0$ and $F_k(\bar{v}) = 1$, which implies $\phi_{k,0} = 0$ and $\phi_{k,P} = 1$. Let $\phi_k = (\phi_{k,1}, ..., \phi_{k,P-1})$ collect remaining free coefficients in the parameterization (16), and $\phi = (\phi_1, ..., \phi_K)$ collect vectors ϕ_k across competition levels $k = 1, ..., K$.

We then enforce the requirement that the coefficients $\phi = (\phi_1, ..., \phi_K)$ imply a set of distributions $F_1, ..., F_K$ consistent with Theorem 1. Toward this end, for each vector of equilibrium entry thresholds $\mathbf{s} \in S$, let the admissible set for ϕ given $\mathbf{s}$, denoted $\Phi(\mathbf{s})$, be the set of vectors $\phi = (\phi_1, ..., \phi_K)$ satisfying all of the following linear inequalities:

M' For each $k = 1, ..., K$, ϕ_k satisfies $0 \leq \phi_{k,1} \leq \phi_{k,2} \leq \cdots \leq \phi_{k,P-2} \leq \phi_{k,P-1} \leq 1$.

O' For all $k = 1, ..., K-1$ and all $u \in [0, 1]$,

$$\sum_{j=1}^{P-1} \phi_{k,j} B_{j,P}(u) \geq \sum_{j=1}^{P-1} \phi_{k+1,j} B_{j,P}(u).$$

D' For each $k = 1, ..., K-1$ and all $u \in [0, 1]$,

$$(1 - s_k) \left(\sum_{j=1}^{P-1} \phi_{k,j} B'_{j,P}(u) + B'_{P,P}(u) \right) \geq (1 - s_{k+1}) \left(\sum_{j=1}^{P-1} \phi_{k+1,j} B'_{j,P}(u) + B'_{P,P}(u) \right).$$

S' For each $k = 1, ..., K-2$ and all $u \in [0,1]$,

$$\sum_{j=1}^{P-1} \phi_{k,j} \left(\frac{1-s_k}{s_{k+1}-s_k} \right) B_{j,P}(u)$$
$$- \sum_{j=1}^{P-1} \phi_{k+1,j} \left(\frac{1-s_{k+1}}{s_{k+1}-s_k} - \frac{1-s_{k+1}}{s_{k+2}-s_{k+1}} \right) B_{j,P}(u)$$
$$+ \sum_{j=1}^{P-1} \phi_{k+2,j} \left(\frac{1-s_{k+2}}{s_{k+2}-s_{k+1}} \right) B_{j,P}(u) \geq 0.$$

Conditions O', D', and S' translate Conditions O, D, and S of Theorem 1 into the space of coefficients ϕ used in our parameterizations of $F_1, ..., F_K$; in practice, we enforce these on a fixed grid in $[0,1]$.[13] Meanwhile, Condition M' implies that each F_k is strictly monotone on $[0, \bar{v}]$, which in turn implies Conditions I and M of Theorem 1.

Let $\psi \equiv (\bar{v}^{-1}, \mathbf{s}, \gamma, \phi)$ denote the full vector of parameters to be estimated. Given the constraints above, ψ will belong to a known compact set Ψ defined by

$$\Psi = \{\psi \equiv (\bar{v}^{-1}, \mathbf{s}, \gamma, \phi) \,|\, \bar{v}^{-1} \in V^{-1}, \mathbf{s} \in S, \gamma \in \Gamma, \phi \in \Phi(\mathbf{s})\}.$$

Letting k_l denote the competition level in auction l, the conditional log-likelihood of observing outcome (e_{il}, b_{il}) in auction l given $N_l = N_{k_l}$ and parameters $\psi \in \Psi$ is

$$\ell_{il}(\psi) = (1 - e_{il}) \ln s_{k_l} + e_{il} \ln(1 - s_{k_l}) + e_{il} \ln g_{k_l}(b_{il}|\psi), \tag{17}$$

where $\ln g_k(\cdot|\psi)$ denotes the log density of equilibrium bids at competition N_k predicted by the model at parameters ψ. To evaluate $\ln g_k(b_{il}|\psi)$, we must first solve for the equilibrium bid function $\beta_k(y|\psi)$ at parameters ψ. Toward this end, let $\tilde{\beta}_k(u|\psi)$ be the solution to the following scale-normalized differential equation in $u \in [0,1]$:

$$\tilde{\beta}_k'(u|\psi) = \tilde{\lambda}\big(u - \tilde{\beta}_k(u|\psi)\big) \cdot \frac{(N_k - 1)(1 - s_k)\tilde{f}_k(u)}{s_k + (1 - s_k)\tilde{F}_k(u)}, \quad \tilde{\beta}_k(0|\psi) = 0. \tag{18}$$

For any candidate $\bar{v}$ and any $y \in [0, \bar{v}]$, we may then evaluate $\beta_k(y|\psi)$ as $\beta_k(y|\psi) = \bar{v}\tilde{\beta}(\bar{v}^{-1}y|\psi)$. This allows us to try multiple candidates for $\bar{v}$ without needing to re-solve (18), as well as focus on candidates such that $\bar{v} \geq b_{il}/\tilde{\beta}_k(1|\psi)$ and thus $\ln g_k(b_{il}|\psi)$ is finite.

[13] Specifically, we enforce O', D', and S' at $u \in \{0.0, 0.1, ..., 0.7, 0.8, 0.85, 0.9, 0.95, 0.99, 1.0\}$. This spacing emphasizes restrictions for u close to 1, which in view of Theorem 1 we expect to be particularly informative.

Moreover, defining u_{il} implicitly by $\bar{v}^{-1}b_{il} \equiv \tilde{\beta}_k(u_{il}|\psi)$, we may express $\ln g_k(b_{il}|\psi)$ as

$$\begin{aligned}
\ln g_k(b_{il}|\psi) &= \ln f_k(\beta_k^{-1}(b_{il}|\psi)) + \ln \beta_k^{-1,\prime}(b_{il}|\psi) \\
&= -\ln \tilde{\lambda}\Big(u_{il} - \bar{v}^{-1}b_{il}\Big) + \ln \bar{v}^{-1} - \ln(N_k - 1) + \ln\Big(s_k + (1 - s_k)\tilde{F}_k\big(u_{il}\big)\Big),
\end{aligned}$$

where the second line follows by substitution from (18). In practice, we compute $\ln g_{k_l}(b_{il}|\psi)$ using a hybrid of grid and exact evaluation, solving (18) using Chebyshev collocation. The resulting algorithm, detailed in Appendix C, yields fast, stable evaluation of both $\ln g_{k_l}(b_{il}|\psi)$ and its analytic gradients in ψ.

Finally, let $\{(e_{il}, b_{il})_{i=1}^{N_l}, N_l\}_{l=1}^{L}$ be a random sample, with asymptotics as $L \to \infty$. We also let $\mathcal{L}(\psi)$ denote the sample log-likelihood derived from (17):

$$\mathcal{L}(\psi) = \sum_{l=1}^{L}\sum_{i=1}^{N_l} \ell_{il}(\psi)$$

We aim to conduct inference on subvectors of $\psi \in \Psi$ based on $\mathcal{L}(\psi)$ without assuming point identification of ψ. We first outline a simple test for risk neutrality based on CCT (2018)'s Procedure 3. We then construct confidence sets on identified sets for subvectors based on CCT (2018)'s Procedure 2. Finally, we discuss potential simplifications introduced by a parametric copula.

4.2 A likelihood ratio test for risk neutrality

We first propose a test for risk neutrality based on Procedure 3 of CCT (2018), which provides a simple approach to constructing robust confidence sets for scalar parameters within potentially set-identified models. Recall that risk neutrality implies $\lambda_0(x) = x$, which is equivalent to $\gamma_j = 0$ for all $j = 1, ..., Q$. For $Q > 1$, this is a compound hypothesis, which requires the more involved procedures described next. For $Q = 1$, however—which is equivalent to linear λ_0 and CRRA U_0—risk neutrality reduces to the simple hypothesis $\gamma_1 = 0$. Furthermore, because linear λ_0 is nested by any higher-degree Bernstein parameterization, testing the null of risk neutrality in a CRRA model with $Q = 1$ is valid within any alternative model of degree $Q > 1$.

Combining these observations leads to the following simple test of risk neutrality. Let $\hat{\psi}_1$ maximize $\mathcal{L}(\psi)$ within the parameter space $\Psi_{Q=1}$ corresponding to the CRRA model with

$Q = 1$. Let $\hat{\psi}_0$ maximize $\mathcal{L}(\psi)$ within $\Psi_{Q=1}$ subject to the null restriction $\gamma_1 = 0$. Consider the standard likelihood ratio test statistic

$$TS_1 = 2\mathcal{L}(\hat{\psi}_1) - 2\mathcal{L}(\hat{\psi}_0). \tag{19}$$

Under the null of risk neutrality and conditions in Theorem 4.4 of CCT (2018), the distribution of the test statistic TS_1 will be asymptotically first-order stochastically dominated by $F_{\chi^2_1}$, the chi-squared distribution with one degree of freedom.[14] Letting $\chi^2_{1,\alpha}$ be the α quantile of $F_{\chi^2_1}$ (i.e., $F_{\chi^2_1}(\chi^2_{1,\alpha}) = \alpha$), one may therefore reject risk neutrality with at least confidence level α whenever $TS_1 \geq \chi^2_{1,\alpha}$. As noted above, this test is valid, although conservative, even when the alternative model of interest involves λ_0 flexibly parameterized with arbitrary $Q > 1$. We thus see this simple likelihood ratio test as a natural diagnostic before considering inference within more flexible models. The simulation result in Section 5 indicates that, within the first-price AS-RA auction framework, our simple test for risk neutrality is powerful even without a continuous instrument z.[15]

4.3 Confidence sets for bid-stage primitives

We now turn to confidence sets for (subvectors of) ψ applicable under flexible parameterizations of bid-stage primitives. Toward this end, we apply Procedures 1 and 2 of CCT (2018), which yield asymptotically valid Monte Carlo confidence sets for the identified sets of ψ and its subvectors, respectively.

To implement these procedures, we require a sample of parameters $\{\psi^b\}_{b=1}^B$ drawn from a quasi-posterior distribution implied by $\mathcal{L}(\psi)$. Letting Π be a prior over Ψ, the quasi-posterior distribution Π_L for ψ given the auction data is defined as

$$d\Pi_L(\psi|Data) = \frac{\exp[\mathcal{L}(\psi)]\, d\Pi(\psi)}{\int_\Psi \exp[\mathcal{L}(\psi)]\, d\Pi(\psi)}\,. \tag{20}$$

[14]In this case, the null hypothesis $\gamma_1 = 0$ is on the edge of the admissible set for γ_1, which leads to a further stochastic reduction in the distribution of TS_1 over and above that in CCT (2018). Even relaxing the admissible set for γ_1 to include negative (risk-seeking) values, TS_1 would be asymptotically dominated by a χ^2_1 distribution under conditions in CCT (2018). The boundary constraint $\gamma_1 \geq 0$ leads TS_1 to be zero, instead of positive, when this constraint binds, which further stochastically reduces TS_1.

[15]Fang and Tang (2014) propose a nonparametric test for risk aversion in ascending auctions which extends to selective entry when the entry cost is observable, while LLZ (2015) use both first price and ascending formats to propose a reduced form test for risk aversion.

To form Π, we use uniform priors over the relevant admissible sets for all elements of ψ except the utility coefficients γ. For this step only, we reparameterize γ as $\tilde{\gamma} = (\gamma_1/\gamma_Q, ..., \gamma_{Q-1}/\gamma_Q, \gamma_Q)$, with admissible set $\tilde{\Gamma}$ defined by $0 \leq \tilde{\gamma}_1 \leq \cdots \leq \tilde{\gamma}_{Q-1} \leq 1$ and $0 \leq \tilde{\gamma}_Q \leq \bar{\gamma}$. We then adopt a uniform prior over $\tilde{\Gamma}$, which ensures that the average slope $\lambda_0(\bar{v})/\bar{v}$ is uniformly distributed on $[1, \bar{\gamma}+1]$ under the prior.[16] Finally, we draw a parameter sample $\{\psi^b\}_{b=1}^B$ from Π_L using an adaptive Sequential Monte Carlo (SMC) algorithm adapting that in CCT (2018), described in detail in Appendix C.

To construct confidence sets for the identified set of the full parameter vector ψ, we apply Procedure 1 of CCT (2018). Specifically, for any $\alpha \in (0, 1)$, let $\xi_{1-\alpha}$ be the $(1-\alpha)$th quantile of $\{\mathcal{L}(\psi^b)\}_{b=1}^B$; i.e., the sample of log-likelihood values obtained by evaluating $\mathcal{L}(\cdot)$ at each posterior parameter draw ψ^b. The set $\hat{\Psi}_\alpha = \{\psi \in \Psi : \mathcal{L}(\psi) \geq \xi_{1-\alpha}\}$ is then a $100\alpha\%$ confidence set for the identified set of ψ. That is, under conditions in CCT Appendix C, $\hat{\Psi}_\alpha$ will cover the identified set for ψ with asymptotic probability α, even in the presence of parameter-dependent support.

For inference on subvectors (or functionals) η of ψ, we apply Procedure 2 in CCT (2018). We refer to CCT (2018) for a full description of this procedure, as well as to Appendix C for details of our implementation. In brief, however, starting from the posterior parameter sample $\{\psi^b\}_{b=1}^B$ obtained above, we compute for each ψ^b a profile criterion $PL(\psi^b)$ defined as follows. For any $\psi, \psi' \in \Psi$, let $D_{KL}(\psi||\psi')$ denote the Kullback-Leibler divergence from the distribution of observables implied by ψ to that implied by ψ'. $PL(\psi^b)$ is then the infimum of the profile log-likelihood $\tilde{L}(\eta') = \sup_{\psi \in \Psi}\{\mathcal{L}(\psi) : \eta(\psi) = \eta'\}$, taken over the set of η' such that there exists some $\psi' \in \Psi$ with $\eta(\psi') = \eta'$ and $D_{KL}(\psi^b||\psi') = 0$. Finally, letting $\xi_{1-\alpha}^\eta$ be the $(1-\alpha)$th quantile of the sample of profile criteria $\{PL(\psi^b)\}_{b=1}^B$, we take the set of subvectors η such that $\tilde{L}(\eta) \geq \xi_{1-\alpha}^\eta$ as a $100\alpha\%$ confidence set for the identified set for η.

[16] In contrast, a uniform prior over the original space Γ would imply that $\lambda(\bar{v})/\bar{v}$ has the same distribution as the highest of Q draws from a uniform distribution over $[1, \bar{\gamma}+1]$, which for $Q > 1$ assigns prior probability approaching zero to values near the risk-neutral boundary $\lambda(\bar{v})/\bar{v} = 1$. For a more general discussion of sieve priors in Bayesian inference, see Ghosal and van der Vaart (2017).

4.4 Inference with a parameteric copula

As an alternative to parameterizing $F_1^0, ..., F_K^0$, one may instead prefer to parameterize F_0 and C_0, especially when the cardinality of the set of observables affecting entry behavior is large. C_0 could be parameterized either flexibly, for example as a two-dimensional Bernstein polynomial, or within a low-dimensional parametric family. In either case, inference may proceed essentially as in Sections 4.2 and 4.3. In view of Theorem 2, however, we also know that (λ_0, F_0) are conditionally identified up to θ_0. When θ_0 is low-dimensional, this fact can substantially simplify inference, as we discuss next.

For concreteness, suppose that C_0 belongs to a single-parameter family with true parameter θ_0; canonical examples include Gaussian copulas and many parametric Archimedean copula families, such as the Gumbel, Clayton, Frank, Joe and Ali-Mikhail-Haq copulas. One can then apply CCT Procedure 3 to obtain a robust $100\alpha\%$ confidence set $\hat{\Theta}_\alpha$ for θ_0 by inverting the relevant profile likelihood ratio statistic. Furthermore, in view of Theorem 2, under any maintained hypothesis $\theta = \theta_0$, we may conduct inference on (λ_0, F_0) as if these are point-identified. This suggests a natural approach to sensitivity analysis: recover estimates $(\hat{\lambda}_\theta, \hat{F}_\theta)$ for (λ_0, F_0) under the hypothesis $\theta = \theta_0$, then trace out the resulting estimates $(\hat{\lambda}_\theta, \hat{F}_\theta)$ across $\theta \in \hat{\Theta}_\alpha$. This approach may be particularly attractive if one wishes to compare a baseline non-selective model with a richer AS-RA alternative, since it indicates both whether selection is empirically important and how this may influence other primitives.

Proceeding to full set inference following CCT (2018), we can again leverage conditional identification of (λ_0, F_0) up to θ_0 to simplify implementation. This is especially true for subvector inference using CCT Procedure 2, which requires one to search over the set of parameters observationally equivalent to a given posterior draw ψ^b. Conditional identification up to θ implies that this equivalent set can be indexed by a subset $\tilde{\Theta}^b \subset \Theta$. In particular, for scalar θ_0, characterizing the observationally equivalent set reduces to line search over θ, regardless of the dimension of the target subvector. This can yield significant computational savings, especially for high-dimensional subvectors.

5 A simulation exercise

Finally, we explore our proposed methods in a Monte Carlo study based on the following true data generating process (DGP). Bidders have CRRA utility: $U_0(x) = U(x;\rho_0) = x^{1-\rho_0}$. We consider both risk neutral ($\rho_0 = 0$) and risk averse ($\rho_0 = 0.5$) baseline models; we also explore testing risk neutrality at intermediate values of ρ_0. Valuations are drawn from a truncated logistic distribution F_0 with mean 0.5 and scale 0.2, truncated on the interval $[0, \bar{v}_0]$ with $\bar{v}_0 = 1$. Dependence between V_i and S_i is captured by a Gumbel copula

$$C(F, s; \theta) = \exp\left\{-[(-\log F)^{\theta} + (-\log s)^{\theta}]^{1/\theta}\right\} \quad \text{for} \quad 1 \leq \theta < \infty,$$

with true parameter value $\theta_0 = 1.5$, corresponding to a Spearman's rank correlation between V_i and S_i of approximately 0.475. N varies exogenously on the set $\mathcal{N} = \{2, 4, 6, 8\}$, so that $K = 4$ and $\mathcal{K} = \{1, ..., 4\}$.

For each DGP considered, we draw 100 sets of Monte Carlo data at each of three sample scales $M \in \{500, 1000, 2000\}$, where for each sample scale M we choose the number of auctions L_k at each competition level k such that we observe approximately M bids for each k: i.e., $L_k = \lceil \frac{M}{N_k(1-s_k)} \rceil$, with $L = \sum_1^4 L_k$. We then apply the procedures in Section 4 to these simulated data. We parameterize ex post distributions $F_k^0(\cdot)$ with Bernstein polynomials of degree $P = 7$ as above. We parameterize $\lambda_0(\cdot)$ as Bernstein polynomials of degree $Q = 1$ (the true CRRA model), $Q = 4$, or $Q = 7$.

We begin by exploring the identified set for $\lambda(x)$ numerically. Figure 1 plots pointwise profiled Kullback-Leibler contour sets for the shape of λ_0 under different combinations of ρ_0 and Q. For the risk neutral model $\rho_0 = 0.0$, these contour sets fall off steeply away from the true value $\lambda(x) = x$, suggesting that (at least for our parameterizations) this case may be close to point identified. In contrast, when $\rho_0 = 0.5$, pointwise profiled KL divergence is numerically flat (with both level and slope less than $10^{-5.5}$) on an interval around each true value $\lambda(x) = 2x$, for both $Q = 4$ and $Q = 7$. Generically, therefore, one should treat parameters as set-identified.

[Figure 1 about here.]

We next explore performance of our testing and inference methods in the CRRA case with $Q = 1$. This can be viewed either as a specification of interest in its own right, or as a device for testing risk neutrality within a more flexible alternative model. Figure 2 explores inference based on CCT Procedures 2 and 3 in a correctly specified CRRA model with $Q = 1$. This figure describes rejection rates for hypothesized values of the average slope $\lambda_0(x)/x \equiv 1/(1-\rho_0)$, for our benchmark risk neutral ($\rho_0 = 0$) and risk averse ($\rho_0 = 0.5$) DGPs with $M = 1000$ bids per auction. Actual confidence rates for both procedures are very close to each other. Since results of CCT (2018) Procedures 2 and 3 should coincide under point identification, this is exactly as we expect.

[Figure 2 about here.]

Meanwhile, Figure 3 presents power curves for our simple test of risk neutrality based on Procedure 3 of CCT (2018). Panels (a) and (b) explore power of this test in our baseline CRRA DGP, under which both null and alternative models are correctly specified. As expected, the test is somewhat conservative when ρ_0 is close to 0, although it displays good power as ρ_0 moves away from 0. Meanwhile, Panels (c) and (d) present power curves obtained when only the null model is correctly specified; for $\rho_0 > 0$, the true bidding process involves CARA, rather than CRRA, utility, although we continue to test within a CRRA ($Q = 1$) specification. Encouragingly, power is comparable to that in the correctly specified CRRA model, suggesting that this simple test can perform quite well even under misspecification of the alternative model.

[Figure 3 about here.]

Finally, we turn to set inference in the full flexible model with $Q > 1$. We apply CCT Procedure 2 to obtain pointwise confidence sets for the identified set for $\tilde{\lambda}_0(x)/x$ at the points $x = 0.4$, $x = 0.6$, and $x = 1$.[17] In constructing these confidence sets, we consider two parameter vectors ψ, ψ' numerically equivalent if $D_{KL}(\psi||\psi') \leq 10^{-5.0}$, where $D_{KL}(\psi||\psi')$ is scaled to reflect the expected loss in sample log-likehood $\mathcal{L}(\psi')$ per bid observed at each N, when true parameters are ψ. Inference will thus be somewhat conservative for true identified sets, although we expect to cover contour sets based on $D_{KL} \leq 10^{-5.0}$ with approximately

[17]in view of Figure 1, we expect that at $x = 1$ only lower bounds on $\tilde{\lambda}_0(x)/x$ will be informative. This lower bound is interesting, however, since the model is globally risk neutral if and only if $\tilde{\lambda}_0(1) = 1$.

correct probability. Note that, since $\tilde{\lambda}_0(x)/x$ now depends on a vector of coefficients γ, inference on $\tilde{\lambda}_0(x)/x$ now requires Procedure 2; the simpler Procedure 3 will be inconsistent.

We first apply CCT Procedure 2 to construct confidence sets for $\tilde{\lambda}_0(x)/x$ within our main flexible specification with $Q = 4$. Figure 4 reports estimated coverage (acceptance) rates for $\tilde{\lambda}_0(x)/x$ for level $\alpha = 0.95$ confidence sets, while Table 1 reports the estimated probabilities with which level $\alpha = 0.95$ and $\alpha = 0.99$ confidence sets cover the $D_{KL} \leq 10^{-5.0}$ and $D_{KL} \leq 10^{-5.5}$ profiled contour sets for $\tilde{\lambda}_0(x)/x$, as well as median confidence intervals for $\tilde{\lambda}_0(x)/x$ based on these confidence sets. As sample size increases, estimated coverage probabilities for $D_{KL} \leq 10^{-5.0}$ contour sets approach nominal confidence levels in almost all cases. When $\rho_0 = 0.5$ and $x = 1$, the low end of the $D_{KL} \leq 10^{-5.0}$ contour set for $\tilde{\lambda}_0(x)/x$ is slightly under-covered; this may reflect the fact that $x = 1$ is well outside the true domain of the argument $v - \beta_k(v)$ to $\tilde{\lambda}_0$, which is approximately $[0, 0.5]$. Even in this case, however, the set of $\tilde{\lambda}_0(x)/x$ thus under-covered is very small, as Figure 4 illustrates for the case of $\alpha = 0.95$. Moreover, as sample size increases, $D_{KL} \leq 10^{-5.5}$ contour sets are covered with at least nominal rates by confidence sets constructed at both $\alpha = 0.95$ and $\alpha = 0.99$.

[Figure 4 about here.]

[Table 1 about here.]

Finally, we turn to our most flexible specification, in which λ_0 is parameterized as a Bernstein polynomial of degree $Q = 7$. We emphasize that this is a very flexible parameterization, which we interpret as a "stress test" of the model's capabilities. Figure 5 plots coverage (acceptance) rates for $\tilde{\lambda}_0(x)/x$ based on CCT Procedure 2 with $\alpha = 0.95$ when $Q = 7$. As above, we consider both risk averse ($\rho_0 = 0.5$) and risk neutral ($\rho = 0$) baseline DGPs, although we compute confidence rates only at $x = 0.4$ and $x = 1.0$. Especially when $\rho_0 = 0.5$ and $x = 0.4$, we find some under-coverage at the low end of the identified set. This under-coverage appears to be driven by over-fitting with a very flexible parameterization for $\lambda_0(\cdot)$. To explore this, we repeated estimation at $\rho_0 = 0.5$ and $x = 0.4$ normalizing $\tilde{\gamma}$ relative to γ_3 and using a marginal exponential prior for γ_3, which serves to regularize the model and prevent over-fitting. As illustrated in panel (c), this alternative prior does indeed resolve the under-coverage issue noted with our baseline uniform prior, suggesting that priors tilted toward risk neutrality are advisable when λ_0 is very flexibly parameterized.

[Figure 5 about here.]

6 Conclusion

This paper studies identification and inference in first price auctions with selective entry and risk averse bidders. We establish results on nonparametric and semiparametric identification of the model primitives based on variation either in the number of potential competitors N or in an instrument z influencing bidders' opportunity costs of entry. Assuming that both utility and the ex ante distribution of potential bidders' private information do not depend on realizations of N and z, we provide a sharp characterization of the set of model primitives consistent with equilibrium bidding behavior. We show that an implication of this result is that risk neutrality is nonparametrically testable even with only variation in N, and we provide a simple likelihood ratio test for risk neutrality, which is useful and important in analyzing auction data given the different economic and policy implications implied by different risk attitudes. Moreover, we show that with only variation in N, the model primitives are nonparametrically set identified, while they can be point identified with an additional continuous instrument z.

Based on these identification results, we propose a new approach to inference in set identified auction models, adapting CCT (2018) to our case. Our inference method is flexible and has the advantage of accommodating both set identified models and models that have parameter dependent support. Noting that parameter dependent support problems are intrinsic to auction models and also appear in other structural microeconometric models such as job search models (Flinn and Heckman (1982)), and that partial identification and inference on identified sets have been among the most active research areas in econometrics recently, our inference method contributes to the literature by providing a flexible and computationally convenient method to deal with both problems. As demonstrated in our simulation study, our inference method has good finite sample performance and has potential to find wide applications.

Appendix A: Theoretical details

For completeness, we first extend the simple AS-RA model presented in Section 2 to accommodate two potentially relevant real-world considerations: nonzero initial wealth for bidders and financial (in addition to opportunity) costs of entry. We then provide a detailed characterization of equilibrium entry and bidding behavior within this extended model, demonstrating in the process how the more general structure considered here collapses in all economically relevant details to that in Section 2.

As above, we assume that potential bidders are risk averse with risk preferences described by some symmetric concave Bernoulli utility function $u(w)$, where w is net post-auction wealth. Net entry costs are given by $c(z) = c_0 + c_1(z)$, where c_0 denotes financial costs of entry and $c_1(z)$ denotes opportunity costs of entry. We further assume that bidders are endowed with common initial wealth $w_0 \geq c_0$.

Following LLZ (2015), we now define a normalized utility function $U(\cdot)$ as a function of the *change* in wealth x derived from bidding, normalized such that a bidder who enters the auction but does not win receives zero normalized utility:

$$U(x) \equiv u(x + w_0 - c_0) - u(w_0 - c_0).$$

For simplicitly, and without loss of generality, we further normalize the scale of utility such that $U(1) = u(1 + w_0 - c_0) - u(w_0 - c_0) \equiv 1$.

As noted by LLZ (2015), centered utility $U(\cdot)$ belongs to the same category of Arrow-Pratt absolute risk aversion (increasing, constant, or decreasing) as initial utility $u(\cdot)$. Furthermore, as we show below, knowledge of normalized utility U is equivalent to joint knowledge of non-normalized (u, w_0, c_0) with respect to characterizing equilibrium entry and bidding behavior. In this sense, the simplified presentation in the text is without loss of generality.

We next provide a detailed derivation of the symmetric monotone Bayesian Nash equilibrium in our entry and bidding game. Recall that in any such equilibrium entry must involve a signal threshold $s_N(z)$ such that bidders with signals $S_i \geq s_N(z)$ elect to enter. We therefore characterize this equilibrium in two steps. First, for any potential entry threshold $\bar{s} \in [0, 1)$, we derive the strategy $\beta(\cdot|N, \bar{s})$ describing a symmetric monotone bidding equilibrium assuming that all bidders entered according to threshold $\bar{s}$. We then characterize the equilibrium entry threshold $s_N(z)$.

A.1 Equilibrium bidding

First consider the Stage 2 bidding problem faced by an entrant with valuation v_i assuming that i's $N-1$ potential rivals all enter according to some (arbitrary) Stage 1 entry threshold $\bar{s} \in [0, 1)$. We seek a strictly increasing bidding strategy $\beta(\cdot|N, \bar{s})$ such that bidder i with valuation v_i optimally bids $\beta(v_i|N, \bar{s})$ when facing $N-1$ rivals who enter according to $\bar{s}$ and bid according to $\beta(\cdot|N, \bar{s})$.

Toward this end, let $F(\cdot|S_j \geq \bar{s})$ denote the c.d.f. of rival j's valuation conditional on choosing to enter at threshold $\bar{s}$:

$$F(y|s_j \geq \bar{s}) = \frac{1}{1-\bar{s}} \int_{\bar{s}}^{1} F(y|t)\, dt.$$

Under Assumptions 1-4, the support of $F(\cdot|S_j \geq \bar{s})$ is a connected interval of the form $[\underline{v}(\bar{s}), 1]$, where the infimum support $\underline{v}(\bar{s})$ is differentiable in $\bar{s}$. Moreover, the density $f(\cdot|S_j \geq \bar{s})$ is locally bounded away from zero for all $v \in (\underline{v}(\bar{s}), 1]$.

Let $F^*(\cdot|N,\bar{s})$ (and $f^*(\cdot|N,\bar{s})$) be the c.d.f. (and the pdf) of the maximum valuation among rival entrants when i's $N-1$ rivals enter according to threshold $\bar{s}$:

$$F^*(y|N,\bar{s}) = [\bar{s} + (1-\bar{s})F(y|S_j \geq \bar{s})]^{N-1},$$

Since $(1-\bar{s})F(y|S_j \geq \bar{s}) = \int_{\bar{s}}^1 F(y|t)\,dt$, we have $\frac{\partial}{\partial \bar{s}} = -F(y|\bar{s})$, and thus $F^*(y|N,\bar{s})$ is increasing in $\bar{s}$ (strictly for y such that $F(y|\bar{s}) < 1$). Furthermore, since $[\bar{s} + (1-\bar{s})F(y|S_j \geq \bar{s})] \leq 1$ (strictly for $y < \bar{v}$), $F^*(y|N,\bar{s})$ is decreasing in N (strictly if $y < \bar{v}$).

Assuming that all potential rivals according to the symmetric monotone strategy $\beta(\cdot|N,\bar{s})$, entrant i submitting bid $b_i \equiv \beta(y_i|N,\bar{s})$ will outbid all potential rivals with probability $F^*(y_i|N,\bar{s})$. The expected profit of entrant i with valuation v_i who bids *as if* her type were y_i is therefore:

$$\begin{aligned}
\pi_N(y_i, v_i; \bar{s}) &\equiv u(v_i - \beta(y_i|N,\bar{s}) + w_0 - c_0)F^*(y_i|N,\bar{s}) + u(w_0 - c_0)(1 - F^*(y_i|N,\bar{s})) \\
&= [u(v_i - \beta(y_i|N,\bar{s}) + w_0 - c_0) - u(w_0 - c_0)]F^*(y_i|N,\bar{s}) + u(w_0 - c_0) \\
&= U(v_i - \beta(y_i|N,\bar{s}))F^*(y_i|N,\bar{s}) + u(w_0 - c_0).
\end{aligned}$$

Taking a first-order condition of the final expression with respect to y_i, enforcing the equilibrium condition $y_i = v_i$, and solving for $\beta'(\cdot|N,\bar{s})$, we conclude that $\beta(\cdot|N,\bar{s})$ must satisfy

$$\begin{aligned}
\beta'(v_i|N,\bar{s}) &= \frac{U(v_i - \beta(v_i|N,\bar{s}))}{U'(v_i - \beta(v_i|N,\bar{s}))}\frac{f^*(v_i|N,\bar{s})}{F^*(v|N,\bar{s})} \\
&= \frac{U(v_i - \beta(v_i|N,\bar{s}))}{U'(v_i - \beta(v_i|N,\bar{s}))}\frac{(N-1)(1-\bar{s})f(v|S_j \geq \bar{s})}{\bar{s} + (1-\bar{s})F(v|S_j \geq \bar{s})}.
\end{aligned}$$

Imposing the boundary condition $\beta(\underline{v}(\bar{s})|N,\bar{s}) = 0$, we ultimately obtain an initial value problem characterizing $\beta(\cdot|N,\bar{s})$:

$$\begin{aligned}
&\beta(\underline{v}(\bar{s})|N,\bar{s}) = 0, \\
&\beta'(v|N,\bar{s}) = \frac{U(v - \beta(v|N,\bar{s}))}{U'(v - \beta(v|N,\bar{s}))}\frac{(N-1)(1-\bar{s})f(v|S_j \geq \bar{s})}{\bar{s} + (1-\bar{s})F(v|S_j \geq \bar{s})}, \quad v \in [\underline{v}(\bar{s}), \bar{v}].
\end{aligned} \tag{21}$$

Arguments in LLZ (2015) show that (21) yields a unique solution $\beta(\cdot|N,\bar{s})$ which is strictly increasing and differentiable in v, strictly increasing in N, and strictly decreasing and continuous in $\bar{s}$. Recalling that $F^*(y|N,\bar{s})$ is increasing in $\bar{s}$ and decreasing in N, expected equilibrium Stage 2 profit

$$\pi_N^*(v_i;\bar{s}) \equiv U(v_i - \beta(v_i|N,\bar{s}))F^*(v_i|N,\bar{s}) + u(w_0 - c_0)$$

will therefore be strictly increasing and continuous in v_i, strictly decreasing in N, and increasing (strictly for $v_i > \underline{v}(\bar{s})$) and continuous in $\bar{s}$.

Observe that the equilibrium bidding function $\beta(\cdot|N,\bar{s})$ depends on the ex ante utility function $u(\cdot)$, bidders' initial wealth w_0, and financial entry costs c_0 only through the normalized post-entry utility function U. For purposes of characterizing equilibrium bidding, knowledge of U is thus equivalent to knowledge of (u, w_0, c_0).

A.2 Equilibrium entry

Now consider the Stage 1 entry decision of potential bidder i with signal s_i facing $N-1$ potential rivals who enter according to $\bar{s}$ and bid according to $\beta(\cdot|N,\bar{s})$. Recall that i must foregо opportunity costs $c_1(z)$ from staying out. Holding the opportunity cost shifter z constant, the *change* in payoff i expects from entry is therefore:

$$\begin{aligned}
&\int_0^{\bar{v}} \pi_N^*(v;\bar{s})\, dF(v|S_i = s_i) - u(w_0 + c_1(z)) \\
&\qquad = \int_0^{\bar{v}} U(v - \beta(v|N,\bar{s}))F^*(v|N,\bar{s})\, dF(v|S_i = s_i) + u(w_0 - c_0) - u(w_0 + c_1(z)).
\end{aligned}$$

Noting that $u(w_0 + c_1(z)) = u(c_0 + c_1(z) + w_0 - c_0)$, we may equivalently rewrite the final line as

$$\Pi(s_i;\bar{s},N) - U(c_0 + c_1(z)),$$

where $\Pi(s_i,\bar{s},N)$ denotes the expected *normalized* post-entry profit of a bidder with signal $S_i = s_i$, facing $N-1$ potential rivals who enter according to threshold $\bar{s}$:

$$\Pi(s_i;\bar{s},N) \equiv \int_0^{\bar{v}} U(v - \beta(v|N,\bar{s}))F^*(v|N,\bar{s})\, dF(v|S_i = s_i).$$

Finally, consider the threshold $s_N(z)$ characterizing *equilibrium* entry at (N,z). In any equilibrium with nontrivial entry (i.e. where $s_N(z)$ is strictly between 0 and 1), this threshold $s_N(z)$ must be such that a bidder with signal $s_i = s_N(z)$ is just indifferent to entry. In view of the final expression above, and recalling that $c(z) \equiv c_0 + c_1(z)$, this in turn implies a break-even condition of the form given in the main text above:

$$\Pi(s_N(z), s_N(z); N) \equiv U(c(z)). \tag{22}$$

Recall from above that bid-stage expected profit $\pi_N^*(v_i,\bar{s})$ is strictly increasing in v_i, strictly increasing in $\bar{s}$ for $v_i > \underline{v}(\bar{s})$, and strictly decreasing in N. Combining these observations with the fact that V_i is stochastically increasing in S_i, and that the expectation of any continuous function of V_i conditional on $S_i = s_i$ is continuous in s_i, it is straightforward to show that pre-entry expected profit $\Pi(s_i;\bar{s},N)$ is also increasing and continuous in s_i, increasing and continuous in $\bar{s}$, and decreasing in N. Hence the breakeven condition above will uniquely determine $s_N(z)$. Furthermore, since $c(z)$ is strictly increasing in z, $s_N(z)$ will be at least weakly increasing in both N and z (strictly if $s_N(z) \in (0,1)$). Finally, note that we can have $s_N(z) = 1$ only if a potential bidder with signal $S_i = 1$ earns nonpositive profit from entering, bidding zero, and winning with certainty: i.e., only if $\int_0^{\bar{v}} U(y)\, dF(y|S_i = 1) \leq U(c(z))$. But this condition does not depend on N. Thus if $s_N(z) < 1$ for any N, $s_{N'}(z) < 1$ for all N'.

Combining this equilibrium threshold $s_N(z)$ with the characterization of equilibrium bidding obtained above, we therefore conclude:

Theorem 3 (Li, Lu and Zhao (2015))**.** *Suppose that $U \in \mathcal{U}$, $F \in \mathcal{F}$, and $C \in \mathcal{C}$. Then there exists a unique symmetric monotone pure strategy Bayesian Nash equilibrium for any $N \in \mathcal{N}$ and $z \in \mathcal{Z}$. The equilibrium bidding strategy $\beta(\cdot|N, s_N(z))$ is the unique solution to the initial value problem* (21) *with $\bar{s} = s_N(z)$. The equilibrium entry threshold $s_N(z)$ is uniquely determined as follows:*

- *If $\Pi(0,0,N) > U(c(z))$, then $s_N(z) = 0$ and all bidders enter.*
- *If $\Pi(1,1,N) < U(c(z))$, then $s_N(z) = 1$ and no bidder enters.*
- *Otherwise, $s_N(z)$ is the unique solution to $\Pi(s_N(z), s_N(z), N) = U(c(z))$.*

Furthermore, $s_N(z)$ is increasing in both N and $c(z)$, strictly if $s_N(z) \in (0,1)$. Finally, if $s_N(z) < 1$, then $s_{N'}(z) < 1$ for any $N' > N$.

Note, once again, that knowledge of the normalized utility function $U(\cdot)$ and the total entry cost $c(z)$ are sufficient to fully characterize all aspects of equilibrium entry and bidding behavior. In particular, it is not necessary to identify initial wealth w_0 or to separate $c(z)$ into financial costs c_0 versus opportunity costs $c_1(z)$. The simplified framing in the main text, in which we begin with normalized utility U and interpret $c(z)$ as a pure opportunity cost, is thus without any essential loss of generality.

A.3 Translating identified sets for bid-stage primitives into bounds on conditional distributions and entry costs

First consider restrictions on (F_0, C_0) imposed by equilibrium bidding, given a fixed candidate λ for λ_0^{-1} belonging to the bid-stage identified set Λ_I^{-1} for λ_0^{-1} defined by Theorem 1. This identified set could be a singleton when λ_0^{-1} is point identified.

Toward this end, let $\mathcal{S} = \{s \in [0,1] : s = s_k(z) \text{ for some } k \in \mathcal{K}, z \in \mathcal{Z}\}$. Recall that if $s_k(z) = s_l(z')$, then by Restriction O we must have $\tilde{v}_k(y|z;\lambda^{-1}) = \tilde{v}_l(y|z';\lambda^{-1})$. With slight abuse of notation, we may therefore define a new collection of candidate selected value c.d.f.s $\tilde{F}(\cdot; s, \lambda^{-1})$ for $s \in \mathcal{S}$ by setting, for each $s \in S$, $\tilde{F}(\cdot; s, \lambda^{-1}) \equiv \tilde{v}_k^{-1}(\cdot|z;\lambda^{-1})$ for some $k \in \mathcal{K}$, $z \in \mathcal{Z}$ such that $s_k(z) = s$. From Theorem 1, the set of $(F, C) \in \mathcal{F} \times \mathcal{C}$ which rationalize bid-stage observables at $\lambda^{-1} \in \Lambda_I^{-1}$ are those for which

$$\tilde{F}(y; s, \lambda^{-1}) = \frac{F(y) - C(F(y), s)}{1 - s} \quad \forall s \in \mathcal{S}, y \in \mathbb{R}. \tag{23}$$

In other words, the set of $(F, C) \in \mathcal{F} \times \mathcal{C}$ rationalizing bid stage observables at λ^{-1} are those which reproduce the distributions of entrant valuations $\tilde{F}(\cdot; s, \lambda^{-1})$ implied by λ^{-1} at each equilibrium entry threshold $s \in \mathcal{S}$.

We next adapt Theorem 3 in GL (2014) to translate the set of candidates $(F, C) \in \mathcal{F} \times \mathcal{C}$ consistent with $\lambda^{-1} \in \Lambda_I^{-1}$ into bounds on candidate conditional c.d.f.s $F(y|S_i = s)$ at each $s \in \mathcal{S}$. Toward this end, let $s_l = \inf \mathcal{S}$ and $s_u = \sup \mathcal{S}$, and for each $s \in \mathcal{S}$, let $t^-(s) = \sup\{t \in \{0\} \cup \mathcal{S} : t < s\}$ and $t^+(s) = \inf\{t \in \{1\} \cup \mathcal{S} : t > s\}$ be the nearest upper and lower neighbors of s in $\mathcal{S}$ (or trivial bounds if no neighbor is available). For each $s \in \mathcal{S}$ such that $s > s_l$, we may then define an identified upper bound $\tilde{F}^+(y|s;\lambda^{-1})$ on the set of $F(y|S_i = s)$ consistent with our original $\lambda^{-1} \in \Lambda_I^{-1}$ as follows:

$$\tilde{F}^+(y|s;\lambda^{-1}) = \lim_{t \uparrow t^-(s)} \frac{\tilde{F}(y; t, \lambda^{-1})(1-t) - \tilde{F}(y; s, \lambda^{-1})(1-s)}{s - t},$$

which we complete with the trivial bound $\tilde{F}^+(y|s_l;\lambda^{-1}) = \mathbb{I}[y \leq 0]$ at $s = s_l$. Analogously, for each $s \in \mathcal{S}$ such that $s < s_u$, we may define an identified lower bound $\tilde{F}^-(y|s;\lambda^{-1})$ on the set of $F(y|S_i = s)$ consistent with λ^{-1} by

$$\tilde{F}^-(y|s;\lambda^{-1}) = \lim_{t \downarrow t^+(s)} \frac{\tilde{F}(y; s, \lambda^{-1})(1-s) - \tilde{F}(y; t, \lambda^{-1})(1-t)}{t - s},$$

which we complete with the nontrivial bound $\tilde{F}^-(y|s_u;\lambda^{-1}) = \tilde{F}(y; s_u, \lambda^{-1})$ at $s = s_u$.

Since $\lambda^{-1} \in \Lambda_I^{-1}$, the candidate distributions $\tilde{F}(y; s, \lambda^{-1})$ satisfy all conditions in Theorem 1. Under these conditions, $\tilde{F}^+(\cdot|s, \lambda^{-1})$ and $\tilde{F}^-(\cdot|s;\lambda^{-1})$ are both proper distributions over V_i which satisfy $\tilde{F}^+(y|s;\lambda^{-1}) \geq F(y|S_i = s) \geq \tilde{F}^-(y|s;\lambda^{-1})$ for every candidate $F(\cdot|S_i = s)$ implied by some $(F, C) \in \mathcal{F} \times \mathcal{C}$ such that $(\lambda^{-1}, F, C) \in \mathcal{I}$. In particular, these inequalities imply that $\tilde{F}^-(\cdot|s;\lambda^{-1})$ first-order stochastically dominates $F(\cdot|S_i = s)$, which in turn first-order stochastically dominates $\tilde{F}^+(\cdot|s;\lambda^{-1})$. Moreover, if $s \in \text{Int}(\mathcal{S})$, so that $t^-(s) = t^+(s) =$

s, then $\tilde{F}^{+}(\cdot|s;\lambda^{-1}) = \tilde{F}^{-}(\cdot|s;\lambda^{-1})$, so that there exists only one candidate $F(\cdot|S_i = s)$ consistent with λ^{-1}. This case is relevant when, in addition to potential competition N, we also observe a continuous entry instrument z. Under our assumptions, $\mathcal{S}$ will then be a union of intervals in $[0,1]$, so that the bounds above will uniquely determine $F(\cdot|S_i = s)$ for almost every $s \in \mathcal{S}$.

Finally, we translate the restrictions on (F, C) described above into bounds $c^{+}(z;\lambda^{-1})$ and $c^{-}(z;\lambda^{-1})$ on the set of cost values $c(z)$ consistent with the hypothesis $\lambda^{-1} = \lambda_0^{-1}$. Toward this end, let $U(x) = \int_1^x 1/\lambda(t)\,dt$ be the unique candidate for U_0 implied by λ^{-1}, and rewrite the breakeven condition entry equilibrium condition (22) as

$$U(c(z)) = \int_0^{v_k(1|z)} \Big\{ U\big(y - b_k(F_k(y|z)|z)\big) \times [s_k(z) + (1 - s_k(z))F_k(v|z)]^{N-1} \Big\}\, dF(v|S_i = s_k(z)). \quad (24)$$

Recall that if $\lambda^{-1} = \lambda_0^{-1}$, then $F_k(\cdot|z) = \tilde{F}(\cdot; s_k(z), \lambda^{-1})$, implying that $F_k(\cdot|z)$ is identified up to λ^{-1}. Thus, given λ^{-1}, all objects on the right-hand side of (24) are identified except $F(v|S_i = s_k(z))$, for which we have an identified lower bound $\tilde{F}^{-}(\cdot|s_k(z);\lambda^{-1})$ which first-order stochastically dominates $F(\cdot|S_i = s_k(z))$, as well as an identified upper bound $\tilde{F}^{+}(\cdot|s_k(z);\lambda^{-1})$ which is first-order stochastically dominated by $F(\cdot|S_i = s)$. Recalling that the integrand in (24) equals bid-stage profit, which is increasing in v, we may plug in $\tilde{F}^{+}(\cdot|s_k(z);\lambda^{-1})$ for $F(\cdot|S_i = s_k(z))$ to obtain an identified lower bound $c^{-}(N_k, z;\lambda^{-1})$ on the set of $c(z)$ consistent with λ^{-1}, and plug in $\tilde{F}^{-}(\cdot|s_k(z);\lambda^{-1})$ to obtain an identified upper bound $c^{+}(N_k, z;\lambda^{-1})$. Since $c(z)$ does not depend on N_k, we may then take intersections across $k \in \mathcal{K}$ to obtain tighter bounds paralleling Theorem 4 of GL (2014), which may be refined to sharp bounds following Appendix A of GL (2014).

Appendix B: Proofs

Proof of Theorem 1. We first establish that for any $(F, C) \in \mathcal{F} \times \mathcal{C}$, the true ex post quantile functions $v_1, v_2, ..., v_K$ satisfy all restrictions in Theorem 1.

Restrictions I, O, and M are straightforward. True quantiles $v_1, v_2, ..., v_K$ must satisfy Restriction I since V_i is stochastically increasing in S_i and the set of entering types (if nonempty) always includes the potential bidder with $S_i = 1$. Similarly, $v_1, v_2, ..., v_K$ must satisfy Restriction O since selected distributions of the form $V_i|S_i \geq s$ are stochastically increasing in the entry threshold s. Finally, $v_1, v_2, ..., v_K$ must satisfy Restriction M since, under Assumptions 1-4, each v_k will be continuous on $[0,1]$ and continuously differentiable for all $a \in [0,1]$ except possibly $a = 0$, with $v_k'(a) = 1/f_k(v_k(a))$ for all a where v_k is differentiable. Finally, since $f_k(y)$ is bounded, we must have $v_k'(a)$ bounded away from zero for all $a \in (0,1]$.

We thus focus on Restrictions D and S. The proof that true $F_1^0, F_2^0, ..., F_K^0$ satisfy these restrictions depends only on properties of the copula C as the entry threshold varies, not whether variation in the entry threshold comes from N or z. For simplicity, we thus present the proof based on variation in N only, suppressing z in notation. The proof with variation in both N and z is identical, but more notationally intensive.

First consider Restriction S. Rearranging the definition of the true c.d.f. F_k^0,

$$F_k^0(y)(1 - s_k) = \int_{s_k}^1 F^0(y|t)dt.$$

Now consider the true distribution of V_i conditional on the event $S_i \in [s_k, s_{k+1}]$:

$$F^0(y|S_i \in [s_k, s_{k+1}]) \equiv \frac{1}{s_{k+1} - s_k} \int_{s_k}^{s_{k+1}} F^0(y|t)dt = \frac{F_k^0(y)(1 - s_k) - F_{k+1}^0(y)(1 - s_{k+1})}{s_{k+1} - s_k}.$$

By stochastic ordering, $F^0(y|S_i \in [s_k, s_{k+1}])$ is decreasing in k. Hence for all $k = 2, ..., K$ we must have restriction S:

$$\frac{F_{k-1}^0(y)(1 - s_{k-1}) - F_k^0(y)(1 - s_k)}{s_k - s_{k-1}} \geq \frac{F_k^0(y)(1 - s_k) - F_{k+1}^0(y)(1 - s_{k+1})}{s_{k+1} - s_k}.$$

Note that Restrictions O and S are jointly required; it is possible for a collection of distributions $F_1, ..., F_K$ to satisfy Restriction O, or equivalently $F_k \geq F_{k+1}$ for all $k < K$, without satisfying restriction S, which follows from the requirement that $V_i|S_i = s$ is stochastically increasing for all $s \in [0, 1]$. To see this, suppose that $F_{k-1}(y) = F_k(y) > F_{k+1}(y) > 0$, which is admissable under Restriction O. Then the LHS of Restriction S above is identically zero, but the RHS satisfies

$$\frac{F_k(y)(1 - s_k) - F_{k+1}(y)(1 - s_{k+1})}{s_{k+1} - s_k} > \frac{F_k(y)(1 - s_k) - F_k(y)(1 - s_{k+1})}{s_{k+1} - s_k} = F_k(y) > 0.$$

Hence restriction S would fail. Analogously, Restriction S does not imply Restriction O.

Finally, consider Restriction D. Note that, from above,

$$\frac{F_k^0(y)(1 - s_k) - F_{k+1}^0(y)(1 - s_{k+1})}{s_{k+1} - s_k} = \frac{1}{s_{k+1} - s_k} \int_{s_k}^{s_{k+1}} F^0(y|t)dt.$$

Since the right-hand side must be at least weakly increasing in y for all $y \in \mathbb{R}$, so must the left-hand side, which in turn implies Restriction D.

We next establish that to any λ^{-1} such that $\tilde{v}_1(\cdot|z; \lambda^{-1})$, ..., $\tilde{v}_K(\cdot|z; \lambda^{-1})$ satisfy all conditions of Theorem 1, there corresponds a structure $(\lambda^{-1}, F, C) \in \mathcal{I}$. Since we are considering fixed λ^{-1}, we suppress λ^{-1} in notation in what follows, writing simply $\tilde{v}_1(\cdot|z)$, ..., $\tilde{v}_K(\cdot|z)$. Note that, by definition, $\tilde{v}_1(\cdot|z)$, ..., $\tilde{v}_K(\cdot|z)$ satisfy (4) at $\lambda^{-1} \in \Lambda_I^{-1} \subset \Lambda^{-1}$. Showing existence of $(F, C) \in \mathcal{F} \times \mathcal{C}$ such that $(\lambda^{-1}, F, C) \in \mathcal{I}$ is thus equivalent to showing existence of $(F, C) \in \mathcal{F} \times \mathcal{C}$ generating the quantile functions $\tilde{v}_1(\cdot|z)$, ..., $\tilde{v}_K(\cdot|z)$ implied by λ^{-1}.

First note that the candidate quantile functions $\tilde{v}_k(\cdot|z)$ inherit all smoothness properties of the true quantile functions $v_k(\cdot|z)$. By construction,

$$\tilde{v}_k(a|z) = b_k(a|z) + \lambda^{-1}(R_k(a|z)),$$

and by definition

$$v_k(a|z) = b_k(a|z) + \lambda_0^{-1}(R_k(a|z)).$$

Since $v_k(a|z)$, $b_k(a|z)$ and $R_k(a|z)$ are continuous on $[0, 1]$, so is $\tilde{v}_k(a|z)$. Since $\lambda^{-1,\prime} \in [0, 1]$ and $v_k(a|z)$ is continuously differentiable on at least $(0, 1]$, $\tilde{v}_k(a|z)$ is also continuous on $[0, 1]$ and continuously differentiable on at least $(0, 1]$. Moreover, since $\tilde{v}_k'(a|z)$ is bounded away from zero for all $a \in (0, 1]$ by Restriction M, $\tilde{v}_k(\cdot|z)$ must be strictly monotone, and hence there exists a unique candidate c.d.f. $\tilde{F}_k(y|z) = \tilde{v}_k^{-1}(a|z)$ implied by $\tilde{v}_k(a|z)$. In view of the properties of $\tilde{v}_k'(a|z)$ this c.d.f. will have a bounded density $\tilde{f}_k(y|z)$ (since $\tilde{v}_k'(a|z)$ is bounded away from zero on $(0, 1]$) which is also locally bounded away from zero except possibly at $a = 0$ (since $\tilde{v}_k'(a|z)$ exists and is bounded everywhere $v_k'(a|z)$ exists and is bounded). Analogously, the distribution of V_i given $S_i \geq s_k(z)$ implied by the candidate quantile function $\tilde{v}_k(a|z)$ is as smooth in z as the true quantile function $v_k(a|z)$.

Now let $\mathcal{S} = \{s \in [0,1] : s = s_k(z) \text{ for some } k \in \mathcal{K}, z \in \mathcal{Z}\}$, and recall that if $s_k(z) = s_l(z')$, then by Restriction O we must have $\tilde{v}_k(y|z) = \tilde{v}_l(y|z')$. With slight abuse of notation, we may therefore define a new collection of candidate quantile functions $\tilde{v}(\cdot; s)$ for $s \in \mathcal{S}$ by setting, for each $s \in S$, $\tilde{v}(\cdot; s) \equiv \tilde{v}_k(\cdot|z)$ for some k, z such that $s_k(z) = s$. Furthermore, as in Appendix A.3, we may define a new collection of candidate post-entry value c.d.f.s for $s \in \mathcal{S}$ by setting $\tilde{F}(\cdot; s) = \tilde{v}^{-1}(\cdot; s)$.

We first construct a candidate F for F_0. Toward this end, let $s_l = \min \mathcal{S}$ be the minimum element of $\mathcal{S}$, and let $s_2 = \inf \mathcal{S}\backslash\{s_l\}$ be the infimum of $\mathcal{S}$ *excluding* s_1. Note that, given this construction, we will have $s_l = s_2$ if a continuous instrument z is available, and $s_l < s_2$ otherwise. If $s_l = 0$, we take $F = \tilde{F}(y; 0)$, which will belong to $\mathcal{F}$ since $\tilde{v}(a; 0)$ will then inherit the density properties of the ex ante quantile function $v_0(a)$, which by construction are generated by $F \in \mathcal{F}$. Otherwise, let $\tilde{F}^-(y|S_i = s_1)$ be the lower bound on $F(y|S_i = s_l)$ derived from $\tilde{F}(\cdot; \cdot)$ following GL as in Appendix A.3:

$$\tilde{F}^-(y|S_i = s_l) = \lim_{t \downarrow s_2} \frac{\tilde{F}(y; s_l)(1 - s_l) - \tilde{F}(y; t)(1 - t)}{t - s_l}.$$

Under Restrictions M, O, D, and S, this lower bound $\tilde{F}^{-1}(y|S_i = s_l)$ will be a proper c.d.f. in y, but generically need not admit a continuous density; for instance, in the Samuelson (1985) model, $\tilde{F}^-(y|S_i = s_l)$ may be a step function. However, since the infimum support of $\tilde{F}^-(y|S_i$ must, by construction, equal the infimum support of $\tilde{F}(y; s_l)$, we can always find a candidate $F(y|S_i = s_l) \geq \tilde{F}^-(y|S_i = s_l)$ with a strictly postive, bounded density everywhere from zero up to any point weakly below the supremum support of $\tilde{F}^-(y|S_i = s_l)$. For any such candidate $F(y|S_i = s_l) \geq \tilde{F}^-(y|S_i = s_l)$, we may then construct a candidate for F as

$$F(y) = s_l F(y|S_i = s_l) + (1 - s_l)\tilde{F}(y; s_l).$$

For any such $F(y|S_i = s_l)$, $F(y)$ will be a strictly increasing distribution with support on $[0, \bar{v}]$, where $\bar{v} = \tilde{v}_1(1; s)$ is the common support of each selected density implied by Condition I. Moreover, since the infimum support of $\tilde{F}^-(y|S_i = s_l)$ is the same as that of $\tilde{F}(y; s_l)$, the restriction $F(y|S_i = s_1) \geq \tilde{F}^-(y|S_i = s_l)$ imposes no constraints below the infimum support of $\tilde{F}^-(y|S_i = s_l)$. We may thus freely choose the density of $F(y|S_i = s_l)$ at this infimum support to ensure that $F \in \mathcal{F}$.

Fixing $F(y)$ as a candidate for F_0, we next construct a candidate C for C_0 as follows. Let $s_u = \sup \mathcal{S}$, and as in Appendix A.3, for any $s \in \mathcal{S}$, let $t^-(s)$ and $t^+(s)$ be the nearest upper and lower neighbors of s in $\mathcal{S}$, adopting the convention that $t^-(s) = t^+(s) = s$ if $s \in \mathcal{S}$. We then define C piecewise as follows:

- For $s \in \mathcal{S}$, we set $C(a, s) \equiv a - (1 - s)\tilde{F}(F^{-1}(a); s)$;
- For $s \in [s_l, s_u]$ such that $s \neq \mathcal{S}$, we construct $C(a, s)$ by linear interpolation (in s) of $C(a, \cdot)$ between the nearest neighbors $C(a, t^-(s))$ and $C(a, t^+(s))$.
- For $s < s_l$, we construct $C(a, \cdot)$ by linear interpolation (in s) between $C(a, s_l)$ and 0:

$$C(a, s) \equiv \frac{s}{s_l}[a - (1 - s_l)\tilde{F}(F^{-1}(a); s_l)];$$

- Finally, for $s > s_u$, we set $C(a, s) \equiv a - (1 - s)\tilde{F}(F^{-1}(a); s_u)$.

By construction, the structure (F, C) reproduces the candidate equilibrium c.d.f.s $\tilde{F}(\cdot; s)$ for all $s \in \mathcal{S}$:

$$\frac{F(y) - C(F(y), s)}{1 - s} = \frac{F(y) - F(y) + (1 - s)\tilde{F}(y; s)}{1 - s} = \tilde{F}(y; s).$$

It only remains, therefore, to show that C is a joint c.d.f. satisfying our assumptions on stochastic ordering. Toward this end, note that C satisfies the limit properties of a distribution: $C(0,s) = C(a,0) = 0$, $C(1,s) = s$, $C(a,1) = a$. Furthermore, C is continuous by construction. We thus need only show that C is 2-increasing and that it satisfies our assumption on stochastic ordering.

Toward this end, we first show that C is 2-increasing. It suffices to restrict attention to rectangles $[a,a'] \times [s,s']$ such that either $s,s' < s_l$, $s,s' > s_u$, or $s,s' \in [s_l,s_l]$. If $s,s' < s_l$, then $C(F(y),s) = \frac{s}{s_l}[F(y) - (1-s_l)\tilde{F}(y;s_l)]$, so that

$$\begin{aligned}\frac{\partial C(F(y'),s)}{\partial s} &= \frac{1}{s_l}[F(y) - (1-s_l)\tilde{F}(y;s_l)] \\ &= [s_l F(y|S_i = s_l) + (1-s_l)\tilde{F}(y;s_l) - (1-s_l)\tilde{F}(y;s_l)] \\ &= s_l F(y|S_i = s_l),\end{aligned}$$

which is clearly nondecreasing in y. Meanwhile, if $s,s' > s_u$, then $\frac{\partial C(F(y,s))}{\partial s} = \tilde{F}(y;s)$, which again is increasing in y. This implies that C is 2-increasing for rectangles with $s,s' < s_l$ or $s,s' > s_u$. Finally, if $s,s' \in [s_l,s_u]$, then $C(a,\cdot)$ is a linear interpolation (in s) between $C(a,t^-(s)) \equiv a - \tilde{F}(F^{-1}(a);t^-(s)) \cdot (1-t^-(s))$ and $C(a,t^+(s)) \equiv a - \tilde{F}(F^{-1}(a);t^+(s)) \cdot (1-t^+(s))$ (recalling that for $s \in \mathrm{Int}(\mathcal{S})$ we have $t^-(s) = t^+(s) = s$). Hence it is sufficient to verify that if $s,s' \in \mathcal{S}$ and $s' > s$,

$$C(a',s') - C(a,s') - C(a',s) + C(a,s) \geq 0.$$

By definition of $C(a,s)$, for all $s \in \mathcal{S}$, we have:

$$C(a,s) = a - \tilde{F}(F^{-1}(a);s) \cdot (1-s).$$

Hence substituting $a = F(y)$, and rearranging the inequality above, C is 2-increasing if:

$$\begin{aligned}& C(F(y'),s') - C(F(y),s') - C(F(y'),s) + C(F(y),s) \geq 0 \\ \Leftrightarrow \quad & -\tilde{F}(y';s')(1-s') + \tilde{F}(y;s')(1-s') + \tilde{F}(y;s)(1-s) - \tilde{F}(y';s)(1-s') \geq 0, \\ \Leftrightarrow \quad & \tilde{F}(y;s)(1-s) - \tilde{F}(y';s)(1-s') \geq \tilde{F}(y';s')(1-s') - \tilde{F}(y;s') \cdot (1-s'),\end{aligned}$$

where the final line is implied by Restriction D. Hence C is 2-increasing.

Finally, we show that C satisfies stochastic ordering. Toward this end, observe that stochastic ordering in s is implied by concavity of $C(a,s)$ in s. This in turn follows if for all $s,s',s'' \in [0,1]$ such that $s < s' < s''$, we have

$$\frac{C(F(y),s') - C(F(y),s)}{s'-s} \geq \frac{C(F(y),s'') - C(F(y),s')}{s''-s'}.$$

For $s,s',s'' \in \mathcal{S}$, we may rewrite this inequality as

$$\frac{(1-s)\tilde{F}(y;s) - (1-s')\tilde{F}(y;s)}{s'-s} \geq \frac{(1-s')\tilde{F}(y;s') - (1-s'')\tilde{F}(y;s'')}{s''-s'}$$

which holds immediately by Restriction S. In view of our linear interpolation scheme for $C(a,s)$, this conclusion extends immediately to s,s',s'' in the interval $[s_l,s_u]$, but not necessarily in $\mathcal{S}$. Finally, for $s < s_l$, $C(F(y),s)$ is linear in s with slope

$$\frac{dC(F(y),s)}{ds} = \lim_{t \downarrow t^+(s_l)} \frac{\tilde{F}(y;s_l)(1-s_l) - \tilde{F}(y;t)(1-t)}{t - s_l},$$

while for $s > s_u$, $C(F(y), s)$ is linear in s with slope $\tilde{F}(y; s_u)$. Again in view of Condition S, it follows that the inequality above also extends to s, s', s'' potentially outside of $[s_l, s_u]$. Hence C satisfies stochastic ordering.

We have thus constructed a candidate structure (F, C) reproducing $\tilde{v}_1, ..., \tilde{v}_K$ and satisfying all properties on (F_0, C_0) except differentiability of C over the domain assumed in Assumption 4, which may not hold since our construction involves kinds at transition points $s \in \mathcal{S}$ which are not in the interior of $\mathcal{S}$. But since, by construction, such nondifferentiabilities can only arise on the boundaries of the equilibrium support $\mathcal{S}$, we may slightly perturbing $C(a, s)$ to smooth transitions at these transition points while still reproducing all observed selected densities. One can construct a copula satisfying the properties above plus differentiability where required; i.e. all conditions required by $\mathcal{C}$. This establishes the claim. □

Proof of Corollary 1. Given in the main text. □

Proof of Corollary 2. For the first statement, suppose there exist distinct $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $s_k(z) = s_l(z') = \bar{s}$, and observe that if the support of $V_i|S_i = s_k(z)$ includes zero, then we must have $R_k(0|z) = 0$. Identification of λ_0^{-1} on its empirical domain $[0, \max_a \max\{R_k(a|z), R_l(a|z')]$ follows directly from GPV (2009) as in the main text. This immediately yields identification of $F_k(\cdot|z)$ (which by construction must be the same as $F_l(\cdot|z)$) through (4). This establishes the first statement in Corollary 2.

For the second statement, further suppose that $s_k(z) \in (0, 1)$ and that $z, z' \in \text{Int}(\mathcal{Z})$. For any $v \in (v_k(0|z), \bar{v})$, our assumptions on C_0 imply $F(v|S_i \geq s)$ is differentiable in s for all $s \leq s_k(z)$, and the conditional distribution $F(v|S_i = s_k(z))$ is related to the selected distribution $F_k(v|z)$ by the identity

$$F(v|S_i = s_k(z)) \equiv -\left[\frac{d}{ds}(1 - s)F(v|S_i \geq s)\right]_{s=s_k(z)}.$$

Recall that $c(z)$ is continuous and strictly monotone in z. Furthermore, by hypothesis, both $z \in \text{Int}(\mathcal{Z})$ and $z' \in \text{Int}(\mathcal{Z})$. Finally, since $c(z)$ is continuous and strictly increasing in z and $s_k(z) = s_l(z) \in (0, 1)$, both $s_k(z)$ and $s_l(z)$ are continuous and strictly increasing in z. Hence for any $\epsilon > 0$, there must exist $\tilde{z}, \tilde{z}' \in \mathcal{Z}$ such that both $s_k(\tilde{z}) \in (s_k(z) - \epsilon, s_k(z))$ and $s_k(\tilde{z}) = s_l(\tilde{z}')$. Choose any such $\tilde{z}, \tilde{z}'$, and note that $F_k(\cdot|\tilde{z})$ is then also identified by the arguments above. Moreover, by definition,

$$\left[\frac{d}{ds}(1 - s)F(v|S_i \geq s)\right]_{s=s_k(z)} = \frac{(1 - s_k(z))F_k(v|z) - (1 - s_k(\tilde{z}))F_k(v|\tilde{z})}{s_k(\tilde{z}) - s_k(\tilde{z})} + o(\epsilon),$$

where our choice of $\tilde{z}, \tilde{z}'$ implies that all terms on the right-hand side are identified. Repeating this construction over any sequence of ϵ such that $\epsilon \downarrow 0$ implies identification of $F(v|S_i = s_k(z))$ for all $v \in (v_k(0|z), \bar{v})$. Since $F(\cdot|S_i = s_k(z))$ is right-continuous, identification of $F(v|S_i = s_k(z))$ for all $v \in (v_k(0|z), \bar{v})$ implies identification of $F(v_k(0|z)|S_i = s_k(z))$, and since $v_k(0|z)$ must be the infimum support of $F(\cdot|S_i = s_k(z))$, we must have $F(v|S_i = s_k(z)) = 0$ for all $v < v_k(0|z)$. Finally, by definition, $F(v|S_i = s_k(z)) = 1$ for $v \geq \bar{v}$. Thus $F(v|S_i = s_k(z))$ is identified for all $v \in [v_k(0|z), \bar{v}]$. □

Proof of Corollary 3. Consider any $k \in \mathcal{K}$, and let $\bar{R}_k(1) \equiv \sup_{z \in \mathcal{Z}} R_k(1|z)$ and $\underline{R}_k(1) \equiv \inf_{z \in \mathcal{Z}} R_k(1|z)$. Temporarily assume that $\underline{R}_k(1) < \bar{R}_k(1)$; we establish that this holds under the conditions of Corollary 3 below. Note that since $c(z)$ is continuous in z, $R_k(1|z)$ is continuous in z. Since we have assumed that z, if continuous, is supported on an interval, for each $r \in (\underline{R}_k(1), \bar{R}_k(1))$ we can find $z' \in \mathcal{Z}$ such that $r = R_k(1|z)$. Moreover, by Restriction I of Theorem 1, we must have for any $\lambda^{-1} \in \Lambda_I^{-1}$:

$$b_k(1|z) - b_l(1|z') = \lambda^{-1}(R_k(1|z')) - \lambda^{-1}(R_k(1|z)). \tag{25}$$

Now pick any $r_0 \in [\underline{R}_k(1), \bar{R}_k(1)]$, and consider any $\epsilon > 0$. From above, we can find $z, z' \in \mathcal{Z}$ such that $R_k(1|z) \neq R_k(1|z')$, $|R_k(1|z) - r_0| < \epsilon$ and $|R_k(1|z') - r_0| < \epsilon$. For any such z, z', taking a first-order Taylor series approximation to $\lambda^{-1}(r)$ around r_0 in (25) will imply

$$\lambda^{-1,\prime}(r_0) = \frac{b_k(1|z) - b_l(1|z')}{R_k(1|z') - R_k(1|z)} + o(\epsilon).$$

The right-hand side of this expression is well-defined and identified up to the residual $o(\epsilon)$. Furthermore, this construction may be repeated for progressively smaller $\epsilon > 0$ to obtain a sequence of identified right-hand terms, which will converge to $\lambda^{-1,\prime}(r_0)$ as $\epsilon \to 0$. It follows that, so long as $\underline{R}_k(1) < \bar{R}_k(1)$, a continuous instrument z will yield identification of $\lambda^{-1,\prime}(r_0)$ for any $r_0 \in [\underline{R}_k(1), \bar{R}_k(1)]$.

It only remains, therefore, to show that the conditions in Corollary 3 imply $\underline{R}_k(1) < \bar{R}_k(1)$. Note that, by definition, $\underline{R}_k(1) \leq \bar{R}_k(1)$, and we can have $\underline{R}_k(1) = \bar{R}_k(1)$ if and only if $R_k(1|z) = R_k(1|z')$ for all $z, z' \in \mathcal{Z}$, which by Condition I of Theorem 1 holds if and only if $b_k(1|z) = b_k(1|z')$ for all $z, z' \in \mathcal{Z}$. But as shown by LLZ (2015), the equilibrium bidding strategy $\beta(\cdot|N, \bar{s})$ is strictly decreasing in $\bar{s}$ for $v > 0$. Since z affects bidding only through $s_k(z)$, we can thus have $b_k(1|z) = b_k(1|z')$ for all $z, z' \in \mathcal{Z}$ only if $s_k(z) = s_k(z')$ for all $z, z' \in \mathcal{Z}$. But since $c(z)$ is strictly increasing in z, we also know that $s_k(z)$ is strictly increasing in z whenever $s_k(z) \in (0, 1)$. Thus, if $s_k(z) \in (0, 1)$ for some $z \in \text{Int}(\mathcal{Z})$, we can find $z' > z$ such that $z' \in \mathcal{Z}$ and $s_k(z') > s_k(z)$, which from above implies that we cannot have $R_k(1|z) = R_k(1|z')$ for all $z, z' \in \mathcal{Z}$.

□

Proof of Corollary 4. Consider any $k, l \in \mathcal{K}$ and $z, z' \in \mathcal{Z}$ such that $R_k(1|z) \neq R_l(1|z')$. With slight abuse of notation, define $\bar{R}_k = R_k(1|z)$, $\bar{R}_l = R_l(1|z')$, $\bar{b}_k = b_k(1|z)$, and $\bar{b}_l = b_l(1|z')$. Suppose WLOG that $\bar{R}_k > \bar{R}_l$, and recall that this implies $\bar{b}_k < \bar{b}_l$.

First suppose that U_0 belongs to the CRRA family. In this case $\lambda_0^{-1}(x) = x/(1 - \gamma_0)$. Equation (11) then simplifies to

$$\bar{b}_k + \bar{R}_k/(1 - \gamma_0) = \bar{b}_k + \bar{R}_l/(1 - \gamma_0).$$

This is a linear equation in $1/(1 - \gamma_0)$, which has a unique solution since $\bar{R}_k > \bar{R}_l$. Since $1/(1 - \gamma_0)$ is monotone, it follows that γ_0 is identified.

Alternatively suppose that U_0 belongs to the CARA family. Then $\lambda_0^{-1}(x) = x$ if $\gamma_0 = 0$, or $\lambda_0^{-1}(x) = [\ln(\gamma_0 x + 1)]/\gamma_0$ otherwise. We consider each case in turn.

First suppose that, in truth, $\gamma_0 = 0$. Then, from Corollary 1, no strictly risk averse model can satisfy (11). Since the CARA model is strictly risk averse for every $\gamma_0 > 0$, it follows that γ_0 is identified.

Now suppose that $\gamma_0 > 0$. Then from above $\lambda^{-1}(x; \gamma) = [\ln(\gamma x + 1)]/\gamma$. Differentiating with respect to x yields $\lambda^{-1,\prime}(x; \gamma) = \frac{1}{\gamma x + 1}$, which is strictly decreasing in γ for $x > 0$. By definition, the change in $\lambda^{-1}(x; \gamma)$ over the interval $[\bar{R}_l, \bar{R}_k]$ is

$$\Delta\lambda^{-1}(\gamma) = \int_{\bar{R}_l}^{\bar{R}_k} \frac{1}{\gamma x + 1} \, dx,$$

which is identified up to γ and strictly decreasing in γ since the integrand is strictly decreasing in γ (bearing in mind that $\bar{R}_l > 0$). Finally, by (11), we must have $\bar{b}_l - \bar{b}_k = \Delta\lambda^{-1}(\gamma_0)$. The right-hand side is identified, and the left-hand side is known up to γ_0 and strictly decreasing in γ_0. It follows that γ_0 is identified. $\square$

Proof of Theorem 2. For each $k \in \mathcal{K}$ and $\theta \in \Theta$, let $\tilde{b}_{k,\theta}(\cdot)$, $\tilde{R}_{k,\theta}(\cdot)$ be the functions obtained when $b_k(\cdot)$, $R_k(\cdot)$ are reindexed according to $h_\theta^k(\cdot)$:

$$\tilde{b}_{k,\theta}(a) \equiv b_k(h_\theta^k(a)),$$
$$\tilde{R}_{k,\theta}(a) \equiv R_k(h_\theta^k(a)).$$

Now consider any $\theta \in \Theta_I$ and $k, l \in \mathcal{K}$ such that Conditions (i) and (ii) of Theorem 2 hold. We first show that, under Condition (ii) of Theorem 2, reindexed bid functions $\tilde{b}_{k,\theta}$, $\tilde{b}_{l,\theta}$ can intersect at most on a closed set of Lebesgue measure zero. We then proceed to establish the main result.

Lemma 1. *Suppose that $\theta \in \Theta_I$ and that the set of points $a \in [0,1]$ such that $(N_k - 1)H(a, s_k; \theta) = (N_l - 1)H(a, s_l; \theta)$ is of Lebesgue measure zero. Let $B_{kl,\theta} = \{a \in [0,1] : \tilde{b}_{k,\theta}(a) = \tilde{b}_{l,\theta}(a)\}$. Then $B_{kl,\theta}$ is a closed set of Lebesgue measure zero.*

Proof. First note that, since $\theta \in \Theta_I$, there must exist some $F_\theta \in \mathcal{F}$ and $\lambda_\theta \in \Lambda$ such that, for all $k \in \mathcal{K}$, the observed equilibrium bid quantile function $b_k(a)$ equals the predicted equilibrium bid quantile function under $(\theta, F_\theta, \lambda_\theta)$. Let $F_{k,\theta}$ denote the selected c.d.f. implied by $(\theta, F_\theta, \lambda_\theta)$, which in view of (1) must satisfy

$$(1 - s_k)F_{k,\theta}(y) = F_\theta(y) - C(F_\theta(y), s_k; \theta).$$

From this, letting $f_{k,\theta}$ be the density of $F_{k,\theta}(y)$, and f_θ be the density of F_θ, it follows that

$$\begin{aligned}\frac{(1 - s_k)f_{k,\theta}(y)}{s_k + (1 - s_k)F_{k,\theta}(y)} &= \frac{1 - C_1(F_{k,\theta}(y), s_k; \theta)}{s_k + F_{k,\theta}(y) - C(F_{k,\theta}(y), s_k; \theta)} f_\theta(y) \\ &= H(F_\theta(y), s_k; \theta) f_\theta(y).\end{aligned}$$

Now let $\beta_{k,\theta}(y)$ denote the equilibrium bid function implied by primitives $(\theta, F_\theta, \lambda_\theta)$ given s_k. For all $y \in [v_\theta(0), v_\theta(1)]$, $\beta_{k,\theta}(y)$ is uniquely defined by the IVP

$$\begin{aligned}\beta_{k,\theta}'(y) &= \lambda_\theta(y - \beta_{k,\theta}(y)) \frac{(N_k - 1)(1 - s_k)f_{k,\theta}(y)}{s_k + (1 - s_k)F_{k,\theta}(y)} \\ &= \lambda_\theta(y - \beta_{k,\theta}(y))(N_k - 1)H(F_\theta(y), s_k; \theta)f_\theta(y),\end{aligned}$$

subject to the boundary condition $\beta_{k,\theta}(v_\theta(1)) = 0$. Furthermore, since $(\theta, F_\theta, \lambda_\theta)$ must reproduce $b_k(a)$, we must have $b_k(a) = \beta_{k,\theta}(v_{k,\theta}(a))$, or equivalently

$$\tilde{b}_{k,\theta}(a) \equiv b_k(h_k(a; \theta)) = \beta_{k,\theta}(v_\theta(a)),$$

where the last identity follows since, by definition of $h_k(a, \theta)$, $v_\theta(a) = v_{k,\theta}(h_k(a; \theta))$.

We now show that $B_{kl,\theta}$ is a closed set of measure zero. Closedness of $B_{kl,\theta}$ follows directly from the fact that $\tilde{b}_{k,\theta}(\cdot)$, $\tilde{b}_{l,\theta}(\cdot)$ are continuous. To show that $B_{kl,\theta}$ is of measure zero, we first show that the set $B_{kl,\theta}^0 = \{a \in [0,1] : \tilde{b}_{k,\theta}(a) = \tilde{b}_{l,\theta}(a) \text{ and } \tilde{b}_{k,\theta}'(a) = \tilde{b}_{l,\theta}'(a)\}$ is of measure zero. We then show that $B_{kl,\theta}$ is also of measure zero.

First consider $B_{kl,\theta}^0$. In view of the identity $\beta_k(v_\theta(a)) = \tilde{b}_{k,\theta}(a)$, we may apply the change of variables $y = v_\theta(a)$ to re-express the FOC defining $\beta_{k,\theta}(y)$ as

$$\tilde{b}_{k,\theta}(a) = \lambda_\theta(v_\theta(a) - \tilde{b}_{k,\theta}(a))(N_k - 1)H(a, s_k; \theta).$$

Moreover, for $a > 0$, we must have $v_\theta(a) > \tilde{b}_{k,\theta}(a)$ and hence $\lambda_\theta(v_\theta(a) - \tilde{b}_{k,\theta}(a)) > 0$. It follows that, for $a > 0$ we can have both $\tilde{b}_{k,\theta}(a) = \tilde{b}_{l,\theta}(a)$ and $\tilde{b}'_{k,\theta}(a) = \tilde{b}'_{l,\theta}(a)$ if and only if $(N_k - 1)H(a, s_k; \theta) = (N_l - 1)H(a, s_l; \theta)$. By hypothesis, the latter can hold on at most a set of measure zero, hence $B^0_{kl,\theta}$ must be of measure zero.

Now consider $B_{kl,\theta}$. By definition, we may partition $B_{kl,\theta}$ into disjoint subsets $B^0_{kl,\theta}$, where $\tilde{b}_{k,\theta}(a)$ and $\tilde{b}_{l,\theta}(a)$ are tangent, and $B_{kl,\theta} \cap (B^0_{kl,\theta})^c$, where $\tilde{b}_{k,\theta}(a) = \tilde{b}_{l,\theta}(a)$ but $\tilde{b}'_{k,\theta}(a) \neq \tilde{b}'_{l,\theta}(a)$. From above, $B^0_{kl,\theta}$ is of measure zero. Meanwhile, for every $a_0 \in B_{kl,\theta} \cap (B^0_{kl,\theta})^c$, we have by definition $\tilde{b}'_{l,\theta}(a) \neq \tilde{b}'_{k,\theta}(a)$. Hence there must exist some $\epsilon > 0$ such that for all $a \in [a_0 - \epsilon, a_0) \cap (a_0, a_0 + \epsilon]$, we have $\tilde{b}_{k,\theta}(a) \neq \tilde{b}_{l,\theta}(a)$. It follows that $B_{kl,\theta} \cap (B^0_{kl})^c$ is at most countable, and thus of Lebesgue measure zero. □

We now establish the main claim that λ^{-1} is point-identified up to θ. Toward this end, define $\bar{r}_k \equiv \max_a \tilde{R}_{k,\theta}(a)$, $\bar{r}_{kl} \equiv \max\{\bar{r}_k, \bar{r}_l\}$ as in the main text, and let functions $\bar{R}_{kl,\theta}(\cdot)$, $\underline{R}_{kl,\theta}(\cdot)$ be the pointwise maximum and minimum of $\tilde{R}_{k,\theta}(\cdot)$, $\tilde{R}_{l,\theta}(\cdot)$ respectively:

$$\bar{R}_{kl,\theta}(a) \equiv \max\{\tilde{R}_{k,\theta}(a), \tilde{R}_{l,\theta}(a)\}, \tag{26}$$

$$\underline{R}_{kl,\theta}(a) \equiv \min\{\tilde{R}_{k,\theta}(a), \tilde{R}_{l,\theta}(a)\}. \tag{27}$$

For each $r \in [0, \bar{r}_{kl}]$, let $\mathcal{A}_{kl,\theta}(r)$ be the set all decreasing sequences $\{\alpha^t\}_{t=1}^\infty$ satisfying the recursive relationship

$$\bar{R}_{kl,\theta}(\alpha^0) \equiv r, \quad \bar{R}_{kl,\theta}(\alpha^t) = \underline{R}_{kl,\theta}(\alpha^{t-1}) \text{ for } t = 1, 2, ... \tag{28}$$

Note the following properties of $\mathcal{A}_{kl,\theta}(r)$:

Lemma 2. *For any $k, l \in \{1, ..., K\}$ and any $r \in [0, \bar{r}_{kl}]$, $\mathcal{A}_{kl,\theta}(r)$ is nonempty. Furthermore, for all sequences $\{\alpha^t\}_{t=1}^\infty \in \mathcal{A}_{kl,\theta}(r)$, $\lim_{t\to\infty} \alpha^t \in B_{kl,\theta}$.*

Proof. First show that $0 \in B_{kl,\theta}$. By Theorem 3, when $v_k(0) = 0$ for all k, which is implied by condition (i) of Theorem 2, we have $\beta(0|N_k, \bar{s}) = 0$ for all $\bar{s} \in [0, 1)$ and $k \in \mathcal{K}$. Hence if $\theta \in \Theta_I$, we must have $R_k(0) = 0$ for all $k \in \mathcal{K}$. Furthermore, for any $k, l \in \mathcal{K}$ and any θ, we have $h_{k,\theta}(0) = h_{l,\theta}(0) = 0$ and therefore $0 \in B_{kl,\theta}$.

Next, following GPV (2009), observe that both $\bar{R}_{kl,\theta}$ and $\underline{R}_{kl,\theta}$ are continuous, with $\bar{R}_{kl,\theta}$ having range $[0, \bar{r}_{kl}]$. Choose any $r_0 \in [0, \bar{r}_{kl}]$. Since $r_0 \in [0, \bar{r}_{kl}]$, by the Intermediate Value Theorem there exists $\alpha \in [0, 1]$ such that $\bar{R}_{kl,\theta} = \alpha$. Choose any such α, set $\alpha_0 = \alpha$, and set $r_1 = \underline{R}_{kl,\theta}(\alpha_0)$. Note that $\bar{R}_{kl,\theta}$ is continuous on $[0, \alpha_0]$, with $\bar{R}_{kl,\theta}(\alpha_0) \geq r_1$. Hence again by the intermediate value theorem there exists $\alpha_1 \in [0, \bar{\alpha_0}]$ such that $\bar{R}_{kl,\theta}(\alpha_1) = r_1$ and $\alpha_1 \leq \alpha_0$. Iterating the argument establishes existence of a decreasing sequence $\{\alpha_t\}_{t=0}^\infty \in \mathcal{A}_{kl,\theta}$.

Finally show that any sequence $\{\alpha_t\}_{t=0}^\infty \in \mathcal{A}_{kl,\theta}$ converges to a limit $\bar{a} \in B_{kl,\theta}$. Clearly, if $\{\alpha_t\}_{t=0}^\infty \in \mathcal{A}_{kl,\theta}$ then $\{\alpha_t\}_{t=0}^\infty$ is a decreasing sequence bounded below by 0. Hence $\{\alpha_t\}_{t=0}^\infty$ converges to some limit $\bar{a}$. Furthermore, by definition, we must have $\lim_{t\to\infty} \bar{R}_{kl,\theta}(\alpha_t) = \lim_{t\to\infty} \underline{R}_{kl,\theta}(\alpha_t)$. Hence $\bar{a} \in B_{kl,\theta}$, establishing the claim.

□

Now let ϕ be any continuous, increasing, zero-at origin function on $[0, \bar{r}_{kl}]$ satisfying the compatibility condition

$$\tilde{b}_{k,\theta}(a) + \phi(\tilde{R}_{k,\theta}(a)) = \tilde{b}_{l,\theta}(a) + \phi(\tilde{R}_{l,\theta}(a)) \, \forall a \in [0, 1]. \tag{29}$$

If no such ϕ exists, then $\theta \notin \Theta_I$, a contradiction. Otherwise, choose any such ϕ and rearrange (29) to obtain for any $a \in [0, 1]$

$$\phi(\tilde{R}_{k,\theta}(a)) - \phi(\tilde{R}_{l,\theta}(a)) = \tilde{b}_{l,\theta}(a) - \tilde{b}_{k,\theta}(a).$$

Since ϕ is continuous, increasing, and satisfies (29), this expression in turn implies

$$\phi(\bar{R}_{kl,\theta}(a)) = |\tilde{b}_{k,\theta}(a) - \tilde{b}_{l,\theta}(a)| + \phi(\underline{R}_{kl,\theta}(a)).$$

Next consider any $r^0 \in [0, \bar{r}_{kl}]$, and let $\alpha^0 = \min\{a \in [0,1] : \bar{R}_{kl,\theta}(a) = r^0\}$. Let $\{\alpha^t\}_{t=0}^{\infty}$ be any sequence in $\mathcal{A}_{kl,\theta}(r^0)$ whose first term is α^0. Recall that by definition $\{\alpha^t\}_{t=0}^{\infty}$ satisfies $\bar{R}_{kl,\theta}(\alpha^{t+1}) = \underline{R}_{kl,\theta}(\alpha^t)$ for all t. Thus for any t

$$\phi(\bar{R}_{kl,\theta}(\alpha^t)) = |\tilde{b}_{k,\theta}(\alpha^t) - \tilde{b}_{l,\theta}(\alpha^t)| + \phi(\bar{R}_{kl,\theta}(\alpha^{t+1})). \tag{30}$$

Noting that $r^0 \equiv \bar{R}_{kl,\theta}(\alpha^0)$ and recursively substituting into (30), we therefore conclude

$$\phi(r^0) = \sum_{t=0}^{\infty} |\tilde{b}_{k,\theta}(\alpha^t) - \tilde{b}_{l,\theta}(\alpha^t)| + \phi(\bar{R}_{kl,\theta}(\lim_{t\to\infty} \alpha^t)), \tag{31}$$

From above, $\lim_{t\to\infty} \alpha^t \in B_{kl,\theta}$. Moreover, if $\alpha^0 \notin B_{kl,\theta}$, then $\lim_{t\to\infty} \alpha^t < \alpha^0$.

We now show that recursive constructions of the form (31) uniquely determine ϕ on $[0, \bar{r}_{kl}]$. Toward this end, recall from Lemma 1 that $B_{kl,\theta}$ is a closed set of Lebesgue measure zero. Hence, letting $B^c_{kl,\theta}$ denote the complement of $B_{kl,\theta}$ in $[0,1]$, $B^c_{kl,\theta}$ is an open set relative to $[0,1]$ with full Lebesgue measure on $[0,1]$. Since $B^c_{kl,\theta}$ is open, it is the union of a countable collection of disjoint open (relative to $[0,1]$) intervals in $[0,1]$. We denote this collection of open intervals by $\{O_m\}_{m\in\mathcal{M}}$, indexed by a countable set $\mathcal{M}$. For each $m \in \mathcal{M}$, let $a_m^+ = \sup O_m$ and $a_m^- = \inf O_m$. Note that, since $0 \in B_{kl,\theta}$, we have $a_m^- \in B_{kl,\theta}$ for all $m \in O_m$, and we also have $a_m^+ \in B_{kl,\theta}$ unless $a_m^+ = 1$.

Now let $\mathcal{A}_I$ be the set of closed intervals $I \subset [0,1]$ (with nonempty interior) such that, for each $I \in \mathcal{A}_I$ both $\bar{R}_{kl,\theta}(a)$ and $\underline{R}_{kl,\theta}(a)$ are strictly increasing on I, *and* I contains at least one element of $B_{kl,\theta}$. Note that since $\tilde{R}'_{k,\theta}(0) > 0$ and $\tilde{R}'_{l,\theta}(0) > 0$, and $0 \in B_{kl,\theta}$, at least one such interval exists. Suppose that there exist two solutions ϕ_1, ϕ_2 to (29). We first show that $\phi_1(\bar{R}_{kl,\theta}(a))$ and $\phi_2(\bar{R}_{kl,\theta}(a))$ must coincide up to a constant κ_I on each interval $I \in \mathcal{A}_I$. We then show that this implies $\phi_1(r^0) = \phi_2(r^0)$ for all $r^0 \in [0, \bar{r}_{kl}]$.

First consider any $I \in \mathcal{A}_I$. By definition, both $\bar{R}_{kl,\theta}(a)$ and $\underline{R}_{kl,\theta}(a)$ are strictly increasing on I, and I contains at least one element $\bar{a} \in B_{kl,\theta}$. Choose any $\alpha^0 \in I$ such that $\alpha^0 \in O_m$ for some $m \in \mathcal{M}$. If $\alpha^0 > \bar{a}$, then since both $\bar{R}_{kl,\theta}$ and $\underline{R}_{kl,\theta}$ are strictly increasing on $[a_m^-, \alpha^0]$ and $\bar{R}_{kl,\theta}(a_m^-) = \underline{R}_{kl,\theta}(a_m^-)$, we may form a decreasing sequence $\{\alpha^t\}_{t=1}^{\infty}$ starting from α^0 such that $\bar{R}_{kl,\theta}(\alpha^{t+1}) = \underline{R}_{kl,\theta}(\alpha^t)$ and $\alpha_t \to a_m^- \geq \bar{a}$. Otherwise, if $\alpha^0 > \bar{a}$, then we may form an *increasing* sequence $\{\alpha^t\}_{t=1}^{\infty}$ such that $\underline{R}_{kl,\theta}(\alpha^{t+1}) = \bar{R}_{kl,\theta}(\alpha^t)$ and $\alpha^t \to a_m^+ \leq \bar{a}$. In either case, for every $a_l, a_u \in I \cap [a_m^-, a_m^+]$, we will ultimately be able to express the integral

$$\int_{\underline{R}_{kl,\theta}(a_l)}^{\bar{R}_{kl,\theta}(a_u)} \phi_i'(r)\, dr$$

as a limit of identified sums of bids by recursively applying (31) and taking appropriate limits. Furthermore, for all r such that $r \in [\underline{R}_{kl,\theta}(a), \bar{R}_{kl,\theta}(a)]$ for some $a \in I$, we may take countable sums of such integrals to express both $\phi_1(r)$ as identified up to $\phi_1(\bar{R}_{kl,\theta}(\bar{a}))$ and $\phi_2(r)$ as identified up to $\phi_1(\bar{R}_{kl,\theta}(\bar{a}))$, since the open sets $\{O_m \cap I\}_{m\in\mathcal{M}}$ have full Lebesgue measure on I. From this, it follows that we must have $\phi_1(\bar{R}_{kl,\theta}(a)) = \phi_2(\bar{R}_{kl,\theta}(a)) + \kappa_I$ for all $a \in I$, where κ_I is an unknown constant potentially varying with $I \in \mathcal{A}_I$. Note that any two intervals I_1, I_2 which have nonempty intersection must have $\kappa_{I_1} = \kappa_{I_2}$. Furthermore, since by definition each $I \in \mathcal{A}_I$ has nonempty interior and thus positive measure, there can be at most countably many mutually disjoint elements of $\mathcal{A}_I$, and thus at most countably many distinct values of κ_I.

Now choose any $r^0 \in [0, \bar{r}_{kl}]$, and let $\alpha^0(r^0) = \alpha^0 = \min\{a \in [0,1] : \bar{R}_{kl,\theta}(a) = \bar{r}_{kl}\}$. Consider the sequence $\{\alpha^t\}_{t=1}^{\infty}$ formed by taking, at every step, $\alpha^{t+1} = \min\{a \in [0,1] : \bar{R}_{kl,\theta}(a) = \underline{R}_{kl,\theta}(\alpha^t)\}$, and let $\underline{a}(r^0) = \lim_{t\to\infty} \alpha^t$.[18] If $\bar{R}_{kl,\theta}(a) < \bar{R}_{kl,\theta}(\underline{a}(r^0))$ for all $a < \alpha^0$, stop. Otherwise, update $\alpha^0 = \min\{a \in [0,1] : \bar{R}_{kl,\theta}(a) = \bar{R}_{kl,\theta}(\underline{a}(r^0))$, form a new sequence α^{t+1} starting from this α^0 as above, let $\underline{a}(r^0)$ be the limit of this sequence, and repeat this process until no further progress is possible. Applying (31) across these sequences, for $i \in \{1,2\}$, we will eventually be able to express $\phi_i(r^0)$ as a limit of identified sums of bids, plus the trailing constant $\phi_i(\bar{R}_{kl,\theta}(\underline{a}(\alpha^0)))$. Moreover, by construction, for any $a < \underline{a}(r^0)$, we must have $\bar{R}_{kl,\theta}(a) < \bar{R}_{kl,\theta}(\underline{a}(\alpha^0))$; since $\bar{R}_{kl,\theta}(a)$ is continuous, this implies that $\bar{R}_{kl,\theta}(a)$ must be strictly increasing on some interval $[\underline{a}(r^0) - \bar{\delta}, \underline{a}(r^0)]$ with $\bar{\delta} > 0$. Furthermore, we have that $\underline{R}_{kl,\theta} < \bar{R}_{kl,\theta}$ almost everywhere, that $\underline{R}_{kl,\theta}(\bar{a}) = \bar{R}_{kl,\theta}(\bar{a})$, and that $\bar{R}_{kl,\theta}(a)$ is strictly increasing on $[\underline{a}(r^0) - \bar{\delta}, \underline{a}(r^0)]$. Hence $\underline{R}_{kl,\theta}(a)$ must also be strictly increasing on $[\underline{a}(r^0) - \underline{\delta}, \underline{a}(r^0)]$ for some $\underline{\delta} > 0$. But then, by definition, $\underline{a}(r^0) \in I$ for some $I \in \mathcal{A}_I$.

Let $\kappa_I(r^0)$ denote the constant associated with this I, and note that $\kappa_I(r^0)$ is a well-defined function of r^0 since every step in the construction of $\underline{a}(r^0)$ is unique, and since every $I \in \mathcal{A}_I$ containing $\underline{a}(r^0)$ must intersect. Moreover, from above, we have $\phi_1(\bar{R}_{kl,\theta}(\underline{a}(r^0))) = \phi_2(\bar{R}_{kl,\theta}(\underline{a}(r^0))) + \kappa_I(r^0)$, implies that we must also $\phi_1(r^0) = \phi_2(r^0) + \kappa_I(r^0)$. Since r^0 was arbitrary, this relationship must hold for all $r^0 \in [0, \bar{r}_{kl}]$. By hypothesis, $\phi_1(r^0) - \phi_2(r^0)$ is a continuous function of r^0, which means that $\kappa_I(r_0)$ must also be continuous. But from above, $\kappa_I(r^0)$ can take at most countably many values. Both of these conditions can hold simultaneously only if $\kappa_I(r^0)$ is constant for all $r^0 \in [0, \bar{r}_{kl}]$. In particular, since $\phi_1(0) = \phi_2(0) = 0$, we must have $\kappa_I(r^0) = 0$ for all $r \in [0, \bar{r}_{kl}]$. From this, we conclude that $\phi_1 = \phi_2$.

Lastly, showing that so long as there exists at least one $k, l \in \mathcal{K}$ satisfying conditions (i) and (ii) at $\theta \in \Theta_I$, there exist unique $(\lambda_\theta^{-1}, F_\theta) \in \Lambda^{-1}[0, \max_{k\in\mathcal{K}} \bar{r}_k] \times \mathcal{F}$ satisfying (4) at θ. By construction, if $\theta \in \Theta_I$, there must exist at least one such $(\lambda_\theta^{-1}, F_\theta)$. By definition, any candidate F_θ rationalizing bid-stage behavior must satisfy (4) at the given k, l which implies in particular that the quantile function $v_\theta(a)$ implied by F_θ must satisfy

$$v_\theta(a) = \tilde{b}_{k,\theta}(a) + \lambda_{kl,\theta}^{-1}(\tilde{R}_{k,\theta}(a)), \forall a \in [0,1].$$

But we have shown that $\lambda_{kl,\theta}^{-1}$ is uniquely identified (up to θ) from which it follows that $v_\theta(a)$ is identified up to θ, or equivalently that F_θ is identified up to θ. Moreover, since $\theta \in \Theta_I$, the *same* $(v_\theta, \lambda_\theta^{-1})$ must satisfy (4) for all $l \in \mathcal{K}$, which implies for any λ_θ^{-1},

$$v_\theta(a) - \tilde{b}_{l,\theta}(a) = \lambda_\theta^{-1}(\tilde{R}_{l,\theta}(a)) \quad \forall a \in [0,1], l \in \mathcal{K}.$$

But we have already shown that v_θ is uniquely identified up to θ, in which case, varying a, we will eventually be able to trace out $\lambda_\theta^{-1}(r)$ for all $r \in [0, \bar{r}_l]$. This holds for any l, implying that λ_θ^{-1} is unique within $\Lambda^{-1}[0, \max_k \bar{r}_k]$. This completes the proof.

□

Appendix C: Computational details

This Appendix provides a detailed description of our numerical implementation of the procedures described in Section 4. Our implementation is in the Julia programming language, and the code is available from the authors upon request.[19]

[18] If $\alpha^0(r^0) \in B_{kl,\theta}$, this may be a constant sequence.

[19] If not published along with the paper, we ultimately intend to make a condensed and annotated version of our code available on Matthew Gentry's website (http://www.matthewgentry.net).

Evaluating the sample likelihood

As described in Section 4, we implement estimation as a conditional maximum likelihood problem, where the likelihood contribution of individual i in auction l is given by the likelihood of observing outcomes (e_{il}, b_{il}) given N_l. Our strategy for evaluating this sample likelihood involves three key elements: partial discretization of observed bids, a Chebyshev collocation strategy for solving equilibrium bid functions, and substitution from the equilibrium bid FOC in evaluating exact (non-discretized) bid observations. Combined, these result in a fast, stable evaluator for the sample likelihood, which additionally yields exact analytic gradients. We next describe each element in turn.

We first translate exact bids b_{il} into partially discretized bids $\ddot{b}_{il}$ as follows. For each competition level $k = 1, ..., K$, let $\bar{b}_k$ denote the maximum bid observed at competition level k. We define a $(J_k + 1)$-element discretized bid grid $B^k = \{B_j\}_{j=0}^{J_k}$ for level k, where J_k is the is the largest integer such that $0.01J_k \leq \bar{b}_k - 0.05$, and each element $B_j = 0.01j$ (so that elements B_j are spaced with step 0.01 on the interval $[0, B_{J_k}]$); bearing in mind that valuations are distributed on $[0, 1]$, this is a relatively fine grid. We then replace each observed bid b_{il} with a partially discretized bid $\ddot{b}_{il}$, defined as $\ddot{b}_{il} = b_{il}$ if $b_{il} > \bar{B}^k$, and $\ddot{b}_{il}$ equal to the upper endpoint of its associated bid interval within the discrete grid B^k otherwise. By discretizing low bids, we greatly reduce the number of distinct points at which the sample log likelihood needs to be evaluated, while by keeping high bids, we preserve exact information for bids close to the maximum, which we expect to be particularly informative in view of our identification results. Through numerical experiments, we found that the maximum B_{J_k} had very little impact on either identified sets or profiled likelihood results.

Let $\ddot{g}_k(\cdot|\psi)$ be the density of discretized bids $\ddot{b}_{il}$ implied by symmetric equilibrium bidding strategies given parameters ψ against competition N_k. After discretizing bids, we may then write the individual log-likelihood function generically as

$$\ell_{il}(\psi) = (1 - e_{il}) \log s_{k_l} + e_{il}[\log(1 - s_{k_l}) + \log \ddot{g}_{k_l}(\ddot{b}_{il}|\psi)], \tag{32}$$

where the first term represents the likelihood contribution of a bidder who does not enter, and the second represents that of a bidder who enters and submits discretized bid $\ddot{b}_{il}$ conditional on entry.

In order to evaluate (32), we must first solve for the equilibrium bid functions $\beta_1(v|\psi), ..., \beta_K(v|\psi)$ implied by parameters ψ. Toward this end, first observe that we can express each $\beta_k(\cdot|\psi)$ as the solution to a scale-normalized differential equation on $[0, 1]$, scaled by the maximum value $\bar{v}$. Let $\tilde{\lambda}$ and $\tilde{F}_k$ be defined as in the main text:

$$\tilde{\lambda}(u) \equiv u + \sum_{j=0}^{Q} \gamma_j B_{j,Q}(u), \quad u \in [0, 1]$$

$$\tilde{F}_k(u) \equiv \sum_{j=0}^{P} \phi_{k,j} B_{j,P}(u), \quad u \in [0, 1],$$

i.e., the shape components of the parameterizations of λ and F_k defined on the interval $[0, 1]$, with both the range and the domain of $\tilde{\lambda}$ and the domain of $\tilde{F}_k$ scaled by $\bar{v}$ to obtain our final parameterizations $\lambda(x) = \bar{v}\tilde{\lambda}^{-1}(\bar{v}^{-1}x)$ and $F_k(x) = \tilde{F}_k(\bar{v}^{-1}x)$. Let $\tilde{\beta}_k(u|\psi)$ denote the solution to the following scale-normalized differential equation (DE) on $[0, 1]$:

$$\tilde{\beta}_k'(u|\psi) = \tilde{\lambda}\big(u - \beta_k(u|\bar{\psi})\big) \cdot \frac{(N_k - 1)(1 - s_k)\tilde{f}_k(u)}{s_k + (1 - s_k)\tilde{F}_k(u)}, \tag{33}$$

subject to the boundary condition $\tilde{\beta}_k(0|\psi) = 0$. Then the true equilibrium bid function $\beta_k(y|\psi)$ on $[0, \bar{v}]$ may be obtained by scaling both the range and the domain of $\tilde{\beta}_k(u|\psi)$ by $\bar{v}$: $\beta_k(y|\psi) = \bar{v}\tilde{\beta}_k(\bar{v}^{-1}y|\psi)$. This has two practical advantages. First, we can try multiple candidates for $\bar{v}$ without needing to re-solve (33). Second, we can guarantee that the log-likelihood is well-defined by restricting attention to $\bar{v} \geq \bar{b}_k/\tilde{\beta}_k(1|\psi)$. The MCMC algorithm we employ below takes advantage of this fact to eliminate "wasted" parameter trials for which $\bar{v}\tilde{\beta}_k(1|\psi) < \bar{b}_k$ and the log-likelihood is negative infinite.

It remains to solve the DE (33) for the scale-normalized bid function $\tilde{\beta}_k(\cdot|\psi)$. We approach this via Chebyshev collocation. For each $k = 1, ..., K$, we first approximate each normalized bid function $\tilde{\beta}_k(\cdot|\psi)$ with an Rth order Chebyshev polynomial:

$$\tilde{\beta}_k(u|\psi) = \sum_{r=1}^{R+1} c_{k,r} T_r(u),$$

where $T_r(u)$ denotes the rth-order Chebyshev polynomial with domain rescaled to the interval $[0, 1]$. We then choose the $R + 1$ unknown coefficients $(c_{k,1}, ..., c_{k,R+1})$ in this approximation to satisfy the system of $R + 1$ Chebyshev collocation equations

$$\sum_{r=1}^{R+1} c_{k,r} T_r'(u_l) \equiv \tilde{\lambda}\left(u_l - \sum_{r=1}^{R+1} c_{k,r} T_r(u_l)\right) \frac{(N_k - 1)(1 - s_k)\tilde{f}_k(u_l)}{s_k + (1 - s_k)\tilde{F}_k(u_l)}, \quad l = 1, ..., R + 1, \qquad (34)$$

where $\{u_l\}_{l=1}^{R+1}$ are the nodes of the collocation system. Since we wish to enforce boundary conditions exactly, we take nodes $u_1 = 0$ and $u_{R+1} = 1$, with remaining interior nodes $u_2, ..., u_R$ equal to the zeros of the $(R - 1)$th order Chebyshev polynomial $T_{R-1}(\cdot)$, rescaled to the domain $[0, 1]$:

$$u_l \equiv \frac{1}{2}\cos\left(\frac{2l - 1}{2(R - 2)}\pi\right) + \frac{1}{2}, \quad l = 2, ..., R - 1.$$

This yields a fast, stable solution to (33) with two additional practical advantages. First, analytical derivatives of each equilibrium bid function $\beta_k(\cdot|\psi)$ in ψ may be obtained by applying the implicit function theorem to the Chebyshev collocations (34). Second, one can reinterpret the Chebyshev collocation equations (34) as auxiliary constraints in a Mathematical Programming with Equilibrium Constraints (MPEC) framing of the MLE problem, as we do in implementing CCT Procedures 2 and 3.

Finally, having solved for each equilibrium bid function $\beta_k(\cdot|\psi)$, we can evaluate the semi-discretized bid density $\ddot{g}_k(\cdot|\psi)$ appearing in the log-likelihood (32). Let $\tilde{u}_k(b|\psi)$ be the inverse of the normalized bid function $\tilde{\beta}_k(\cdot|\psi)$, which in view of our Chebyshev collocation strategy we may define implicitly as the solution to

$$\bar{v}^{-1}b = \sum_{r=1}^{R+1} c_{k,r} T_r(\tilde{u}_k(b|\psi)).$$

We express $\ddot{g}_k(\ddot{b}_{il}|\psi)$ in terms of $\tilde{u}_k(\ddot{b}_{il}|\psi)$ and other parameters as follows. If $\ddot{b}_{il} \leq B_{J_k}$, we have $b_{il} \in (B_{j-1}, B_j]$ or equivalently $\ddot{b}_{il} = B_j$ for some $j = 1, ..., J_k$, so that $\ddot{g}_k$ takes the discrete form

$$\ddot{g}_k(\ddot{b}_{il}|\psi) = G_k(B_j|\psi) - G_k(B_{j-1}|\psi) = \tilde{F}_k(\tilde{u}_k(B_j|\psi)) - \tilde{F}_k(\tilde{u}_k(B_{j-1}|\psi)).$$

Meanwhile, for $\ddot{b}_{il} > \bar{B}^k$, we have $\ddot{b}_{il} = b_{il}$, in which case $\ddot{g}_{k_l}(\ddot{b}_{il}|\psi)$ is equal to the original equilibrium bid density $g_k(b_{il}|\psi)$, and we have

$$\ln \ddot{g}_k(\ddot{b}_{il}|\psi) = \ln f_k(\beta_k^{-1}(\ddot{b}_{il}|\psi)) + \ln \beta_k^{-1,\prime}(\ddot{b}_{il}|\psi),$$

which after substitution from the normalized bid DE (33) we may re-express as

$$\ln \ddot{g}_k(\ddot{b}_{il}|\psi) = -\ln \tilde{\lambda}\Big(\tilde{u}_k(\ddot{b}_{il}|\psi) - \bar{v}^{-1}\ddot{b}_{il}\Big) + \ln \bar{v}^{-1} - \ln(N_k - 1) + \ln\Big(s_k + (1-s_k)\tilde{F}_k\big(\tilde{u}_k(\ddot{b}_{il}|\psi)\big)\Big).$$

In both cases, we may express $\ln \ddot{g}_k(\cdot|\psi)$ as a closed form up to the normalized inverse bid function $\tilde{u}_k(b|\psi)$. In turn, given our Chebyshev collocation strategy for solving equilibrium bids, evaluation of $\tilde{u}_k(\ddot{b}_{il}|\psi)$ reduces to finding the unique root of a polynomial equation on a bounded interval, which can be achieved quickly and easily using either Newton-Raphson or other numerical root-finding equations. Combining these methods, we ultimately obtain a log-likelihood which can be evaluated very efficiently (a few hundredths of a second in our simulations), and for which analytic gradients in the parameters ψ can easily be obtained by applying the implicit function theorem.

Evaluating Kullback-Leibler divergence

We evaluate Kullback-Leibler divergence $D_{KL}(\psi||\psi')$ from ψ to ψ' using essentially the same partial grid strategy as above, but now integrating with respect to the true density of observables at ψ. Specifically, let $\ddot{g}_k(\cdot|\psi)$ be the discretized bid density defined above. First consider the KL divergence between the distribution of observables $(e_i, \ddot{b}_i)$ conditional on N_k implied by parameters ψ to that implied by parameters ψ'. Simplifying the definition of KL divergence, we may express this as follows:

$$D_{KL,k}(\psi||\psi') = s_k(\log s_k - \log s'_k) + (1-s_k)(\log(1-s_k) - \log(1-s'_k)) + (1-s_k)\int_{\ddot{B}} \Big[\log \ddot{g}_k(\ddot{B}|\psi) - \log \ddot{g}_k(\ddot{B}|\psi')\Big]\, d\ddot{G}_k(\ddot{B}|\psi). \quad (35)$$

For ease of interpretation, we rescale each $D_{KL,k}(\psi||\psi')$ by $1/(1-s_k)$, such that $D_{KL,k}(\psi||\psi')$ now represents the expected log-likelihood loss per bid observed at competition N_k, when the log-likelihood is evaluated at parameters ψ' and true parameters are ψ. We then compute overall scaled KL divergence as $D_{KL}(\psi||\psi') = \sum_{k=1}^{K} D_{KL,k}(\psi||\psi')$. The resulting scaled measure $D_{KL}(\psi||\psi')$ reflects the average loss in sample log-likelihood at parameters ψ' per unit of sample scale M; i.e., per bid observed at each competition level. So, for example, if scaled $D_{KL}(\psi||\psi') = 10^{-4}$, then a sample of $M = 10000$ bids per competition level would be required to generate an expected sample log likelihood loss of -1 at parameters ψ' relative to true parameters ψ. In our main specifications, we take scaled $D_{KL}(\psi||\psi') \leq 10^{-5.0}$ as numerically equivalent, implying that $M = 100,000$ bids per competition level would generate an average loss of -1.

MPEC algorithms for profile likelihood and KL divergence

We solve profiled likelihood and Kullback-Leibler problems using a Mathematical Programming with Equilibrium Constraints (MPEC) algorithm inspired by Su and Judd (2012). Specifically, let $\mathbf{c}_k$ denote the vector of coefficients in the Chebyshev approximation to $\beta_k(\cdot)$, and $\mathbf{c} = \{\mathbf{c}_1, ..., \mathbf{c}_K\}$ collect these coefficients across competition levels k. For example, in computing the profiled likelihood function $\tilde{L}_\eta(\eta_0)$, we solve

$$\tilde{L}_\eta(\eta_0) = \max_{\psi, c} \mathcal{L}(\psi, \mathbf{c})$$

subject to the following constraints:

Profile constraints $\eta(\psi) = \eta_0$;

Admissible parameters $\psi \in \Psi$;

Maximum bid $\sum_{r=1}^{R+1} c_{k,r} T_{r,R}(1) \geq \bar{v}^{-1}\bar{b}_k$ for all $k = 1, ..., K$;

MPEC For each $k = 1, ..., K$, $(\mathbf{c}_k, \psi)$ satisfy the Chebyshev collocation system (34).

This MPEC approach is mathematically equivalent to a nested fixed point strategy in which the Chebyshev collocation equations (34) are solved exactly for each trial of parameters ψ. In our simulations, however, we found the MPEC framing to be several times faster on average. In conjunction with state of the art numeric optimizers such as KNITRO or IPOPT, and analytic gradients derived from the log-likelihood above, we are ultimately able to solve the profiled MLE problems with very high efficiency via this MPEC algorithm. In our simulations, the median profiled MLE problem is solved in less than 5 seconds, and almost all are solved in less than 10 seconds.

Recall that, for a given vector of equilibrium entry thresholds $\mathbf{s} \in S$, the vector $\phi = (\phi_1, ..., \phi_K)$ must belong to an admissible set $\Phi(\mathbf{s})$ defined in Section 4. Fixing $\mathbf{s} \in S$, $\Phi(\mathbf{s})$ is defined by a collection of linear inequality restrictions on the vector ϕ, which facilitates the reflection proposal distribution which we use in blockwise Metropolis-Hastings sampling of ϕ, which we describe below. Unfortunately, however, the restrictions defining $\Phi(\mathbf{s})$ are not linear in $\mathbf{s}$. Consequently, when solving the MPEC problem above, which involves joint search over $(\mathbf{s}, \phi)$, we need a computationally convenient representation of the unconditional admissible set Ψ.

We achieve this by introducing a $(K-1)$-element vector of auxiliary variables $W = (W_2, ..., W_K)$ defined by $W_k = (1 - s_k)/(s_{k+1} - s_k) \geq 0$ for each $k = 2, ..., K$. We can then express Conditions D', S', and O' in the definition of $\Phi(\mathbf{s})$ as quadratic inequality constraints in the vectors ϕ and W. Furthermore, for each $k = 2, ..., K$, we can reexpress the definition of W_k as

$$W_k(s_{k+1} - s_k) + s_k = 1, \tag{36}$$

i.e. a quadratic equality constraint in W and $\mathbf{s}$. We can thus reframe nonlinear, nonquadratic constraints on $(\mathbf{s}, \phi)$ as quadratic constraints in $(\mathbf{s}, \phi, W)$. This is a significant advantage for high-performance nonlinear solvers such as KNITRO which are programmed to exploit simplifications due to quadratic constraints.

Sequential Monte Carlo algorithm for sampling $\{\psi^b\}_{b=1}^B$

Finally, we describe the Sequential Monte Carlo (SMC) algorithm we use for obtaining posterior samples of $\{\psi^b\}_{b=1}^B$, the key first step in Procedures 1 and 2 of CCT (2018). SMC algorithms in general consider particle approximations to the posterior, which associate each draw ψ^b with a sample weight w^b. As in CCT (2018), we use an adaptive SMC algorithm proposed by Herbst and Schorfheide (2014), which starts with draws from the prior and gradually increases weight on the sample log likelihood until targeting the posterior. As in CCT, this adaptive SMC algorithm has two practical advantages. First, it tends to acheive good mixing even when the posterior is multi-modal. Second, each iteration of the algorithm involves many highly parallel operations, the SMC algorithm may be run with little additional computational time.

Each iteration of this adaptive SMC algorithm involves three main steps: Correction, which updates weights on w^b to reflect an increased weight on the sample log-likelihood; Selection, which resamples $\{\psi^b\}_{b=1}^B$ according to weights $\{w^b\}_{b=1}^B$ when weights become too unequal; and Mutation, which runs B separate and independent MCMC chains starting from (potentially resampled) $\{\psi^b\}_{b=1}^B$ and targeting the tempered posterior to obtain starting draws for the next iteration. Our implementation of these steps closely follows CCT (2018); we refer readers to their Appendix A for a detailed explanation. However, some aspects of our implementation of the MCMC sampler in the mutation step are distinctive. We briefly describe these next.

Starting from a given parameter draw $\psi = (\bar{v}^{-1}, \gamma, \phi, \mathbf{s})$, we obtain a mutated parameter draw ψ' by running 4 iterations of the following blockwise Metropolis-Hastings (MH) algorithm. We consider three core parameter blocks: utility parameters γ, distribution parameters ϕ, and entry thresholds $\mathbf{s}$. Each MCMC iteration involves three substeps, each involving update of one block of core parameters along with inverse of maximum valuation $\bar{v}^{-1}$. We draw new core parameters from the following blockwise proposal distributions.

Utility parameters γ Draw an innovation ϵ_γ from $N(0, \hat{\Sigma}_\gamma)$, then perturb $\gamma' = \gamma + \epsilon_\gamma$, with reflection at the boundaries of the feasible set Γ to ensure that $\gamma' \in \Gamma$. Since Γ involves only linear inequality constraints, results in Mohasel Afshar and Domke (2015) imply that the resulting blockwise proposal density q_γ for γ satisfies symmetry: $q_\gamma(\gamma'|\gamma) = q_\gamma(\gamma|\gamma')$. We thus do not need to account for this density explicitly in computing the MH acceptance probability.

Distribution parameters ϕ Draw an innovation ϵ_ϕ from $N(0, \hat{\Sigma}_\phi)$, then perturb $\phi' = \phi + \epsilon_\phi$, with reflection at the boundaries of the feasible set $\Phi(\mathbf{s})$ to ensure that $\phi' \in \Phi(\mathbf{s})$. Since, given $\mathbf{s}$, $\Phi(\mathbf{s})$ involves only linear inequality constraints, results in Mohasel Afshar and Domke (2015) again imply that resulting blockwise proposal density q_ϕ is symmetric and does not appear explicitly in the MH acceptance probability.

Entry thresholds s Draw an innovation ϵ_s from $N(0, \hat{\Sigma}_s)$, then update $\mathbf{s}' = \mathbf{s} + \epsilon_s$. Since the feasible set for $\mathbf{s}$ given other parameters is not affine, we do not employ reflection in this step. Rather, we simply reject $\mathbf{s}'$ if this leads to $\psi' \notin \Psi$.

In addition, in each block, after proposing a new set of core parameters, we propose a new candidate $\bar{v}^{-1,\prime}$ from the conditional (tempered) posterior of $\bar{v}^{-1}$ given the new proposed parameters. Thus, for example, in the utility parameter update step, we first draw a new candidate γ' from the proposal density $q_\gamma(\cdot|\gamma)$. We then draw a new candidate $\bar{v}^{-1,\prime}$ from $q_{\bar{v}^{-1}}(\cdot|\gamma', \phi, \mathbf{s})$, a log-linear approximation to the (tempered) posterior density of $\bar{v}^{-1}$ given $\gamma', \phi, \mathbf{s}$. Finally, we accept the new candidates $(\bar{v}^{-1,\prime}, \gamma')$ with probability given by the usual MH acceptance ratio

$$\frac{\Pi(\psi')}{\Pi(\psi)} \left(\frac{q_{\bar{v}^{-1}}(\bar{v}^{-1}|\gamma, \phi, \mathbf{s})}{q_{\bar{v}^{-1}}(\bar{v}^{-1,\prime}|\gamma', \phi, \mathbf{s})} \frac{q_\gamma(\gamma|\gamma')}{q_\gamma(\gamma'|\gamma)} \right), \tag{37}$$

where the leading term is the ratio of (tempered) posterior distributions and the term in parentheses is the density of proposing $(\bar{v}^{-1}, \gamma)$ from $(\bar{v}^{-1,\prime}, \gamma')$ relative to that of proposing $(\bar{v}^{-1,\prime}, \gamma')$ from $(\bar{v}^{-1}, \gamma)$. Further, by symmetry of q_γ, the ratio $\frac{q_\gamma(\gamma|\gamma')}{q_\gamma(\gamma'|\gamma)}$ cancels out in this acceptance probability as noted above.

By redrawing $\bar{v}^{-1}$ from its posterior in each proposal block, we ensure that proposed parameters always yield a maximum predicted bid at least as high as the maximum observed bid at each competition level, eliminating "wasted" parameter draws where the log-likelihood is negative infinite. Recalling, as noted above, that we may evaluate the equilibrium bid functions at multiple $\bar{v}^{-1}$ without re-solving the differential equation (33), this is a notable computational savings. Furthermore, bearing in mind that the proposal $q_{\bar{v}^{-1}}(\cdot|\gamma, \phi, \mathbf{s})$ approximates the conditional density of $\bar{v}^{-1}$ given $\gamma, \phi, \mathbf{s}$, the MH acceptance ratio (37) approximates the ratio of the marginal posteriors of γ relative to γ' conditional on other parameters, integrating out $\bar{v}^{-1}$.

In practice, following CCT, we set blockwise proposal variances in each MCMC update equal to the estimated variance of the relevant parameters in the previous SMC iteration, adjusted by a scale factor to ensure an acceptance rate of approximately 0.35 in each block. This scale factor is updated adaptively as in CCT; we refer interested readers to their discussion for further details. We run 200 iterations of the tempered SMC iteration with tempering, followed by a further 20 iterations targeting the true posterior. Through inspection of many sub-cases, we verify that the average log-likelihood remains stable over these final 20 iterations, indicating that the underlying SMC algorithm has converged to a steady state.

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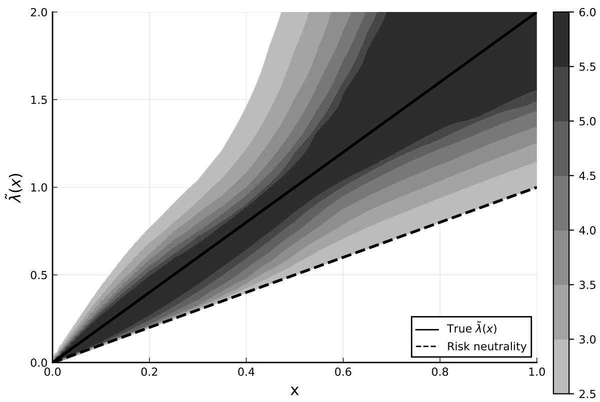
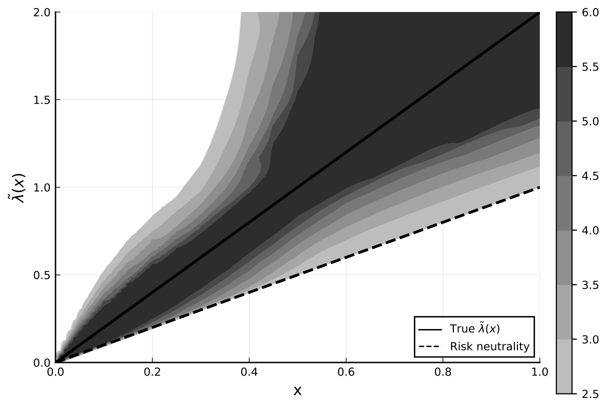
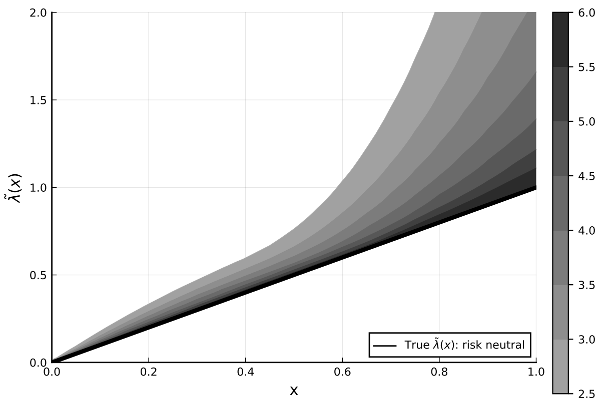
Figure 1: Pointwise Kullback-Leibler contour sets for $\tilde{\lambda}_0(x)$



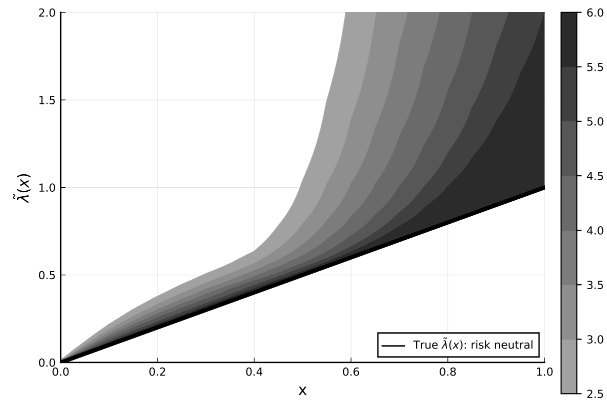
(a) $\rho_0 = 0.5$, $Q = 4$



(b) $\rho_0 = 0.5$, $Q = 7$



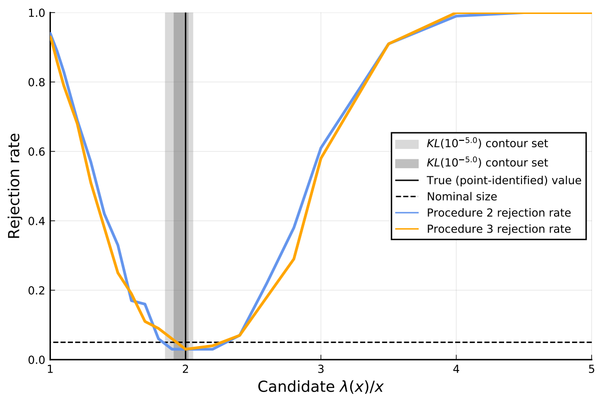
(c) $\rho_0 = 0.0$, $Q = 4$



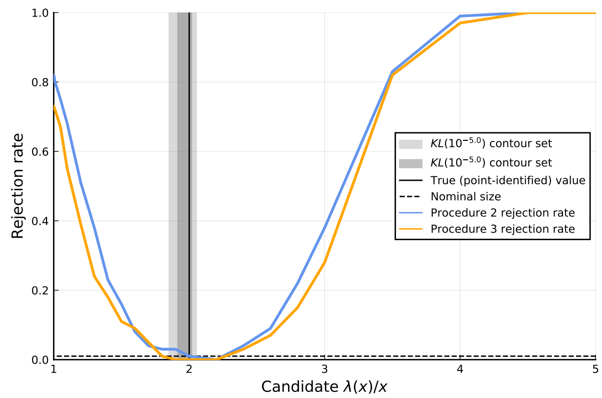
(d) $\rho_0 = 0.0$, $Q = 7$

Notes: Range and scale of $\lambda_0(x)$ are normalized as if $\bar{v} = 1$, to best reflect the shape of the underlying Bernstein polynomial parameterization. Contour values reflect the inverse order of magnitude of the pointwise profiled Kullback-Leibler divergence $\inf_{\psi \in \Psi}\{D_{KL}(\psi_0||\psi) : \tilde{\lambda}(x) = y\}$, where $D_{KL}(\psi_0||\psi)$ is scaled to represent the expected difference in log-likelihood per unit of sample scale. For example, a contour value above 5.0 implies that there is a parameter vector $\psi \in \Psi$ with $\tilde{\lambda}(x) = y$ such that $D_{KL}(\psi_0||\psi) \leq 10^{-5.0}$.

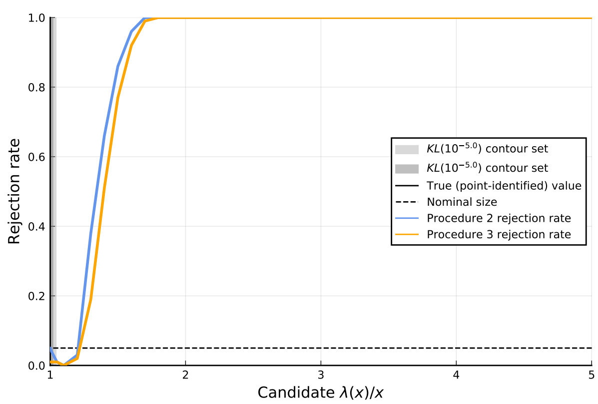
Figure 2: Rejection rates for candidate values of the average slope $\lambda_0(x)/x$ based on CCT Procedures 2 and 3 in a correctly specified CRRA model with sample scale $M = 1000$.



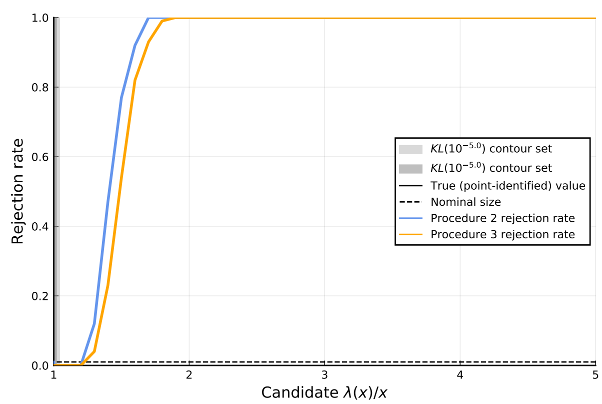
(a) True $\rho_0 = 0.5$, nominal size 0.05



(b) True $\rho_0 = 0.5$, nominal size 0.01



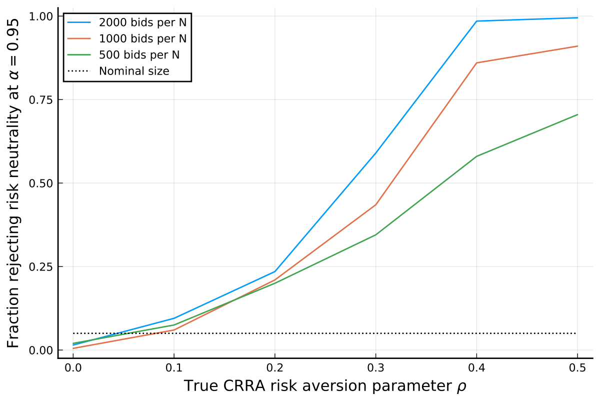
(c) True $\rho_0 = 0.5$, nominal size 0.05



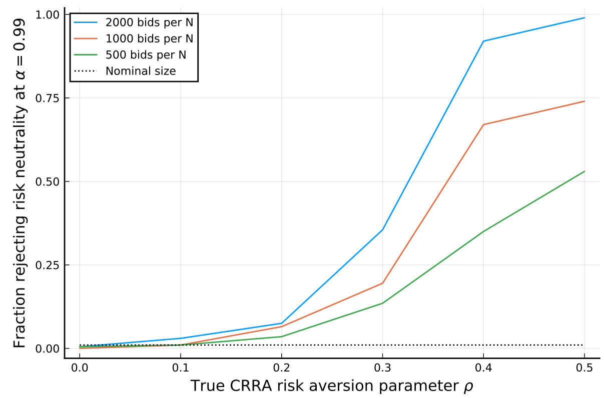
(d) True $\rho_0 = 0.5$, nominal size 0.01

Notes: Rejection rates are estimated based on 100 Monte Carlo simulations. In this CRRA specification, $\lambda_0(x)/x$ is point-identified. CCT Procedures 2 and 3 are both efficient. In implementing CCT Procedure 2, we scale $D_{KL}(\psi_0||\psi)$ to represent the expected difference in log-likelihood per bid observed at each competition level. We treat two parameter vectors ψ, ψ' as numerically equivalent if $D_{KL}(\psi||\psi') \leq 10^{-5.0}$. The grey shaded area represents the $D_{KL}(\psi||\psi') \leq 10^{-5.0}$ contour set for $\lambda_0(x)/x$ obtained at true parameters ψ_0.

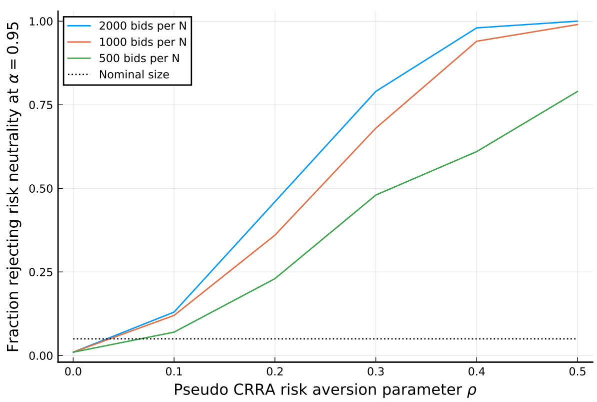
Figure 3: Power curves for testing risk neutrality ($\rho_0 = 0$) as proposed in Section 4.2 (based on CCT Procedure 3), as a function of risk aversion ρ and sample scale M



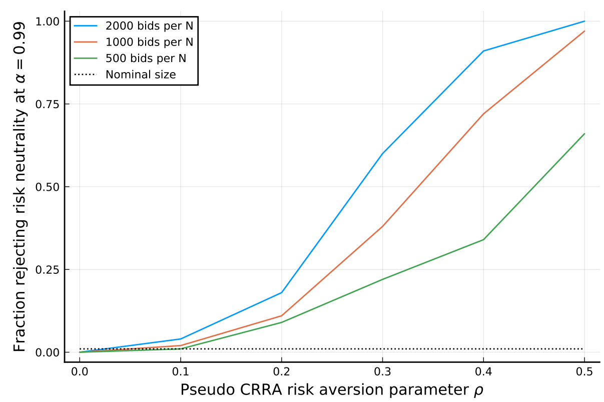
(a) CRRA U_0, nominal size 0.05



(b) CRRA U_0, nominal size 0.01



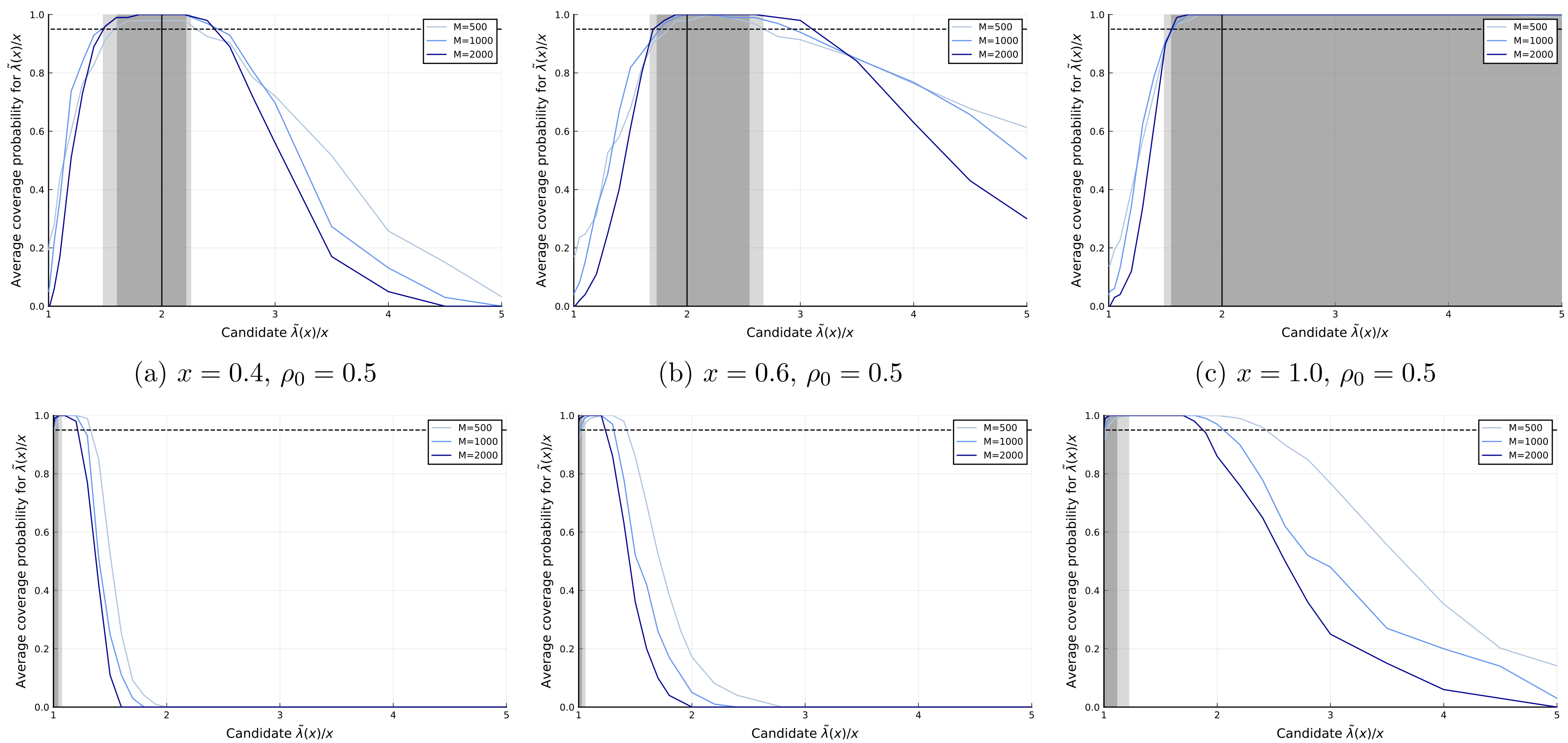
(c) CARA U_0, nominal size 0.05



(d) CARA U_0, nominal size 0.01

Notes: Rejection rates are estimated based on 100 Monte Carlo simulations. To render x-axes comparable across panels, we calibrate CARA utility parameters such that $\lambda_0(0.5) = 0.5/(1-\rho)$, where ρ is a pseudo-true CRRA parameter indexed on the x axis. This ensures that the average slope of $\lambda(x)$ in each CARA model is similar to the corresponding CRRA model over the interval $[0, 0.5]$, which is roughly the empirically relevant domain of the argument $v - \beta(v)$ in $\lambda(v - \beta_k(v))$. To isolate the effect of misspecified $\lambda_0(x)$, we simulate bidding data under both CARA and CRRA models at CRRA entry thresholds.

Figure 4: Estimated average coverage (acceptance) rates for candidate values of $\tilde{\lambda}_0(x)/x$ based on level $\alpha = 0.95$ confidence sets derived from CCT Procedure 2, parameterizing λ_0 flexibly as a Bernstein polynomial of degree $Q = 4$.

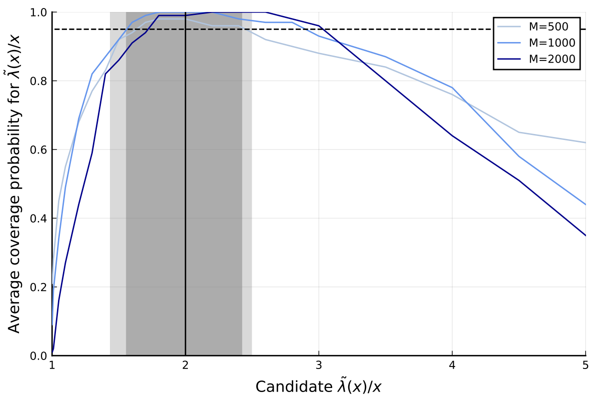


(a) $x = 0.4$, $\rho_0 = 0.5$ (b) $x = 0.6$, $\rho_0 = 0.5$ (c) $x = 1.0$, $\rho_0 = 0.5$

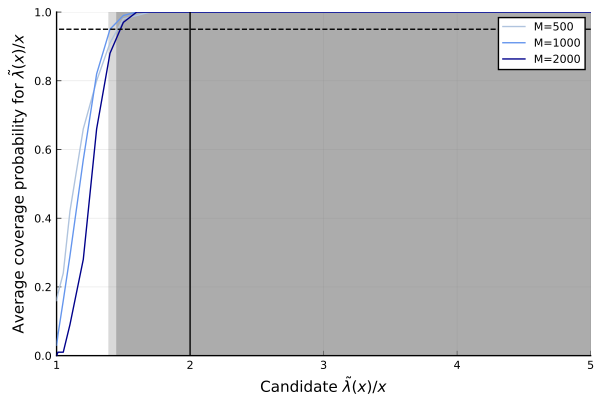
(d) $x = 0.4$, $\rho_0 = 0.0$ (e) $x = 0.6$, $\rho_0 = 0.0$ (f) $x = 1.0$, $\rho_0 = 0.0$

Notes: Estimated based on 100 Monte Carlo simulations. M is the sample scale (i.e., the average number of bids observed at each competition level). In implementing CCT Procedure 2, we treat two parameter vectors ψ, ψ' as numerically equivalent if $D_{KL}(\psi||\psi') \leq 10^{-5.0}$. The light grey shaded area in each figure represents the $D_{KL}(\psi||\psi') \leq 10^{-5.0}$ contour set for $\lambda_0(x)/x$ at true parameters ψ_0, while the dark grey area represents the $D_{KL}(\psi||\psi') \leq 10^{-5.5}$ contour set. The dashed horizontal line is nominal level $\alpha = 0.95$, and the solid vertical line is true $\lambda_0(x)/x$.

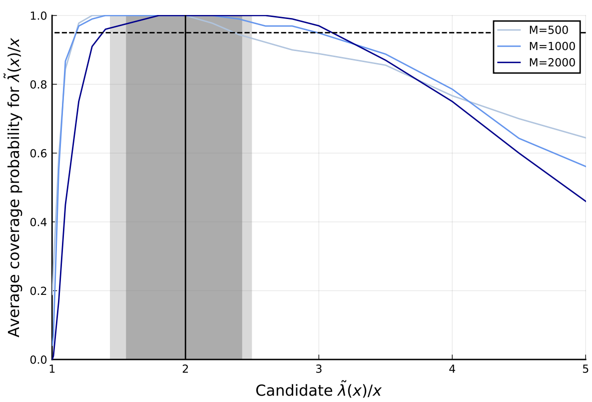
Figure 5: Estimated average coverage (acceptance) rates for candidate values of $\tilde{\lambda}_0(x)/x$ based on level $\alpha = 0.95$ confidence sets derived from CCT Procedure 2, parameterizing λ_0 flexibly as a Bernstein polynomial of degree $Q = 7$.



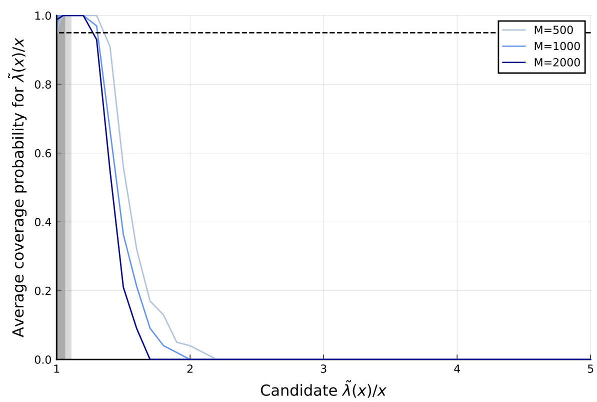
(a) $x = 0.4$, $\rho_0 = 0.5$, uniform prior



(b) $x = 1.0$, $\rho_0 = 0.5$, uniform prior



(c) $x = 0.4$, $\rho_0 = 0.5$, regularizing prior



(d) $x = 0.4$, $\rho_0 = 0.0$, uniform prior

Notes: Estimated based on 100 Monte Carlo simulations. M is the sample scale (i.e., the average number of bids observed at each competition level). In implementing CCT Procedure 2, we treat two parameter vectors ψ, ψ' as numerically equivalent if $D_{KL}(\psi||\psi') \leq 10^{-5.0}$. The light grey shaded area in each figure represents the $D_{KL}(\psi||\psi') \leq 10^{-5.0}$ contour set for $\lambda_0(x)/x$ at true parameters ψ_0, while the dark grey area represents the $D_{KL}(\psi||\psi') \leq 10^{-5.5}$ contour set. The dashed horizontal line is nominal level $\alpha = 0.95$, and the solid vertical line is true $\lambda_0(x)/x$.

Table 1: Estimated average coverage probabilities for profiled $D_{KL} \leq 10^{-5.0}$ and $D_{KL} \leq 10^{-5.5}$ contour sets for $\tilde{\lambda}_0(x)/x$ based on CCT Procedure 2, and median confidence intervals for $\tilde{\lambda}_0(x)/x$ implied by these confidence sets.

			Nominal confidence $\alpha = 0.95$			Nominal confidence $\alpha = 0.99$			
			Fraction cover D_{KL}		Median	Fraction cover D_{KL}		Median	True
ρ_0	x	M	$\leq 10^{-5.0}$	$\leq 10^{-5.5}$	Conf Int	$\leq 10^{-5.0}$	$\leq 10^{-5.5}$	Conf Int	$\tilde{\lambda}_0(x)/x$
0.5	0.4	500	0.87	0.94	$[1.151, 3.523]$	0.98	0.98	$[1.060, 3.781]$	2.0
		1000	0.95	0.99	$[1.136, 3.220]$	0.99	1.00	$[1.073, 3.407]$	2.0
		2000	0.94	0.99	$[1.197, 3.037]$	0.99	1.00	$[1.150, 3.150]$	2.0
	0.6	500	0.84	0.91	$[1.295, 5.517]$	0.97	0.98	$[1.156, 6.000]$	2.0
		1000	0.91	0.91	$[1.308, 5.100]$	0.97	0.98	$[1.178, 5.678]$	2.0
		2000	0.94	0.96	$[1.466, 4.341]$	0.98	0.99	$[1.356, 4.807]$	2.0
	1.0	500	0.86	0.91	$[1.251, 6.000]$	0.98	0.98	$[1.142, 6.000]$	2.0
		1000	0.90	0.95	$[1.246, 6.000]$	0.98	0.98	$[1.149, 6.000]$	2.0
		2000	0.89	0.96	$[1.362, 6.000]$	0.97	1.00	$[1.297, 6.000]$	2.0
0.0	0.4	500	0.95	0.95	$[1.000, 1.503]$	0.98	0.98	$[1.000, 1.608]$	1.0
		1000	0.95	0.95	$[1.000, 1.400]$	0.99	0.99	$[1.000, 1.477]$	1.0
		2000	0.96	0.96	$[1.000, 1.370]$	1.00	1.00	$[1.000, 1.424]$	1.0
	0.6	500	0.92	0.92	$[1.000, 1.725]$	0.97	0.97	$[1.000, 1.911]$	1.0
		1000	0.93	0.93	$[1.000, 1.528]$	0.98	0.98	$[1.000, 1.657]$	1.0
		2000	0.98	0.98	$[1.000, 1.455]$	1.00	1.00	$[1.000, 1.549]$	1.0
	1.0	500	0.92	0.92	$[1.000, 3.635]$	0.95	0.95	$[1.000, 4.369]$	1.0
		1000	0.92	0.92	$[1.000, 2.879]$	0.97	0.97	$[1.000, 3.489]$	1.0
		2000	0.96	0.96	$[1.000, 2.597]$	1.00	1.00	$[1.000, 2.994]$	1.0

Notes: Estimates based on 100 Monte Carlo simulations. M is the sample scale (i.e., the average number of bids observed at each competition level). Columns labeled "Fraction cover D_{KL}" denotes the fraction of simulations in which level-α confidence sets based on CCT Procedure 2 cover profiled Kullback-Leibler contour sets for $\tilde{\lambda}_0(x)/x$ based on $D_{KL}(\psi_0||\psi) \leq 10^{-5.0}$ and $D_{KL}(\psi_0||\psi) \leq 10^{-5.5}$ respectively. Columns labeled "Median Conf Int" describes the median, across simulations, of lower and upper endpoints for level-α confidence sets for $\tilde{\lambda}_0(x)/x$. In implementing CCT Procedure 2, we treat two parameter vectors ψ, ψ' as numerically equivalent if $D_{KL}(\psi||\psi') \leq 10^{-5.0}$.