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Zvika Neeman

Aniko Öry (Oery)

Jungju Yu

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THE BENEFIT OF COLLECTIVE REPUTATION

By

Zvika Neeman, Aniko Öry, and Jungju Yu

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

The Benefit of Collective Reputation

Zvika Neeman, Aniko Öry, Jungju Yu*

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Abstract

We study a model of collective reputation and use it to analyze the benefit of collective brands. Consumers form beliefs about the quality of an experience good that is produced by one firm that is part of a collective brand. Consumers' limited ability to distinguish among firms in the collective and to monitor firms' investment decisions creates incentives to free-ride on other firms' investment efforts. Nevertheless, we show that collective brands induce stronger incentives to invest in quality than individual brands under two types of circumstances: if the main concern is with quality control and the baseline reputation of the collective is low, or if the main concern is with the acquisition of specialized knowledge and the baseline reputation of the collective is high. We also contrast the socially optimal information structure with the profit maximizing choice of branding if branding is endogenous. Our results can be applied to country-of-origin, agricultural appellation, and other collective brands.

*Neeman: Tel Aviv University (e-mail: zvika@post.tau.ac.il). Öry: Yale School of Management (e-mail: aniko.oery@yale.edu). Yu: Yale School of Management (e-mail: jungju.yu@yale.edu). We thank Joyee Deb, Anthony Dukes, Johannes Hörner, Larry Samuelson, Jiwoong Shin, K. Sudhir, Robert Zeithammer for helpful comments. We also thank seminar participants at McGill University, the FTC and Marketing Science economic conference on consumer protection, the SICS conference at Berkeley, the University of Munich, and Yale University.

1 Introduction

A product's country of origin indicates something about its quality. How does such a collective brand operate, and how can it sustain its brand value? Why, for example, does the car manufacturer Volkswagen advertise "The power of German engineering" and so many successful Chinese suppliers emphasize their country of origin, while the German appliance manufacturer Bosch uses the non-country specific slogan "made for life"? Collective brands are also prevalent in many other domains, such as in the form of appellations.

A brand, defined as "a unique design, sign, symbol, words, or combination of these, employed in creating an image that identifies a product and differentiates it from its competitors,"[1] can be thought of as a means to build a good reputation. When building reputation, a firm faces a moral hazard problem; its investment in quality is unobservable to current consumers, and the reputational return on its investment can only be collected in the future.

The benefits of good reputation differ in a collective brand and an individual brand. At first glance, collective brands may seem like a bad idea. If several firms operate under one brand name, each firm has an incentive to free-ride on other firms? investments. Moreover, a firm?s investment in its own quality creates a weaker impact on the brand value of a collective brand because consumers are uncertain whether quality is generated by the firm itself or one of the other firms in the brand. Thus, the "precision" of the signal that is generated by a firm's investment in quality is lower in a collective brand, weakening the incentive to invest in quality.

Nevertheless, under some circumstances, a collective brand can serve as a commitment device for investment in high quality. If a brand is very successful (possibly as a result of previous large investments), then a firm might be discouraged from additional investment because the returns from it become small. The firm can afford to rest on its laurels, so to speak. Analogously, if a brand develops a bad reputation (possibly as a result of no investment), then returns on investment are also low, and the firm might give up investment

[1] This definition is taken from BusinessDictionary.Com.

altogether. As we show, collective brands can mitigate these "discouragement effects" faced by individual firms after very good or very bad histories by making extreme beliefs about the value of the brand less likely.

Exactly how extreme the beliefs about a brand can be depends both on the structure of signals that consumers obtain about firms? investments in quality and the baseline reputation of firms in the industry. For example, in manufacturing, quality control is important, and consumers can easily learn that a firm is incompetent (or has failed to invest) when a product has been observed to have a low quality. In contrast, in industries that require some exclusive knowledge about, for example, a technology, a good quality realization reveals that a firm possesses that technology and is a "competent type."

We analyze a model of reputation that can incorporate both individual brands and collective brands with multiple firms in the vein of Mailath and Samuelson (2001). The model has the following features. Time is discrete. There are two types of firms, competent and incompetent. In every period, only competent firms have the option to invest in quality. Consumers observe the qualities of past products, which are noisy signals of past investment decisions. Given these features, competent types can differentiate themselves from incompetent types by investing over time and producing higher quality products. If consumers believe that competent types invest, then they infer that a firm with good signals is indeed more likely to be competent. As a result, they are willing to pay more for goods produced by firms with better past signals. This, in turn, provides an incentive for a competent firm to invest in quality.

Accordingly, we define a firm's reputation as the consumers' posterior belief that it is competent. The best possible equilibrium from a welfare point of view is the one in which competent firms invest after each and every history, and we call the equilibrium in which this is the case the *reputational equilibrium*. In most of the paper, we restrict our analysis to the properties of this equilibrium. As pointed out by Mailath and Samuelson (2001), such an equilibrium exists only if beliefs are bounded. If beliefs are not bounded, then as the competent type continues to invest and to generate favorable signals, consumers eventually learn almost perfectly that the firm must be competent. This destroys the firm's incentive to invest, which leads to a collapse of the reputational equilibrium. This cannot happen in our model because we assume that consumers' memory is finite and limited to the last T periods only, as in Moav and Neeman (2010).[2] In the main part of the paper, we focus on the case where $T = 2$, but we show that our results hold for any finite $T \geq 2$.[3]

The timing of our model of collective reputation, which is a natural extension of the basic model used in the literature, is the following. Firms' types are independently drawn from a given distribution once and for all. In each period, a short-lived consumer is randomly matched with one firm. If firms establish themselves as individual brands, then consumers observe firm-specific past signals. If firms establish themselves as a collective brand, then consumers observe signals only at the brand level, and they cannot tell whether the signal is generated by the firm with which they have been matched in any specific period.

The reputational equilibrium exists in these environments if the benefit of investment exceeds its cost after every possible history. A firm has both short-run and long-run incentives to invest or refrain from investment. In the *short-run*, a firm may want to exploit its current reputation. In the *long-run*, a firm may want to free-ride on future efforts by itself and other members of the brand. Collective reputation can improve the short-run incentives to invest because the best possible collective reputation is weaker than the best possible individual reputation, and so the incentive to cash in on existing good reputation is weaker. However, the very fact that the best possible individual reputation is better than the best possible collective reputation also implies that individual reputation induces stronger incentives to invest in the long-run. It allows the firm to establish a stronger individual reputation in the

[2]Mailath and Samuelson (2001) assume instead that firms exit the market and are replaced by new firms whose types are drawn from some distribution. Another alternative is to assume that a firm's type changes randomly over time, as discussed in Holmström (1999). In this paper, we make the assumption of bounded memory because it allows us to compute the necessary and sufficient conditions for the existence of the reputational equilibrium in a closed form, which we use to compare two models of reputation (individual and collective). This assumption captures the nature of the market's limited memory and/or the inattention paid to the very distant past.

[3]The case where $T = 1$ is too simple and does not allow a firm to develop an history dependent reputation.

future, and the firm does not have the option of free-riding on another firm?s investments.

In the case of "quality control," the discouragement effect is stronger for individual brands after a firm has produced low quality because in this case consumers infer that the firm is likely to be incompetent, regardless of what else is observed or will be observed in the future. We show that in this case, collective reputation can help overcome the firm?s moral hazard problem if the baseline reputation for the firm or brand is low. This is because if baseline reputation is low, then observation of a good signal in the next period moves beliefs significantly in the case of a collective brand. It indicates that, despite the fact that one firm has been shown to be incompetent, other firms in the brand may be competent.

In the case of "exclusive knowledge," we show that a collective brand induces stronger incentives to invest than an individual brand if the baseline reputation for the brand is high. The reason is that it is easier for a competent individual firm to re-establish its good reputation after cashing in on an individual brand than for a firm that is part of a collective brand. Even if consumers observe high signals, it is still possible that some members of the brand are incompetent. This implies that firms who own individual brands have a stronger incentive to shirk than firms that belong to a collective brand.

It is important to note that in order for collective reputation to function as a commitment device, investment decisions cannot be made too frequently, or equivalently, the discount factor cannot be too large. This is because a firm would have a stronger incentive to free-ride on future efforts by itself or other members of the brand with collective reputation (long-run incentive). Thus, the short-run advantage of collective reputation can only outweigh the long-run free-riding incentive if firms do not care too much about the future.

Finally, we also address the issue of brand formation. If firms can freely choose with whom to brand, then it is important to understand whether the commitment value of a collective brand is sufficiently strong to encourage a competent firm to brand with an incompetent firm. We show that in an economy with high base reputation, a competent firm in an industry that requires exclusive knowledge always wants to brand with other firms irrespective of whether they are competent or not. In contrast, in the quality control case, firms do never want to brand with other firms in an economy with low base repuation, even though it is socially optimal. This suggests that in developing countries, government enforcement of country of origin labeling can be useful for quality control industries, if moral hazard is a major concern.

The paper is structured as follows. In the next section we discuss the related literature. In Section 3, we set up the model and discuss the equilibrium concept. Section 4 presents an example that highlights the main trade-offs and intuition of our analysis. In Section 5, we investigate the existence of the reputational equilibrium for individual and collective brands separately and then show under which circumstances collective brands can serve as a commitment device. We analyze firms? incentives to brand together in Section 6 and the case where $T \geq 2$ in Section 7. Finally, Section 8 concludes with a discussion of the interpretation and implications of our results. All proofs are relegated to the Appendix.

2 Related Literature

Our work is related to the theoretical economics literature on reputation, as well as to the literature on umbrella branding, country-of-origin and career concerns.

The theoretical work on reputation that is most relevant to our paper is Mailath and Samuelson (2001). They study the case of an individual reputation, and consider a firm that sells an experience good to consumers over an infinite discrete time horizon where the firm's investment decision is unobserved by consumers. Only the competent (as opposed to inept) type of the firm can invest in quality, and so the production of high quality products provides a noisy signal of the firm's investment decision. Mailath and Samuelson (2001) assume that in every period the firms exits the market with a certain probability and can unobservably sell its reputation to a new entrant. This formulation makes it hard to explicitly calculate the threshold cost below which the firm invests in quality so we follow Moav and Neeman (2010) and assume instead that consumers have a finite memory, and extend the standard model of individual reputation to accommodate collective brands.

Research that identifies the benefits of collective reputation is scarce. One exception is Fishman et al. (2014) In their model an individual firm can only generate one signal per period. Hence a collective brand that includes many firms can send many signals in every period and so provide better information to consumers. Fishman et al. (2014) show that this informational benefit outweighs firms' incentive to free-ride on other firms' investment efforts as long as the number of brand members is not too large. However, they consider a two-period model where firms' investment decision is made once-and-for-all in the first period, thus they abstract away from issues of commitment and dynamic trade-offs, which are the we focus of our analysis.

To the best of our knowledge, Tirole (1993) is the first to formalize an analytical model of collective reputation in context of a large organization. There, the group reputation is an aggregate of individual reputations. Each member's reputation is determined by its noisily observed past behavior as well as the group's track record. The complementarity between the group's reputation and current incentives of its members can give rise to multiple steady state equilbria. Given a good track record of the group, members have incentives to maintain the good reputation, and hence a "low corruption" steady state arises. But, starting with a bad record, the group is locked into a "high corruption" steady state.[4]

Collective reputation has also been studied in the context of umbrella branding in which an existing brand name is extended to a new product line. Wernerfelt (1988), Choi (1998), Cabral (2000), Miklós-Thal (2012), and Moorthy (2012) have examined the incentives that a monopolist has to signal quality by pooling reputation for different products. Others have considered settings where free-riding incentives are more pronounced. Zhang (2015) examines country-of-origin labelling. He shows that the ability to free-ride on other firms' quality investments implies that high quality firms have an incentive to dissociate themselves from the country-of-origin, which in turn mitigates free-riding and can improve the reputation

[4] Levin (2009) extends Tirole (1993) by considering the case where the cost of high effort evolves stochastically over time.

for the group. Bar-Isaac (2007) investigates an overlapping generations model in the moral hazard-in-teams framework á la Holmstrom (1982). With a career concern present, senior entrepreneurs looking to sell the firm in the next period have an incentive to work hard and to hire young juniors who need to build reputation. Finally, Fleckinger (2014) considers collective reputation under Cournot competition where consumers only learn the average quality in the market. He studies the effect of the number of firms on welfare, and shows that quality is decreasing in the number of firms whereas quantity increases.

3 The Model

Basics. We consider a market with two firms that produce a vertically differentiated experience good, that can be of either good (G) or bad (B) quality, at zero cost. In every period, $t \in \{\dots, -1, 0, 1, \dots\}$, one short-lived consumer with unit demand arrives and is *randomly matched* with one of the firms.[5]

Each firm is competent (C) with probability μ or incompetent (I) with probability $1-\mu$, independently of the other firm. We interpret μ as the *baseline reputation* of firms in the economy. Firms' types are unobservable to consumers. After being matched with a consumer, a competent firm can invest by incurring a cost of $c > 0$ in order to increase the probability that its product is of good quality.[6] If a competent firm invests in period t, then its product has good quality (G) with probability π_H. If it does not invest, then the product has good quality with probability π_L, with $\pi_L < \pi_H$. An incompetent firm cannot invest and also produces good quality with probability π_L. Consumers do not observe firm's investment decisions, but they can observe the quality of goods produced in the last two periods through word-of-mouth or consumer reviews.[7] Using this information, they update their beliefs about the type of the firm they are matched with. After the investment decision firms make a take-

[5] Thus, we abstract away from (price) competition between the two firms.

[6] The qualitative analysis and results would not change if we assumed that investments are made prior to being matched. This specification simplifies the exposition of the firm's Bellman equation.

[7] Alternatively, one can think of a long-lived consumer with limited memory.

it-or-leave-it offer to the consumer.[8] The consumer can either accept or reject the firm's offer and then it leaves the market.

Payoffs. We normalize the payoff of a consumer who does not buy the good to 0. If a consumer accepts a price p, she receives a payoff of $1-p$ if the good is of good quality, and $-p$ otherwise. A firm that sells in period t at price p_t receives a payoff or profit of $v_t = p_t - c$ at t if it invests at t and $v_t = p_t$ if it does not. Firms discount future payoffs by $\delta \in (0,1)$.

Information Structure. In every period, a consumer is assigned randomly to a firm. With a *collective brand*, consumers cannot distinguish between the identities of the two firms. This means that consumers obtain a signal about the brand in every period. If a firm maintains an *individual brand*, a consumer does not receive a signal about the firm that he is not matched with. Notice that the matching process ensures that the two firms sell the same expected quantities under the two regimes. We assume that firms know each other's types.

It follows that the set of histories for an individual brand is

$$\mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}^2$$

where $\varnothing$ represents a failure to match. The set of histories for a collective brand is

$$\mathcal{H}^{\text{col}} = \{G, B\}^2.$$

We denote the quality of the good that was produced n periods ago by h_{t-n} and the history at time t by $\mathbf{h}_t \equiv h_{t-2}h_{t-1} \in \mathcal{H}^b$ $(b \in \{\text{ind}, \text{col}\})$.

Equilibrium. We are interested in *stationary equilibria* in which strategies depend only on the histories specified above. A stationary equilibrium is given by an investment and pricing strategy of firms, a purchasing strategy of consumers, and consumers' beliefs over

[8] This assumption guarantees that all surplus goes to the firm in equilibrium, which simplifies the analysis. Firms must receive some surplus to have reputational concerns and for our results to hold.

the type of the firm they are matched with. For simplicity, we assume that consumers always purchase the good if it gives them a nonnegative expected payoff given their beliefs.

In the case of an *individual brand*, posterior beliefs given a history $\mathbf{h} \in \mathcal{H}^{\text{ind}}$ are given by a probability $Pr^{\text{ind}}(C|\mathbf{h})$ that the firm is competent. In the case of a *collective brand*, posterior beliefs are given by a probability distribution over the pair of types of the two firms. We denote the posterior belief that the two firms' types are $s \in \{C, I\}^2$ given history $\mathbf{h} \in \mathcal{H}^{\text{col}}$ by $\eta_s(h)$ and so in the case of a collective brand the posterior belief that the matched firm is competent given history $\mathbf{h}$ is $\Pr^{\text{col}}(C|\mathbf{h}) = \eta_{CC}(\mathbf{h}) + \frac{1}{2}(\eta_{CI}(\mathbf{h}) + \eta_{IC}(\mathbf{h}))$. We define the reputation of a brand – both individual and collective – to be consumers' posterior beliefs. In equilibrium, each player's strategy maximizes its payoffs given other players' strategies and beliefs, and posterior beliefs are derived from the realized histories and the firms' strategies by Bayes' rule whenever possible.

For most of the paper we focus our attention on the stationary equilibrium in which competent firms invest in quality whenever they are matched with a consumer, after each and every history. We call this the **reputational equilibrium**. This equilibrium is *socially optimal* if and only if

$$\Delta\pi \equiv \pi_H - \pi_L \geq c, \tag{1}$$

which we assume to be satisfied throughout the paper. Note that this equilibrium is also the brand profit-maximizing equilibrium in that case.

The game allows for the existence other equilibria. For example, a "no investment" equilibrium, in which a competent firm never invests in quality, always exists. We discuss other equilibria in Section 6 where we discuss endogenous brand formation.

4 Example

In order to intuitively understand the trade-offs present in our model, it is useful to think about the following illustrative example.[9] Consider two drivers, Adam and Brian, who work for a company that provides limousine services. In every period, a consumer who needs the service arrives and the company randomly assigns her to either Adam or Brian. After the ride, the consumer posts a review about the quality of the ride on the company's website, which displays the last two reviews given by consumers. The company can decide as a policy to reveal or conceal the names of the drivers on the reviews. In the former policy, new consumers can check past records of individual drivers, so drivers are building their reputation individually (individual brand). In the latter, consumers cannot distinguish between two drivers' records, so they are building a collective reputation (collective brand).

Each driver's competency type is drawn independently so that he is competent with a probability μ. Only competent drivers can exert effort at a cost $c > 0$ to provide a good transportation service (G) with probability $\pi_H \in (0, 1)$. If they do not exert effort, they are indistinguishable from the incompetent drivers in that they always provide a bad transportation service (B), i.e., $\pi_L = 0$. Consumers receive a payoff of 1 from a good transportation service and 0 otherwise. This set-up corresponds to the "exclusive knowledge" environment in which a good outcome reveals the driver's competence. Since the company makes a take-it-or-leave-it price offer to consumers, it can extract the entire consumer surplus by charging their willingness to pay:

$$p^b(\mathbf{h}) = \Pr^b(C|\mathbf{h})\pi_H + (1 - \Pr^b(C|\mathbf{h}))\pi_L = \Pr^b(C|\mathbf{h})\pi_H$$

where $b \in \{\text{ind}, \text{col}\}$ denotes the company's policy on whether to reveal names of drivers, and $Pr^b(C|\mathbf{h})$ is the posterior probability that consumers assign to the driver being competent

[9] We thank Robert Zeithammer for suggesting this example.

given history $\mathbf{h}$. For simplicity, suppose $\mu = \frac{1}{2}$ and let π_H approach 1.[10]

History $\mathbf{h}$	Individual brand	Collective brand
GG	$Pr^{\text{ind}}(C\vert\mathbf{h}) = 1$	$Pr^{\text{col}}(C\vert\mathbf{h}) \to_{\pi_H \to 1} \frac{5}{6}$
$G\varnothing, \varnothing G$	$Pr^{\text{ind}}(C\vert\mathbf{h}) = 1$	—
GB, BG	$Pr^{\text{ind}}(C\vert\mathbf{h}) = 1$	$Pr^{\text{col}}(C\vert\mathbf{h}) \to_{\pi_H \to 1} \frac{1}{2}$
$B\varnothing, \varnothing B$	$Pr^{\text{ind}}(C\vert\mathbf{h}) \to_{\pi_H \to 1} 0$	—
BB	$Pr^{\text{ind}}(C\vert\mathbf{h}) \to_{\pi_H \to 1} 0$	$Pr^{\text{col}}(C\vert\mathbf{h}) \to_{\pi_H \to 1} \frac{1}{6}$

Table 1: Consumers' beliefs $Pr^b(C|\mathbf{h})$, $b \in \{\text{ind}, \text{col}\}$

As demonstrated in Table 1, consumers' posterior beliefs are different for an individual and a collective brand. Because $\pi_L = 0$, a good outcome reveals the firm's competence. So, a consumer's belief about an individual brand reaches 1 for any histories that contain a G, and she pays the full price. But, in a collective brand, the type of the other firm remains unknown. Also, consumers do not observe a firm's identity, so uncertainty remains about which of the two firms the consumer will be matched with. Therefore, a consumer's belief is bounded away from 1 and 0 even after the best and worst outcomes, respectively.

To study when and how a collective brand provides more incentives to invest in quality, we examine when an individual and a collective brand sustain the reputational equilibrium. Without loss of generality, we focus on Adam's incentives. To make an investment decision relevant for him, we assume that Adam is competent. We find conditions under which investing is always his optimal decision by ruling out profitable single deviations. Suppose that Adam is visited in period t and is endowed with a history $\mathbf{h}_t = h_{t-2}G$ where $h_{t-2} \in \{G, B\}$. He can either exert effort or deviate.

First, suppose Adam is an individual brand and exerts effort. This investment in quality made in period t only affects payoffs in periods $t+1$ (short-run) and $t+2$ (long-run), as its outcome will be observed starting in period $t+1$ and forgotten by period $t+3$. Given $\pi_H \approx 1$, Adam's history observed in period $t+1$ will be $\mathbf{h}_{t+1} = GG$. He makes a sale if he is visited, which happens with a probability $\frac{1}{2}$. So, he expects to get $\frac{1}{2}(p^{\text{ind}}(GG) - c)$.

[10]As we take the limit for $\pi_H \to 1$, beliefs are equivalent to prices. That is, $\lim_{\pi_H \to 1} p^b(\mathbf{h}) = \Pr^b(C|h)$, which are listed in Table 1.

In period $t+2$, Adam is visited with a probability $\frac{1}{2}$ and the history can be $\mathbf{h}_{t+2} = GG$ or $G\varnothing$, depending on whether he was visited in period $t+1$. So, the long-run expected payoff is $\frac{1}{4}(p^{\text{ind}}(GG) - c) + \frac{1}{4}(p^{\text{ind}}(G\varnothing) - c)$. Plugging in the beliefs from the table, the expected payoff from investment is

$$\frac{\delta}{2} \cdot \left(p^{\text{ind}}(GG) - c\right) + \frac{\delta^2}{4} \cdot \left(p^{\text{ind}}(GG) - c\right) + \frac{\delta^2}{4} \cdot \left(p^{\text{ind}}(G\varnothing) - c\right) \quad = \quad \left(\delta + \delta^2\right) \cdot \frac{1-c}{2}.$$

If Adam does not exert effort in period t, he saves the cost of effort at an expense of worse reputation in the subsequent two periods. Given the history $\mathbf{h}_t = h_{t-2}G$ and $\pi_L = 0$, $\mathbf{h}_{t+1} = GB$ deterministically, and $\mathbf{h}_{t+2} = BG$ or $B\varnothing$, depending on whether he is visited in period $t+1$. Thus, the expected payoff from a deviation is given by

$$\frac{\delta}{2} \cdot \left(p^{\text{ind}}(GB) - c\right) + \frac{\delta^2}{4}\left(p^{\text{ind}}(BG) - c\right) + \frac{\delta^2}{4}\left(p^{\text{ind}}(B\varnothing) - c\right) \quad = \quad \delta \cdot \frac{1-c}{2} + \delta^2 \cdot \frac{1-2c}{4}.$$

Subtracting this payoff from $(\delta+\delta^2)\frac{1-c}{2}$, we find $\frac{\delta^2}{4}$ to be the *benefit* or *return from investment* in period t. As long as this benefit is greater than the investment cost c, a deviation after the history $h_{t-2}G$ is not profitable.

It is important to note that the short-run payoffs are the same whether Adam invests or not in period t. Adam's short-run concern is that by investing today, he can be paid a higher price by improving $\mathbf{h}_{t+1} = h_{t-1}h_t$. But, recall that Adam is endowed with the history ($\mathbf{h}_t = h_{t-2}G$), which includes a good outcome. Then, regardless of his investment decision, $\mathbf{h}_{t+1} = GG$ or GB. In either case, a consumer's belief about Adam in period $t+1$ is 1, so Adam is paid the full price. Thus, the short-run incentive for investment is zero. This underscores the severe short-run moral hazard problem when Adam builds reputation by himself; Adam wants to exploit his current reputation when it is very high.

The long-run incentive for investment remains positive because Adam wants to improve $\mathbf{h}_{t+2}$ by investing today. If he knew that h_{t+1} was G, Adam would deviate and save the investment cost now, because he will be paid the full price in period $t+2$ independent of his investment decision today. In other words, there is an incentive to free-ride on effort exerted by his future-self. But since he might not be visited in period $t+1$ while being visited in period $t+2$, which happens with a probability $\frac{1}{4}$, Adam has an incentive to invest in period t.

Similarly, it can be shown that the return on investment after a history $h_{-2}B$, with $h_{-2} \in \{G, B\}$, is $\frac{\delta}{2} + \frac{\delta^2}{4}$. At this point, we have considered all two-period histories and conclude that the reputational equilibrium can be sustained if and only if the cost of investment is less than or equal to the minimum of return from the investment, i.e.

$$c \leq \min \left\{ \frac{\delta^2}{4}, \frac{\delta}{2} + \frac{\delta^2}{4} \right\} = \frac{\delta^2}{4}.$$

Next, we consider the case in which Adam and Brian are evaluated anonymously, which corresponds to the case of a collective brand. Suppose that both are competent. We rule out profitable single deviations for Adam, given Brian always exerts effort.

First we assume that the brand is endowed with a history $\mathbf{h}_t = h_{t-2}G$ and suppose that Adam is visited in period t. If he invests, he produces G with a probability 1 and his expected payoff in period $t+1$ (short-run) is $\frac{1}{2} \cdot \left(p^{\text{col}}(GG) - c\right)$. Then, in period $t+1$, since both drivers are competent, the brand produces a G independent of who provides the service. Therefore, $\mathbf{h}_{t+2} = GG$, and Adam's expected payoff in that period (long-run) is again $\frac{1}{2} \cdot \left(p^{\text{col}}(GG) - c\right)$. In total, Adam expects to receive

$$\frac{\delta}{2}\left(p^{\text{col}}(GG) - c\right) + \frac{\delta^2}{2}\left(p^{\text{col}}(GG) - c\right) \quad = \quad \delta \cdot \frac{5-6c}{12} + \delta^2 \cdot \frac{5-6c}{12}.$$

If Adam deviates in period t by not exerting effort, $\mathbf{h}_{t+1} = GB$ and $\mathbf{h}_{t+2} = BG$, resulting in his expected payoff being:

$$\frac{\delta}{2} \cdot \left(p^{\text{col}}(GB) - c\right) + \frac{\delta^2}{2} \cdot \left(p^{\text{col}}(BG) - c\right) \quad = \quad \delta \cdot \frac{1-2c}{4} + \delta^2 \cdot \frac{1-2c}{4}.$$

All in all, the return on investment for Adam is $\frac{\delta+\delta^2}{6}$.

In contrast to the individual brand, the short-run return on investment in the collective brand does not vanish. As an anonymous driver in a group, Adam is never fully revealed to be competent. Even if he is endowed with a good history, Adam has an incentive to exert effort in order to be paid a higher price in period $t+1$. More explicitly, if Adam invests in period t, he can receive a price $p^{\mathrm{col}}(GG) = \frac{5}{6}$ instead of $p^{\mathrm{col}}(GB) = \frac{1}{2}$.

Similarly, one can show that the return on investment given a history $\mathbf{h}_t = h_{t-2}B$ with $h_{t-2} \in \{G, B\}$ is also $\frac{\delta+\delta^2}{6}$. Thus, the reputational equilibrium exists if and only if the cost of investment c is less than or equal to $\frac{\delta+\delta^2}{6}$. And this is always greater than $\frac{\delta^2}{4}$, the return on investment for an individual brand.

Consequently, if the reputational equilibrium exists for an individual brand, it also exists for a collective brand. But, the converse is not true. In particular, if $c \in (\frac{\delta^2}{4}, \frac{\delta+\delta^2}{6})$, it only exists for a collective brand. This finding holds even in the case in which there is an incompetent driver in a collective brand.[11] So, the average quality of limousine service can be improved if the company promotes drivers' collective reputation by concealing the identity of each driver.

Put differently, the collective brand provides drivers with commitment power to invest in quality mainly by alleviating an individual brand's moral hazard problem. Finally, the interaction between the short-run and long-run incentives implies that a collective brand induces more investment efforts for a small δ, i.e., when short-run incentives are more important. This is the case when producers are not too patient or investments are made infrequently.

We have thus far considered Adam's investment incentives when the type of brands are provided exogenously. Now, we examine whether Adam would like to form a collective brand with Brian for the sake of the commitment power, for example even if Brian is incompetent. Suppose $c \in \left(\frac{\delta^2}{4}, \frac{\delta+\delta^2}{6}\right)$ such that the reputational equilibrium exists only for a collective brand. Then, Adam must first determine which equilibrium is feasible if he refuses to brand

[11] We do not consider the case where both drivers are incompetent because then for no driver is investing a relevant decision. Only competent drivers can invest.

with Brian.

One candidate stationary equilibrium is one in which Adam invests if and only if he produced good quality in the previous period. In such an equilibrium, Adam will not invest after outcome B or $\varnothing$, which makes the set $\{B, \varnothing\}$ an absorbing set with expected long-run equilibrium profits equal to $\pi_L = 0$.

Another candidate stationary equilibrium is one in which Adam invests if and only if the last period's outcome is either G or $\varnothing$. In such an equilibrium, if it exists, the short-run incentive to invest after the realization B is larger than the short-run incentive to invest after the realization G. This is the case because one good quality realization fully reveals that Adam is competent, so additional investment makes no difference. Hence, this cannot be a stationary equilibrium. Similarly, it is possible to show that no other stationary equilibrium exists, except for the stationary equilibrium in which Adam never invests. Thus, if we restrict attention only to stationary equilibria, the highest long-run profit that Adam can hope for if he refuses to brand with Brian is 0.

Consequently, if $c \in \left(\frac{\delta^2}{4}, \frac{\delta+\delta^2}{6}\right)$ then Adam always benefits from branding with Brian – even if Brian is incompetent. This last conclusion is not true in general as we show in Section 6 below. The reason it fails to hold is that if μ is low the consumers' willingness to pay for the good may be lower than c, which can make investment so unattractive to Adam that he would not want to invest even if investment is socially optimal.

The rest of the paper provides more detailed insights about when collective brands are more attractive than individual brands from a social and private point of view by elaborating on the role of the baseline reputation μ and the signal structure.

5 Reputational Equilibrium

As we exhibited in the example, the reputational equilibrium exists if and only if the cost of investment is less than a threshold, which is equivalent to its benefit. We compare the threshold for an individual and a collective brand and identify conditions under which a collective brand sustains a reputational equilibrium for a larger range of investment costs.

5.1 Individual Brand

In a reputational equilibrium, in which the firm invests after every history, the equilibrium price after history $\mathbf{h}_t = h_{t-2}h_{t-1} \in \mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}^2$ is given by

$$p^{\text{ind}}(\mathbf{h}_t) = \Pr^{\text{ind}}(C|\mathbf{h}_t) \cdot \pi_H + (1 - \Pr^{\text{ind}}(C|\mathbf{h}_t)) \cdot \pi_L. \tag{2}$$

The stationary structure of the model allows us to express all the equilibrium prices, namely $p^{\text{ind}}(GG)$, $p^{\text{ind}}(GB)$, $p^{\text{ind}}(BG)$, and $p^{\text{ind}}(BB)$, in terms of the basic parameters of the model. For example, $p^{\text{ind}}(GG) = \frac{\mu\pi_H^3+(1-\mu)\pi_L^3}{\mu\pi_H^2+(1-\mu)\pi_L^2}$ because $\Pr^{\text{ind}}(C|GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2+(1-\mu)\pi_L^2}$.

Lemma 1. *The reputational equilibrium exists for an individual brand if and only if the cost of investment c satisfies*

$$c \leq \hat{\mathbf{c}}^{ind} \;\; \equiv \min_{h_{t-1}\in\{G,B,\varnothing\}} \bar{c}^{ind}(h_{t-1})$$

where $\bar{c}^{ind}(h_{t-1})$ denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$:

$$\begin{aligned}\bar{c}^{ind}(h_{t-1}) \equiv \;\; & \Delta\pi \cdot \Big[\tfrac{\delta}{2}\left(p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B)\right) + \tfrac{\delta^2}{4}\big(\pi_H(p^{ind}(GG) - p^{ind}(BG)) + \\ & (1-\pi_H)(p^{ind}(GB) - p(BB)) + p^{ind}(G\varnothing) - p^{ind}(B\varnothing)\big)\Big].\end{aligned} \tag{3}$$

Since investing should be the firm's optimal decision after all histories, the equilibrium exists if and only if the cost is less than the minimum of returns on investment across feasible histories. Given the most recent outcome h_{t-1}, the benefit from investment, $\bar{c}^{\text{ind}}(h_{t-1})$, consists of a sum of price premiums that are obtained in the next two periods. The firm enjoys a positive return on investment at t only if it leads to a better outcome (i.e., G instead of B) and if it is visited again in at least one of the next two periods. Accordingly, the return on investment must be multiplied by $\Delta\pi$; the *short-run benefit* obtained in period $t+1$ must be multiplied by $\frac{\delta}{2}$; and the *long-run benefit* obtained in period $t+2$ must be multiplied by $\frac{\delta^2}{4}$. We also note from equation (3) that $\bar{c}^{\text{ind}}(h_{t-1})$ does not depend on the entire history $\mathbf{h}_t = h_{t-2}h_{t-1}$ because h_{t-2} will be forgotten by period $t+1$ and is thus irrelevant to the return from the investment.

In the short-run, the firm's investment in period t can improve $\mathbf{h}_{t+1} = h_{t-1}h_t$, which allows for a price premium $p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)$. But, had the firm reached a very high or low reputation with h_{t-1}, the firm's additional investment would have a small influence on the reputation, thus yielding a small price premium. So, the moral hazard problem arises through the firm's short-run incentive to exploit its current reputation.

In the long-run, an investment in period t can improve $\mathbf{h}_{t+2} = h_t h_{t+1}$. The given history, h_{t-1} is now irrelevant. Recall that under the reputational equilibrium, the firm is supposed to invest in period $t+1$, so that the realized outcome h_{t+1} is G, B, and $\varnothing$ with probabilities $\frac{\pi_H}{2}$, $\frac{1-\pi_H}{2}$, and $\frac{1}{2}$, respectively. Accordingly, the expected long-run benefit from investment at t is a weighted sum over price premiums of a form $p^{\text{ind}}(Gh_{t+1}) - p^{\text{ind}}(Bh_{t+1})$, where each term is weighted by the probability of h_{t+1}. If the firm expects additional investment at $t+1$, which would make h_{t+1} more likely to be G, it has less incentive to exert effort at t. In other words, the firm has an incentive to free-ride on its own future investment efforts, which in turn reduces its long-run incentives to invest at t.

Figure 1 depicts $\hat{\mathbf{c}}^{\text{ind}}$ as a function of the baseline reputation μ. The three dotted curves represent $\bar{c}^{\text{ind}}(h_{t-1})$ for $h_{t-1} \in \{G, B, \varnothing\}$ and the solid line represents $\hat{\mathbf{c}}^{\text{ind}}$. So, for any fixed cost c, the values of μ where $\hat{\mathbf{c}}^{\text{ind}}$ lies above c sustain the reputational equilibrium. One can see that $\hat{\mathbf{c}}^{\text{ind}}$ converges to zero at extreme values of μ, which implies that the reputational equilibrium is harder to sustain. For extreme values of μ, consumers' beliefs are not much influenced by the brand's history. Thus, the firm has weak incentives to incur the investment cost. This commitment problem under extreme beliefs has been pointed out by Mailath and Samuelson (2001). If μ is large, the firm is tempted to"rest on its laurels," which we call

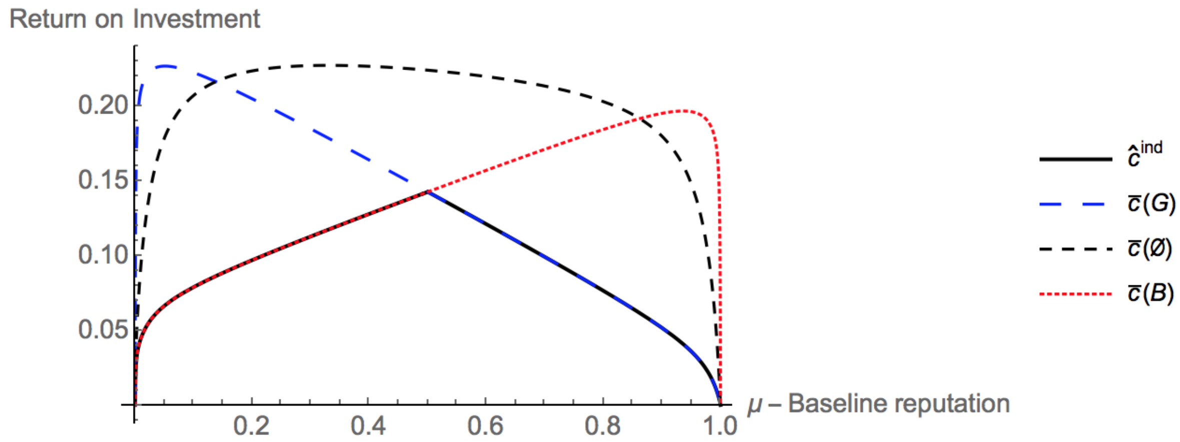


Figure 1: Return on Investment (ROI) after each history and the minimum

$$\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.4$$

the *complacency effect*. This effect is the strongest after good outcomes, i.e., $h_{t-1} = G$, which generate the most extreme posterior beliefs. Thus, $\hat{\mathbf{c}}^{\mathrm{ind}}$ is attained for $h_{t-1} = G$. If μ is small, then the firm simply gives up on improving its reputation, or the *discouragement effect*. Then, the reputation plunges more following a bad outcome. So, $h_{t-1} = B$ provides the lowest return from investment. The following lemma provides the binding condition for $\hat{\mathbf{c}}^{\mathrm{ind}}$:

Lemma 2. *If μ is sufficiently large, i.e.*

$$\mu \geq \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H) + \pi_L(1-\pi_L)}$$

then $\hat{\mathbf{c}}^{ind} = \bar{c}^{ind}(G)$. Otherwise, $\hat{\mathbf{c}}^{ind} = \bar{c}^{ind}(B)$.

5.2 Collective Reputation

Consumers facing a collective brand cannot distinguish between the identities of individual firms. They care about the expected quality of a randomly matched firm. Thus, by jointly generating the group's history observed by consumers, the firms share a common, collective reputation. Upon observing a history $\mathbf{h}_t \in \mathcal{H}^{\text{col}} = \{G, B\}^2$, a consumer forms beliefs about the quality of a random firm in the group, and is willing to pay

$$p^{\text{col}}(\mathbf{h}_t) = \Pr^{\text{col}}(C|\mathbf{h}_t) \cdot \pi_H + (1 - \Pr^{\text{col}}(C|\mathbf{h}_t)) \cdot \pi_L, \tag{4}$$

which, as in the case of an individual brand, can be expressed in terms of the primitives of the model. However, since posterior beliefs are different from those for an individual brand, prices are also different. For example, $\Pr^{\text{col}}(C|GG) = \frac{\mu^2\pi_H^2+\mu(1-\mu)(\frac{(\pi_H)^2}{4}+\frac{\pi_H\pi_L}{2}+\frac{(\pi_L)^2}{4})}{\mu^2\pi_H^2+2\mu(1-\mu)(\frac{(\pi_H)^2}{4}+\frac{\pi_H\pi_L}{2}+\frac{(\pi_L)^2}{4})+(1-\mu)^2\pi_L^2}$.

Recall that firms' competency is known to firms, but not to consumers. Payoffs depend on the type of the other member, denoted $\theta \in \{C, I\}$, and so do returns from investment.

Lemma 3. *The reputational equilibrium exists for a collective brand if and only if the cost of investment c satisfies*

$$c \leq \hat{\mathbf{c}}^{col} \quad \equiv \min_{h_{t-1}\in\{G,B\},\, \theta\in\{C,I\}} \bar{c}^{col}(h_{t-1}, \theta)$$

where

$$\begin{aligned}\bar{c}^{col}(h_{t-1},\theta) \equiv \quad & \Delta\pi \cdot \big[\tfrac{\delta}{2}\left(p^{col}(h_{t-1}G) - p^{col}(h_{t-1}B)\right) + \\ & \tfrac{\delta^2}{4}\left((\pi_H + \pi(\theta))(p^{col}(GG) - p^{col}(BG)) + (2 - \pi_H - \pi(\theta))(p^{col}(GB) - p^{col}(BB))\right)\big],\end{aligned} \tag{5}$$

denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$ and where $\pi(\theta)$ denotes type θ's probability of producing high quality upon exerting effort, that is $\pi(\theta) = \pi_H$ if $\theta = C$ and $\pi(\theta) = \pi_L$ if $\theta = I$.

Analogously to the individual brand, the short-run benefit from investment at t, obtained in period $t+1$, is given by $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)$. The long-run benefit from investment at t, obtained in period $t+2$, is a weighted sum of price premiums of the form $p^{\text{col}}(Gh_{t+1}) - p^{\text{col}}(Bh_{t+1})$. The realization of h_{t+1} depends on the effort provided by the brand in that period, which depends on which firm is visited and the type of the other firm. If $\theta = C$, then both members of the brand are competent and invest on equilibrium path in period $t+1$ if matched. But, if $\theta = I$, then the brand invests only if the consumer at $t+1$ is matched with the competent firm. So, the probability that $h_{t+1} = G$ is $\frac{\pi_H + \pi(\theta)}{2}$ where $\pi(\theta) = \pi_H$ if $\theta = C$, and $\pi(\theta) = \pi_L$ otherwise.

A collective brand with two competent firms generates more good outcomes than an individual brand because the group invests in every period, whereas an individual brand does not invest when it is not matched with a consumer. A member of a collective brand is thus tempted to *free-ride* on the future investment of other members; today's deviation can be undone by tomorrow's effort. This consideration suggests that a collective brand would have weaker long-run incentives to invest than an individual brand.

If μ is large, the firm faces the commitment problem due to the complacency effect. This problem becomes worse if the brand is endowed with a good outcome $h_{t-1} = G$ and expects more effort in the future, $\theta = C$. Thus, $(h_{t-1}, \theta) = (G, C)$ attains $\hat{\mathbf{c}}^{\text{col}}$. If μ is small, the commitment problem arises through the discouragement effect. This problem is more severe if the brand is endowed with a bad outcome $h_{t-1} = B$ and expects less effort in the futre, $\theta = I$. Consequently, $(h_{t-1}, \theta) = (B, I)$ attains $\hat{\mathbf{c}}^{\text{col}}$.

Lemma 4. *For μ close to 1, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(G, C)$, and for μ close to 0, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(B, I)$.*

The lemma identifies the binding condition for the threshold level for extreme values of μ, which are the regions of focus in our further analysis.[12] Figure 2 illustrates $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ for $(h_{t-1}, \theta) = (G, C)$ and (B, I) for given parameters.

5.3 Comparing Individual and Collective Brands

Having characterized the reputational equilibrium for an individual and a collective brand, we now compare the two types of branding. That is, we identify sufficient conditions for $\hat{\mathbf{c}}^{\text{ind}} < \hat{\mathbf{c}}^{\text{col}}$ to be satisfied. We focus our analysis on two special signal structures that are suggestive of two different industry types:

[12]Binding conditions for intermediate values of μ are more complicated and are analyzed in the lemma's proof in the appendix.

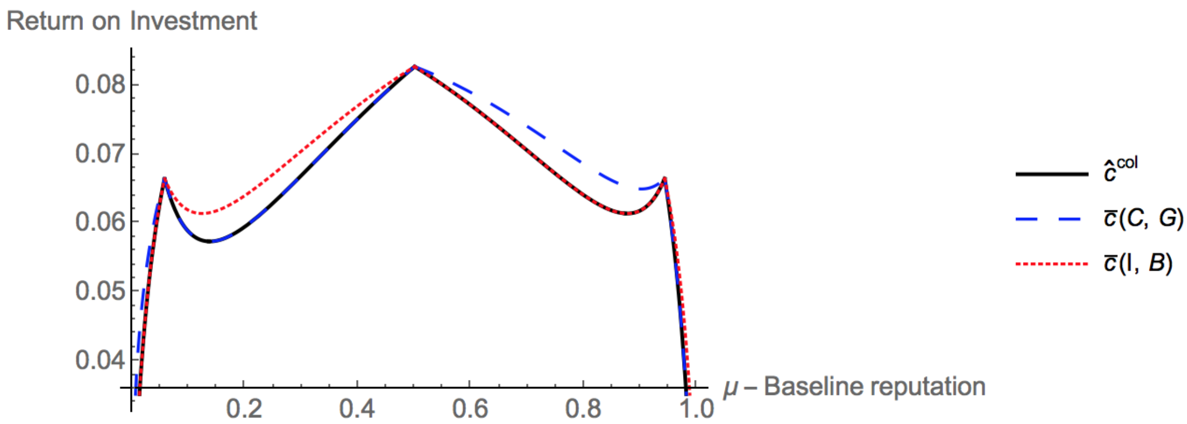


Figure 2: Return on Investment under (G, C) and (B, I)
$\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.4$

1. **"Exclusive knowledge"** $(\pi_L \approx 0)$: If π_L is very small, then it is nearly impossible to produce good quality without investment. In this case, an observation of good quality reveals that a firm is competent. This structure fits industries where competence represents a possession of a special technology or expertise that is required of producing high quality (e.g., for watches, automobiles, electronics, agriculture, etc.).

2. **"Quality control"** $(\pi_H \approx 1)$: If π_H is very large, then a firm can almost always produce good quality if it invests. In this case, an observation of bad quality reveals that a firm is incompetent. This assumption fits industries where competence represents the ability to perform effective quality control, such as in the case of the manufacturing of generic products (e.g., nuts and bolts, widgets, etc.).

In these two special cases, either a good or bad signal reveals the firm's type completely. That there is a revealing signal should be bad for short-run incentives, and especially so for an individual brand; when endowed with an appropriate outcome ($h_{t-1} = G$ or B), the firm's type is known, so the short-run incentive for investment vanishes. For example, for exclusive knowledge, the short-run premium $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) = 1 - 1 = 0$, and for quality control, $p^{\text{ind}}(BG) - p^{\text{ind}}(BB) = 0 - 0 = 0$. However, the problem is mitigated for a collective brand because of consumers' limited observability. Though an outcome may reveal the type of a particular firm, the type of the other firm remains unknown. Since consumers do know with which firm in the group they are matched with, the short-run premium $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)$ is strictly positive for all h_{t-1}.

Since short-run incentives for investment are greater for a collective brand, for small enough δ, $\hat{\mathbf{c}}^{\text{col}} > \hat{\mathbf{c}}^{\text{ind}}$ is satisfied. However, as noted before, an individual brand induces stronger long-run incentives. This is because there is less future effort to free-ride on. Also, since the firm is alone, the firm may not have an opportunity to recover the damaged reputation after a deviation. Therefore, δ cannot be too large for $\hat{\mathbf{c}}^{\text{col}} > \hat{\mathbf{c}}^{\text{ind}}$ to be true.

In the two extreme cases described above, the fact that one observation reveals competence or incompetence is bad for short-run incentives: strong evidence of competence induces complacency, and strong evidence of incompetence is discouraging. We show that in both of these cases, a collective brand may induce stronger incentives to invest than an individual brand. The reason is that the most extreme beliefs about a collective brand are less extreme than the most extreme beliefs about an individual brand, because there is always a possibility that other members of the collective brand are not as competent or incompetent as those whose type has been revealed. Consequently, complacency and discouragement are also less extreme in a collective brand, which can induce stronger incentives to invest.

This effect is particularly strong in the exclusive knowledge case if baseline beliefs μ are high. The intuition is the following: If prior beliefs are close to 1, observing a good realization moves beliefs further up - in the individual brand all the way to 1 - but posterior beliefs for a collective and individual brand stay close. After such a good signal, beliefs do not shift in the individual case even if a bad realization is observed. However, for a collective brand, beliefs can drop a lot after a bad realization - far below the prior, i.e. incentives to invest in the collective brand are much stronger than in an individual brand when the discouragement effect is the strongest for an individual brand.

Finally, notice that a small δ ensures that short-run incentives dominate long-run incentives, which favors a collective brand relative to an individual brand. The following

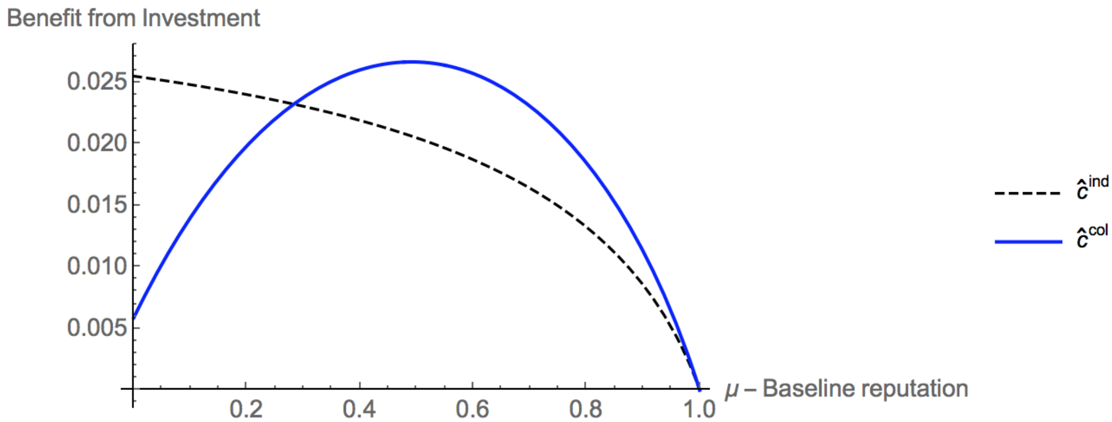


Figure 3: Comparison of Returns on Investment with $\pi_L = 0$
$\pi_H = 0.7,\ \delta = 0.4$

proposition formalizes this intuition.

On the flip side, if μ is small under the exclusive technology, an individual brand is provided with very strong incentive to invest and improve its reputation. By producing one good outcome, the brand can fully reveal its competence. On the other hand, a collective brand's incentive declines as μ decreases. The limited observation of consumers limit the extent to which the firms can improve the collective reputation by investing in quality.

Proposition 1 (Exclusive knowledge)**.** *Suppose that π_L is sufficiently close to* 0*.*

1. *A collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$) if consumers' prior belief μ about the firm's type is sufficiently high and δ is not too large.*

2. *An individual brand sustains a reputational equilibrium for higher investment costs than a collective brand ($\hat{\mathbf{c}}^{col} < \hat{\mathbf{c}}^{ind}$) for sufficiently low μ.*

Figure 3 illustrates the return on investment for given parameter values. It shows that a collective brand dominates an individual brand for a wide range of priors μ.

The same intuition applies for the case of quality control. In this case, a collective brand can alleviate the moral hazard problem of an individual brand the best if priors are low and the firms discount time a lot in one period.

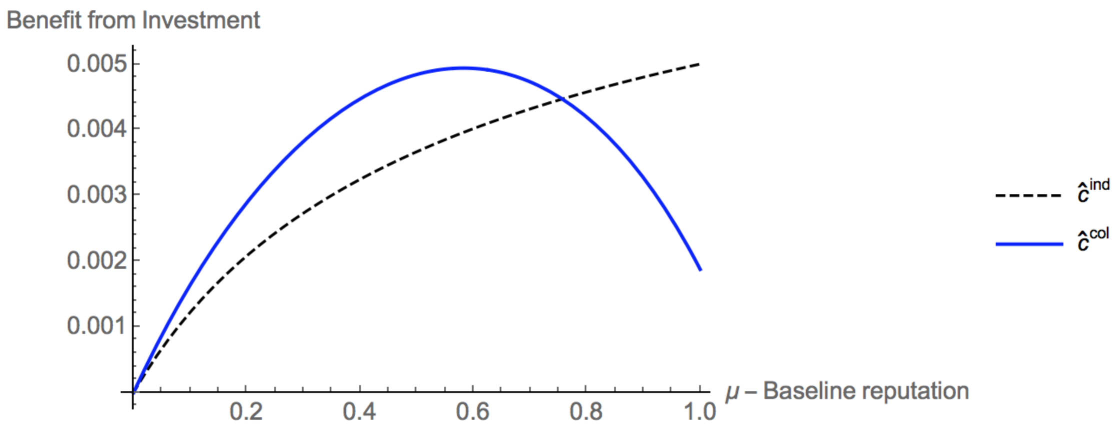


Figure 4: Comparison of Returns on Investment with $\pi_H = 1$
$\pi_L = 0.5,\ \delta = 0.2$

Proposition 2 (Quality control). *Suppose that π_H is sufficiently close to 1. Then, the following holds:*

1. *A collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$) if consumers' prior belief μ about the firm's type is sufficiently low and δ is not too large.*

2. *An individual brand sustains a reputational equilibrium for higher investment costs than a collective brand ($\hat{\mathbf{c}}^{col} < \hat{\mathbf{c}}^{ind}$) for sufficiently high μ.*

Figure 4 depicts the return on investment for collective and individual brands in the case for given parameter values.

6 Brand Formation

Until now, we treated the brand structure as exogenously given which is a realistic assumption in some applications: A country might legally require labeling of the country of origin and the appellation label is determined by the physical location of the production site. We have shown that collective brands can be a socially superior form of reputation building as they can serve as a commitment device for investment. However, the wedge between the payoffs of a firm and social welfare implies that the firm may sometimes prefer to remain alone and establish an individual reputation, in particular if the other firm is incompetent. In this section we identify conditions under which competent firms prefer to group with another (competent or incompetent) firm to staying alone.

To this end, we restrict attention to the parameter regions we identified in Propositions 1 and 2, so that a collective brand indeed induces more investment incentives than an individual brand, i.e., $\hat{\mathbf{c}}^{\text{ind}} < c < \hat{\mathbf{c}}^{\text{col}}$.

6.1 Stationary Equilibria for Individual Brands

We first investigate which stationary equilibria exist for an individual brand when investment costs c are so high that the reputational equilibrium does not exist. Recall that the set of relevant histories is given by $\mathcal{H}^{\text{ind}} = \{G, \varnothing, B\}^2$. A stationary equilibrium strategy specifies a mapping from the set of relevant histories into a decision of whether to invest or not. It can be characterized by a subset $\mathcal{S} \subset \mathcal{H}^{\text{ind}}$ of histories after which a competent firm invests.

As we noted in Lemma 1, the return from investment only depends on the outcome of the previous period, i.e., there can be at most $2^3 = 8$ candidates for stationary equilibria. The *no investment equilibrium*, where competent firms never invest, always exists. The reputational equilibrium is represented by the set $\mathcal{S} = \mathcal{H}^{\text{ind}}$ and the no investment equilibrium by $\mathcal{S} = \varnothing$.

Six other candidate stationary equilibria remain to be considered: $\mathcal{S} = \{G, \varnothing\}, \{G\}, \{\varnothing, B\}$, $\{B\}, \{G, B\}$, and $\{\varnothing\}$. In the next lemma, we identify which equilibria exist under exclusive technology and quality control. In the former, we show that the no investment equilibrium is the unique equilibrium, while in the latter, we show that the additional equilibrium $\mathcal{S} = \{G, \varnothing\}$ always exists and the equilibrium $\mathcal{S} = \{G\}$ sometimes exist.

Lemma 5. *Suppose that the reputational equilibrium exists for a collective brand but not for an individual brand, that is* $\hat{\mathbf{c}}^{ind} < c < \hat{\mathbf{c}}^{col}$.

1) Exclusive knowledge: If π_L *is close to* 0 *and* μ *is close to* 1*, then the only equilibrium that exists for an individual brand is the "no investment" equilibrium where a competent firm* *never invests.*

2) Quality control: If π_H *is close to* 1 *and* μ *is close to* 0*, then*

i) $\mathcal{S} = \{G, \varnothing\}$ *exists for* $\hat{\mathbf{c}}^{ind} < c < \hat{\mathbf{c}}^{col}$*.*

ii) $\mathcal{S} = \{G\}$ *exists if and only if* $\delta > \frac{2\pi_L^2}{1+2\pi_L^2}$*.*

Note that in that case, $\hat{\mathbf{c}}^{ind} < c < \hat{\mathbf{c}}^{col}$ *if and only if* $\delta < \frac{2\pi_L}{3+\pi_L}$*.*

Why do the two equilibria $\mathcal{S} = \{G, \varnothing\}$ and $\mathcal{S} = \{G\}$ do not exist under exclusive knowledge? For both equilibria, the firm's optimal decision following a bad outcome is to not invest. Knowing this, consumers pay a low price π_L to the firm who produced a bad quality in the previous period. At the same time, a firm that just produced a good outcome is maximally rewarded with a price π_H because it reveals the firm's competence. Given this low profit followed by a bad outcome, the firm is tempted to exert effort and produce a good outcome, in expectation of higher profits in the future. Since deviation is an attractive option, sustaining this equilibrium requires cost of investment c that is larger than $\hat{\mathbf{c}}^{ind}$, i.e., $c \notin (\hat{\mathbf{c}}^{ind}, \hat{\mathbf{c}}^{ind})$. Therefore, $\mathcal{S} = \{G, \varnothing\}$ and $\{G\}$ do not exist.

In the case of quality control, the equilibrium $\mathcal{S} = \{G, \varnothing\}$ always exists and $\mathcal{S} = \{G\}$ may exist for sufficiently large δ. This is because a deviation following a bad outcome is less attractive, supporting the equilibrium for $c \in (\hat{\mathbf{c}}^{ind}, \hat{\mathbf{c}}^{ind})$. In particular, a good outcome is not revealing so that the reward from an investment is low relative to the cost.

6.2 Profits and Endogenous Brand Formation

Next, we compare the firm's best feasible expected profit in each equilibrium, which is determined by the investment strategy and stationary distribution over realized histories. In any stationary equilibrium, the brand's history evolves according to the equilibrium investment strategy and the signal realizations. For example, consider the stationary equilibrium characterized by $\mathcal{S} = \{G, \varnothing\}$. This equilibrium induces a transition matrix

$$\begin{pmatrix} \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

where the first row describes the probabilities of outcomes $G, \varnothing$, and B, respectively, when the most recent history is G, the second row describes the same respective probabilities if the most recent history is $\varnothing$, and the third row describes these same probabilities if the most recent history is B. We denote the probability of outcome G in the stationary equilibrium $\mathcal{S} = \{G, \varnothing\}$ by $\Pr_C^{\mathcal{S}}(G)$. Since the outcome $\varnothing$ is realized in every period with probability $1/2$, the probability of B is $\Pr_C^{\mathcal{S}}(B) = \frac{1}{2} - \Pr_C^{\mathcal{S}}(G)$. Stationarity implies that $\Pr_C^{\mathcal{S}}(G)$ satisfies the following equation:

$$\Pr_C^{\mathcal{S}}(G) = \Pr_C^{\mathcal{S}}(G) \cdot \frac{\pi_H}{2} + \left(\frac{1}{2} - \Pr_C^{\mathcal{S}}(G)\right) \cdot \frac{\pi_L}{2} + \frac{1}{2} \cdot \frac{\pi_H}{2} \tag{6}$$

which yields $\Pr_C^{\mathcal{S}}(G) = \frac{\pi_L + \pi_H}{2(2+\pi_L+\pi_H)}$. Finally, the stationary probability that a competent firm produces a history GG is $\Pr_C^{\mathcal{S}}(GG) = \frac{\pi_L+\pi_H}{2(2+\pi_L+\pi_H)} \cdot \frac{\pi_H}{2}$.

An incompetent firm produces the same history GG with a probability $\Pr_I^{\mathcal{S}}(GG) = (\frac{\pi_L}{2})^2$. Given these stationary probabilities for different types of firms, consumers' posterior belief upon observing a history $\mathbf{h} = GG$ is obtained using Bayes' rule, and is denoted by $\hat{\mu}^{\mathcal{S}}(\mathbf{h})$. It follows that the firm's profit conditional on the history GG is

$$\hat{\mu}^{\mathcal{S}}(GG) \cdot \pi_H + (1 - \hat{\mu}^{\mathcal{S}}(GG)) \cdot \pi_L - c,$$

because the equilibrium requires the firm to invest after $\mathbf{h} = GG$. The *average expected per-period profit* of the firm is then a sum of profits over all possible histories, each weighted according to its stationary probability. For an arbitrary equilibrium $\mathcal{S}$, the expected profit is:

$$
\begin{aligned}
\Pi^{\mathcal{S}} \;=\; & \underbrace{\sum_{h_1 \in \mathcal{S},\, h_2 \in \mathcal{H}} \Pr_C^{\mathcal{S}}(h_2 h_1) \cdot \left(\hat{\mu}^{\mathcal{S}}(h_2 h_1) \cdot \pi_H + (1 - \hat{\mu}^{\mathcal{S}}(h_2 h_1)) \cdot \pi_L - c \right)}_{\text{invest}} \\
& + \underbrace{\sum_{h_1 \notin \mathcal{S},\, h_2 \in \mathcal{H}} \Pr_C^{\mathcal{S}}(h_2 h_1) \cdot \pi_L}_{\text{not invest}}.
\end{aligned}
$$

The first line of this equation is the expected payoff from investment, where the firm is paid $\hat{\mu}^{\mathcal{S}} \cdot \pi_H + (1 - \hat{\mu}^{\mathcal{S}}) \cdot \pi_L$, and incurs the investment cost c. The second line describes the firm's profit when it does not invest and is paid π_L.

The average profit for a collective brand under the reputational equilibrium is computed similarly. The price that the consumer pays after a history $h_2 h_1$, $p(h_2 h_1)$, is described in (4). The stationary distribution over histories depends on the type of the collective brand $\omega \in \{CC, CI, II\}$. The stationary probability $\Pr_\omega^{\mathcal{S}}(h_2 h_1)$ of history $h_2 h_1$ for a brand ω can be calculated using the corresponding transition probabilities. For example, for a history $\mathbf{h} = GG$,

$$
\Pr_{CC}(GG) = \pi_H^2, \quad \Pr_{CI}(GG) = \left(\tfrac{\pi_H + \pi_L}{2}\right)^2, \quad \Pr_{II}(GG) = \pi_L^2
$$

Finally, the expected average profit of a collective brand of type ω is

$$
\Pi_\omega^{\text{col}} = \sum_{\mathbf{h}} \Pr_\omega(\mathbf{h}) \cdot p(\mathbf{h}) - c.
$$

Proposition 3. *Suppose that the reputational equilibrium exists for a collective brand but not for an individual brand, that is* $\hat{\mathbf{c}}^{ind} < c < \hat{\mathbf{c}}^{col}$.

1) Exclusive knowledge: If π_L *is close to* 0 *and consumers' prior belief* μ *is close to* 1, *then a competent firm prefers forming a collective brand with another firm to establishing an individual brand, regardless of the type the other firm.*

2) Quality control: If π_H is close to 1 and μ is close to 0, a competent firm always weakly prefers to not brand.

Proposition 3 shows that for industries that require exclusive knowledge and have high baseline reputation ($\mu \approx 1$), the commitment value of a collective brand induces competent firms to want to brand with another firm regardless of its competency whenever it is socially optimal to do so (i.e., $\hat{\mathbf{c}}^{\text{ind}} < c < \hat{\mathbf{c}}^{\text{col}}$). The reason is that the social planner's incentives are almost perfectly internalized when $\mu \approx 1$.

In the case of quality control, if μ is close to 0, a competent firm is better off as an individual brand in the long run. Though a collective brand provides commitment power for investment, with μ very small, the cost of an investment is not worth the reward. The intuition is similar to the one in lemons markets. In a reputational equilibrium, consumers are willing to pay $\tilde{\mu} \cdot \pi_H + (1 - \tilde{\mu}) \cdot \pi_L$ if their posterior is $\tilde{\mu}$, and the firm incurs the cost c. If the firm is expected to not invest in equilibrium, consumers pay π_L. Then, roughly speaking, the firm favors to be in the reputational equilibrium if $\tilde{\mu} \cdot (\pi_H - \pi_L) > c$ "on average". This condition is violated for a small $\tilde{\mu}$, in which case the firm is better off not investing.[13] Thus, with endogenous brand formation, there is no investment in quality in the market, even though investment is socially optimal.

Note that as long as the $\{G, \varnothing\}$ equilibrium exists, profits of a competent firm are always higher for an individual brand than for a collective brand playing the reputational equilibrium.[14] The reason is that in the $\{G, \varnothing\}$ equilibrium, the state in which the competent firm does not invest, i.e., a B outcome, occurs with zero probability if $\pi_H \approx 1$. Thus, for π_H close to 1 (but not equal to 1), the collective brand yields strictly higher welfare than the individual brand, while the firm prefers to be in an individual brand. Thus, the social planner can improve total welfare by imposing collective brands such as country-of-origin labeling.

[13] In the literature on adverse selection, this condition is frequently called the lemons condition.

[14] We have proven in Lemma 5 that the equilibrium exists for small μ, but this statement is ture independently of μ.

7 T-period Memory

In this section, we show that our results are robust to longer T- period memory of consumers. In fact, our results become stronger in the following sense: the range of discount factors δ for which collective brands provide commitment value (i.e., $\hat{\mathbf{c}}^{\text{ind}} < \hat{\mathbf{c}}^{\text{col}}$) is non-empty for all T and in the limit, as T tends to infinity, it is larger than for 2-period memory. In particular, in the case of exclusive knowledge, the set increases monotonically in T and converges to the unit interval.

In general, with a longer memory, each individual investment becomes less important. Thus, the benefit of investment decreases in T both for individual and collective brands. However, the benefit of investment in individual brands is more adversely affected. The intuition is identical to that for the 2-period memory. With a longer memory, an individual brand can reach more extreme reputations following a sequence of good or bad outcomes.

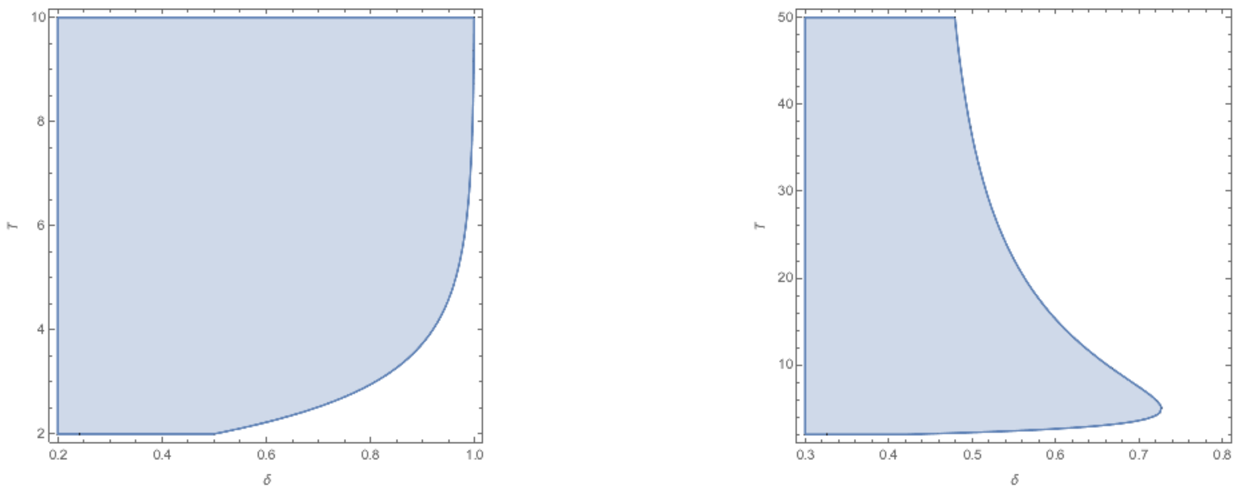
Here, we only present the main results and briefly compare them to the 2-period memory model described in Section 5. The detailed analysis of this case we have deferred in Appendix B. The following proposition provides a generalization of Propositions 1 and 2.

Proposition 4. *(i) Exclusive knowledge: If π_L is sufficiently close to 0 and μ is sufficiently close to 1, then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\hat{\mathbf{c}}^{ind} < \hat{\mathbf{c}}^{col}$) if either $\delta < \frac{1}{2}$, or ($\delta > \frac{1}{2}$ and $(2\delta)^T > \frac{\delta}{1-\delta}$).*
(ii) Quality control: If π_H is sufficiently close to 1 and μ is sufficiently close to 0, then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand if

$$\frac{1}{2} \cdot \frac{1 - \left(\frac{\delta(1+3\pi_L)}{2\pi_L}\right)^T}{1 - \frac{\delta(1+3\pi_L)}{2\pi_L}} > \left(\frac{\delta(1+\pi_L)}{\pi_L}\right)^{T-1}.$$

The following corollary follows immediately from this proposition.

Corollary 1. *(i) Exclusive knowledge: If π_L is sufficiently close to 0 and μ is sufficiently close to 1, then the set of δ for which $\hat{\mathbf{c}}^{ind}\hat{\mathbf{c}}^{col}$ increases monotonically in T and converges to $[0, 1]$.*



(a) $\pi_L = 0$ and $\mu = 1$

(b) $\pi_H = 1$, $\mu = 0$ and $\pi_L = 4/5$

Figure 5: Region in the T-δ space where $\hat{\mathbf{c}}^{\text{ind}} > \hat{\mathbf{c}}^{\text{col}}$.

(ii) Quality control: If π_H is sufficiently close to 1 and μ is sufficiently close to 0, then there is a $\bar{T} > 2$ such that for all $T > \bar{T}$, the set of δ that satisfies $\hat{\mathbf{c}}^{ind} > \hat{\mathbf{c}}^{col}$ decreases and converges to $(0, \frac{\pi_L}{1+\pi_L})$. for any $\delta \in (0, \frac{\pi_L}{1+\pi_L}]$, there is a large enough $\tilde{T}$ such that for all $T > \tilde{T}$, $\hat{\mathbf{c}}^{ind} > \hat{\mathbf{c}}^{col}$. Otherwise, if $\delta \in (\frac{\pi_L}{1+\pi_L}, 1)$, there is a large enough $\hat{T}$ such that for all $T > \hat{T}$, $\hat{\mathbf{c}}^{ind} > \hat{\mathbf{c}}^{col}$.

Figure 5 exhibits the range of parameters for which a collective brand induces stronger incentives to invest than an individual brand. For the case of exclusive knowledge, a larger δ requires a correspondingly larger memory T for collective brand to outperform an individual brand, but for the case of quality control, longer memory requires δ not to be too large. In particular, as T tends to infinity, it must be that $\delta \in (0, \pi_L/(1 + \pi_L)))$.

8 Conclusion

While we need to make some technical simplifications and focus on limiting cases, we believe that we can highlight novel benefits of collective reputation building and the mechanisms behind it. To conclud, we summarize the economic implications and interpretation of the results in the context of country of origin labelling. Recall that the prior belief μ can be interpreted as the *base reputation* of firms in a country.

Propositions 1 and 2 imply that in countries with strong base reputation (high μ) country of origin labelling contributes to social welfare by improving firms' ability to commit to invest in quality in industries with exclusive knowledge such as French wine, Swiss watches, German automobiles, Japanese electronics, US software, etc. In contrast, producers of generic products such as screws, basic clothes, etc., in such countries should advertise their own brand only. The exact opposite conclusion applies in countries with a weak base reputation (low μ). In such countries, social welfare is maximized when manufacturers of generic goods label their country of origin while manufacturers of specialized goods refrain from it.

These theoretical results are consistent with anecdotal evidence. For example the collective brand "Made in China" is advertised by subsuppliers on platforms such as 'Made-in-China.com',' while successful high-tech companies such as Huawei try to build their own brand names. On the other hand, German sub-suppliers of generics such as ThyssenKrupp count on their own brand reputation. This distinction can also explain the choice of the label "Made in Germany" versus "Made in Europe," which is supposedly indicative of a lower base reputation (lower μ).

As discussed in Section 6, the implementation of the optimal branding strategy might require some government regulation if the base reputation of firms is low. Indeed, the regulation of the labeling of country of origin is an important issue in many countries. The standard argument is that firms should be required to label their product with certain information for consumer protection. The insights developed here suggest that the type of labeling, in particular the inclusion of country of origin, may affect the incentives of firms to invest in quality positively.

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A Appendix: Proofs

A.1 Proofs of Section 5

Proof. [Proof of Lemma 1] The posterior beliefs $\hat{\mu}^{\text{ind}}$ about the quality of the product after observing history $\mathbf{h}_t = h_{t-2}h_{t-1}$ are given by

$$\hat{\mu}^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}, \qquad \hat{\mu}^{\text{ind}}(GB) = \hat{\mu}^{\text{ind}}(BG) = \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)},$$

$$\hat{\mu}^{\text{ind}}(BB) = \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}, \qquad \hat{\mu}^{\text{ind}}(G\varnothing) = \hat{\mu}(\varnothing G) = \frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L}$$

$$\hat{\mu}^{\text{ind}}(\varnothing\varnothing) = \mu, \qquad \hat{\mu}^{\text{ind}}(B\varnothing) = \hat{\mu}^{\text{ind}}(\varnothing B) = \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}.$$

The reputational equilibrium exists if and only if a competent firm invests whenever visited following all histories. Let us denote the value function when visited and not visited by $V^{\text{ind}}(\mathbf{h}_t)$ and $W^{\text{ind}}(\mathbf{h}_t)$, respectively. Also, let $V^{\text{ind}}(\mathbf{h}_t; \text{not})$ denote the payoff to a competent firm from a single deviation. Then, $V^{\text{ind}}(\mathbf{h}_t) \geq V^{\text{ind}}(\mathbf{h}_t; \text{not})$ for all $\mathbf{h}_t$, which is equivalent to:

$$c \leq \bar{c}^{\text{ind}}(h_{-1}) \equiv \frac{\delta(\pi_H - \pi_L)}{2} \cdot \underbrace{\left(V^{\text{ind}}(h_{-1}G) - V^{\text{ind}}(h_{-1}B) + W^{\text{ind}}(h_{-1}G) - W^{\text{ind}}(h_{-1}B)\right)}_{\equiv \bar{d}^{\text{ind}}(h_{-1})}.$$

Note that $\bar{d}^{\text{ind}}(h_{-1})$ can potentially depend on c. Using $W(h_{-2}h_{-1}) = \frac{\delta}{2}\left(V(h_{-1}\varnothing) + W(h_{-1}\varnothing)\right)$

we can calculate

$$\begin{aligned}V^{\text{ind}}(h_{-1}G) - V^{\text{ind}}(h_{-1}B) &= p^{\text{ind}}(h_{-1}G) - p^{\text{ind}}(h_{-1}B) \\ &+\frac{\delta}{2}\pi_H(\underbrace{V^{\text{ind}}(GG) - V^{\text{ind}}(BG)}_{=p^{\text{ind}}(GG)-p^{\text{ind}}(BG)} + \underbrace{W^{\text{ind}}(GG) - W^{\text{ind}}(BG)}_{=0}) \\ &+\frac{\delta}{2}(1-\pi_H)(\underbrace{V^{\text{ind}}(GB) - V^{\text{ind}}(BB)}_{=p^{\text{ind}}(GB)-p^{\text{ind}}(BB)} + \underbrace{W^{\text{ind}}(GB) - W^{\text{ind}}(BB)}_{=0}).\end{aligned}$$

Similarly, $W^{\text{ind}}(h_{-1}G) - W^{\text{ind}}(h_{-1}B) = \frac{\delta}{2}(p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))$. □

Proof. [Proof of Lemma 2] First, note that

$$\begin{aligned}p^{\text{ind}}(GG) - p^{\text{ind}}(GB) &= (\pi_H - \pi_L)\cdot\left(\frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2} - \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)}\right) \\ &= \frac{\mu(1-\mu)\pi_H\pi_L(\pi_H-\pi_L)^2}{\Pr(GG)\cdot\Pr(GB)},\end{aligned}$$

and

$$\begin{aligned}p^{\text{ind}}(GB) - p^{\text{ind}}(BB) &= (\pi_H - \pi_L)\cdot\left(\frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} - \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}\right) \\ &= \frac{\mu(1-\mu)(1-\pi_H)(1-\pi_L)(\pi_H-\pi_L)^2}{\Pr(GB)\cdot\Pr(BB)}.\end{aligned}$$

Finally,

$$\begin{aligned}p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing) &= (\pi_H - \pi_L)\cdot\left(\frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L} - \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}\right) \\ &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2}{\Pr(G)\cdot\Pr(B)} \geq \min\{p^{\text{ind}}(GG) - p^{\text{ind}}(GB),\ p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\}.\end{aligned}$$

Hence, the minimum is attained at $h_{-1} = G$ if and only if

$$\frac{\pi_H \pi_L}{\Pr(GG) \cdot \Pr(GB)} \leq \frac{(1-\pi_H)(1-\pi_L)}{\Pr(GB) \cdot \Pr(BB)}$$
$$\Leftrightarrow \Pr(BB) \cdot \pi_H \pi_L \leq \Pr(GG) \cdot (1-\pi_H)(1-\pi_L)$$
$$\Leftrightarrow \pi_H \pi_L (\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2) \leq (1-\pi_H)(1-\pi_L)(\mu \pi_H^2 + (1-\mu)\pi_L^2)$$
$$\Leftrightarrow \mu \pi_H (1-\pi_H) \geq (1-\mu)\pi_L(1-\pi_L)$$

This inequality holds if and only if $\mu \geq \bar{\mu} \equiv \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H)+\pi_L(1-\pi_L)}$. □

Proof. [Proof of Lemma 3] Let us denote the present discounted expected equilibrium profit of a competent firm when branding with a θ-type firm after history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$ by $V(\mathbf{h}_t, \theta)$ if it is visited and $W(\mathbf{h}_t, \theta)$ when it is not visited. Also, let the continuation payoff after no investment (assuming the firm follows the equilibrium strategy after the deviation) be $V(\mathbf{h}_t, \theta; \text{not})$. Then, a reputational equilibrium exists if and only if $V(\mathbf{h}_t, \theta) \geq V(\mathbf{h}_t, \theta; \text{not})$ for all $\mathbf{h}_t, \theta$. A competent firm invests after a history $\mathbf{h}_t$ if and only if

$$c \leq \bar{c}^{\text{col}}(h_{t-1}) \equiv \delta \cdot \frac{\pi_H - \pi_L}{2} \cdot \underbrace{(V(h_{t-1}G, \theta) - V(h_{t-1}B, \theta) + W(h_{t-1}G, \theta) - W(h_{t-1}B, \theta))}_{\equiv \bar{d}^{\text{col}}(h_{t-1}, \theta)}.$$

First, note that for all $q_1, q_2, x \in \{G, B\}$, we have that $V(q_1 x, \theta) - V(q_2 x, \theta) = p^{\text{col}}(q_1 x) - p^{\text{col}}(q_2 x)$ and $W(q_1 x, \theta) - W(q_2 x, \theta) = 0$. Using this, we can calculate

$$\begin{aligned}
V(h_{-1}G, \theta) - V(h_{-1}B, \theta) &= p^{\text{col}}(h_{-1}G) - p^{\text{col}}(h_{-1}B) \\
&+ \frac{\delta \pi_H}{2}(V(GG, \theta) - V(BG, \theta)) + \frac{\delta(1-\pi_H)}{2}(V(GB, \theta) - V(BB, \theta)) \\
&+ \frac{\delta \pi_H}{2}(W(GG, \theta) - W(BG, \theta)) + \frac{\delta(1-\pi_H)}{2}(W(GB, \theta) - V(BB, \theta)) \\
&= p^{\text{col}}(h_{-1}G) - p^{\text{col}}(h_{-1}B) + \frac{\delta \pi_H}{2}(p^{\text{col}}(GG) - p^{\text{col}}(BG)) + \\
&\quad \frac{\delta(1-\pi_H)}{2}(p^{\text{col}}(GB) - p^{\text{col}}(BB))
\end{aligned}$$

Likewise,

$$\begin{aligned}W(h_{-1}G,\theta)-W(h_{-1}B,\theta)=&\frac{\delta\pi(\theta)}{2}(V(GG,\theta)-V(BG,\theta))+\frac{\delta(1-\pi(\theta))}{2}(V(GB,\theta)-V(BB,\theta))\\&+\frac{\delta\pi(\theta)}{2}(W(GG,\theta)-W(BG,\theta))+\frac{\delta(1-\pi(\theta))}{2}(W(GB,\theta)-W(BB,\theta))\\=&\frac{\delta\pi(\theta)}{2}(p^{\text{col}}(GG)-p^{\text{col}}(BG))+\frac{\delta(1-\pi(\theta))}{2}(p^{\text{col}}(GB)-p^{\text{col}}(BB))\end{aligned}$$

where $\pi(\theta)=\pi_L$ if $\theta=I$ and π_H if $\theta=C$. □

Proof. [Proof of Lemma 4] As noted in Section 3, upon observing a history $\mathbf{h}_t\in\mathcal{H}^{\text{col}}$, a consumer places a probability $\eta_s(\mathbf{h}_t)$ on the group's type $s\in\{CC,CI,IC,II\}$. These beliefs are given by:

$$\begin{aligned}\eta_{CC}(GG)&=\frac{\mu^2\pi_H^2}{\mu^2\pi_H^2+2\mu(1-\mu)\left(\frac14\pi_H^2+\frac12\pi_H\pi_L+\frac14\pi_L^2\right)+(1-\mu)^2\pi_L^2},\\ \eta_{CI}(GG)&=\eta_{IC}(GG)\\&=\frac{\mu(1-\mu)\left(\frac14\pi_H^2+\frac12\pi_H\pi_L+\frac14\pi_L^2\right)}{\mu^2\pi_H^2+2\mu(1-\mu)\left(\frac14\pi_H^2+\frac12\pi_H\pi_L+\frac14\pi_L^2\right)+(1-\mu)^2\pi_L^2},\\ \eta_{II}(GG)&=1-\eta_{CC}(GG)-2\eta_{CI}(GG),\\ \eta_{CC}(GB)&=\frac{\mu^2\pi_H(1-\pi_H)}{\mu^2\pi_H(1-\pi_H)+2\mu(1-\mu)\frac14\left(\pi_H(1-\pi_H)+\pi_H(1-\pi_L)+\pi_L(1-\pi_H)+\pi_L(1-\pi_L)\right)+(1-\mu)^2\pi_L(1-\pi_L)},\\ \eta_{CI}(GB)&=\eta_{IC}(GB)\\&=\frac{\mu(1-\mu)\frac14\left(\pi_H(1-\pi_H)+\pi_H(1-\pi_L)+\pi_L(1-\pi_H)+\pi_L(1-\pi_L)\right)}{\mu^2\pi_H(1-\pi_H)+2\mu(1-\mu)\frac14\left(\pi_H(1-\pi_H)+\pi_H(1-\pi_L)+\pi_L(1-\pi_H)+\pi_L(1-\pi_L)\right)+(1-\mu)^2\pi_L(1-\pi_L)},\\ \eta_{II}(GB)&=1-\eta_{CC}(GB)-2\eta_{CI}(GB),\\ \eta_{CC}(BB)&=\frac{\mu^2(1-\pi_H)^2}{\mu^2(1-\pi_H)^2+2\mu(1-\mu)\left(\frac14(1-\pi_H)^2+\frac12(1-\pi_H)(1-\pi_L)+\frac14(1-\pi_L)^2\right)+(1-\mu)^2(1-\pi_L)^2},\\ \eta_{CI}(BB)&=\eta_{IC}(BB)\\&=\frac{\mu(1-\mu)\left(\frac14(1-\pi_H)^2+\frac12(1-\pi_H)(1-\pi_L)+\frac14(1-\pi_L)^2\right)}{\mu^2(1-\pi_H)^2+2\mu(1-\mu)\left(\frac14(1-\pi_H)^2+\frac12(1-\pi_H)(1-\pi_L)+\frac14(1-\pi_L)^2\right)+(1-\mu)^2(1-\pi_L)^2},\\ \eta_{II}(BB)&=1-\eta_{CC}(BB)-2\eta_{CI}(BB).\end{aligned}$$

Then, the consumer's posterior belief is about the firm being competent is given by $\Pr(\mathbf{h}_t)=\eta_{CC}(\mathbf{h}_t)+\frac12(\eta_{CI}(\mathbf{h}_t)+\eta_{IC}(\mathbf{h}_t))$ and thus, the price differentials are given by:

$$\begin{aligned}p^{\text{col}}(GG)-p^{\text{col}}(GB)&=(\pi_H-\pi_L)\cdot\left(\frac{\mu^2\pi_H^2+\frac14\mu(1-\mu)(\pi_H+\pi_L)^2}{\Pr(GG)}-\frac{\mu^2\pi_H(1-\pi_H)+\frac14\mu(1-\mu)(\pi_H+\pi_L)(2-\pi_H-\pi_L)}{\Pr(GB)}\right)\\&=\frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2+2\mu(\pi_H-\pi_L)\pi_L+\pi_L(\pi_H+\pi_L)\right)}{4\cdot\Pr(GG)\cdot\Pr(GB)}\\&\to_{\pi_L\to0}\frac{(1-\mu)\mu\pi_H}{(1+\mu)(1-\pi_H+1-\mu\pi_H)}\\ p^{\text{col}}(GB)-p^{\text{col}}(BB)&=\frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2-2\mu(\pi_H-\pi_L)(1-\pi_L)+(1-\pi_L)(2-\pi_H-\pi_L)\right)}{4\cdot\Pr(GB)\cdot\Pr(BB)}\\&\to_{\pi_L\to0}\frac{(1-\mu)\pi_H\left((1-\mu\pi_H)^2+1-\pi_H\right)}{((1-\mu\pi_H)^2+\mu(1-\pi_H)^2+1-\mu)\,(1-\pi_H+1-\mu\pi_H)}.\end{aligned}$$

Thus, $p^{\mathrm{col}}(GG) - p^{\mathrm{col}}(GB) \leq p^{\mathrm{col}}(GB) - p^{\mathrm{col}}(BB)$ if and only if

$$\frac{\mu^2(\pi_H-\pi_L)^2+2\mu(\pi_H-\pi_L)\pi_L+\pi_L(\pi_H+\pi_L)}{\Pr(GG)} \leq \frac{\mu^2(\pi_H-\pi_L)^2-2\mu(\pi_H-\pi_L)(1-\pi_L)+(1-\pi_L)(2-\pi_H-\pi_L)}{\Pr(BB)}$$
$$= \frac{\mu^2(\pi_H-\pi_L)^2+2\mu(\pi_H-\pi_L)\pi_L+\pi_L(\pi_H+\pi_L)-(2\mu-1)(\pi_H-\pi_L)+2(1-\pi_H-\pi_L)}{\Pr(GG)-\mu(\pi_H-\pi_L)\pi_L+\frac{1}{2}}$$

This condition can be re-written as $\frac{A}{C} \leq \frac{A+B}{C+D}$, where $A = \mu^2(\pi_H-\pi_L)^2+2\mu(\pi_H-\pi_L)\pi_L+\pi_L(\pi_H+\pi_L)$, $B = -(2\mu-1)(\pi_H-\pi_L)+2(1-\pi_H-\pi_L)$, $C = \Pr(GG)$, and $D = -\mu(\pi_H-\pi_L)\pi_L+\frac{1}{2}$. This holds if and only if $AD \leq BC$ which can be rewritten as

$$(\pi_H-\pi_L)\big(\underbrace{\mu^2(2\mu-3)(\pi_H-\pi_L)^2+\mu(2-\pi_H-\pi_L)(\pi_H+\pi_L)-2(1-\pi_L)\pi_L}_{=f(\mu,\pi_H,\pi_L)}\big) \geq 0.$$

Note that if $\mu = 1$, the LHS is equivalent to $2(1-\pi_H)\pi_H \geq 0$ and if $\mu = 0$, it is equivalent to $-2(1-\pi_L)\pi_L \leq 0$. As f is a continuous function, by the Intermediate Value Theorem, it vanishes at least once for some value of μ between 0 and 1. The question is whether it is zero more than once. To this end, note that f is increasing in μ if and only if

$$\frac{\partial f}{\partial \mu} = \mu(\mu-1) + \frac{(\pi_H+\pi_L)(2-\pi_H-\pi_L)}{6(\pi_H-\pi_L)^2} > 0 \Leftrightarrow (1-\mu)\mu < \frac{(\pi_H+\pi_L)(2-\pi_H-\pi_L)}{6(\pi_H-\pi_L)^2}$$

First, if $\pi_H - \pi_L$ is small, RHS is large and the condition holds always, so f crosses 0 at a single point. If $\pi_H - \pi_L$ is sufficiently large, then the condition holds for small and large values of μ. Then, though $f(0, \pi_H, \pi_L) < 0$, it increases in μ for μ close to 0, at which point f may cross 0 for the first time. Then, for intermediate values of μ, f decreases and may cross 0 one more time. Then, lastly f increases for large values of μ and cross 0 again.

Although we do not fully identify necessary and sufficient conditions for $f \geq 0$, under symmetric signals with $\pi_L = 1 - \pi_H < \frac{1}{2}$, $f(\mu, \pi_H, 1-\pi_H) \geq 0$ if and only if $\frac{1}{2} < \pi_H \leq \frac{3+\sqrt{6}}{6}$, $\frac{1}{2} \leq \mu \leq 1$ or $\frac{3+\sqrt{6}}{6} < \pi_H < 1$ and $\mu \in [\frac{1}{2} - \frac{\sqrt{1-12\pi_H+12\pi_H^2}}{2}, \frac{1}{2}] \cup [\frac{1}{2} + \frac{\sqrt{1-12\pi_H+12\pi_H^2}}{2}, 1]$. This coincides with the patter described above. □

Proof. [Proof of Proposition 1] From Lemma 2 it follows that $\bar{c}^{\mathrm{ind}}(G)$ determines the cutoff cost for all parameters if $\pi_L = 0$ and $0 < \pi_H < 1$. Also, though not straightforward from Lemma 4, it implies that $\hat{c}^{\text{col}} = \bar{c}^{\text{col}}(G;C)$ for $\pi_L = 0$ for high and low values of μ. This is because $p^{\text{col}}(GG) - p^{\text{col}}(GB) \leq p^{\text{col}}(GB) - p^{\text{col}}(BB)$ if and only if either $\mu < \frac{1}{2}$, or $\mu \geq \frac{1}{2}$ and $\pi_H < \frac{2}{1+3\mu-2\mu^2}$. So, we compare $\hat{c}^{\text{ind}}(G) = \delta \cdot \frac{\pi_H - \pi_L}{2} \cdot \bar{c}^{\text{ind}}(G)$ and $\hat{c}^{\text{col}}(G;C) = \delta \cdot \frac{\pi_H - \pi_L}{2} \cdot \bar{c}^{\text{col}}(G;C)$.

First, for an individual brand,

$$\begin{aligned}
\lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G) &= \lim_{\pi_L \to 0} \left(1 + \frac{\delta \pi_H}{2}\right) \big(\underbrace{p(GG) - p(GB)}_{\to 0}\big) \\
&\quad + \frac{\delta(1-\pi_H)}{2}\big(p(GB) - p(BB)\big) + \frac{\delta}{2}(p(G\varnothing) - p(B\varnothing)) \\
&= \delta \pi_H (1-\mu) \left(\frac{1-\pi_H}{2} \cdot \frac{1}{1 - \mu \pi_H (2 - \pi_H)} + \frac{1}{2} \cdot \frac{1}{1 - \mu \pi_H} \right) \\
&:= \pi_H (1-\mu) \cdot X^{\text{ind}}(\mu, \pi_H)
\end{aligned}$$

where $X^{\text{ind}}(\mu, \pi_H) \underset{\mu \to 1}{\to} \frac{\delta}{1-\pi_H}$. For a collective brand,

$$\begin{aligned}
\lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G;C) &= \lim_{\pi_L \to 0} (1 + \delta \pi_H)(p^{\text{col}}(GG) - p^{\text{col}}(GB)) + \delta(1-\pi_H)(p^{\text{col}}(GB) - p^{\text{col}}(BB)) \\
&= (1 + \delta \pi_H) \cdot \frac{(1-\mu)\mu \pi_H}{(1+\mu)(2 - (1+\mu)\pi_H)} \\
&\quad + \delta \cdot (1 - \pi_H) \cdot \frac{(1-\mu)\pi_H \left(2 - \pi_H(1 + \mu(2 - \mu\pi_H))\right)}{\left((1-\mu\pi_H)^2 + \mu(1-\pi_H)^2 + 1 - \mu\right)\left(2 - (1+\mu)\pi_H\right)} \\
&:= (1-\mu)\pi_H \cdot X^{\text{col}}(\mu, \pi_H).
\end{aligned}$$

As μ approaches 1, we get

$$\begin{aligned}
\lim_{\mu \to 1} X^{\text{col}}(\mu, \pi_H) &= (1 + \delta \pi_H) \frac{\pi_H}{2(2 - 2\pi_H)} + \delta(1 - \pi_H) \frac{(1-\pi_H)(2-\pi_H)}{4(1-\pi_H)^3} \\
&= \frac{1 + \delta \pi_H}{4(1-\pi_H)} + \frac{\delta(2-\pi_H)}{4(1-\pi_H)} \\
&= \frac{1 + 2\delta}{4(1-\pi_H)}.
\end{aligned}$$

So, $\lim_{\mu \to 1} X^{\text{col}} \geq \lim_{\mu \to 1} X^{\text{ind}}$ if and only if $\frac{1+2\delta}{4(1-\pi_H)} \geq \frac{\delta}{1-\pi_H}$, i.e., if and only if $\delta < \frac{1}{2}$. for sufficiently small μ collective brands are better whenever $\delta \leq \frac{1}{2}$. As $\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\bar{c}^{\text{col}}}{\pi_H(1-\mu)} \geq \lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\bar{c}^{\text{ind}}}{\pi_H(1-\mu)}$, we have proven that for μ close to 1, we can find π_L close to 0 such that $\hat{\mathbf{c}}^{\text{col}} \geq \hat{\mathbf{c}}^{\text{ind}}$.

If μ is close to 0,

$$\begin{aligned}
\lim_{\mu \to 0} X^{\text{ind}} &= \frac{\delta(2-\pi_H)}{2} \\
\lim_{\mu \to 0} X^{\text{col}} &= \frac{\delta(2-\pi_H)(1-\pi_H)}{2(2-\pi_H)}
\end{aligned}$$

Clearly, $\delta \cdot \frac{2-\pi_H}{2} > \delta \cdot \frac{1-\pi_H}{2}$ for all $\pi_H \in (0,1)$. So, for a μ close 0, there is a π_L close to 0 such that $\hat{\mathbf{c}}^{\text{ind}} \geq \hat{\mathbf{c}}^{\text{col}}$. $\square$

Proof. [Proof of Proposition 2] Here, set $\pi_H = 1$ and $0 < \pi_L < 1$. It follows from Lemma 2 that $\hat{\mathbf{c}}^{\text{ind}} = \overline{\mathbf{c}}^{\text{ind}}(B)$. Similarly, $\hat{\mathbf{c}}^{\text{col}} = \overline{c}^{\text{col}}(B; I)$ for high and low values of μ.

With an individual brand

$$\begin{aligned}
\lim_{\pi_H \to 1} \overline{c}^{\text{ind}}(B) &= \lim_{\pi_H \to 1} (p^{\text{ind}}(GB) - p^{\text{ind}}(BB)) + \frac{\delta}{2}(p^{\text{ind}}(GG) - p^{\text{ind}}(GB) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)) \\
&= \frac{\delta}{2}\left(\frac{\mu(1-\pi_L)}{\mu + (1-\mu)\pi_L^2} + \frac{\mu(1-\pi_L)}{\mu + (1-\mu)\pi_L}\right) \\
&:= \frac{\mu(1-\pi_L)}{2} \cdot Y^{\text{ind}}(\mu, \pi_L).
\end{aligned}$$

Under a collective brand,

$$\begin{aligned}
\lim_{\pi_H \to 1} \overline{c}^{\text{col}}(B; I) &= \lim_{\pi_H \to 1} p^{\text{col}}(BG) - p^{\text{col}}(BB) + \\
&\quad \frac{\delta}{2} \cdot ((1+\pi_L) \cdot (p^{\text{col}}(GG) - p^{\text{col}}(GB)) + (1-\pi_L)(p^{\text{col}}(GB) - p^{\text{col}}(BB))) \\
&= \frac{\mu(1-\pi_L)}{2} \cdot Y^{\text{col}}(\mu, \pi_L)
\end{aligned}$$

where $Y^{\text{col}}(\mu, \pi_L) = \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta + 2\pi_L + 3\delta\pi_L) + \mu(2+\delta+4(1+\delta)\pi_L - (10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L + \delta(-1+3\pi_L))}{(2-\mu)(\mu(1-\pi_L)+2\pi_L)(\mu(1+\mu)+2(1-\mu)\mu\pi_L+(2-\mu)(1-\mu)\pi_L^2)}$.
To make a comparison for μ close to 0, it is sufficient to compare Y^{ind} and Y^{col} in that region:

$$\begin{aligned}
\lim_{\mu \to 0} Y^{\text{ind}}(\mu, \pi_L) &= \frac{\delta(1+\pi_L)}{\pi_L^2} \\
\lim_{\mu \to 0} Y^{\text{col}}(\mu, \pi_L) &= \frac{2\pi_L + \delta(1+3\pi_L)}{4\pi_L^2}
\end{aligned}$$

So, $\lim_{\mu\to 0} Y^{\text{col}} > \lim_{\mu\to 0} Y^{\text{ind}}$ if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$. Thus, for μ close to 0 and π_H close to 1, $\hat{\mathbf{c}}^{\text{col}} \geq \hat{\mathbf{c}}^{\text{ind}}$, if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$.

If μ is close to 1,

$$\lim_{\mu\to 1} Y^{\text{ind}} = 2\delta > \lim_{\mu\to 1} Y^{\text{col}} \frac{1}{2}\delta(1+\pi_L)$$

So, $\lim_{\mu\to 1} \bar{c}^{\text{ind}}(B) > \lim_{\mu\to 1} \bar{c}(B; I)$. Therefore, for any μ close to 1, we can find π_H close to 1 such that $\hat{\mathbf{c}}^{\text{ind}} \geq \hat{\mathbf{c}}^{\text{col}}$ □

A.2 Proofs of Section 6

Proof. [Proof of Lemma 5] For all equilibria other than the reputational equilibrium and no investment equilibrium, the firm sometimes invest and other times not. This implies the cost of investment cannot be too small or too large and the necessary and sufficient condition for the existence of each of any equilibrium $\mathcal{S}$ is given by $\underline{C}^{\mathcal{S}}(\mu, \pi_H, \pi_L) < c < \overline{C}^{\mathcal{S}}(\mu, \pi_H, \pi_L)$ for some $\underline{C}^{\mathcal{S}}, \overline{C}^{\mathcal{S}}$. Repeating the steps taken to find the cutoff levels $\hat{\mathbf{c}}^{\text{col}}$ for collective and $\hat{\mathbf{c}}^{\text{ind}}$ for individual brands, we obtain the following expressions for cutoffs for each equilibrium.

Exclusive knowledge: First, consider the case of exclusive knowledge, i.e. $\pi_L = 0$ and μ close to 1. Note that $\lim_{\mu\to 1}\lim_{\pi_L\to 0} \hat{\mathbf{c}}^{\text{col}} = 0$ and $\lim_{\mu\to 1}\lim_{\pi_L\to 0} \hat{\mathbf{c}}^{\text{ind}} = 0$.

For any equilibrium $\mathcal{S} = \{G, \varnothing\}$ or $\{G\}$,

$$\begin{aligned}
\underline{C}^{\mathcal{S}} &= \frac{\delta\Delta\pi}{2}\Big(p(BG) - p(BB) \\
&\quad + \frac{\delta}{2}(\pi_H \cdot (p(GG) - p(BG)) + (1-\pi_H)\cdot(p(GB) - p(BB)) + p(G\varnothing) - p(B\varnothing))\Big) \quad (7)\\
\overline{C}^{\mathcal{S}} &= \frac{\delta\Delta\pi}{2}\Big(p(GG) - p(GB) \\
&\quad + \frac{\delta}{2}(\pi_L \cdot (p(GG) - p(BG)) + (1-\pi_L)\cdot(p(GB) - p(BB)) + p(G\varnothing) - p(B\varnothing))\Big) \quad (8)
\end{aligned}$$

Then,

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0} \underline{C}^{\mathcal{S}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0} \overline{C}^{\mathcal{S}} = \frac{\delta\pi_H^2}{2} > 0.$$

Therefore, there is no $c > 0$ such that $\hat{c}^{\text{ind}}(\mu, \pi_H, 0) < c < \hat{c}(\mu, \pi_H, 0)$ for μ close to 1 that satisfy $\underline{C}^{\mathcal{S}}(\mu, \pi_H, 0) < c < \overline{C}^{\mathcal{S}}(\mu, \pi_H, 0)$.

Likewise, for equilibria $\mathcal{S} = \{B, \varnothing\}$ or $\{B\}$, we find that

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \underline{C}^{\mathcal{S}} = \lim_{\mu \to 1} \lim_{\pi_L \to 0} \overline{C}^{\mathcal{S}} = -\frac{\delta \pi_H^2}{2} < 0.$$

Therefore, these equilibria only exist when the investment cost is negative, and hence do not exist in our setup.

Then, the only two equilibria that need to be checked are $\mathcal{S} = \{G, B\}$ and $\{\varnothing\}$. These two equilibria demonstrate strategies non-monotonic in the firm's reputation in the sense that the firm takes the same action following a good and bad outcome, but a different one following an empty outcome. We can show that these do not exist. First, suppose $\mathcal{S} = \{G, B\}$ so that the firm invests unless its recent outcome is $\{\varnothing\}$. That the firm finds it optimal to invest following a good outcome implies $c < \frac{\pi_H - \pi_L}{2}(V^{\mathcal{S}}(GG) - V^{\mathcal{S}}(GB))$, and the same about a bad outcome $c < \frac{\pi_H - \pi_L}{2}(V^{\mathcal{S}}(BG) - V^{\mathcal{S}}(BB))$. On the other hand, the firm should not invest following an empty outcome, and therefore $c > \frac{\pi_H - \pi_L}{2}(V^{\mathcal{S}}(\varnothing G) - V^{\mathcal{S}}(\varnothing B))$. Therefore, the equilibrium exists if

$$\frac{\Delta\pi}{2}(V^{\mathcal{S}}(\varnothing G) - V^{\mathcal{S}}(\varnothing B)) < c < \min_{x \in \{G, B\}} \frac{\Delta\pi}{2}(V^{\mathcal{S}}(xG) - V^{\mathcal{S}}(xB)).$$

Clearly, this is only possible if $V^{\mathcal{S}}(\varnothing G) - V^{\mathcal{S}}(\varnothing B) < \min_{x \in \{G,B\}} V^{\mathcal{S}}(xG) - V^{\mathcal{S}}(xB)$. Since the firm's investment decision only depends on the most recent outcome, the future payoffs in $V^{\mathcal{S}}(yG)$ and $V^{\mathcal{S}}(yB)$ are independent of $y \in \{G, \varnothing, B\}$. Therefore, the inequality entires hinges on the immediate prices, and thus holds if and only if $p^{\mathcal{S}}(\varnothing G) - p^{\mathcal{S}}(\varnothing B) < \min_{x \in \{G,B\}} p^{\mathcal{S}}(xG) - p^{\mathcal{S}}(xB)$. If $\pi_L = 0$, the right-hand side is zero for $x = G$, as one good outcome reveals the firm to be competent. Therefore, the inequality cannot hold. If $\pi_H = 1$, $x = B$ causes the right-hand side to vanish. Therefore, such non-monotonic equilibria do not exist.

We can similarly show the non-existence of the equilibrium $\mathcal{S} = \{\varnothing\}$: The equilibrium exists if and only if

$$\max_{x \in \{G,B\}} \frac{\Delta\pi}{2}(V^{\mathcal{S}}(xG) - V^{\mathcal{S}}(xB)) < c < \frac{\Delta\pi}{2}(V^{\mathcal{S}}(\varnothing G) - V^{\mathcal{S}}(\varnothing B)).$$

There exists some c that satisfies the condition if and only if $\max_{x\in\{G,B\}} p^{\mathcal{S}}(xG) - p^{\mathcal{S}}(xB) < p^{\mathcal{S}}(\varnothing G) - p^{\mathcal{S}}(\varnothing B)$. But both the left- and right-hand side are zero, as the firm does not invest following a good or bad outcome. Therefore, such equilibrium cannot exist if $\pi_L = 0$ or $\pi_H = 1$.

Quality control: Now consider the region where $\pi_H = 1$ and μ is close to 0. We already ruled out existence of two equilibria: $\mathcal{S} = \{G, B\}$, and $\{\varnothing\}$. We now show that, for $\hat{\mathbf{c}}^{\text{ind}} < c < \hat{\mathbf{c}}^{\text{col}}$, $\{B, \varnothing\}$ do not in the limit, by verifying $\overline{C}^{\mathcal{S}} < \hat{\mathbf{c}}^{\text{ind}}$. (Recall an equilibrium $\mathcal{S}$ exists if and only if $\underline{C}^{\mathcal{S}} < c < \overline{C}^{\mathcal{S}}$.)

From equations 7 and 8, for an equilibrium $\mathcal{S} = \{B\}$ and $\mathcal{S} = \{B, \varnothing\}$

$$
\begin{aligned}
\lim_{\pi_H\to 1} \overline{C}^{\{B\}} &= \frac{\delta^2\mu(1-\pi_L)^3(1+\pi_L)}{4(\mu(1-\pi_L)^2 + (3-\pi_L)\pi_L} > 0 \\
\lim_{\pi_H\to 1} \overline{C}^{\{B,\varnothing\}} &= \frac{\delta^2\mu(1-\pi_L)^3}{2\mu(2-\pi_L)(1-\pi_L) + 2(3-\pi_L)\pi_L} > 0
\end{aligned}
$$

Note that

$$
\begin{aligned}
\lim_{\pi_H\to 1} \hat{c}^{\text{ind}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))}{4(\pi_L - \mu(1-\pi_L))(\pi_L^2 + \mu(1-\pi_L^2))}, \\
\lim_{\pi_H\to 1} \overline{C}^{\{B,\varnothing\}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(1-\pi_L)}{2\mu(2-\pi_L)(1-\pi_L) + 2(3-\pi_L)\pi_L}, \\
\lim_{\pi_H\to 1} \overline{C}^{\{B\}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(1-\pi_L)(1+\pi_L)}{4(\mu(1-\pi_L)^2 + (3-\pi_L)\pi_L}.
\end{aligned}
$$

Thus, $\lim_{\pi_H\to 1} \hat{\mathbf{c}}^{\text{ind}}$, $\lim_{\pi_H\to 1} \overline{C}^{\{B,\varnothing\}}$ and $\lim_{\pi_H\to 1} \overline{C}^{\{B\}}$ converge to zero as μ approaches 0. In order to make a comparison in a close neighborhood of $\mu = 0$, we eliminate a common factor and compare the remaining terms at $\mu = 0$:

$$
\begin{aligned}
\lim_{\mu\to 0}\lim_{\pi_H\to 1} \frac{\delta(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))}{4(\pi_L - \mu(1-\pi_L))(\pi_L^2 + \mu(1-\pi_L^2))} &= \frac{\delta(1+\pi_L)}{4\pi_L^2}, \\
\lim_{\mu\to 0} \frac{\delta(1-\pi_L)}{2\mu(2-\pi_L)(1-\pi_L) + 2(3-\pi_L)\pi_L} &= \frac{\delta(1-\pi_L)}{2(3-\pi_L)\pi_L}, \\
\lim_{\mu\to 0} \frac{\delta(1-\pi_L)(1+\pi_L)}{4(\mu(1-\pi_L)^2 + (3-\pi_L)\pi_L} &= \frac{\delta(1-\pi_L^2)}{4(3-\pi_L)\pi_L}.
\end{aligned}
$$

For all values of π_L, $\frac{\delta(1+\pi_L)}{4\pi_L^2} > \frac{\delta(1-\pi_L)}{2(3-\pi_L)\pi_L}$ and $\frac{\delta(1+\pi_L)}{4\pi_L^2} > \frac{\delta(1-\pi_L^2)}{4(3-\pi_L)\pi_L}$, which proves the non-existence of these equilibria.

Next, we show the existence of two equilibria, $\mathcal{S} = \{G\}$ and $\{G, \varnothing\}$ in the relevant parameter region. We show this by showing that, in that region, the interval $(\underline{C}^{\mathcal{S}}, \overline{C}^{\mathcal{S}})$ contains $(\hat{\mathbf{c}}^{\text{ind}}, \hat{\mathbf{c}}^{\text{col}})$, i.e. $\underline{C}^{\mathcal{S}} < \hat{\mathbf{c}}^{\text{ind}}$ and $\overline{C}^{\mathcal{S}} > \hat{\mathbf{c}}^{\text{col}}$. If $\mathcal{S} = \{G\}$,

$$\begin{aligned}
\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \underline{C}^{\{G\}}}{\delta\mu(1-\pi_L)^2} &= \lim_{\mu\to 0} \frac{\delta(1-\mu)(2-\pi_L)(1+\pi_L) + 2(\pi_L + \pi_L^2 + \mu(2-\pi_L-\pi_L^2))}{4(1-(1-\mu)\pi_L)(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))} \\
&= \frac{2\pi_L + \delta(2-\pi_L)}{4(1+\pi_L)\pi_L} \\
\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \overline{C}^{\{G\}}}{\delta\mu(1-\pi_L)^2} &= \frac{(4 + (1-\mu)\pi_L(4 + \delta(2-\pi_L)(1+\pi_L)))}{4(1-(1-\mu)\pi_L)(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))} \\
&= \frac{4 + \delta(2-\pi_L)\pi_L}{4(1+\pi_L)\pi_L}.
\end{aligned}$$

If $\mathcal{S} = \{G, \varnothing\}$,

$$\begin{aligned}
\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \underline{C}^{\{G,\varnothing\}}}{\delta\mu(1-\pi_L)^2} &= \frac{\delta}{4(\mu + (1-\mu)\pi_L^2)} = \frac{\delta}{4\pi_L^2} \\
\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \overline{C}^{\{G,\varnothing\}}}{\delta\mu(1-\pi_L)^2} &= \frac{2 + \delta\pi_L}{4(\mu + (1-\mu)\pi_L^2)} = \frac{\delta\pi_L + 2}{4\pi_L^2}.
\end{aligned}$$

From Proposition 1 we also know that:

$$\begin{aligned}
\lim_{\mu\to 0} \lim_{\pi_H\to 1} \frac{\hat{c}^{\text{ind}}}{\delta\mu(\pi_H-\pi_L)^2} &= \frac{\delta(1+\pi_L)}{4\pi_L^2}, \\
\lim_{\mu\to 0} \lim_{\pi_H\to 1} \frac{\hat{c}^{\text{col}}}{\delta\mu(\pi_H-\pi_L)^2} &= \frac{2\pi_L + \delta(1+3\pi_L)}{16\pi_L^2}.
\end{aligned}$$

Recall that we are focusing on the case that $\hat{c}^{\text{col}} > \hat{c}^{\text{ind}}$, i.e., $\delta < \frac{2\pi_L}{3+\pi_L}$. Now we can compare $\hat{c}^{\text{col}}$ with $\overline{C}^{\{G,\varnothing\}}$ and $\overline{C}^{\{G\}}$, and $\hat{c}^{\text{ind}}$ with $\underline{C}^{\{G,\varnothing\}}$ and $\underline{C}^{\{G\}}$.

First, $\hat{c}^{\text{col}} < \overline{C}^{\{G,\varnothing\}}$ and $\hat{c}^{\text{ind}} > \underline{C}^{\{G,\varnothing\}}$, implying that the $\{G,\varnothing\}-$equilibrium exists whenever $\delta < \frac{2\pi_L}{3+\pi_L}$. Furthermore, $\hat{c}^{\text{col}} < \overline{C}^{\{G\}}$ holds whenever $\delta < \frac{2\pi_L}{3+\pi_L}$. In contrast, $\hat{c}^{\text{ind}} > \overline{C}^{\{G\}}$ holds if and only if $\delta > \frac{2\pi_L^2}{1+2*\pi_L}$. □

Proof. [Proof of Proposition 3] Under the case of exclusive knowledge, the only equilibrium for an individual brand is the "no investment" equilibrium. In this equilibrium, its average profits are given by $\lim_{\mu\to 1} \lim_{\pi_L\to 0} \Pi^{\text{ind}} \approx \pi_L \approx 0$. In a collective brand, regardless of the other firm's competency, the firm's average profit in a reputation equlibrium is given by $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\Pi^{\text{col}} = \pi_H - c$. Therefore, the firm always prefers branding with another firm to staying alone as long as $c < \pi_H$ with is the case by assumption.

For quality control industries, two equilibria can exist for an individual firm: $\mathcal{S} = \{G, \varnothing\}$, $\{G\}$. We compare profits under these equilibria to identify the best alternative to the reputational equilibrium. $\lim_{\mu\to 0}\lim_{\pi_H\to 1}\Pi^{\{G,\varnothing\}} = \pi_L - c$ and $\lim_{\mu\to 0}\lim_{\pi_H\to 1}\Pi^{\{G\}} = \pi_L - \frac{\pi_L}{1+\pi_L}\cdot c$. Both these profits are less than $\lim_{\mu\to 0}\lim_{\pi_H\to 1}\Pi^{\varnothing} = \pi_L$, and therefore an individual brand's best alternative is the bad equilibrium. If it forms a group with another firm, regardless of the type, $\lim_{\mu\to 0}\lim_{\pi_H\to 1}\Pi^{\text{col}} = \pi_L - c$ under the reputational equilibrium. For a collective brand too, the no investment equilibrium exists, and there the firm secures a profit of π_L. Therefore, the firm is indifferent between two types of brand. $\square$

A.3 Proof of Section 7

Proof. [Proof of Proposition 4] For the good news case with $\pi_L = 0$, we compare the cutoff levels we obtained by taking limit of μ to 1. $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ in this region if $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta Z_C^{\text{col}}(G^{T-1})}{1-\mu} > \lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta Z^{\text{ind}}(G^{T-1})}{1-\mu}$.

$$\frac{\pi_H}{2^{T+1}(1-\pi_H)}\cdot\frac{1-(2\delta)^T}{1-2\delta} \quad > \quad \frac{\pi_H}{2(1-\pi_H)}\cdot\delta^{T-1}. \tag{9}$$

This holds if and only if either $\delta \leq \frac{1}{2}$, or $\delta > \frac{1}{2}$ and $(2\delta)^T > \frac{\delta}{1-\delta}$. Because $(2\delta)^T$ is increasing in T for $\delta > \frac{1}{2}$, there is a large enough $\bar{T}(\delta)$ such that for all $T > \bar{T}(\delta)$, (21) holds. This implies that a longer memory expands the region of δ under which a collective brand sustains the reputational equilibrium better. For example, if $T = 2$, it is supported for $\delta \in (0, \frac{1}{2})$, if $T = 3$, $\delta \in (0, \frac{1+\sqrt{5}}{4} \approx 0.809)$, and if $T = 4$, $\delta \in (0, 0.919)$.

For the bad news case with $\pi_H = 1$, the cutoff level for a collective brand is greater than that of an individual brand for μ close to 0 if $\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\Delta Z_I^{\text{col}}(B^{T-1})}{\mu} > \lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\Delta Z^{\text{ind}}(B^{T-1})}{\mu}$.

$$\begin{aligned}
\frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} &> \frac{\delta^{T-1}(1-\pi_L)}{2^T\pi_L}\cdot(\frac{1+\pi_L}{\pi_L})^{T-1} \\
\frac{1}{2}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} &> \delta^{T-1}\cdot(\frac{1+\pi_L}{\pi_L})^{T-1}. \qquad (10)
\end{aligned}$$

□

Proof. [Proof of Corollary 1]

The first result regarding the case of exclusive technology ($\pi_L = 0$ and μ close to 1) follows immediately from the equation (21).

For the second result, we investigate the equation (22). Note that the left-hand side converges as T goes to ∞ if and only if $\frac{\delta(1+3\pi_L)}{2\pi_L} < 1$, which is equivalent to $\delta < \frac{2\pi_L}{1+3\pi_L}$. Likewise, the right-hand side converges if and only if $\delta < \frac{\pi_L}{1+\pi_L}$. And, $\frac{\pi_L}{1+\pi_L} < \frac{2\pi_L}{1+3\pi_L}$.

So, first, if $\delta < \frac{\pi_L}{1+\pi_L}$, (22) holds for a large enough T. It is straight-forward to show the same is true for $\delta = \frac{\pi_L}{1+\pi_L}$. Second, if $\frac{\pi_L}{1+\pi_L} < \delta < \frac{2\pi_L}{1+3\pi_L}$, the left-hand side converges, but the right-hand side goes to ∞. Therefore, (22) does not hold for large Ts. Lastly, if $\delta \geq \frac{2\pi_L}{1+3\pi_L}$, both sides diverge. Because $\frac{1+\pi_L}{\pi_L} > \frac{1+3\pi_L}{2\pi_L}$, the right-hand side is greater for a large enough T. This completes the proof. □

B Appendix: $T-$Period Memory

In this section, we extend our analysis to a T-period memory for $T > 2$. With a $T-$period memory, a relevant history at period t is of the form $\mathbf{h}_t \in \mathcal{H}^{\text{ind}} := \{G, \varnothing, B\}^T$ for an individual brand and $\mathbf{h}_t \in \mathcal{H}^{\text{col}} := \{G, B\}^T$ for a collective brand. The history consists of outcomes produced in the previous T periods, $\mathbf{h}_t = h_{t-T}h_{t-T+1}\cdots h_{t-1}$. As time proceeds, consumers' new history consists of the most recent outcomes from $\mathbf{h}_t$ and new outcomes. Let us denote the n most recent outcomes by $\mathbf{h}_t^n = h_{t-n}\cdots h_{t-1}$ for any $1 \leq n \leq T$.

As in Section 5, we start by finding conditions under which the reputational equilibrium exists for an individual and a collective brand. Then, we compare the respective parameter regions to find where the equilibrium exists under a collective, but not under an individual brand. The analysis is similar to that in Section 5, so to avoid redundancy, we omit repetitive details.

B.1 Individual brand

In a reputational equilibrium, a competent firm must find it optimal to invest after anyl history. To rule out profitable deviations, we consider the firm's investment decision at period t (also often referred as today) when the firm will invest whenever visited in the future. By investing, it can add a $h_t = G$ to the history $\mathbf{h}_t$ with a greater probability, which will be remembered in the next T periods. $k+1$ periods after period t, consumers would have forgotten the $k+1$ oldest outcomes, and $k+1$ new outcomes are added to the relevant history

$$\mathbf{h}_{t+k+1} = \mathbf{h}_t^{T-k-1} \underbrace{h_t h_{t+1} \cdots h_{t+k}}_{\text{new outcomes}} = \mathbf{h}_t^{T-k} h_t \mathbf{h}_{t+k+1}^{k-1}.$$

The new outcomes are denoted by $h_t \mathbf{h}_{t+k+1}^k$, where h_t is the result of the focal investment decision. To simplify the notation and to distinguish the known (old) outcomes and those to be realized, we denote future outcomes $\mathbf{h}_{t+k+1}^k$. Then, conditional on realizing the future outcomes $\mathbf{f}$, the benefit of investing in period t comes from a probabilistic improvement in the history from $\mathbf{h}_t^{T-k-1} B \mathbf{h}_{t+k+1}^k$ to $\mathbf{h}_t^{T-k-1} G \mathbf{h}_{t+k+1}^k$. This allows the firm to receive a higher price $p^{\text{ind}}(\mathbf{h}_t^{T-k-1} G \mathbf{h}_{t+k+1}^k) - p^{\text{ind}}(\mathbf{h}_t^{T-k-1} B \mathbf{h}_{t+k+1}^k)$. The total expected benefit from a decision to invest today then is a sum of such price differences, weighted according to the probability of realizing $\mathbf{h}_{t+k+1}^k$ and accounting for an appropriate discounting.

So, we can compute the benefit of an investment for each history. Then, the reputational equilibrium exists if and only if the cost of investment is less than the minimum of benefits over all histories. We summarize this in the next lemma, which is a general statement of Lemma 1.

Lemma 6. *For an individual firm, there exists a constant $\hat{\mathbf{c}}^{ind} > 0$ such that the reputatoinal equilibrium exists if and only if $c \leq \hat{\mathbf{c}}^{ind}$ where*

$$\hat{\mathbf{c}}^{ind} = \min_{\mathbf{h}_t^{T-1}} \bar{c}^{ind}(\boldsymbol{h}_t^{T-1}) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f} \in \{G, \varnothing, B\}^k} Pr(\mathbf{f}) (p(\mathbf{h}_t^{T-k-1} G \mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B \mathbf{f})) \Big). \quad (11)$$

Proof. [Proof of Lemma 6] As in Lemma 1, we obtain an expression for the cutoff in terms of price differences. We find it useful to define a new value function before the consumer's visit. Let $Z(\mathbf{h}_t)$ be the expected payoff to the firm in equilibrium:

$$Z^{\text{ind}}(\mathbf{h}_t) \quad \equiv \tfrac{1}{2}(p(\mathbf{h}_t) - c) + \delta \quad \left(\frac{\pi_H}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + \frac{1-\pi_H}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B) + \frac{1}{2} \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}\varnothing) \right).$$

As the consumer visits the firm with probability $\frac{1}{2}$, the firm's expected period-t profit is $\frac{1}{2}(p(\mathbf{h}_t)-c)$. The expected future payoff depends on the realized outcome in the current period. The firm produces outcomes G, B, $\varnothing$ with probabilities $\frac{\pi_H}{2}, \frac{1-\pi_H}{2}, \frac{1}{2}$, respectively.

Once the firm is visited, it should be optimal for the firm to invest always, i.e., $V(\mathbf{h}_t) \geq V(\mathbf{h}_t; \text{not})$ for all $\mathbf{h}_t \in \mathcal{H}^{\text{ind}}$ where

$$\begin{aligned} V^{\text{ind}}(\mathbf{h}_t) &= p(\mathbf{h}_t) - c + \delta(\pi_H \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_H) \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)), \\ V^{\text{ind}}(\mathbf{h}_t; \text{not}) &= p(\mathbf{h}_t) + \delta(\pi_L \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L) \cdot Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)). \end{aligned}$$

By investing in quality, the firm is able to produce a good outcome with a greater probability π_H, which improves the future payoffs. Then, the condition for the existence of the reputational equilibrium can be expressed as a cutoff-rule; the invest cost is always less than its benefit. So,

$$c \leq \hat{\mathbf{c}}^{\text{ind}} := \delta(\pi_H - \pi_L) \cdot \min_{\mathbf{h}_t^{T-1} \in \{G, \varnothing, B\}^{T-1}} \Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}), \tag{12}$$

where $\Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}) := Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) - Z^{\text{ind}}(\mathbf{h}_t^{T-1}B)$. The firm is able to receive a higher price in the next T periods due to the good outcome produced today. For this reason, $\Delta Z(\mathbf{h}_t^{T-1})$ is a present-discounted weighted-sum of price premiums, as we saw in the analysis for two-period memory:

The future payoff, conditional on producing a good outcome, is

$$Z^{\text{ind}}(\mathbf{h}_t^{T-1}G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\{G,\varnothing,B\}^k} \Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{first } T \text{ periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\{G,\varnothing,B\}^T}\Pr(\mathbf{g})(p(\mathbf{g})-c)}_{\text{after } T \text{ periods}}.$$

$$= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}G\mathbf{f})\right)-c\right)$$

$$+\frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{g})=i,N_B(\mathbf{g})=j} p(\mathbf{g})\right)-c\right).$$

Given a history $\mathbf{h}_t^{T-1}G$, the relevant history k periods later becomes $\mathbf{h}^{T-k-1}G\mathbf{f}$. That is, consumers replace oldest k memories with a new memory realized throughout k periods, i.e., $\mathbf{f} \in \mathcal{H}^k$. Conditional on the realization of $\mathbf{f}$, the firm's period-profit is $p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c$. This realization occurs with a probability denoted by $\Pr(\mathbf{f})$. Accouting for these probabilities and discounting, we obtain the first double sum in the equation. Once T periods have passed and consumers no longer remember the good outcome of the investment made in period t, the firm's relevant history can be any $\mathbf{g} \in \mathcal{H}^T$. So, we obtain the second double sum by weighting and discounting each period-profit appropriately. The firm receives a period-profit if and only if the consumer visits, and therefore we divide the whole expression by 2.

To compute $\Pr(\mathbf{f})$, counting the number of good, bad and empty histories is just enough, as the order of each outcome does not matter. Let $N_h(\mathbf{h}_t)$ for $h \in \mathcal{H}$ and $\mathbf{h}_t \in \mathcal{H}^T$ be the count of an outcome of type h in the T-period history $\mathbf{h}_t$. For example, $N_G(G\varnothing G) = 2$, $N_B(G\varnothing G) = 0$ and $N_\varnothing(G\varnothing G) = 1$. Suppose $N_G(\mathbf{f}) = i$, $N_B(\mathbf{f}) = j$, and $N_\varnothing(\mathbf{f}) = l$, respectively, such that $i+j+l = k$. Then, $\Pr(\mathbf{f}) = (\frac{\pi_H}{2})^i \cdot (\frac{1-\pi_H}{2})^j(\frac{1}{2})^l$. The next two lines in the equation are results of simply plugging in these probabilities.

Likewise, the future payoff to the firm if it produced a bad outcome would be

$$Z^{\text{ind}}(\mathbf{h}_t^{T-1}B) = \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}B\mathbf{f})\right)-c\right)$$

$$+\frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{N_G(\mathbf{g})=i,N_B(\mathbf{g})=j} p(\mathbf{g})\right)-c\right).$$

Therefore, subtracting the two gives

$$\begin{aligned}\Delta Z^{\text{ind}}(\mathbf{h}_t^{T-1}) &= \tfrac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\{G,\varnothing,B\}^k}\Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-p(\mathbf{h}^{T-k-1}B\mathbf{f}))\\ &= \tfrac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\tfrac{\pi_H}{2})^i(\tfrac{1-\pi_H}{2})^j(\tfrac{1}{2})^l\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j}(p(\mathbf{h}^{T-1-k}G\mathbf{f})-p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\end{aligned}$$

Plugging this into (12) completes the proof. □

To obtain an explicit expression for $\hat{c}^{\text{ind}}$, we need to uncover the minimum operator by identifying the binding history for different parameter regions. As in the two-period memory case, we focus on two special signal structures: exclusive knowledge ($\pi_L = 0$) and quality control ($\pi_H = 1$). The former provides an environment where building an extremely high level of reputation is easy for a competent firm, as one good outcome completely reveals its type. Therefore, we can attain the minimum by choosing a history that has a lasting damage to the firm's incentives. This implies that any history $\mathbf{h}_t^{T-1}$ with $h_{t-1} = G$ does the job. Since the most recent outcome in the history is good, consumers know perfectly the firm's type to be good until $t = T-1$. This eliminates all the benefits to be realized until period $t+T-1$. The only expression that survives in equation (11) is the very last period $(t+T)$ when $h_{t-1} = G$ will have been forgotten. As this benefit is discounted by δ^T, a longer history clearly hurts investment incentives for an individual brand.

Under the structure of quality control ($\pi_H = 1$), one bad outcome completely reveals a firm to be an incompetent type. Then, similarly, any history with $h_{t-1} = B$ attains the minimum because it puts a bad stamp on the brand for until period $t+T-1$. Then, all benefits other than ones to be realized in the very last period $(t+T)$, again discounted by δ^T.

Therefore, $\lim_{\pi_L\to 0}\hat{\mathbf{c}}^{\text{ind}} = \lim_{\pi_L\to 0}\bar{c}^{\text{ind}}(\mathbf{h}_t)$ where $h_{t-1} = G$, and $\lim_{\pi_H\to 1}\hat{\mathbf{c}}^{\text{ind}} = \lim_{\pi_H\to 1}\bar{c}^{\text{ind}}(\mathbf{g}_t)$ where $g_{t-1} = B$. We state next lemma with characterization of the cutoff once we take limits for μ, as we will use these cutoffs for comparison later.[15]

Lemma 7. *(i) In an the environment with exclusive knowledge ($\pi_L = 0$), a history in which the most recent outcome is G attains $\hat{\mathbf{c}}^{ind}$. If μ is close to 0,*

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\hat{\mathbf{c}}^{ind}}{1-\mu} = \frac{\delta^T\pi_H^2}{2(1-\pi_H)} \tag{13}$$

[15] Please see the appendix for the cutoffs prior to taking limits of μ.

(ii). In an environment with quality control ($\pi_H = 1$), a history in which the most recent outcome is B attains $\hat{\mathbf{c}}^{ind}$. If μ is close to 0,

$$\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\hat{\mathbf{c}}^{ind}}{\mu} = \frac{\delta^T(1-\pi_L)^2}{2^T\pi_L}\cdot(\frac{1+\pi_L}{\pi_L})^{T-1}. \tag{14}$$

Proof. [Proof of Lemma 7] (Identify the binding constraint for two cases and then compute the cutoff-level.) As the exact cutoff level involves a minimum operator, we need to compare $\Delta Z(\mathbf{h}_t^{T-1})$ for all $\mathbf{h}_t^{T-1} \in \{G, B, \varnothing\}$. Obtaining an explicit formula for it is not feasible. Instead, we focus on two special signal structures–$\pi_L = 0$, $\pi_H \in (0,1)$ and $\pi_H = 1$, $\pi_L \in (0,1)$.

First, suppose $\pi_L = 0$, $\pi_H \in (0,1)$. This is the case of exclusive technology where a good outcome reveals the firm to be competent. So, $\mu(\mathbf{h}) = 1$ if and only if $N_G(\mathbf{h}) \geq 1$. Here, the price $p(\mathbf{h}) = \pi_H \cdot \mu(\mathbf{h})$. So,

$$\begin{aligned} p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) &= \pi_H \cdot (\mu(\mathbf{h}^{T-1-k}G\mathbf{f}) - \mu(\mathbf{h}^{T-1-k}B\mathbf{f})) \\ &= \pi_H \cdot (1 - \mu(\mathbf{h}^{T-1-k}B\mathbf{f})). \end{aligned}$$

This vanishes if and only if $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) \geq 1$, i.e. there is at least one good outcome in this history. To find a history that minimizes $\Delta Z(\cdot)$, we want as many of the price difference as possible to vanish. For this purpose, it suffices to have $h_{-1} = G$. Recall h_{-1} is the outcome produced a period before the focal investment decision. So, the good outcome reveals the firm's competence until it is forgotten T periods later. So, with $h_{-1} = G$, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) = 0$ for all $\mathbf{f} \in \mathcal{H}^k$ for $0 \leq k \leq T-2$. For $k = T-1$, h_{-1} is forgotten and the relevant price premium is $p(G\mathbf{f}) - p(B\mathbf{f})$. So, for $h_{-1} = G$,

$$\Delta Z^{\text{ind}}(\mathbf{h}) \to_{\pi_L \to 0} \frac{1}{2}\cdot\delta^{T-1}\sum_{\mathbf{f}\in\mathcal{H}^{T-1}}\Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f}))$$

That is, all benefits other than the one realized in the last period vanish. And, this part is independent of $\mathbf{h}$, the history at the time of investment decision. Therefore, $h_{-1} = G$ indeed attains the minimum for $\Delta Z(\cdot)$.

Clearly, $p(G\mathbf{f}) - p(B\mathbf{f})$ again vanishes for any $N_G(\mathbf{f}) \geq 1$. Therefore, terms that survive in the equation above are $\mathbf{f}$ of length $T-1$ that only consist of B and/or $\varnothing$. Therefore,

$$
\begin{aligned}
\Delta Z^{\text{ind}}(\mathbf{h}) \quad &\to_{\pi_L \to 0} \quad \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(\hat{\mu}(GB^j \varnothing^{T-1-j}) - \hat{\mu}(B^{j+1}\varnothing^{T-1-j}) \right) \right) \\
&= \quad \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(1 - \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1} + 1 - \mu} \right) \right) \\
&= \quad \frac{\pi_H(1-\mu)}{2^T} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right).
\end{aligned}
$$

The first equality holds because $\hat{\mu}(GB^j\varnothing^{T-1-j}) = 1$ because a good history causes a full revelation, and $\hat{\mu}(B^{j+1}\varnothing^{T-1-j}) = \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1}+1-\mu}$. Simply plugging into (12) proves the lemma for $\pi_L = 0$ and $\pi_H \in (0,1)$. In particular, $\lim_{\mu\to 1} \lim_{\pi_L \to 0} \frac{\Delta Z^{\text{ind}}(\mathbf{h})}{1-\mu} = \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1}$.

Now, consider the case where $\pi_H = 1$ and $\pi_L \in (0,1)$. Here, a bad outcome is revealing of a firm's incompetence. Therefore, $\mu(\mathbf{h}) = 0$ if and only if $N_B(\mathbf{h}) \geq 1$, and $p(\mathbf{h}) = \pi_L$. We omit details for this case, as it is very similar to the previous case.

From (B.1), $h_{-1} = B$ attains the minimum for $\Delta Z^{\text{ind}}(\cdot)$. Then, all price premiums other than the ones to be realized in the last period vanish. Therefore,

$$
\begin{aligned}
\Delta Z^{\text{ind}}(\mathbf{h}) \quad &\to_{\pi_H \to 1} \quad \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f})) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)}{2} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1}{2})^j (\frac{1}{2})^{T-1-j} \left(\hat{\mu}(G^{j+1}\varnothing^{T-1-j}) - \hat{\mu}(BG^j\varnothing^{T-1-j}) \right) \right) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)\mu}{2^T} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \right).
\end{aligned}
$$

Plugging this into (12) completes the proof. In particular, $\lim_{\mu\to 0} \lim_{\pi_H \to 1} \frac{\Delta Z^{\text{ind}}(\mathbf{h})}{\mu} = \frac{\delta^{T-1}(1-\pi_L)}{2^T \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1}$. □

As we see in equations (13) and (14), the expected benefit to be realized in the last period is a weighted sum, depending on realization of $\mathbf{f}$, the future outcomes following the focal investment decision at period t. The price differences are of the form $p^{\text{ind}}(G\mathbf{f}) - p^{\text{ind}}(B\mathbf{f})$, where $\mathbf{f} \in \{G, \varnothing, B\}^{T-1}$. Under $\pi_L = 0$, if any outcome in $\mathbf{f}$ is G, the difference vanishes, as one good outcome reveals the firm to be competent. So, the summation accounts for the cases where $\mathbf{f} \in \{\varnothing, B\}^{T-1}$, i.e. only bad or empty outcomes constitute $\mathbf{f}$. Likewise, under $\pi_H = 1$, the price difference vanishes if and only if there is a B in $\mathbf{f}$. So, (14) sums over the cases $\mathbf{f} \in \{G, \varnothing\}^{T-1}$.[16]

B.2 Collective brand

A longer memory may have a similar adverse effect on short-run incentives for collective brands. However, as we saw in the analysis of the two-period model, consumers' limited observability for a collective brand alleviates this problem; as consumers cannot observe history at firm-level, they can never learn perfectly about the types of two firms in the group. Therefore, a competent firm can always improve the brand reputation by investing in quality.

The next lemma establishes the necessary and sufficient condition for the existence of reputational equilibrium. Let $\Pr(\mathbf{f};\theta)$ for $\mathbf{f} \in \{G, B\}^k$ and $\theta \in \{C, I\}$ with $0 \leq k \leq T$ be the probability that the brand of type θ produces a sequence of outcome $\mathbf{f}$ in k periods if a competent firm always invests.

Lemma 8. *For a competent firm within a collective brand, there exists a constant $\hat{c} > 0$ such that a* **RE** *exists if and only if $c \leq \hat{c}$ where*

$$\hat{\mathbf{c}}^{col} = \min_{\boldsymbol{h}_t^{T-1},\theta} \bar{c}^{col}(\boldsymbol{h}_t^{T-1}, \theta) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f} \in \{G,B\}^k} Pr(\mathbf{f};\theta)(p(\mathbf{h}_t^{T-k-1} G\mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B\mathbf{f})) \Big), \tag{15}$$

where $\mathbf{h}_t^{T-1} \in \{G, B\}^{T-1}$ and $\theta \in \{C, I\}$.

Proof. [Proof of Lemma 8] As this lemma is a straightforward generalization of lemma 3, we omit many details. Also, we adopt notation from the proof for 6. Let $Z_\theta^{\mathrm{col}}(\mathbf{h}_t)$ denote the payoff to a competent firm of a collective brand before the customer's visit. $\theta \in \{C, I\}$ denotes the other firm's type, which determines the brand's type, $s \in \{CC, CI\}$.

[16] See more details in the appendix.

$$Z^{\text{col}}_\theta(\mathbf{h}_t) \;\equiv\; \underbrace{\frac{1}{2}(p(\mathbf{h}_t)-c)}_{\text{expected current period}} \;+\delta\; \left(\underbrace{\frac{\pi_H+\pi(\theta)}{2}\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t G)+(1-\frac{\pi_H+\pi(\theta)}{2})\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t B)}_{\text{future profit}}\right).$$

In the current period the firm makes $p(\mathbf{h}_t)-c$ if visited and 0 otherwise. In the next period, the brand will face a history $\mathbf{h}^{T-1}_t G$ or $\mathbf{h}^{T-1}_t B$ depending on today's investment outcome, which also depends on the type of the other firm. So, on average, the firm produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B otherwise.

Once the firm is visited, it should be optimal for the firm to invest always. After a history $\mathbf{h}_t$, a firm's payoff conditional on being visited is denoted by $V^{\text{col}}_\theta(\mathbf{h}_t)$. Then, we need $V^{\text{col}}_\theta(\mathbf{h}_t) \geq V^{\text{col}}_\theta(\mathbf{h}_t;\text{not})$ for all $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$.

$$V^{\text{col}}_\theta(\mathbf{h}_t) = p(\mathbf{h}_t)-c+\delta(\pi_H\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t G)+(1-\pi_H)\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t B)),$$
$$V^{\text{col}}_\theta(\mathbf{h}_t;\text{not}) = p(\mathbf{h}_t)+\delta(\pi_L\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t G)+(1-\pi_L)\cdot Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t B)).$$

This is equivalent to

$$c \leq \bar{\mathbf{c}}^{\text{col}} := \delta(\pi_H-\pi_L)\cdot \min_{\mathbf{h}^{T-1}_t\in\{G,B\}^{T-1}} \Delta Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t), \tag{16}$$

where $\Delta Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t) := Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t G) - Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t B)$.

The future payoff, conditional on producing an outcome of either G or B, is

$$Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\Pr(\mathbf{g};\theta)(p(\mathbf{g})-c)}_{\text{After } T \text{ Periods}},$$
$$Z^{\text{col}}_\theta(\mathbf{h}^{T-1}_t B) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}B\mathbf{f})-c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\Pr(\mathbf{g};\theta)(p(\mathbf{g})-c)}_{\text{After } T \text{ Periods}}$$

In each period, the brand produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B with the complementary probility. Therefore, for any $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, if $N_G(\mathbf{h}_t) = i$ and $N_B(\mathbf{h}_t) = j = t - i$, $\Pr(\mathbf{f};\theta) = (\frac{\pi_H+\pi(\theta)}{2})^i(1 - \frac{\pi_H+\pi(\theta)}{2})^j$.

Therefore, subtracting the two gives

$$\begin{aligned} \Delta Z_\theta^{\text{col}}(\mathbf{h}_t^{T-1}) &= \frac{1}{2} \cdot \sum_{k=0}^{T-1} \delta^k \sum_{\mathbf{f}\in\mathcal{H}^k} \Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \qquad (17) \\ &= \frac{1}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} (\frac{\pi_H+\pi(\theta)}{2})^i(1-\frac{\pi_H+\pi(\theta)}{2})^j \left(\sum_{N_G(\mathbf{f})=i} (p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})) \right) \right) \end{aligned}$$

Plugging this into (16) completes the proof. □

This lemma generalizes lemma 3. The cutoff now depends on the type of the other firm, as it affects realization of future outcomes $\mathbf{f}$ through $\Pr(\mathbf{f};\theta)$. Also, prices here are different from those in the individual brand because conditional on a history, posterior beliefs are different.

None of these price differences in equation (15) vanish even for $\pi_L = 0$ and $\pi_H = 1$. And, which history and type provide the binding constraint is less clear for a collective brand. To characterize the cutoff level, we take limits for μ, the prior belief.

First, for $\pi_L = 1$, consider μ close to 1. Then, a good outcome is informative. However, the informativeness of each additional good outcome must be decreasing. For example, having one good outcome compared to none is quite desirable, as it reveals the existence of at least one competent firm. But, having a fifth good outcome in the history in addition to an existing four is not as appealing, as consumers already believe with a high probability that both firms are competent. So, in this parameter region, the binding constraint would be provided by an environment that produces as many good outcomes as possible. Naturally, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would do the job.

Second, for $\pi_H = 1$, let focus on μ close to 0. Then, while a bad outcome is informative, it's informativeness decreases as there are more bad outcomes in the history. So, the binding condition would be provided by $\mathbf{h}_t^{T-1} = B^{T-1}$ and $\theta = I$, as together they produce as many bad outcomes as possible in the brand's history.

Then, we can compute the cutoff levels explicitly:

Lemma 9. *(i) Under the environment of exclusive technology ($\pi_L = 0$), if μ is close to* 1*, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(G^{T-1}, C)$ and*

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\hat{\mathbf{c}}^{col}}{1-\mu} = \frac{\delta \pi_H^2}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta} \tag{18}$$

(ii) Under the quality control ($\pi_H = 1$), if μ is close to 0*, $\hat{\mathbf{c}}^{col} = \bar{c}^{col}(B^{T-1}, I)$ and*

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\hat{\mathbf{c}}^{col}}{\mu} = \frac{\delta(1-\pi_L)^2}{2^{T+1}\pi_L} \cdot \frac{1-(\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L}} \tag{19}$$

Proof. [Proof of Lemma 9] The exact cutoff levels in lemma 8 is a discounted sum of price premiums over T periods. It is not feasible to obtain an explicit expression for general parameter regions. So, we again focus on two parameter regions: $\pi_L = 0$ and $\pi_H = 1$. Before we shift our focus to these cases, we find it useful to understand posterior beliefs denoted by $\eta(\cdot)$. Facing a collective brand, consumers update beliefs over types of the brand, $s \in \{CC, CI, IC, II\}$, and use this to compute the probability of visiting a competent firm: $\eta(\cdot) = \eta_{CC}(\cdot) + \frac{1}{2}(\eta_{CI}(\cdot) + \eta_{IC}(\cdot))$. So, $\eta(\mathbf{h}_t)$, if $N_G(\mathbf{h}_t) = i$, is

$$\eta(\mathbf{h}_t) = \frac{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + \mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i}}{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + 2\mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i} + (1-\mu)^2 \cdot \pi_L^i (1-\pi_L)^{T-i}} \tag{20}$$

It is infeasible to obtain an explicit expression for $\Delta Z_\theta(\cdot)$, not to mention the overall cutoff, $\bar{c}^{\text{col}}$. As we did in previous analyses, we i) focus on two signal structures ($\pi_L = 0$ and $\pi_H = 1$), ii) identify the binding history and the brand type, and iii) obtain a lower bound $\underline{c}^{\text{col}}$ for the cutoff.

First, consider the case $\pi_L = 0$. Then, after a history $\mathbf{h}_t$, the consumer pays $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) \cdot \pi_H$. The reputational benefit realized in each period is the price difference made available by one more good outcome in the history, and thus is of a form $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, where $N_G(\mathbf{h}^{T-1-k}G\mathbf{f}) = N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) + 1$. And, here we claim that this difference is decreasing in i for a large enough μ. That is, when $\pi_L = 0$ and μ is large, the price premium reduces as the number of good outcomes becomes large. If this were true, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would provide the minimum for $\Delta Z_\theta(\mathbf{h}_t^{T-1})$, as these two conditions both places the brand under histories with more good outcomes. We formally state this and prove:

Claim 1. Suppose $\pi_L = 0$ and μ is close to 1. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is decreasing in i. Then, the price premium from the investment is low when there are many good outcomes in the history. So, $\boldsymbol{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ attains the minimum for $\Delta Z_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

The intuition is the following. As long as there is a good outcome in the history, consumers believe the brand has either one or two competent firms. But, as they see more good outcomes, they become more convinced that both firms are competent. As more good outcomes resolve consumers' uncertainty, the price difference becomes small. Mathematically,

$$\begin{aligned}
\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2) &= \frac{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}} \\
&\quad - \frac{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&= \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&\quad - \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}
\end{aligned}$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{1-\mu}$ to a limit as $\mu \to 1$,

$$\begin{aligned}
\lim_{\mu\to 1} \frac{\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)}{1-\mu} &= \frac{\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\pi_H^{i}(1-\pi_H)^{T-i}} - \frac{\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\pi_H^{i+1}(1-\pi_H)^{T-i-1}} \\
&= \frac{1}{(1-\pi_H)2^{i+1}}(\frac{1-\frac{\pi_H}{2}}{1-\pi_H})^{T-i-1},
\end{aligned}$$

which is clearly decreasing in i. Therefore, for any positive integer T, there is a $\bar{\mu}$ close enough to 1 so that the difference in beliefs (and thus prices) is decreasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$\begin{aligned}
\lim_{\mu \to 1} \frac{\Delta Z_C^{\mathrm{col}}(G^{T-1})}{1-\mu} &= \frac{\pi_H}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} \pi_H^i (1-\pi_H)^j \left(\sum_{N_G(\mathbf{f})=i} (\eta(\mathbf{h}^{T-1-k} G\mathbf{f}) - \eta(\mathbf{h}^{T-1-k} B\mathbf{f})) \right) \right) \\
&= \frac{\pi_H}{2(1-\pi_H)} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} \pi_H^i (1-\pi_H)^j \binom{k}{i} \frac{1}{2^{T-k+i}} \left(\frac{1-\frac{\pi_H}{2}}{1-\pi_H}\right)^j \right) \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \sum_{k=0}^{T-1} (2\delta)^k \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta}
\end{aligned}$$

Next, we consider the case $\pi_H = 1$. Then, the price consumer pays after a history $\mathbf{h}_t$ is $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) + (1-\eta(\mathbf{h}_t))\pi_L$. In this setting, a bad outcome is very informative, as it reveals existence of an incompetent firm in the brand. And, intuitively as there are more bad outcomes in the history, informativeness of each bad outcome decrease. Therefore, the price premium to be realized k period after the focal investment decision conditional on the new outcomes $\mathbf{f}$ is $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, and this decreases in i, where $i = N_G(\mathbf{h}^{T-1-k}B\mathbf{f})$. We state it formally in the next claim.

Claim 2. Suppose $\pi_H = 0$ and μ is close to 0. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is increasing in i. Then, $\boldsymbol{h}_t^{T-1} = B^{T-1}$ and $\theta = I$ attains the minimum for $\Delta Z_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

$$\begin{aligned}
\eta(G^T) &= \frac{\mu^2 + \mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T}{\mu^2 + 2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T + (1-\mu)^2 \cdot \pi_L^T} \\
\eta(\mathbf{h}) &= \frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, $\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2) =$

$$\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1} + (1-\mu)^2\cdot\pi_L^{i+1}(1-\pi_L)^{T-i-1}}$$
$$-\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i}(\frac{1-\pi_L}{2})^{T-i}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i}(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^{i}(1-\pi_L)^{T-i}}$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{\mu}$ to a limit as $\mu \to 0$,

$$\begin{aligned}
\lim_{\mu\to 1}\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{\mu} &= (\frac{1+\pi_L}{2\pi_L})^{i+1}\frac{1}{2^{T-i-1}} - (\frac{1+\pi_L}{2\pi_L})^{i}\frac{1}{2^{T-i}} \\
&= \frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}.
\end{aligned}$$

This is clearly increasing in i. Therefore, there is a $\bar{\mu}_{\pi_H=1}$ close enough to 0 so that the difference in beliefs (and thus prices) is increasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$\begin{aligned}
\lim_{\mu\to 0}\frac{\Delta Z}{\mu} &= \frac{1-\pi_L}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(\frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}})\right)\right) \\
&= \frac{1-\pi_L}{2^{T+1}}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\binom{k}{i}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}\right) \\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\sum_{k=0}^{T-1}\frac{\delta^k}{2^k}\left(\frac{1+3\pi_L}{\pi_L}\right)^k \\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})}
\end{aligned}$$

□

Even in the limits, benefits of investment for a collective brand do not vanish, and the cutoff turns out to be a sum of what turns out to be a finite geometric sequence. Unlike the cutoff for an individual brand, the cutoff is not discounted by δ^T, so it decreases in T as a much slower rate. This highlights the advantage of collective brands over individual ones.

B.3 Comparing Individual and Collective Brands

It remains to find out when $\hat{\mathbf{c}}^{\mathrm{col}}$ is greater than $\hat{\mathbf{c}}^{\mathrm{ind}}$ for two special parameter regions by comparing equations (13) and (18), and (14) and (19).

Proposition 5. *(i) If $\pi_L = 0$, and μ close to 1, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$ if either*

$$\delta < \frac{1}{2}, \text{ or } \delta > \frac{1}{2} \text{ and } (2\delta)^T > \frac{\delta}{1-\delta}. \tag{21}$$

(ii) If $\pi_H = 1$, and μ close to 0, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$ if

$$\frac{1}{2} \cdot \frac{1 - \left(\frac{\delta(1+3\pi_L)}{2\pi_L}\right)^T}{1 - \frac{\delta(1+3\pi_L)}{2\pi_L}} > \left(\frac{\delta(1+\pi_L)}{\pi_L}\right)^{T-1} \tag{22}$$

Proposition 4 shows that for any history length, we can find regions where a collective brand sustains the reputational equilibrium better than an individual brand does. In fact, a longer memory expands the region of δ that supports our result for the case of $\pi_L = 0$. This is because when $\delta > \frac{1}{2}$, $2\delta > 1$, so the left-hand side increases in T. For example, if $T = 2$, $\hat{\mathbf{c}}^{\mathrm{col}} > \hat{\mathbf{c}}^{\mathrm{ind}}$ for $\delta \in (0, \frac{1}{2})$, if $T = 3$, $\delta \in (0, \frac{1+\sqrt{5}}{4} \approx 0.809)$, and if $T = 4$, $\delta \in (0, 0.919)$.

The case of $\pi_H = 1$ is more complicated. Because the left-hand side is always increasing in T, (22) is more likely to hold if $\frac{\delta(1+\pi_L)}{\pi_L} \leq 1$. Otherwise, if $\frac{\delta(1+\pi_L)}{\pi_L} > 1$, the right-hand side diverges as T goes to infinity. So, in order for the condition to hold, the left-hand side must diverge at a faster rate. The left-hand side converges if and only if $\delta < \frac{2\pi_L}{1+3\pi_L}$. So, if $\frac{\pi_L}{1+\pi_L} < \delta < \frac{2\pi_L}{1+3\pi_L}$, the condition holds only for a small enough T. If $\delta > \frac{2\pi_L}{1+3\pi_L}$, we can show that the condition cannot hold for T too large.

Corollary 2. *If $\pi_L = 0$ and μ close to 1, for any $\delta \in (0, 1)$, there is a large enough $\bar{T}$ such that for all $T > \bar{T}$, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$. If $\pi_H = 1$ and μ close to 0, for any $\delta \in (0, \frac{\pi_L}{1+\pi_L}]$, there is a* *large enough* $\tilde{T}$ *such that for all* $T > \tilde{T}$, $\hat{\mathbf{c}}^{col} > \hat{\mathbf{c}}^{ind}$. *Otherwise, if* $\delta \in (\frac{\pi_L}{1+\pi_L}, 1)$, *there is a large enough* $\hat{T}$ *such that for all* $T > \hat{T}$, $\hat{\mathbf{c}}^{ind} > \hat{\mathbf{c}}^{col}$.