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(NON)RANDOMIZATION: A THEORY OF QUASI-EXPERIMENTAL EVALUATION OF SCHOOL QUALITY

By

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(Non)Randomization:
A Theory of Quasi-Experimental Evaluation of School Quality

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November 27, 2016

Abstract

In centralized school admissions systems, rationing at oversubscribed schools often uses lotteries in addition to preferences. This partly random assignment is used by empirical researchers to identify the effect of entering a school on outcomes like test scores. This paper formally studies if the two most popular empirical research designs successfully extract a random assignment. For a class of data-generating mechanisms containing those used in practice, I show: One research design extracts a random assignment under a mechanism if and almost only if the mechanism is strategy-proof for schools. In contrast, the other research design does not necessarily extract a random assignment under any mechanism.

Keywords: Matching Market Design, Natural Experiment, Program Evaluation, Random Assignment, Quasi-Experimental Research Design, School Effectiveness

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1 Introduction

The spread of choice and quasi-markets in public education is giving more families the option to attend a school other than their neighborhood default. As choice has proliferated, school assignment has grown increasingly centralized and algorithmic in order to respect heterogeneous preferences and various priorities based on family background. Centralized assignment mechanisms solve the problem of matching the limited supply of school seats to the demand for them by using centralized algorithms. Such mechanisms are employed in Boston, Charlotte, Denver, New Orleans, Newark, New York City, San Francisco, Washington, D.C., and numerous Asian and European countries. Well-designed centralized assignment provides a transparent way to achieve a fair and efficient school seat allocation, while narrowing the scope for strategic behavior (Abdulkadiroğlu and Sönmez, 2003).

Moreover, centralized assignment generates valuable data for empirical research on education. In particular, when a school is oversubscribed, mechanisms often use random lotteries to ration limited seats. This generates quasi-experimental variation in school assignment that opens the door to a variety of impact evaluations. Researchers used such variation to study schools in the Bay Area (Bergman, 2016), Boston (Angrist et al., 2016), Charlotte-Mecklenburg (Hastings et al., 2009; Deming, 2011; Deming et al., 2014), Denver (Abdulkadiroğlu et al., 2016), and New York (Bloom and Unterman, 2014; Abdulkadiroğlu et al., 2014b).[1]

Centralized assignment mechanisms combine lotteries, preferences, and priorities into complex stratified randomized experiments. Empirical research designs based on such mechanisms therefore need to condition on appropriate objects to isolate random components of their data-generating mechanisms. Yet, the above empirical work provides only a limited foundation for how their research designs extract a conditionally random assignment.[2] Although the theoretical market design literature has analyzed the welfare and strategic properties of mechanisms, it has so far little guidance to offer empirical researchers.

[1] See also Hastings et al. (2012). Other studies use similar regression-discontinuity-style tie-breaking rules to evaluate college majors in Norway (Kirkeboen et al., 2016) and in Chile (Hastings et al., 2013), daycare in Italy (Fort et al., 2016), privately managed public schools in Trinidad and Tobago (Beuermann et al., 2016), as well as popular selective schools in Ghana (Ajayi, 2013), Kenya (Lucas and Mbiti, 2014), Romania (Pop-Eleches and Urquiola, 2013), Trinidad and Tobago (Jackson, 2010, 2012), and the U.S. (Abdulkadiroğlu et al., 2014a; Dobbie and Fryer, 2014). Narita (2015) uses lottery-based randomization to identify a structural model of evolving demand for schools.

[2] An exception is Abdulkadiroğlu et al. (2016). See the literature review at the end of this introduction for the relationship between my paper and theirs. The other papers simply check empirical "covariate balance." That is, they compare the treatment and control groups by baseline characteristics or covariates that are fixed at the time of treatment assignment and not used for it. If the two groups' covariates are similar (covariates are balanced), it is interpreted as not rejecting randomization. Covariate balance is necessary but not sufficient for randomization.

This paper studies when widely-used empirical research designs successfully extract conditionally random assignment of students to schools. I focus on the two most popular empirical research designs that are applicable to any centralized mechanism that assigns students to schools by combining: (1) applicants' rank-ordered preferences over schools, (2) applicants' priority statuses (e.g., walk zone) at schools, and (3) lottery numbers for breaking ties in priority status. Each of the empirical examples above uses one of these research designs to extract a random assignment.

The first research design, which I call the *first-choice research design*, focuses on applicants who rank a given treatment school first and are in the "marginal" priority group at the school where some students are assigned to the treatment school while others are not. Within this *first-choice subsample*, some applicants are assigned to the treatment school while others are not, though all students rank it first and share the same priority; thus it appears that treatment assignment is determined solely by lottery numbers. Based on this idea, the first-choice research design assumes that applicants are randomly assigned to the treatment school conditional on being in the first-choice subsample. It then compares the outcomes (e.g., test scores) of students in the first-choice subsample who are assigned to the treatment school against those who are not. The outcome difference between the two groups is interpreted as a causal effect of the treatment school.

Despite its intuitive construction, the first-choice research design extracts a random assignment only under a condition.[3] For a class of data-generating mechanisms containing those used in practice, I show: The first-choice research design extracts a conditionally random assignment (i.e., applicants in the first-choice subsample share the same assignment probability at the treatment school) for a mechanism if and almost only if the mechanism is strategy-proof for schools.[4] Here I treat strategy-proofness as an algorithmic property that does not depend on any assumption about school behavior in reality. This result has important implications for applied research. It justifies the first-choice research design for mechanisms that are known to be strategy-proof for schools, such as the Boston (immediate acceptance) mechanism (Ergin and Sönmez, 2006). It also suggests that attention should be

[3]How can the first-choice design fail to extract a random assignment? To gain intuition, imagine the treatment school A has only one seat, and the first-choice subsample contains two students, 1 and 2. Student 1 ranks only A while 2 ranks other schools below A. When 2 has a better lottery number than 1, 1 is rejected by A and does not apply for any other school. When 1 has a better lottery number, 2 is rejected by A and then applies for other schools, potentially crowding out other students there. These crowded-out students may apply for A, which may crowd student 1 out of A. Such chain reactions of rejections and new applications dilute 1's, but not 2's, treatment assignment probability at A. As a result, 1 and 2 may have different treatment assignment probabilities, a potential problem with the first-choice design. Section 3.1 provides a more precise example.

[4]The if part is exactly true. The almost-only-if part means that the first-choice design sometimes fails to extract a random assignment for any non-strategy-proof mechanisms used in the above empirical studies.

paid to the research design for other widely-used mechanisms that are not strategy-proof for schools, such as the deferred acceptance mechanism, a mechanism used in Charlotte, and the top trading cycles mechanism.

By contrast to the above partial justification for the first-choice design, no similar sufficient condition is obtainable for another popular research design, which I call the *qualification instrumental variable (IV) research design*. Unlike the first-choice research design (trying to make assignments random by focusing on a subset of students), the qualification IV research design considers all students. It then codes a supposedly random instrumental variable for non-random assignments. The IV is based on "qualification," i.e., whether a student's lottery number is better than the worst number offered a seat at the treatment school (conditional on priority). I find that even in the simple case with no priorities and unit school capacities, the qualification IV research design does not necessarily extract a random assignment for any mechanism (within my mechanism class), i.e., applicants may not share the same conditional probability of qualification at the treatment school. This shows a contrast between the qualification IV design and the first-choice design, as summarized in Table 1.[5]

Table 1: Summary of the main results

Do empirical research designs always extract a random assignment?	1st choice research design	Qualification IV research design
Under mechanisms strategy-proof for schools (e.g., Boston mechanism)	✓	×
Under other mechanisms (e.g., deferred acceptance, Charlotte, and top trading cycles mechanisms)	×	×

Before I move on to the analysis, two remarks are in order about the initial result using strategy-proofness for schools. First, I do *not* assume that schools have preferences or are strategic in reality. This is because my analysis treats strategy-proofness not as a desideratum or incentive compatibility constraint but rather as an algorithmic property, which turns out to enable an empirical research design to extract a random assignment. Therefore, the

[5]This comparison is based on the set of mechanisms under which each design always extracts a random assignment. In fact, even if I fix a particular mechanism, the same point can be made in the following sense: The above result for the qualification IV implies that for any mechanism, the qualification IV may not extract a random assignment even in the simple case with no priorities and unit school capacities. By contrast, in that simple case, the first-choice design sometimes extracts a random assignment under several mechanisms, as shown below. Therefore, even conditional on a particular mechanism, the first-choice design is weakly more likely to extract a random assignment.

empirical implications of my theoretical result are free from any assumption about school behavior or preference. In addition, I need no assumption on student behavior (e.g., truthful preference reporting) because I study how to extract a random assignment conditional on any reported preferences in data.

Second, the initial result — strategy-proofness for schools is sufficient for the first-choice research design to extract a random assignment — has an additional empirical implication. Particularly, it provides an asymptotic support for the first-choice research design even for mechanisms that are not strategy-proof in general. This is because such non-strategy-proof mechanisms like deferred acceptance are known to be approximately strategy-proof for schools in certain large markets with many students and schools (Roth and Peranson (1999) and subsequent studies).[6] This may explain why the first-choice design appears to extract a random assignment in empirical applications even for non-strategy-proof mechanisms. Viewed differently, the existing empirical justification for the first-choice research design in the form of covariate balance regressions may suggest the empirical relevance of theoretical results on strategy-proofness in large markets.

The rest of this paper is organized as follows. After a literature review, the next section introduces my model. Section 3 defines the first-choice empirical research design and gives conditions under which the research design extracts a random assignment. Section 4 analyzes the alternative qualification IV research design and compares it with the first-choice research design. Section 5 confirms that my results are robust to a variety of modifications to the definitions of research designs and randomization. Finally, Section 6 summarizes the empirical implications of my theoretical results and suggests an agenda for further research.

Related Literature

This paper theoretically studies the empirical practice in the above-mentioned econometric evaluations of school effectiveness. My analysis reveals the connection between their empirical strategies and theoretical market design studies, especially those on strategy-proofness. In addition, Abdulkadiroğlu et al. (2016) is closely related. They develop a large-sample framework based on an asymptotic approximation assuming a growing number of students and school seats. They use their model to propose an improvement over the first-choice and qualification IV research designs and apply the improved design to evaluate charter schools in Denver. They also confirm that the first-choice and qualification IV research designs extract a random assignment for many mechanisms in the limit of their large market sequence. In contrast, the current paper allows for general finite markets and provides conditions for

[6]See, among others, Immorlica and Mahdian (2005); Kojima and Pathak (2009); Azevedo and Budish (2013); Lee (2016); Ashlagi et al. (2016).

the first-choice or qualification IV research design to extract a random assignment in a finite sample. These conditions allow me to compare the two research designs, as in Table 1. I also provide additional large market justifications for the first-choice research design in large market models different from Abdulkadiroğlu et al. (2016)'s.

2 Framework

I use a model of school-student assignment with coarse school priorities and lotteries. There are a finite set I of students and a finite set S of schools. Each student $i \in I$ has a strict preference $\succ_i$ over $S \cup \{\emptyset\}$, where $\emptyset$ denotes the outside option of the student. This $\succ_i$ is i's reported preference recorded in the data; I do not make any assumption about whether $\succ_i$ is truthful or not. School s is said to be acceptable for student i if $s \succ_i \emptyset$. A preference profile for all students is denoted by $\succ_I \equiv (\succ_i)_{i \in I}$. Each school s has a capacity c_s. Schools also grant students various coarse priorities. $\rho_{is} \in \{1, ..., K\}$ denotes student i's priority at school s where $\rho_{is} < \rho_{js}$ means s prioritizes i over j. Motivated by public school applications, I assume every student is acceptable to every school. The number of possible priority statuses K may change as the number of students $|I|$ changes. Priorities may be coarse in the sense that it is possible that $\rho_{is} = \rho_{js}$ for some $i \neq j$. Let $\rho_s \equiv (\rho_{is})_{i \in I}$. Denote the **type** of student i by $\theta_i = (\succ_i, (\rho_{is})_{s \in S})$. I call $X \equiv (I, S, \succ_I, (c_s)_{s \in S}, (\rho_s)_{s \in S})$ an **assignment problem**.

2.1 Generalized Deferred Acceptance Mechanisms

A (stochastic) mechanism maps each assignment problem into a distribution over matchings between students and schools. Mechanisms usually use lotteries to break indifferences or ties in priority and then use the resulting strict priorities to create a matching. A random variable R_{is} denotes student i's lottery number at school s. Assume that at each school, R_{is} is iid across students according to $U[0, 1]$. For the correlation of lottery numbers across different schools, I consider two focal regimes. Under a "single tie breaker" (STB), each student has a single lottery number used by all schools, i.e., $R_{is} = R_{is'}$ always holds for all i, s, and s'. Under a "multiple tie breaker" (MTB), each student has an independent lottery number at each school, i.e., R_{is} and $R_{is'}$ are independent for all i and $s \neq s'$.[7] Let $r_{is} \in [0, 1]$ denote i's realized lottery number at school s and let $R \equiv (R_{is})_{i \in I, s \in S}$, and $r \equiv (r_{is})_{i \in I, s \in S}$. When referring to any r, I assume $r_{is} \neq r_{js}$ for all students i, j, and school s.

[7] In reality, most school districts use STB, though some cities like Washington, D.C., New Orleans, and Amsterdam use MTB. It is possible but requires messier notation to extend my analysis to any structure in between STB and MTB where some schools use a common lottery while others use independent ones.

To define mechanisms of interest, I first introduce the following (student-proposing) **deferred acceptance (DA) algorithm** (Gale and Shapley, 1962). The DA algorithm produces an assignment by using any given strict student preferences and strict school priority orders as follows.

- Step 1: Each student i applies to her most preferred acceptable school (if any). Each school tentatively keeps the highest-ranking students up to its capacity, and rejects every other student.

In general, for any subsequent step $t \geq 2$,

- Step t: Each student i who was not tentatively matched to any school in Step $t-1$ applies to her most preferred acceptable school that has not rejected her (if any). Each school tentatively keeps the highest-ranking students up to its capacity from the set of students tentatively matched to this school in previous step $t-1$ and the students newly applying, and rejects every other student.

The algorithm terminates at the first step at which no student applies to any school. Each student tentatively kept by a school at that step is allocated a seat in that school, resulting in an assignment. I use this algorithm to define a class of mechanisms of interest as follows.[8]

Definition 1. A **generalized deferred acceptance (gDA) mechanism** φ is a mechanism that can be expressed as the following procedure. Take any assignment problem as given.

(1) Draw lottery numbers r according to its lottery regime (STB or MTB).

(2) For each student i and school s, compute the modified priority

$$\rho_{is}^{\varphi} \equiv f^{\varphi}(\rho_{is}) + g^{\varphi}(rank_{is}),$$

where $f^{\varphi} : \mathbb{N} \to \mathbb{N}$ ($\mathbb{N}$ is the set of positive integers) is a strictly increasing function, $rank_{is}$ is the preference rank of school s in student i's preference $\succ_i$, and $g^{\varphi} : \mathbb{N} \to \mathbb{N}$ is a weakly increasing function. Define school s's ex post strict modified priority order $\succ_{r_s}^{\varphi}$ over students by $i \succ_{r_s}^{\varphi} i'$ if $\rho_{is}^{\varphi} + r_{is} < \rho_{i's}^{\varphi} + r_{i's}$.

(3) Given $\succ_I$ and $(\succ_{r_s}^{\varphi})_{s \in S}$, run the DA algorithm to produce an assignment, where each school s's priority order is given by $\succ_{r_s}^{\varphi}$.

[8] Others also use similar classes of mechanisms. See, for example, Ergin and Sönmez (2006); Pathak and Sönmez (2008); Agarwal and Somaini (2015); Abdulkadiroğlu et al. (2016).

gDA mechanisms are parametrized by the lottery regime (STB or MTB) and the modified priority function $(f^{\varphi}, g^{\varphi})$. This gDA class includes most of the mechanisms used in empirical research as I now show.

Deferred Acceptance Mechanism

Given an assignment problem and realized lottery numbers, the **deferred acceptance (DA) mechanism** (Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003) makes a matching through the DA algorithm in which schools' strict priorities are induced by $\rho_{is} + r_{is}$. The DA mechanism makes no modification to priorities, and it corresponds to the gDA mechanism with $(f^{\varphi}(m) = m, g^{\varphi}(n) = 0)$.

Boston (Immediate Acceptance) Mechanism

Given an assignment problem and realized lottery numbers, the **Boston (immediate acceptance) mechanism** (Abdulkadiroğlu and Sönmez, 2003; Ergin and Sönmez, 2006) is defined through the following immediate acceptance algorithm.

- Step 1: Each student i applies to her most preferred acceptable school (if any). Each school accepts its highest-priority (with respect to $\rho_{is} + r_{is}$) students up to its capacity and rejects every other student.

In general, for any step $t \geq 2$,

- Step t: Each student who has not been accepted by any school applies to her most preferred acceptable school that has not rejected her (if any). Each school accepts its highest-priority (with respect to $\rho_{is} + r_{is}$) students up to its remaining capacity and rejects every other student.

The algorithm terminates at the first step in which no student applies to any school. Each student accepted by a school at some step of the algorithm is allocated a seat in that school. The immediate acceptance algorithm differs from the DA algorithm in that when a school accepts a student at a step, in the immediate acceptance algorithm, the student is guaranteed that school, while in the deferred acceptance algorithm, that student may be later displaced by another student with a better priority status.

The Boston mechanism can be interpreted as modifying priorities so that each school prioritizes students ranking it higher over students ranking it lower, and it is known that the Boston mechanism is a gDA mechanism with $(f^{\varphi}(m) = m, g^{\varphi}(n) = (K+1)n)$ (Ergin and Sönmez, 2006). Under this $(f^{\varphi}(m), g^{\varphi}(n))$, any school's modified priority order is lexicographic in preference ranks and priority statuses. That is, $i \succ^{\varphi}_{r_s} i'$ for all i and i' with

$rank_{is} < rank_{i's}$ regardless of the original priorities ρ_{is} and $\rho_{i's}$ and lottery numbers r_{is} and $r_{i's}$; $i \succ^{\varphi}_{r_s} i'$ for all i and i' with $rank_{is} = rank_{i's}$ and $\rho_{is} < \rho_{i's}$ regardless of lottery numbers r_{is} and $r_{i's}$.

Charlotte Mechanism

The mechanism used in Charlotte is the same as the Boston mechanism except that each school respects the walk zone priority ahead of preference ranks so that every student is guaranteed a seat at her walk zone school (Hastings et al., 2009; Deming, 2011; Deming et al., 2014). Assume without loss of generality that $\rho_{is} = 1$ means i has walk zone priority at s. The **Charlotte mechanism** is a gDA mechanism with $(f^{\varphi}(m) = m + 1\{m > 1\}[K + (K + 1)|S|], g^{\varphi}(n) = (K+1)n)$. Under this $(f^{\varphi}(m), g^{\varphi}(n))$, any school's modified priority order is lexicographic in the walk zone priority status, preference ranks, and other (non-walk-zone) priority statuses. That is, $i \succ^{\varphi}_{r_s} i'$ for all i and i' with $\rho_{is} = 1$ and $\rho_{i's} > 1$; $i \succ^{\varphi}_{r_s} i'$ for all i and i' with $1\{\rho_{is} = 1\} = 1\{\rho_{i's} = 1\}$ and $rank_{is} < rank_{i's}$; $i \succ^{\varphi}_{r_s} i'$ for all i and i' with $1\{\rho_{is} = 1\} = 1\{\rho_{i's} = 1\}$, $rank_{is} = rank_{i's}$, and $\rho_{is} < \rho_{i's}$.

3 First-Choice Empirical Research Design

As explained in the introduction, many empirical studies use data from gDA mechanisms to identify and estimate the causal effect of assignment to a treatment school on outcomes such as test scores, crime rates, college attendance, and earnings.[9] Their empirical research designs fall into two categories. I start with analyzing one of them and move on to the other in Section 4.

To describe the first empirical strategy, fix any gDA mechanism φ and assignment problem X that generates the data at hand. Following the standard notation in econometrics, let $D_{is}(r) = 1$ if student i is assigned the treatment school s under (realized or counterfactual) lottery number profile r; $D_{is}(r) = 0$ otherwise. I consider the set of students who rank s first and are in s's "marginal priority group," where some students are assigned s but others are not though all of them share the same priority at s. That is, define

$First_s(r)$

[9]Many of the studies mentioned in the introduction investigate the effect of a group of schools rather than an individual school. My analysis extends to such group-level treatments too. Also, when the effect of interest is that of attendance or enrollment rather than assignment, the analyst would see attendance or enrollment as the endogenous treatment and use assignment as an instrument for the treatment. The analyst would then use an instrumental variable method to estimate the effect of attendance or enrollment. My analysis is applicable to such instrumental variable settings. See footnote 13.

$$\equiv \{i \in I | rank_{is} = 1 \text{ and } \exists i' \text{ such that } rank_{i's} = 1, \rho_{is} = \rho_{i's}, \text{ and } D_{is}(r) \neq D_{i's}(r)\}.^{10}$$

Let r_0 be the realized profile of lottery numbers in the data.

The first widely-used empirical strategy, which I call the first-choice research design, compares the outcomes of students with $D_{is}(r_0) = 1$ against those with $D_{is}(r_0) = 0$ within $First_s(r_0)$.[11] The outcome difference between the two groups is then interpreted as the causal effect of being assigned to school s for students in $First_s(r_0)$. The idea is that since all students in $First_s(r_0)$ rank s first and share the same priority at s, whether they get an offer from s should be determined solely by their lottery numbers and hence independent of students' covariates or choices correlated with outcomes. Therefore, offers from s within $First_s(r_0)$ are thought of as being randomly assigned in a randomized controlled trial. Albeit intuitive, for the first-choice research design to identify a causal effect by this logic, assignments to s within $First_s(r_0)$ have to be indeed random and not confounded by non-random preferences or priorities. This requirement is formalized as the following concept.

Definition 2. The first-choice research design extracts a random assignment for a gDA mechanism φ if at any assignment problem X, for any school s and all potential lottery realizations r and all students $j, k \in First_s(r)$,

$$P(D_{js}(R) = 1) = P(D_{ks}(R) = 1).$$

An equivalent requirement is

$$P(D_{is}(R) = 1 | i \in First_s(r), \theta_i = \theta) = P(D_{is}(R) = 1 | i \in First_s(r)),$$

for any student type θ for which the left-hand-side conditional probability is well-defined. $P(D_{is}(R) = 1 | i \in First_s(r), \theta_i = \theta)$ means the probability of assignment to s for an arbitrary student of type θ in $First_s(r)$.

This property requires that conditional on being in $First_s(r_0)$, offers from s are random and independent of students' preferences and priorities summarized by θ_i. In the econometric

[10] Since $rank_{is} = rank_{i's} = 1$ holds and $f^\varphi(\cdot)$ is strictly increasing by definition, $\rho_{is} = \rho_{i's}$ is equivalent to

$$(f^\varphi(\rho_{is}) + g^\varphi(1) \equiv) \rho^\varphi_{is} = \rho^\varphi_{i's} (\equiv f^\varphi(\rho_{i's}) + g^\varphi(1)).$$

I could therefore replace $\rho_{is} = \rho_{i's}$ with $\rho^\varphi_{is} = \rho^\varphi_{i's}$ in the definition of $First_s(r)$ without changing anything in the following analysis. Finally, note that it is possible $First_s(r) = \emptyset$ for some or even all r.

[11] Applications of the first-choice research design include Hastings et al. (2009); Deming (2011); Abdulkadiroğlu et al. (2014b); Bloom and Unterman (2014); Deming et al. (2014); Angrist et al. (2016).

terminology, this requires that the propensity score (Rosenbaum and Rubin, 1983) is constant across all students in $First_s(r_0)$.[12] Only under this conditionally random assignment are the treatment and control groups in $First_s(r_0)$ comparable with each other. Econometric program evaluation methods require this conditional independence for the first-choice research design to identify a causal treatment effect (Heckman and Vytlacil (2007) chapters 8 and 9, Manski (2008) chapters 3 and 7, Angrist and Pischke (2009) chapter 3.2).[13]

3.1 Motivating Example

Despite its intuitive construction, the first-choice research design may fail to extract a random assignment. Consider the following example.

Example 1. There are applicants 1, 2, 3, and schools A and B with the following preferences and priorities:

$$\succ_1 : A, B, \emptyset$$
$$\succ_2 : A, \emptyset$$
$$\succ_3 : B, A, \emptyset$$
$$\rho_A : 3, \{1, 2\}$$
$$\rho_B : 1, \{2, 3\},$$

where $\succ_1$: $A, B, \emptyset$ means 1 prefers A over B and both schools are acceptable for 1; and $\rho_A : 3, \{1, 2\}$ means that A prioritizes 3 over 1 and 2 and is indifferent between 1 and 2. The capacity of each school is 1. The treatment school is A.

In this example, the first-choice research design does not extract a random assignment for A for the DA mechanism (with no priority modification). Under the DA mechanism, 1 is assigned to A when 1 has a better lottery number than 2 at A. Otherwise, 3 is assigned to A. Each of the two cases occurs with equal probability 0.5. Thus,

$$P(D_{1A}(R) = 1) = 1/2 \neq 0 = P(D_{2A}(R) = 1),$$

[12] It is possible to define random assignment conditional on being in random $First_s(R)$, where R are random lottery numbers and $First_s(R)$ is a random set. My result is robust to using such an alternative definition; see Section 5.1.

[13] When assignment within $First_s(r_0)$ is used as an instrument for an endogenous treatment such as enrollment, Definition 2 is interpreted as a conditional independence requirement for the instrument. For the instrument to identify a causal effect, it usually needs to additionally satisfy properties such as "exclusion" or "monotonicity." See Heckman and Vytlacil (2007); Manski (2008); Angrist and Pischke (2009). In this case, Definition 2 becomes a necessary condition. See also Sections 4 and 5.2 for related discussions.

despite having

$$First_A(r) = \begin{cases} \{1,2\} & \text{if } r_{1A} < r_{2A} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, the first-choice research design does not extract a random assignment for the DA mechanism.[14]

The above problem may bias treatment effect estimates. Imagine that school A has no real treatment effect, and student 1 ranks more schools than student 2 because student 1 is more eager and higher achieving (regardless of whether she attends A). Whenever $First_A(r) = \{1,2\}$, student 1 gets the seat at A and student 2 does not. Comparing 1 and 2 within $First_A(r) = \{1,2\}$, the researcher is likely to mistakenly conclude A has a positive achievement effect. This raises the question: For what mechanisms does the first-choice research design extract a random assignment as desired?

3.2 Strategy-proofness for Schools

The success or failure of the first-choice research design turns out to be linked to a seemingly unrelated property of mechanisms. So far, I have treated priorities and lottery numbers as public information. In this section, I depart from this assumption and imagine a hypothetical thought experiment in which schools have priorities and lottery numbers as their private information, and the priorities and lottery numbers represent school preferences; I come back to the interpretation of this thought experiment at the end of this section. Suppose a gDA mechanism asks schools to report priorities and lottery numbers. Their reports are not necessarily truthful. The gDA mechanism then uses the reported priorities and lottery numbers to create a matching.

Given any assignment problem $(I, S, \succ_I, (c_s)_{s\in S})$, let $\varphi(\rho, r) \equiv (\varphi_s(\rho, r))_{s\in S}$ be the assignment produced by a gDA mechanism φ when the reported priorities and lottery numbers are (ρ, r). School s's preference $\succ_s$, which is defined over *the set of subsets of* I, is said to be responsive with respect to (c_s, ρ_s, r_s) (Roth and Sotomayor, 1992) if

(1) For any $i, i' \in I$, if $\rho_{is} + r_{is} < \rho_{i's} + r_{i's}$, then for any $I' \subseteq I \setminus \{i, i'\}$, $I' \cup \{i\} \succ_s I' \cup \{i'\}$,

(2) $\emptyset \succ_s I'$ for any $I' \subseteq I$ with $|I'| > c_s$, and

[14] It is possible to create a similar counterexample even when there are no priorities as long as ties are broken by MTB. Also, Section 5.2 demonstrates the first-choice research design may fail even if I modify it to the more refined version that pools applicants who rank the treatment school first and share the same priority at *every* school. Conditioning on the whole preference list is not a solution, either.

(3) For any $I' \subseteq I$ with $|I'| < c_s$ and any $i \in I \setminus I'$, it holds $I' \cup \{i\} \succ_s I'$.

I use these concepts to define the following property.

Definition 3. A gDA mechanism φ is **strategy-proof for school** s at assignment problem X if given X, for any priority and lottery number profile (ρ^*, r^*), any preference $\succ_s$ responsive with respect to (c_s, ρ_s^*, r_s^*), and any (ρ_s', r_s'),

$$\varphi_s(\rho^*, r^*) \succeq_s \varphi_s((\rho_s', r_s'), (\rho_{-s}^*, r_{-s}^*)),$$

where $\succeq_s$ is the weak preference associated with $\succ_s$.[15] A gDA mechanism φ is strategy-proof for schools if it is strategy-proof for every school s at every X.

Though my setting is stochastic due to lotteries, this usual definition of strategy-proofness is a non-stochastic, ex post concept. The standard behavioral interpretation of this concept is that no school ever has a preference manipulation that is profitable with respect to its true preference. It is crucial to note, however, that I am *not* concerned with this usual interpretation. As will become clearer in the next section, in contrast to usual studies on strategy-proofness, my analysis treats strategy-proofness not as a desideratum or an incentive compatibility requirement. Instead I use strategy-proofness as a purely algorithmic property and am interested not in strategy-proofness itself but in its implications for empirical research. As a result, I do not need to assume anything about school behavior. Therefore, the following usual questions about strategy-proofness for schools are all irrelevant for my analysis: Do schools have preferences? Are school preferences consistent with priorities? Do schools ever "game" the system?

3.3 Sufficiency: Strategy-proofness Generates Natural Experiments

Strategy-proofness for schools turns out to be sufficient for the first-choice research design to extract a random assignment.

Theorem 1. *The first-choice research design extracts a random assignment for a gDA mechanism φ if φ is strategy-proof for schools.*

The proof is in Appendix A.1. Combined with existing results on strategy-proofness for schools, Theorem 1 provides positive results for the first-choice research design for some of the gDA mechanisms. (I describe another key implication in Section 3.5.)

[15] The domain for (ρ_s^*, r_s^*) and (ρ_s', r_s') is a subset of $\{1, ..., K\}^{|I|} \times [0, 1]^{|I|}$ such that no two students share the same value of $\rho_{is} + r_{is}$ at any school, where $|I|$ is the number of students. This implies every student is acceptable to every school in any (ρ_s^*, r_s^*) and (ρ_s', r_s'). ρ^* may or may not be the same as ρ in X. I do not require either r_s^* or r_s' to be consistent with φ's lottery structure.

Corollary 1. *a) The first-choice research design extracts a random assignment for the Boston mechanism with any lottery regime.*
b) The first-choice research design extracts a random assignment for the DA mechanism with STB when there are no priorities ($\rho_{is} = \rho_{js}$ for all i, j, and s). This mechanism is often called random serial dictatorship.

Proof. (a) follows from Theorem 1 and Ergin and Sönmez (2006)'s Theorem 2 that the Boston mechanism is strategy-proof for schools. (b) follows from the proof of Theorem 1 and the fact that for the DA mechanism, truth-telling is optimal for any school s when all the other schools report the same preference as s's true preference. See Appendix A.2 for details. □

Before providing intuition, I illustrate Theorem 1 with the Boston mechanism. Consider Example 1 in Section 3.1 and a thought experiment where schools have private preferences and the mechanism asks schools to report their preferences. First of all, school A is never matched with student 3 since 3 ranks A second and the seat at A is always filled by one of the two students who rank A first. A is thus matched with either 1 or 2. When A's true preference is such that $1 \succ_A 2$, A is matched with the more preferred student 1 by truth-telling.[16] When A's true preference is with $2 \succ_A 1$, A is matched with the more preferred student 2 by truth-telling. Therefore, unlike with the DA mechanism discussed in Section 3.1, there is no profitable preference manipulation for A; the Boston mechanism is strategy-proof for A in Example 1 (Ergin and Sönmez, 2006).

As it should be by strategy-proofness and Theorem 1, the first-choice research design extracts a random assignment for A in Example 1 for the Boston mechanism. Note that $First_A(r) = \{1, 2\}$ for all r since only 1 and 2 rank A first with the same priority and only one of them with a better lottery number is assigned A under any r. Enumerating all lottery outcomes shows that 1 and 2 share the same assignment probability of $1/2$ at A, i.e., $P(D_{1A}(R) = 1) = P(D_{2A}(R) = 1) = 1/2$. Therefore the first-choice research design extracts a random assignment.

Intuition for the Proof

More generally, the intuition for Theorem 1 is as follows. Readers who are not interested in the proof may skip the remainder of this section and jump to Section 3.4. A sufficient condition for the first-choice research design to extract a random assignment for school s is that as in randomized controlled trials, any permutation or shuffle of lottery numbers

[16] Since A's capacity is 1, I do not need to distinguish its preference over sets of students and its priority order over individual students.

r_{is} within $First_s(r)$ translates into the corresponding permutation of assignments $D_{is}(r)$. Let me name this the "Fisher property" after Ronald Fisher, the inventor of randomized experiments. Strategy-proofness for schools turns out to guarantee this Fisher property, as the following two-step argument illustrates. For simplicity, ignore priority and consider the case with unit school capacities and MTB (school-specific independent lotteries).

The first step of the intuition is summarized as "a preference manipulation is a lottery number permutation." It is a retrospectively obvious observation that with no priority and MTB, each school's strict priority is pinned down solely by lottery numbers at the school. Any preference manipulation by school s in the hypothetical thought experiment (used for defining strategy-proofness) corresponds to a permutation of lottery numbers at s in the real world. This step does not use strategy-proofness for schools.

The second step is summarized as "an *unprofitable* preference manipulation is an *assignment* permutation." This second step is more subtle and claims that when a preference manipulation by school s is unprofitable for s, the associated permutation of lottery numbers r_{is} results in the corresponding permutation of assignments $D_{is}(r)$. To get an idea of this, consider students $i_0, i_1 \in First_s(r)$ such that s prefers/prioritizes i_1 over i_0 in the true preference or the associated default lottery number r_s. Assume that i_1 is assigned to s while i_0 is not under r_s. Also, suppose that there is no third student j whom s prefers over i_0 but dis-prefers over i_1, i.e., $r_{i_1 s} < r_{js} < r_{i_0 s}$.

Now suppose that in the thought experiment, school s manipulates its preferences by swapping students i_1 and i_0. That is, school s reports r'_s with $r'_{i_0 s} = r_{i_1 s}$ and $r'_{i_1 s} = r_{i_0 s}$ but reports all other lottery numbers honestly. What does it mean that the data-generating gDA mechanism is strategy-proof for schools and the manipulation is unprofitable for s? It implies under the unit-capacity assumption that following this misreport, school s's single seat must be assigned to student i_0. To see this, consider how the DA algorithm operates inside the gDA mechanism. In the first round, students i_0 and i_1 will apply to school s (as it is their first choice by $i_0, i_1 \in First_s(r_0)$). School s's single seat will be tentatively assigned to i_0, who has a better (reported) lottery number than i_1 by $r'_{i_0 s} = r_{i_1 s} < r_{i_0 s} = r'_{i_1 s}$. (Recall the assumption made in the last paragraph that $r_{i_1 s} < r_{i_0 s}$, and student i_1 is assigned to school s absent school s's manipulation.) In later rounds, school s would only reject i_0 in favor of some student $j(\neq i_0, i_1)$ with (true) better lottery number $(r_{js} =) r'_{js} < r'_{i_0 s} (= r_{i_1 s})$. But if school s were to get such a student j, then school $s's$ preference manipulation would be profitable, contrary to strategy-proofness for schools. Therefore, student i_0 will be assigned to school s, as claimed.[17]

[17]Note that this reasoning depends on the assumption that i_1 and i_0 rank s first. In fact, Section 4 shows that the intuition described here breaks down if I extend the first-choice research design to another design

This is the desired Fisher property: School s's preference manipulation of switching students i_0 and i_1, which corresponds to permuting i_0 and i_1's lottery numbers at s ($r_{i_0 s}$ and $r_{i_1 s}$), results in the corresponding permutation of i_0 and i_1's assignments at s ($D_{i_0 s}(r)$ and $D_{i_1 s}(r)$). Therefore, if a gDA mechanism is strategy-proof for schools, any preference manipulation is unprofitable and so the Fisher property holds, i.e., any lottery permutation within $First_s(r)$ translates into the corresponding permutation of assignments. This enables lottery permutations to induce a successful randomized experiment within $First_s(r)$, as already explained above.

This intuition is still far from a complete proof, however. For instance, Theorem 1 allows for additional complications like priorities, non-unit capacities, and STB, all of which the above intuition ignores. When lotteries are correlated across schools as in STB, a lottery permutation affects multiple schools simultaneously. This destroys the exact correspondence between a lottery permutation in the real world and a unilateral preference manipulation by a single school in the thought experiment, posing additional challenges. Nevertheless, the proof in Appendix A.1 shows that the conclusion generally holds.

3.4 Almost Necessity

Theorem 1 shows that strategy-proofness for schools is sufficient for the first-choice research design to extract a random assignment. Strategy-proofness turns out to be not only sufficient but also nearly necessary in the following sense.

Proposition 1. *Even with unit school capacities ($c_s = 1$ for all s), the first-choice research design does not extract a random assignment for the DA, Charlotte, and "top trading cycles" mechanisms (with any lottery regime), all which are known to be not strategy-proof for schools.*

To see this, consider the DA mechanism in Example 1. Imagine A's true preference is $3 \succ_A 1 \succ_A 2$ while B's is $1 \succ_B 2 \succ_B 3$. Under these true preferences, A is matched with 1. If A misreports $3 \succ'_A 2 \succ'_A 1$, however, A is matched with 3, the most preferred student with respect to $\succ_A$. Therefore, the DA mechanism is not strategy-proof for A in Example 1. This reconfirms the classic result that the DA mechanism is not strategy-proof for schools (Roth and Sotomayor, 1992).

Intuitively, school A benefits from manipulating its preference and rejecting 1 by the following chain reaction of rejections and applications. After being rejected by A, student 1

that contains students who do not rank s first. For the extended research design, strategy-proofness for schools is shown to be no longer sufficient for a random assignment even with no priorities and unit school capacities.

next applies for B, which results in B's rejecting 3. Student 3, the most preferred student for A, then applies for and benefits A. The same chain reaction causes the first-choice research design to fail. Depending on schools ranked below A, different applicants cause different chain reactions that have different effects on assignment probabilities at A. As I showed in Section 3.1, this can cause applicants in $First_A(r_0)$ to have different assignment probabilities at A.[18]

Similarly, the first-choice research design does not extract a random assignment for the Charlotte mechanism, another mechanism that is not strategy-proof for schools. Suppose that 3's priority at A and 1's priority at B in Example 1 are walk zone priorities. In this case, the Charlotte mechanism coincides with the DA mechanism in Example 1. The Charlotte mechanism is therefore manipulable by schools and the first-choice research design fails in the same way as for the DA mechanism. Section 5.3 also shows the same failure of the first-choice design for yet another mechanism (the top trading cycles mechanism) that is not strategy-proof for schools. In these senses, strategy-proofness for schools is almost necessary for the first-choice research design to extract a random assignment.[19]

Empirical Illustration

Denver Public Schools use the usual DA mechanism for unified public and charter school admissions. As shown above, the DA mechanism is not strategy-proof for schools and may not extract a random assignment via the first-choice research design. To see whether the first-choice research design extracts a random assignment in Denver, I use the data from its DA mechanism in school year 2011-2012 as follows.

(1) Taking student preferences, school priorities, and capacities as fixed, I simulate the DA mechanism by drawing counterfactual lottery numbers one million times. This gives me an approximate assignment probability $\hat{P}(D_{is}(R) = 1)$ for each student i and school s, i.e., the empirical frequency of student i's being assigned to s over the one million simulations.[20]

[18] In contrast, for the Boston mechanism analyzed in the last section, such chain reactions do not affect assignments to A. By its construction, for the Boston mechanism, each school is forced to prioritize students ranking it higher over students ranking it lower. As a result, chain reactions caused by student i at schools ranked below A involve only students who rank A lower than student i does. Since A rejects i, A also rejects any students ranking A lower than i and thus A never accepts any student involved in chain reactions i causes. Thus, different chain reactions caused by different students have the same effect on assignments at A, that is, no effect at all. This is the reason why the Boston mechanism is strategy-proof for schools and the first-choice research design extracts a random assignment for it.

[19] On the other hand, strategy-proofness for schools turns out to be not exactly necessary. See Appendix B.2 for details.

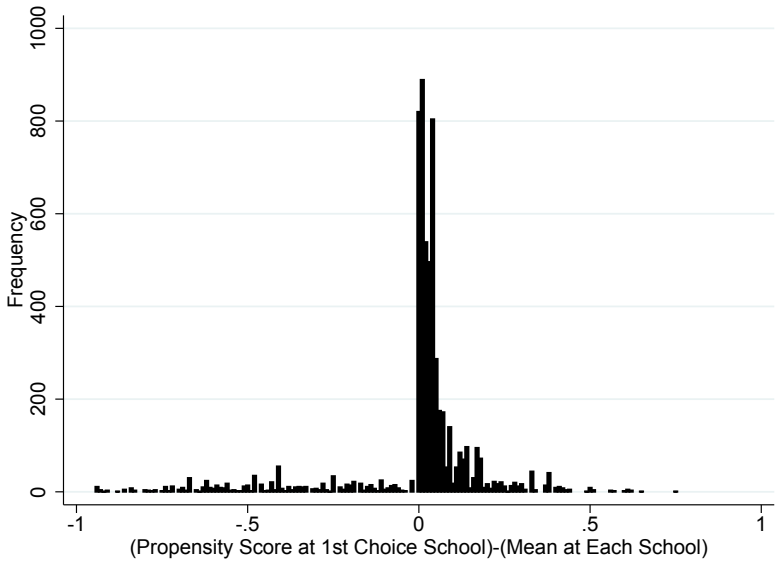
[20] In Denver, each school is divided into multiple sub-schools (called "buckets") with their own priorities

(2) For each school s and each student i in the realized first-choice subsample $First_s(r_0)$ (if any), I demean i's assignment probability by subtracting the mean of assignment probabilities at s across all students in $First_s(r_0)$. That is, I compute

$$\hat{P}^{demean}(D_{is}(R)=1) = \hat{P}(D_{is}(R)=1) - \frac{\Sigma_{j\in First_s(r_0)}\hat{P}(D_{js}(R)=1)}{|First_s(r_0)|}.$$

(3) I plot this assignment probability deviation $\hat{P}^{demean}(D_{is}(R)=1)$ across all schools s and all students in $First_s(r_0)$.

Figure 1: The First Choice Design Fails for the DA Mechanism



Notes: I simulate the DA mechanism with STB by drawing counterfactual lottery numbers one million times. This gives us an approximate assignment probability. For each school s and each student i in $First_s(r_0)$, I demean i's assignment probability by subtracting the mean of assignment probabilities at s across all students in $First_s(r_0)$. I plot the demeaned assignment probability across all students in $\cup_{s\in S}First_s(r_0)$.

The resulting histogram is in Figure 1. If the 1st choice strategy extracts a random assignment, $\hat{P}(D_{is}(R)=1) \approx \hat{P}(D_{js}(R)=1)$ for all s and all $i,j \in First_s(r_0)$ and so the assignment probability deviation $\hat{P}^{demean}(D_{is}(R)=1)$ would be almost 0 (up to simulation errors) for all s and all i in $First_s(r_0)$. As the figure shows, however, there are many values of $\hat{P}^{demean}(D_{is}(R)=1)$ that are far from 0. The mean is almost 0 (by construction) but

and capacities. Buckets correspond to schools in my theoretical model. Below I use "schools" to mean buckets. See Abdulkadiroğlu et al. (2016) for more details of the Denver school admissions system.

the standard deviation is around 0.19. This provides an empirical illustration of the theoretical necessity of strategy-proofness for schools. I leave for future research whether these assignment probability deviations result in any serious bias in treatment effect estimates.

3.5 Explaining Empirical Regularities

My analysis shows that the first-choice research design extracts a random assignment for a gDA mechanism if and almost only if the mechanism is strategy-proof for schools. This justifies its use for strategy-proof mechanisms such as the Boston mechanism. However, care should be taken when using it for non-strategy-proof mechanisms such as the DA, Charlotte, and top trading cycles mechanisms.

There remains the puzzle, however, that even under the DA and Charlotte mechanisms, the first-choice research design often receives empirical support for randomization. In particular, such applications usually find that in the first-choice subsample $First_s(r_0)$, observable covariates of students with $D_{is}(r_0) = 1$ and $D_{is}(r_0) = 0$ are similar, a standard check of a necessary condition for randomization.[21] How can I resolve the tension between their empirical validity findings and my theoretical result?

A potential resolution is hinted by Theorem 1, the sufficiency of strategy-proofness for schools. Unlike small counterexamples like Example 1, empirical work is only done with data with at least hundreds of students. Though the DA and Charlotte mechanisms are not strategy-proof for schools in general, they are often approximately so in certain large markets with many students and schools, as has been shown empirically and theoretically (Roth and Peranson, 1999; Immorlica and Mahdian, 2005; Kojima and Pathak, 2009; Azevedo and Budish, 2013; Lee, 2016; Ashlagi et al., 2016). The reason is that as the number of students and schools grows, chain reactions of rejections and applications at schools ranked below a manipulating school — which make the DA and Charlotte mechanisms manipulable in Example 1 — become less likely to come back to the manipulating school and benefit it. Existing empirical settings with hundred or thousands of students may therefore be subject to large market forces that make the DA and Charlotte mechanisms almost non-manipulable by schools. If so, Theorem 1 suggests the first-choice research design approximately extracts a random assignment even for the DA and Charlotte mechanisms.

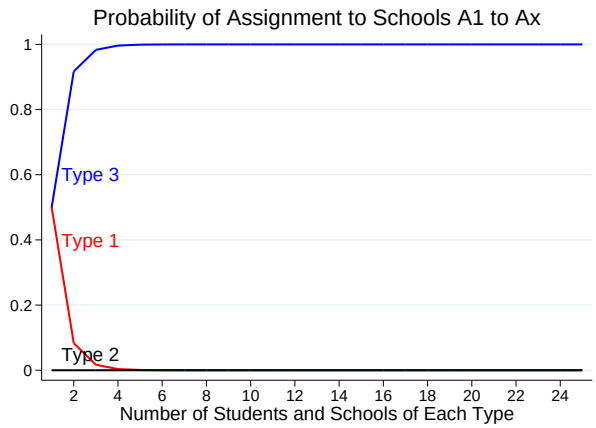
To see the effect of such large market forces, Figure 2 plots assignment probabilities at A for two types of expansions of Example 1 and the DA mechanism (equivalent to the Charlotte mechanism in this example).[22] A computer program to implement this simulation is available

[21] See, for example, Hastings et al. (2009)'s Table VI, Deming (2011)'s Table I, Abdulkadiroğlu et al. (2014b)'s Table 2, Deming et al. (2014)'s Table A2.

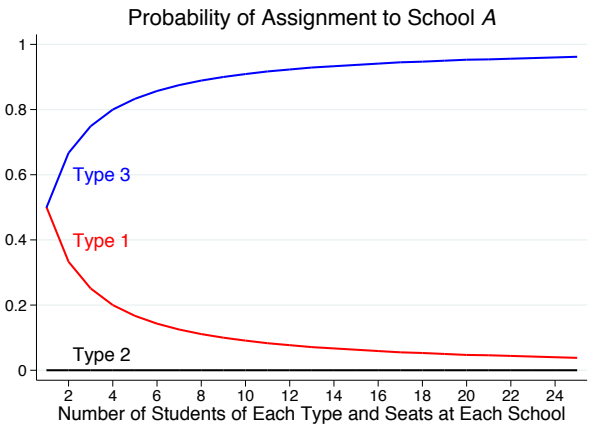
[22] Figure 2b is of more interest in that it is closer to the models of the above papers on strategy-proofness

Figure 2: The Counterexample Evaporates in Large Markets

(a) Increasing Students and Number of Schools



(b) Increasing Students and School *Size*



Notes: In Panel 2a, for each value of the x axis, I create an expansion of Example 1 with $2x$ schools $A_1, ..., A_x, B_1, ..., B_x$ with one seat each, and $3x$ students such that there are x students with each of the following three preferences:

$\succ_1 : A_1, B_1, A_2, B_2, ..., A_x, B_x, \emptyset$
$\succ_2 : A_1, ..., A_x, \emptyset$
$\succ_3 : B_1, A_1, B_2, A_2, ..., B_x, A_x, \emptyset$
$\rho_{A_1}, ..., \rho_{A_x} : \{\text{students with } \succ_3\}, \{\text{students with } \succ_1 \text{ or } \succ_2\}$
$\rho_{B_1}, ..., \rho_{B_x} : \{\text{students with } \succ_1\}, \{\text{students with } \succ_2 \text{ or } \succ_3\}.$

In Panel 2b, for each value of the x axis, I create another expansion of Example 1 with x seats at each of schools A and B, and $3x$ students such that there are x students of each of the three types. For each scenario, I approximate the assignment probabilities by simulating the DA mechanism (equivalent to the Charlotte mechanism in this example) with STB 100000 times.

upon request. This figure reveals that as the market size grows, the discrepancy between student types 1 and 2's assignment probabilities at A disappears, implying that breaks in randomization under the first-choice research design become smaller and smaller. This may explain why the first-choice research design often appears to extract a random assignment in empirical applications even for mechanisms that are not strategy-proof for schools. At the same time, the existing empirical support for the first-choice research design for the DA and Charlotte mechanisms (recall footnote 21) can be re-interpreted as suggesting that large market forces emphasized by the above theoretical market design papers are empirically relevant. Theorem 1 thus provides a delicate asymptotic justification for the research design even for some mechanisms that are not strategy-proof for schools in general.[23]

in large markets. For Figure 2a, Abdulkadiroğlu et al. (2016) complementarily show that in its limit, the first-choice design extracts a random assignment under the DA mechanism.

[23]This suggests that many empirical studies using the first-choice design for non-strategy-proof mechanisms

4 Qualification Instrumental Variable Research Design

While the previous sections focus on the first-choice research design, several empirical studies use an alternative research design, which I call the qualification instrumental variable (IV) research design.[24] Unlike the first-choice research design (which tries to make assignments random by focusing on a subset of students), the qualification IV research design considers all students and tries to code a random instrumental variable for non-random assignments. Define the qualification IV by

$$Z_{is}(r) \equiv 1\{\rho^{\varphi}_{is} + r_{is} \leq \max\{\rho^{\varphi}_{js} + r_{js} | D_{js}(r) = 1\}\}.$$

If there is no j with $D_{js}(r) = 1$, then define $Z_{is}(r) = 1$ for all i. The qualification IV for a student at a school is turned on if her realized priority rank at the school is better than that of some student assigned to the school. Note that $Z_{is}(r) = 1$ is possible even for students who do not apply to school s.

The qualification IV looks random conditional on ρ^{φ}_{is} and is likely to be correlated with assignment $D_{is}(r)$ since i never gets assigned when she is not qualified ($Z_{is}(r) = 0$) but she can get assigned when she is qualified ($Z_{is}(r) = 1$). Based on this idea, the qualification IV research design instruments for assignment $D_{is}(r)$ by qualification $Z_{is}(r)$ conditional on ρ^{φ}_{is} and estimates treatment effects by Two Stage Least Square or other instrumental variable models (Heckman and Vytlacil (2007) chapter 4, Manski (2008) chapter 3, Angrist and Pischke (2009) chapter 4).[25] For this research design to identify a causal effect, the qualification IV needs to be random conditional on ρ^{φ}_{is}, as formalized in the following definition.

Definition 4. The qualification IV research design extracts a random assignment for a gDA mechanism φ for school s at assignment problem X if given φ and X, for all modified priority ρ and student type θ,

---

provide an unexpected set of empirical settings where treatment assignment is asymptotically random (but not exactly random in a finite sample). For inference, therefore, it is appropriate to use recent econometric program evaluation methods based on asymptotically random treatment assignment, such as Canay et al. (2014) and Belloni et al. (2014).

[24] For empirical examples of the qualification IV design, see Pop-Eleches and Urquiola (2013); Dobbie and Fryer (2014); Lucas and Mbiti (2014).

[25] That is, for outcome Y_i of interest and the realized lottery outcome r_0 in the data, the qualification IV research design uses the following Two Stage Least Square regression or a similar IV model:

$Y_i = \alpha_2 + \beta_2 D_{is}(r_0) + \Sigma_{k=1}^{K} \gamma_2^k 1\{\rho^{\varphi}_{is} = k\} + \epsilon_{2i}$ (second stage regression)
$D_{is}(r_0) = \alpha_1 + \beta_1 Z_{is}(r_0) + \Sigma_{k=1}^{K} \gamma_1^k 1\{\rho^{\varphi}_{is} = k\} + \epsilon_{1i}$ (first stage regression)

$$P(Z_{is}(R) = 1 | \rho_{is}^{\varphi} = \rho, \theta_i = \theta) = P(Z_{is}(R) = 1 | \rho_{is}^{\varphi} = \rho).$$

An equivalent requirement is

$$P(Z_{js}(R) = 1) = P(Z_{ks}(R) = 1)$$

for all students $j, k \in I$ with $\rho_{js}^{\varphi} = \rho_{ks}^{\varphi}$. The qualification IV research design extracts a random assignment for a gDA mechanism φ if it does so for every school s at every problem X.

This property requires that conditional on modified priority status ρ_{is}^{φ}, being qualified for s is random and independent of students' preferences and priorities summarized by θ_i. Only under this conditionally random assignment does the qualification IV Z_{is} generate exogenous or random variation in assignment D_{is}.[26] It turns out that no gDA mechanism satisfies the above property even in the simple case with no priorities and unit capacities.

Proposition 2. *Consider any sets of at least three students and at least three schools. Even with no priorities ($\rho_{is} = \rho_{js}$ for all students i, j, and school s) and unit school capacities ($c_s = 1$ for all s), there exist student preference profiles at which every student ranks some schools and the following holds: There is no gDA mechanism with any lottery regime for which the qualification IV research design extracts a random assignment.*

The proof is in Appendix A.3. I illustrate this result by an example.

Example 2. There are applicants 1, 2, 3, 4, 5 and schools A, B, C with the following preferences and priorities:

$\succ_1 : B, A, \emptyset$
$\succ_2 : B, \emptyset$
$\succ_3 : C, A, \emptyset$
$\succ_4, \succ_5 : C, \emptyset$
$\rho_A, \rho_B, \rho_C : \{1, 2, 3, 4, 5\}$.

The capacity of each school is 1. The treatment school is A.

[26] Definition 4 for the qualification IV design may appear to be incomparable with Definition 2 for the first-choice design. However, Appendix B.1 shows that these two definitions are special cases of a unified definition of a random assignment under general empirical research designs, including the first-choice and qualification IV designs. Hence, it is legitimate to use Definitions 2 and 4 to compare the two research designs.

Students 1 and 3 share the same modified priority ρ^{φ}_{iA} for any gDA mechanism φ: Both students 1 and 3 rank A second and have the same priority at A so that $\rho^{\varphi}_{1A} \equiv f^{\varphi}(\rho_{1A}) + g^{\varphi}(rank_{1A}) = f^{\varphi}(\rho_{3A}) + g^{\varphi}(rank_{3A}) \equiv \rho^{\varphi}_{3A}$, which I denote by ρ. Nevertheless, enumerating all possible lottery orders shows that for any gDA mechanism, we have

$$P(Z_{iA}(R) = 1|\rho^{\varphi}_{iA} = \rho, \theta_i = \theta_1) = 2/3 \neq 5/6 = P(Z_{iA}(R) = 1|\rho^{\varphi}_{iA} = \rho, \theta_i = \theta_3) \text{ for STB}$$
$$P(Z_{iA}(R) = 1|\rho^{\varphi}_{iA} = \rho, \theta_i = \theta_1) = 2/3 \neq 3/4 = P(Z_{iA}(R) = 1|\rho^{\varphi}_{iA} = \rho, \theta_i = \theta_3) \text{ for MTB.}$$

A computer program to implement this computation is available upon request. Therefore, even with no priorities and unit capacities, the qualification IV research design does not extract a random assignment for any gDA mechanism.[27]

Intuitively, the qualification IV research design fails in this example because students 1 and 3 experience different levels of competition at their first-choice schools B and C, respectively, before applying for A. Let me consider the following cases.

Case i: Neither student 1 nor 3 applies for A, i.e., 1 and 3 are assigned B and C, respectively. In this case, no student applies for A, and A is undersubscribed. Both 1 and 3 are therefore qualified for A.

Case ii: Only student 1 applies for A. In this case, 1 is always assigned A and qualified for A. By $\rho^{\varphi}_{1A} = \rho^{\varphi}_{3A}$ shown above, student 3 is qualified for A if and only if 3 has a better lottery number than 1 at A.

Case iii: Only student 3 applies for A. In this case, 3 is always assigned A and qualified for A. Student 1 is qualified for A if and only if 1 has a better lottery number than 3 at A.

Case iv: Both students 1 and 3 apply for A. In this case, only one of 1 and 3 with a better lottery number is assigned A and qualified for A.

For simplicity, consider the MTB lottery regime. Cases i and iv are ignorable since they do not cause any difference between 1's and 3's qualification probabilities at A. Conditional

[27]Since both $\rho^{\varphi}_{1A} = \rho^{\varphi}_{3A}$ and $\rho_{1A} = \rho_{3A}$, the counterexample works even if I use original priorities to define an alternative qualification IV as $Z'_{is}(r) \equiv 1\{\rho_{is} + r_{is} \leq \max\{\rho_{js} + r_{js}|D_{js}(r) = 1\}\}$. Also, note that students 1 and 3 share the same priority at all schools in the above example. Thus, the qualification IV research design may fail even if I modify it to the more refined version that conditions on having the same priority at all schools. Finally, the qualification IV research design does not extract a random assignment even for the top trading cycles mechanism, as shown in Section 5.3.

on Case *ii*, student 1 is qualified for sure while 3 is qualified with probability 0.5, the probability that 3 has a better lottery number than 1 for A. Likewise, conditional on Case *iii*, student 3 is qualified for sure, but 1 is qualified only with probability 0.5. Crucially, Case *iii* is more likely to happen than Case *ii* since 3's first choice (C) is more competitive than 1's (B) and so 3 is more easily rejected by the first choice and more likely to apply for A. As a result, 3 is more likely to be qualified for A than 1 due to differential competition at their first-choice schools, as the proof in Appendix A.3 makes it precise. The proof also generalizes this observation to any lottery regime and any market size. Section 5.2 shows that the problem with the qualification IV persists even if I modify its definition, e.g., by changing the priority cutoff $\max\{\rho^{\varphi}_{js} + r_{js} | D_{js}(r) = 1\}$ to a constant number.

The above discussion illustrates a general point that students may have different qualification probabilities depending on which schools they rank higher than the treatment school. This does not matter for students in the first-choice subsample $First_s(r_0)$ since everybody in $First_s(r_0)$ ranks the same schools above the treatment school, that is, no school at all. Therefore, the above trouble does not happen to the first-choice research design focusing on the first-choice subsample $First_s(r_0)$. In this sense, there are more threats to the qualification IV design than to the first-choice design.

Proposition 2 therefore sheds light on a contrast between the qualification IV and the first-choice research designs. Unlike the first-choice research design, strategy-proofness for schools is no longer sufficient for the qualification IV research design to extract a random assignment, and it may extract an unintended broken random assignment not only for the DA or top trading cycles mechanism but also for the Boston mechanism and random serial dictatorship discussed in Corollary 1.

5 Discussion

5.1 Alternative Definition of a Random Assignment

Definition 2 of a "random assignment" requires that all students in realized fixed set $First_s(r_0)$ share the same assignment probability (propensity score). A possible alternative definition treats $First_s(R)$ as random and requires that

$$P(D_{is}(R) = 1 | i \in First_s(R), \theta_i = \theta) = P(D_{is}(R) = 1 | i \in First_s(R))$$

for all i for whom these probabilities are defined. Recall that R denotes the random (not realized) lottery number profile. This alternative definition requires that $D_{is}(R)$ is independent

of type θ_i or the propensity score as a confounder conditional on random event $i \in First_s(R)$. This independence conditional on a random event or statistic is reminiscent of Chamberlain (1980) and Rosenbaum (1984)'s conditional logit panel frameworks, where the treatment distribution is independent of individual heterogeneity conditional on the random empirical frequency of being treated in the past.

All of my arguments extend to this alternative definition. See Appendix A.1 (especially Remark 1) for why Theorem 1 is correct even under the alternative definition. The discussion about the Boston mechanism under Example 1 in Section 3.3 goes through even under the alternative definition since $P(D_{iA}(R) = 1 | i \in First_A(R), \theta_i = \theta) = 1/2$ for $i = 1, 2$ and is independent of θ. The analysis of the DA and Charlotte mechanisms in Example 1 in Sections 3.1 and 3.4 also remains essentially the same since

$$P(D_{iA}(R) = 1 | i \in First_A(R), \theta_i = \theta_1) = 1 \neq 0 = P(D_{iA}(R) = 1 | i \in First_A(R), \theta_i = \theta_2),$$

where θ_1 and θ_2 denote student types having $\succ_1$ and $\succ_2$, respectively. This shows that the first-choice design does not extract a random assignment.

5.2 Alternative Definitions of Research Designs

Section 4 shows a potential problem with the qualification IV $Z_{is}(r) \equiv 1\{\rho^{\varphi}_{is} + r_{is} \leq \max\{\rho^{\varphi}_{js} + r_{js} | D_{js}(r) = 1\}\}$, where $\max\{\rho^{\varphi}_{js} + r_{js} | D_{js}(r) = 1\}$ is a random priority cutoff that varies as the lottery outcome changes. One may expect a modification of the qualification IV to solve the problem. For any constant $\pi \in [0, K+1]$, define the **constant cutoff qualification IV** by

$$Z^{\pi}_{is}(r) \equiv 1\{\rho^{\varphi}_{is} + r_{is} \leq \pi\}.$$

In practice, the econometrician would define $\pi \equiv \max\{\rho^{\varphi}_{js} + r_{0js} | D_{js}(r_0) = 1\}$ where r_0 is the realized lottery numbers in the data. The constant cutoff qualification IV trivially extracts a random assignment since $P(Z^{\pi}_{is}(R) = 1 | \rho^{\varphi}_{is} = \rho, \theta_i = \theta) = P(r_{is} \leq \pi - \rho^{\varphi}_{is} | \rho^{\varphi}_{is} = \rho, \theta_i = \theta) = \pi - \rho$, which is constant and independent of θ conditional on $\rho^{\varphi}_{is} = \rho$.

However, the constant cutoff qualification IV entails new problems other than randomness. First, when using $\pi \equiv \max\{\rho^{\varphi}_{js} + r_{0js} | D_{js}(r_0) = 1\}$, I define or select an instrument depending on the realized data. Such data-dependent model selection often makes standard inference invalid (Leamer, 1978). In addition, perhaps more importantly, the constant cutoff qualification IV may violate other requirements for a valid IV than random assignment or

independence even with no priority. To see this, consider the following example.

Example 3. There are applicants 1, 2, 3, 4 and schools A and B with the following preferences and priorities:

$$\succ_1, \succ_2 : A, B, \emptyset$$
$$\succ_3, \succ_4 : B, \emptyset$$
$$\rho_A, \rho_B : \{1, 2, 3, 4\}.$$

The capacity of each school is 1. The treatment school is A. Without loss of generality, set $\rho^{\varphi}_{1A} = \rho^{\varphi}_{2A} = 0$.

Consider two lottery outcomes at school A:

- $r_A \equiv (r_{1A}, r_{2A}, r_{3A}, r_{4A})$ with $r_{1A} < r_{2A} < r_{3A}, r_{4A}$ and $r_{2A} \leq 0.5$
- $r'_A \equiv (r'_{1A}, r'_{2A}, r'_{3A}, r'_{4A})$ with $r'_{3A}, r'_{4A} < r'_{2A} < r'_{1A}$ and $r'_{2A} > 0.5$.

Fix any lottery numbers r_B at school B. Clearly, $Z^{0.5}_{1A}(r_A, r_B) = Z^{0.5}_{2A}(r_A, r_B) = 1$ and $Z^{0.5}_{1A}(r'_A, r_B) = Z^{0.5}_{2A}(r'_A, r_B) = 0$. On the other hand, for any gDA mechanism, $D_{1A}(r_A, r_B) = 1, D_{1A}(r'_A, r_B) = 0, D_{2A}(r_A, r_B) = 0$, and $D_{2A}(r'_A, r_B) = 1$. This violates the "monotonicity" requirement for $Z^{0.5}_{iA}$ as an instrument for D_{iA}: Endogenous variables D_{1A} and D_{2A} move in the opposite directions in response to the same change in the IV from $Z^{0.5}_{iA} = 1$ to $Z^{0.5}_{iA} = 0$. Monotonicity is required by many modern IV models with heterogeneous behavior and treatment effects (Heckman and Vytlacil (2007) chapter 4, Manski (2008) chapter 3, Angrist and Pischke (2009) section 4.4). Therefore, while the constant cutoff qualification IV always extracts a random assignment, it may not be able to identify a causal effect due to monotonicity violations.

Furthermore, since $First_A(r) = \{1, 2\}$ for all r, this monotonicity violation persists even if I restrict the sample to $First_A(r_0)$. Also, since both $\rho^{\varphi}_{1A} = \rho^{\varphi}_{2A}$ and $\rho_{1A} = \rho_{2A}$, the counterexample works even if I use original priorities to define an alternative constant cutoff qualification IV as $Z^{0.5}_{is}(r) \equiv 1\{\rho_{iA} + r_{is} \leq 0.5\}$. Finally, note that 1 and 2 share the same priority at all schools. Thus the constant cutoff qualification IV research design may fail to satisfy monotonicity even if I modify it to the more refined version that conditions on having the same priority at all schools.

Example 3 also shows that yet another potential modification of the qualification IV does not extract a random assignment either. For any positive integer m, define the **constant rank qualification IV** by

$$Z_{is}^{m\text{-th}}(r) \equiv 1\{r_{is} \leq m\text{-th}(\{r_{js}|j \in I\})\},$$

where m-th$(\cdot)$ is the m-th order statistic.[28] The constant rank qualification IV extracts a random assignment since $P(Z_{is}^{m\text{-}th}(R) = 1|\rho_{is}^{\varphi} = \rho, \theta_i = \theta) = m/|I|$ is independent of θ. However,

- $Z_{1A}^{2nd}(r_A, r_B) = Z_{2A}^{2nd}(r_A, r_B) = 1 (= Z_{1A}^{0.5}(r_A, r_B) = Z_{2A}^{0.5}(r_A, r_B))$ and
- $Z_{1A}^{2nd}(r'_A, r_B) = Z_{2A}^{2nd}(r'_A, r_B) = 0 (= Z_{1A}^{0.5}(r'_A, r_B) = Z_{2A}^{0.5}(r'_A, r_B))$.

Therefore, by the same reason for $Z_{iA}^{0.5}$, potential IV Z_{iA}^{2nd} violates monotonicity regardless of whether I restrict the sample to $First_A(r_0)$.[29] The above discussion also shows that the simplest possible IV, the random number r_i itself, suffers from the same monotonicity violation.

Finally, going back to the original first-choice and qualification IV research designs, they might fail to extract a random assignment even if I modify them to the more refined version that conditions on sharing the same priority at *every* school. Consider the following modification of Example 1.

Example 4. There are applicants 1, 2, 3, and schools A and B with the following preferences and priorities:

$\succ_1 : A, B, \emptyset$
$\succ_2 : A, \emptyset$
$\succ_3 : B, A, \emptyset$
$\rho_A : 3, \{1, 2\}$
$\rho_B : \{1, 2, 3\},$

where ρ_{3A} is walk zone priority. The indifferences in the school priorities are broken by STB. The capacity of each school is 1. The treatment school is A.

The only difference from Example 1 is ρ_B: School B is now indifferent among all students. In Example 4, students 1 and 2 rank A first and share the same priority at both A and B. However, students 1 and 2 do not share the same assignment probability at A for the DA or

[28] Other possible definitions are $Z_{is}^{m\text{-th}}(r) \equiv 1\{\rho_{is}^{\varphi} + r_{is} \leq m\text{-}th(\{\rho_{js}^{\varphi} + r_{js}|j \in I, \rho_{js}^{\varphi} = \rho_{is}^{\varphi}\})\}$ or $1\{\rho_{is} + r_{is} \leq m\text{-th}(\{\rho_{js} + r_{js}|j \in I, \rho_{js} = \rho_{is}\})\}$. The discussion below applies to these alternative definitions too.

[29] All of the above points in this Section 5.2 apply to the top trading cycles mechanism, as shown in the next Section 5.3.

Charlotte mechanism: Under the DA mechanism,

$$First_A(r) = \begin{cases} \emptyset & \text{if } r_2 < r_1 < r_3 \\ \{1,2\} & \text{otherwise,} \end{cases}$$

where r_i is student i's lottery number used by both schools. Nevertheless, enumerating all lottery outcomes shows that

$$P(D_{iA}(R) = 1|\theta_i = \theta_1) = P(Z_{iA}(R) = 1|\theta_i = \theta_1) = 1/2$$
$$\neq 1/3 = P(D_{iA}(R) = 1|\theta_i = \theta_2) = P(Z_{iA}(R) = 1|\theta_i = \theta_2).$$

Thus neither the first-choice or qualification IV research design extracts a random assignment for the DA mechanism even if they condition on sharing the same priority at every school. For the first-choice research design, this point remains true even if using the alternative random assignment criterion in Section 5.1 since $P(D_{iA}(R) = 1|i \in First_A(R), \theta_i = \theta_1) = 3/5 \neq 2/5 = P(D_{iA}(R) = 1|i \in First_A(R), \theta_i = \theta_2)$.

Likewise, conditioning on the whole preference list is not a solution. Consider yet another modification of Example 1.

Example 5. There are applicants 1, 2, 3, and schools A and B with the following preferences and priorities:

$$\succ_1 : A, B, \emptyset$$
$$\succ_2 : A, B, \emptyset$$
$$\succ_3 : B, A, \emptyset$$
$$\rho_A : 3, \{1, 2\}$$
$$\rho_B : 1, 3, 2,$$

where ρ_{3A} and ρ_{1B} are walk zone priority. The capacity of each school is 1. The treatment school is A.

The key difference from Example 1 is $\succ_2$: Students 1 and 2 share the same preference list, i.e., $\succ_1$=$\succ_2$. Nevertheless, the first-choice or qualification IV research design does not extract a random assignment for the DA or Charlotte mechanism: 1 is assigned to A when 1 has a better lottery number than 2 at A. Otherwise, 3 is assigned to A. Each of the two cases occurs with equal probability 1/2. Thus,

$$P(D_{1A}(R)=1)=P(Z_{1A}(R)=1)=1/2\neq 0=P(Z_{2A}(R)=1)=P(D_{2A}(R)=1),$$

despite having

$$First_A(r)=\begin{cases}\{1,2\} & \text{if } r_{1A}<r_{2A}\\ \emptyset & \text{otherwise.}\end{cases}$$

Therefore, the first-choice or qualification IV research design does not necessarily extract a random assignment for the DA or Charlotte mechanism even if one additionally conditions on the entire preference list.

5.3 Top Trading Cycles Mechanism

Some cities such as New Orleans and San Francisco have used a mechanism outside the generalized DA class studied in this paper. This mechanism, the top trading cycles mechanism, is also advocated by matching market design researchers as a Pareto efficient mechanism that is strategy-proof for *students* (Abdulkadiroğlu and Sönmez, 2003).

Definition 5. The **top trading cycles (TTC) mechanism** creates a matching through the following procedure. Take any assignment problem as given.

(1) Draw r according to its lottery regime (STB or MTB).

(2) Define s's ex post strict priority order $\succ_{r_s}$ over students by $i \succ_{r_s} i'$ if $\rho_{is}+r_{is}<\rho_{i's}+r_{i's}$.

(3) Given $\succ_I$ and $(\succ_{r_s})_{s\in S}$, run the following **top trading cycles algorithm** (Shapley and Scarf, 1974).

- Step $t\geq 1$: Each student i points to her most preferred acceptable remaining school (if any). Students who do not point to any school are assigned to $\emptyset$. Each school s points to its most preferred student. As there are a finite number of schools and students, there exists at least one cycle, i.e., a sequence of distinct schools and students $(i_1,s_1,i_2,s_2,\ldots,i_L,s_L)$ such that student i_1 points to school s_1, school s_1 points to student i_2, student i_2 points to school s_2, …, student i_L points to school s_L, and, finally, school s_L points to student i_1. Every student i_l $(l=1,\ldots,L)$ in any cycle is assigned to the school she is pointing to. Any student who has been assigned a school seat or the outside option as well as any school s which has been assigned students such that the number of them is equal to its capacity c_s is removed. If no student remains, the algorithm terminates. Otherwise, it proceeds to the next step.

This algorithm terminates in a finite number of steps because at least one student is matched with a school (or $\emptyset$) at each step and there are only a finite number of students.

It is possible to apply the first-choice research design to data from the TTC mechanism. However, the TTC implementation of the first-choice research design turns out not to extract a random assignment, as stated in Proposition 1.[30] In Example 1, the TTC mechanism always assigns 1 to A under all lottery realizations r (regardless of whether the lottery structure is STB or MTB); no randomization occurs. Therefore, $First_A(r) = \{1, 2\}$ for all r, but $P(D_{is}(R) = 1|\theta_i = \theta_1) = 1 \neq 0 = P(D_{is}(R) = 1|\theta_i = \theta_2)$. Thus the first-choice research design does not extract a random assignment for the TTC mechanism.[31] The same logic also implies that the qualification IV research design fails in this example. In fact, the qualification IV research design for the TTC mechanism does not extract a random assignment even without priorities. It can be seen from Example 2, since there are no priorities and the top trading cycles mechanism is equivalent to the DA mechanism.

This failure of the first-choice research design is related to the fact that the TTC mechanism is not strategy-proof for schools. In Example 1, imagine A's true preference is $3 \succ_A 2 \succ_A 1$ while B's is $1 \succ_B 2 \succ_B 3$. Under these true preferences, A is matched with 1. If A misreports $2 \succ'_A 1, 3$, however, A is matched with 2, who is preferred to 1 under $\succ_A$. Therefore, the TTC mechanism is not strategy-proof for A in Example 1. This is a reconfirmation of the well-known fact that the TTC mechanism is not strategy-proof for schools, and provides further support for the necessity of strategy-proofness for schools for successful randomization under the first-choice research design.

6 Conclusion

The above analysis provides a formal basis for understanding when and why the two popular empirical research designs do or do not extract a random assignment. The first-choice design does so for mechanisms that are strategy-proof for schools; the design may break down for other mechanisms, but the problem approximately goes away in certain large markets where these mechanisms become approximately strategy-proof. On the other hand, the qualification IV design does not necessarily extract a random assignment for any mechanism,

[30] It is possible to show that the more refined version of the first-choice design which conditions on having the same priority at every school extracts a random assignment for the TTC mechanism. This makes a contrast with the DA and Charlotte mechanisms, for which even the more refined first-choice design does not extract a random assignment (recall Section 5.2).

[31] This point remains the same under the alternative definition of extracting a random assignment in Section 5.1: $P(D_{iA}(R) = 1|i \in First_A(R), \theta_i = \theta_1) = 1 \neq 0 = P(D_{iA}(R) = 1|i \in First_A(R), \theta_i = \theta_2)$.

thus suggesting a difference between the two research designs. Table 1 in the introduction provides a summary of the main results.

This paper takes a step toward deciphering theoretical structures hidden in empirical research designs exploiting market design with lotteries. This opens the door to several open questions. For example, while my framework assumes the use of random lottery numbers for tie-breaking, some existing empirical studies use data with regression-discontinuity-style tie-breaking by admissions test scores. I would expect the point of the current paper to be valid even in regression discontinuity situations, but it is open how to extend this paper's results to a regression discontinuity setting. Also, my results point to the importance of strategy-proofness for schools within the mechanism class I focus on. It is thus important to characterize or axiomatize mechanisms that are strategy-proof for schools in the class. It is also a technical open question to use Theorem 1 and existing results on strategy-proofness in the large to formally justify the first-choice research design in large markets.

An even more ambitious agenda is to design assignment mechanisms that enable as informative causal inference as possible (subject to welfare and strategic considerations). For example, it is intriguing to compare the Boston, DA, and top trading cycles mechanisms with different lottery regimes by their capabilities for quasi-experimental information production. The contrast between Corollary 1 and Proposition 1 is a step toward such a comparison. Finally, the empirically most important direction is to see if possible randomization failures (recall Propositions 1 and 2) cause significant biases in treatment effect estimates in real data. I leave these challenging directions for future research.

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Appendix (For Online Publication)

A Proofs

A.1 Proof of Theorem 1

Preliminaries

I use the following lemma to prove Theorem 1.

Lemma 1. *a) The following holds for any assignment problem and any gDA mechanism φ that is strategy-proof for schools. For each lottery number profile r, let $\delta_s(r)$ be any permutation of r_s that switches only i' and i'' such that $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$ and $\min\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} > \rho^{\varphi}_{is} + r_{is}$ for all i with $D_{is}(r) = 1$. If there are no such i' and i'', let $\delta_s(r) = r_s$. Then $\varphi(r) = \varphi(\delta_s(r), r_{-s})$, where $\varphi(r)$ is a shorthand for $\varphi(\rho, r)$, the assignment produced by a gDA mechanism φ when the reported priorities and lottery numbers are (ρ, r).*

b) The following holds for any assignment problem and any gDA mechanism φ. For each lottery number profile r, let $\delta_s(r)$ be any permutation of r_s that switches only two students i' and i'' such that $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$ and there exists i with $D_{is}(r) = 1$ such that $\max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{is} + r_{is}$. If there are no such i' and i'', let $\delta_s(r) = r_s$. Then $\varphi(r) = \varphi(\delta_s(r), r_{-s})$.

Proof of Lemma 1. a) For any assignment problem, a deterministic **assignment** (or a matching) is a vector μ that assigns each school s a set of at most c_s students $\mu_s \subset I$, and assigns each student i a seat at a school or the outside option $\mu_i \in S \cup \{\emptyset\}$. A matching μ is **individually rational** if $\mu_i \succeq_i \emptyset$ for every $i \in I$. With the notation $\rho^{\varphi} \equiv (\rho^{\varphi}_{is})_{i \in I, s \in S}$, μ is (ρ^{φ}, r)**-blocked** by $(s, i) \in S \times I$ if $s \succ_i \mu_i$ and either $|\mu_s| < c_s$ or there exists $\bar{i} \in I$ such that $i \succ^{\varphi}_{r_s} \bar{i}$, i.e., $\rho^{\varphi}_{is} + r_{is} < \rho^{\varphi}_{\bar{i}s} + r_{\bar{i}s}$. A matching μ is (ρ^{φ}, r)**-stable** if it is individually rational and not (ρ^{φ}, r)-blocked by (s, i).

Fact 1. *(Roth and Sotomayor (1992)'s Theorem 5.8) For any assignment problem X, any lottery number profile r, any gDA mechanism φ, any (ρ^{φ}, r)-stable matching μ, any school s, any preference $\succ^{\varphi}_s$ responsive with respect to (c_s, ρ^{φ}, r), it holds $\mu_s \succeq^{\varphi}_s \varphi_s(\rho, r)$, where $\varphi_s(\rho, r)$ is the set of students assigned to s in the outcome of φ under X and r.*

Fact 2. *(Roth and Sotomayor (1989)'s Theorem 4) For any assignment problem X, any lottery number profile r, and any gDA mechanism φ, let μ and μ' be (ρ^{φ}, r)-stable matchings with $\mu_s \succ^{\varphi}_s \mu'_s$ for some preference $\succ^{\varphi}_s$ responsive with respect to $(c_s, \rho^{\varphi}_s, r_s)$. Then, for any $i \in \mu_s$ and $i' \in \mu'_s \setminus \mu_s$, it holds $i \succ^{\varphi}_{r_s} i'$, i.e., $\rho^{\varphi}_{is} + r_{is} < \rho^{\varphi}_{i's} + r_{i's}$.*

Fact 3. *Under any assignment problem, any lottery number profile r, any gDA mechanism φ, any $\delta_s(r)$ satisfying the conditions in the statement of Lemma 1.a, every (ρ^φ, r)-stable matching is also $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable.*

Proof of Fact 3 . Let μ be a (ρ^φ, r)-stable matching. $i', i'' \notin \mu_s$ by Facts 1 and 2 and the assumption that $i', i'' \notin \varphi_s(\rho, r)$. The only potential $(\rho^\varphi, (\delta_s(r), r_{-s}))$-blocking pairs against μ are (s, i') and (s, i'') since the only difference between r and $(\delta_s(r), r_{-s})$ is the positions of i' and i'' in r_s. Since (1) both μ and $\varphi(\rho, r)$ are (ρ^φ, r)-stable and (2) $\min\{\rho^\varphi_{i's} + r_{i's}, \rho^\varphi_{i''s} + r_{i''s}\} > \rho^\varphi_{is} + r_{is}$ for all i with $D_{is}(r) = 1$, Facts 1 and 2 imply that for any $i \in \mu_s$, $\rho^\varphi_{is} + r_{is} < \min\{\rho^\varphi_{i's} + r_{i's}, \rho^\varphi_{i''s} + r_{i''s}\} = \min\{\rho^\varphi_{i's} + \delta_{i's}(r), \rho^\varphi_{i''s} + \delta_{i''s}(r)\}$, where the last equality comes from $\rho^\varphi_{i's} = \rho^\varphi_{i''s}$. Neither (s, i') nor (s, i'') thus $(\rho^\varphi, (\delta_s(r), r_{-s}))$-blocks μ. □

Fact 4. *Under any assignment problem, any lottery number profile r, any gDA mechanism φ that is strategy-proof for schools, any $\delta_s(r)$ satisfying the conditions in the statement of Lemma 1.a, every $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable matching is also (ρ^φ, r)-stable.*

Proof of Fact 4 . Suppose to the contrary that μ is $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable but not (ρ^φ, r)-stable. I show that it contradicts the assumption that φ is strategy-proof for schools.

Step 1.1. Either student i' or i'' (but not both) is in μ_s.

Proof of Step 1.1. Case i: Suppose that neither i' nor i'' (but not both) is in μ_s. By the $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stability of μ, neither (s, i') or (s, i'') does $(\rho^\varphi, (\delta_s(r), r_{-s}))$-block μ. Either (s, i') or (s, i'') does (ρ^φ, r)-blocks μ, since (1) μ is $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable and so individually rational and (2) the only difference between r and $(\delta_s(r), r_{-s})$ is the positions of i' and i'' in r_s. That is, there exists $i \in \mu_s$ such that $\min\{\rho^\varphi_{i's} + r_{i's}, \rho^\varphi_{i''s} + r_{i''s}\} < \rho^\varphi_{is} + r_{is} < \min\{\rho^\varphi_{i's} + \delta_{i's}(r), \rho^\varphi_{i''s} + \delta_{i''s}(r)\} = \min\{\rho^\varphi_{i's} + r_{i's}, \rho^\varphi_{i''s} + r_{i''s}\}$, a contradiction. *Case ii*: Suppose that both i' and i'' are in μ_s. Since μ is not (ρ^φ, r)-stable but is individually rational (by its $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stability), there exists i such that $s \succ_i \mu_i$ and $\rho^\varphi_{is} + r_{is} < \max\{\rho^\varphi_{i's} + r_{i's}, \rho^\varphi_{i''s} + r_{i''s}\} = \max\{\rho^\varphi_{i's} + \delta_{i's}(r), \rho^\varphi_{i''s} + \delta_{i''s}(r)\}$, a contradiction to the $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stability of μ. □

Step 1.2. $\varphi_s(r) \neq \varphi_s(\delta_s(r), r_{-s})$.

Proof of Step 1.2. $i', i'' \notin \varphi_s(r)$ (by assumption), either i' or i'' is in μ_s (by Step 1.1), and Fact 3 implies that (ρ^φ, r)-stable $\varphi_s(r)$ is $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable. Thus, Facts 1 and 2 implies that for any $\succ_s^{\varphi\delta}$ responsive with respect to $(c_s, \rho_s^\varphi, (\delta_s(r), r_{-s}))$, it holds that $\varphi_s(r) \succ_s^{\varphi\delta} \mu_s$, where I use the fact that μ is not (ρ^φ, r)-stable and so $\mu \neq \varphi(r)$. Fact 1 implies that for any $\succ_s^{\varphi\delta}$ responsive with respect to $(c_s, \rho_s^\varphi, (\delta_s(r), r_{-s}))$, it holds that $\mu_s \succeq_s^{\varphi\delta} \varphi_s(\delta_s(r), r_{-s})$. The two preference relations $\varphi_s(r) \succ_s^{\varphi\delta} \mu_s$ and $\mu_s \succeq_s^{\varphi\delta} \varphi_s(\delta_s(r), r_{-s})$ jointly imply that for any $\succ_s^{\varphi\delta}$ responsive with respect to $(c_s, \rho_s^\varphi, (\delta_s(r), r_{-s}))$, $\varphi_s(r) \succ_s^{\varphi\delta} \varphi_s(\delta_s(r), r_{-s})$, implying $\varphi_s(r) \neq \varphi_s(\delta_s(r), r_{-s})$. □

Step 1.3. $\varphi_s(r) \succ_s^\varphi \varphi_s(\delta_s(r), r_{-s})$ for some preference $\succ_s^\delta$ responsive with respect to $(c_s, \rho_s, \delta_s(r))$, a contradiction to the assumption that φ is strategy-proof for schools.

Proof of Step 1.3. Take any $\succ_s$ responsive with respect to (c_s, ρ_s, r_s). If $\varphi_s(\delta_s(r), r_{-s}) \succ_s \varphi_s(r)$, then it is a contradiction to the assumption that φ is strategy-proof for schools. This implies $\varphi_s(r) \succ_s \varphi_s(\delta_s(r), r_{-s})$ since $\varphi_s(r) \neq \varphi_s(\delta_s(r), r_{-s})$ as shown in Step 1.2. Note that $\succ_s$ is also responsive with respect to $(c_s, \rho_s, \delta_s(r))$ since (1) the only difference between r and $(\delta_s(r), r_{-s})$ is the positions of i' and i'' in r_s, and (2) $i', i'' \notin \varphi_s(r)$ by assumption. Therefore, $\varphi_s(r) \succ_s^\varphi \varphi_s(\delta_s(r), r_{-s})$ for some $\succ_s^\delta$ responsive with respect to $(c_s, \rho_s, \delta_s(r))$. □

□

Facts 3 and 4 imply that under any assignment problem, any lottery number profile r, any gDA mechanism φ that is strategy-proof for schools, any $\delta_s(r)$ satisfying the conditions in the statement of Lemma 1.a, the set of (ρ^φ, r)-stable matchings coincides with the set of $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable matchings. Each student or school except s has the same set of responsive preferences over these common stable matchings both under r and $(\delta_s(r), r_{-s})$. School s also has the same preference over these stable matchings both under r and $(\delta_s(r), r_{-s})$ since (1) the only difference between r and $(\delta_s(r), r_{-s})$ is the positions of i' and i'' in r_s, and (2) $\min\{\rho_{i's}^\varphi + r_{i's}, \rho_{i''s}^\varphi + r_{i''s}\} > \rho_{is}^\varphi + r_{is}$ for all i with $D_{is}(r) = 1$ and thus $D_{i's}(r) = D_{i''s}(r) = 0$, which in turn implies $i', i'' \notin \mu_s$ for any (ρ^φ, r)- or $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable matching μ. Therefore, the school pessimal (ρ^φ, r)-stable matching is the same as the school pessimal $(\rho^\varphi, (\delta_s(r), r_{-s}))$-stable matching, i.e., $\varphi(r) = \varphi(\delta_s(r), r_{-s})$.

b) Under r or $(\delta_s(r), r_{-s})$, let t_0 be the step in the DA algorithm at which either i' or i'' or both first apply for s. If there is no such a step t_0 under either r or $(\delta_s(r), r_{-s})$, then the DA algorithm works in the same way until its end both under r and $(\delta_s(r), r_{-s})$, completing the proof. Assume the existence of such a step t_0 under both r and $(\delta_s(r), r_{-s})$. Until step $t_0 - 1$, the DA algorithm operates in the same way both under r and $(\delta_s(r), r_{-s})$ since the only

difference between the two situations is the positions of i' and i'' in r_s. t_0 is thus common to r and $(\delta_s(r), r_{-s})$. Let I_{st_0} be the set of students who are kept by s from step $t_0 - 1$ or newly apply for s in step t_0. I_{st_0} is again the same between r and $(\delta_s(r), r_{-s})$. There are a few cases to consider.

Case I: Both i' and i'' apply for s at step t_0. Under r, s tentatively accepts both i' and i'' by the assumption that there exists i with $D_{is}(r) = 1$ such that $\max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{is} + r_{is}$. Under $(\delta_s(r), r_{-s})$, s again tentatively accepts both i' and i''. This is because $\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_0}} = \{\rho^{\varphi}_{is} + \delta_i(r_s)\}_{i \in I_{st_0}}$ (recall $i', i'' \in I_{st_0}$ by assumption and I_{st_0} is the same between r and $(\delta_s(r), r_{-s})$) and the above fact that s tentatively accepts both i' and i'' under r, which jointly imply that $\max\{\rho^{\varphi}_{i's} + \delta_{i's}(r), \rho^{\varphi}_{i''s} + \delta_{i''s}(r)\} = \max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_0}})$ where c_s-th$(\cdot)$ is the c_s-th order statistic. The DA algorithm also works in the same way for the remaining steps.

Case II: Only one of i' or i'' applies for s at step t_0. Without loss of generality, suppose only i' applies for s at step t_0. Under r, s tentatively accepts i' by the assumption that there exists i with $D_{is}(r) = 1$ such that $\rho^{\varphi}_{i's} + r_{i's} \leq \max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{is} + r_{is}$.

Case II.A: $r_{i's} > r_{i''s}$ (and so $\delta_{i's}(r) < \delta_{i''s}(r)$). Under $(\delta_s(r), r_{-s})$, s also tentatively accepts i' by the following reason. By the above fact that s tentatively accepts i' under r, it holds that $\rho^{\varphi}_{i's} + r_{i's} \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_0}})$, implying $\rho^{\varphi}_{i's} + \delta_{i's}(r) < \rho^{\varphi}_{i's} + r_{i's} \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + \delta_{is}(r)\}_{i \in I_{st_0}})$.

Case II.B: $r_{i's} < r_{i''s}$. Under $(\delta_s(r), r_{-s})$, s also tentatively accepts i' by the following reason. Suppose not. Then $c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_0}}) \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + \delta_i(r_s)\}_{i \in I_{st_0}}) < \rho^{\varphi}_{i's} + \delta_{i's}(r) = \rho^{\varphi}_{i''s} + r_{i''s}$, where the first inequality $r_{i's} < r_{i''s}$ and the last equality uses the assumption $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$. Let's call $c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_0}})$ the *tentative cutoff for school s at step t_0*, which is common between r and $(\delta_s(r), r_{-s})$ since I_{st_0} is the same between r and $(\delta_s(r), r_{-s})$ and $r_{is} = \delta_{is}(r)$ for all $i \in I_{st} \setminus i'$. Since the tentative cutoff is monotonically decreasing in steps, the above inequality implies that for all i with $D_{is}(r) = 1$, $\rho^{\varphi}_{is} + r_{is} < \rho^{\varphi}_{i's} + \delta_{i's}(r) = \rho^{\varphi}_{i''s} + r_{i''s}$, contradicting the assumption that there exists i with $D_{is}(r) = 1$ such that $\rho^{\varphi}_{i''s} + r_{i''s} \leq \max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{is} + r_{is}$.

In all cases, the DA algorithm works in the same way at step t_0 under r and $(\delta_s(r), r_{-s})$. In Case I, the DA algorithm also works in the same way for the remaining steps. In Case II, let t_1 be the step in the DA algorithm at which i'' first applies for s. If there is no such a step t_1 under either r or $(\delta_s(r), r_{-s})$, then the DA algorithm works in the same way until its end both under r and $(\delta_s(r), r_{-s})$, completing the proof. Assume the existence of such a step t_1 under both r and $(\delta_s(r), r_{-s})$. Until step $t_1 - 1$, the DA algorithm operates in the same way both under r and $(\delta_s(r), r_{-s})$ since the only difference between the two situations is the positions of i' and i'' in r_s. t_1 is thus common to r and $(\delta_s(r), r_{-s})$. Let I_{st_1} be the set

of students who are kept by s from step $t_1 - 1$ or newly apply for s in step t_1. I_{st_1} is again the same between r and $(\delta_s(r), r_{-s})$. I again consider the following two cases.

Case II.A (Continued): $r_{i's} > r_{i''s}$. Under $(\delta_s(r), r_{-s})$, s also tentatively accepts i'' by the following reason. Suppose not. Then $c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_1}}) \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + \delta_i(r_s)\}_{i \in I_{st_1}}) < \rho^{\varphi}_{i''s} + \delta_{i''s}(r) = \rho^{\varphi}_{i's} + r_{i's}$. Since the tentative cutoff is monotonically decreasing in steps, the above inequality implies that for all i with $D_{is}(r) = 1$, $\rho^{\varphi}_{is} + r_{is} < \rho^{\varphi}_{i's} + \delta_{i's}(r) = \rho^{\varphi}_{i''s} + r_{i''s}$, contradicting the assumption that there exists i with $D_{is}(r) = 1$ such that $\rho^{\varphi}_{i''s} + r_{i''s} \leq \max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{is} + r_{is}$.

Case II.B (Continued): $r_{i's} < r_{i''s}$. By the above fact that s tentatively accepts i'' under r, it holds that $\rho^{\varphi}_{i''s} + r_{i''s} \leq c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in I_{st_1}})$, implying $\rho^{\varphi}_{i''s} + \delta_{i''s}(r) < c_s\text{-th}(\{\rho^{\varphi}_{is} + \delta_{is}(r)\}_{i \in I_{st_1}})$.

In both cases, the DA algorithm works in the same way at step t_1 under r and $(\delta_s(r), r_{-s})$. Since both i_0 and i_1 have already applies for s by step t_1 or never apply for s, the DA algorithm also works in the same way for the remaining steps. □

Main Proof

Suppose that the first-choice research design does not extract a random assignment for gDA mechanism φ for some school s at some assignment problem X. Fix φ, s, and X throughout. For each lottery number profile r, define students $i_0(r)$ and $i_1(r)$ by

- $i_0(r), i_1(r) \in First_s(r)$
- $D_{i_0(r)s}(r) = 0$ and $D_{i_1(r)s}(r) = 1$
- $r_{i_0(r)s} \leq r_{is}$ for all $i \in First_s(r)$ with $D_{is}(r) = 0$
- $r_{i_1(r)s} \geq r_{is}$ for all $i \in First_s(r)$ with $D_{is}(r) = 1$.

If there are no two students satisfying the conditions, let $i_0(r) = i_1(r) = \emptyset$. With this convention, $i_0(r)$ and $i_1(r)$ are uniquely well-defined for all r. (If there are two $\hat{i}_0(r) \neq \tilde{i}_0(r)$ satisfying the conditions, then $r_{\hat{i}_0(r)s} > r_{\tilde{i}_0(r)s}$ and $r_{\hat{i}_0(r)s} < r_{\tilde{i}_0(r)s}$, a contradiction. The same logic holds for $i_1(r)$ too.) This proof uses the following equivalent representation of gDA mechanism φ.

Definition 6. Algorithm `2STAGES`(r) operates in the following way.

(1) Same as in Definition 1.

(2) Same as in Definition 1.

(3) Run the following sub-algorithm $\texttt{STAGE1}(r)$: Remove $i_0(r)$ and $i_1(r)$ from X (without changing anything else) and run the DA algorithm on the remaining subproblem where schools' strict priorities are given by $(\succ^{\varphi}_{r_s})$.

(4) Starting from the output of $\texttt{STAGE1}(r)$ as the initial tentative assignment, run the following sub-algorithm $\texttt{STAGE2}(r)$: Include $i_0(r)$ and $i_1(r)$ and run the DA algorithm where schools' strict priorities are given by $(\succ^{\varphi}_{r_s})$.

By McVitie and Wilson (1970)'s order irrelevance result, $\texttt{2STAGES}(r)$ and $\varphi(r)$ (the simplified notation for $\varphi(\rho, r)$) produce the same matching for all r. Let $t-1$ be the last step of $\texttt{STAGE1}(r)$ at which $\texttt{STAGE1}(r)$ stops and $\mu_{t-1}(r) \equiv (\mu_{st-1}(r))_{s\in S}$ be the tentative matching at the end of step $t-1$. Start counting $\texttt{STAGE2}(r)$'s steps without resetting the step index, and let t be the first step of $\texttt{STAGE2}(r)$. (Note that t implicitly depends on r.)

For each lottery number profile r, define $\sigma^*(r) = (\sigma^*_{s'}(r))_{s'\in S}$ as the following permutation of r. If $i_0(r) = i_1(r) = \emptyset$, then $\sigma^*(r) = r$. Otherwise, if gDA mechanism φ uses MTB, $\sigma^*(r)$ is obtained by switching only $i_0(r)$ and $i_1(r)$ only in r_s, i.e.,

- $\sigma^*_{i_0(r)s}(r) = r_{i_1(r)s}$
- $\sigma^*_{i_1(r)s}(r) = r_{i_0(r)s}$
- $\sigma^*_{is}(r) = r_{is}$ for all $i \neq i_0(r), i_1(r)$
- $\sigma^*_{is'}(r) = r_{s'}$ for all $s' \neq s$.

If φ uses STB, $\sigma^*(r)$ is obtained by switching $i_0(r)$ and $i_1(r)$ in $r_{s'}$ for all s', i.e., for all s'

- $\sigma^*_{i_0(r)s'}(r) = r_{i_1(r)s'}$
- $\sigma^*_{i_1(r)s'}(r) = r_{i_0(r)s'}$
- $\sigma^*_{is'}(r) = r_{is'}$ for all $i \neq i_0(r), i_1(r)$.

I say two students i' and i'' are **consecutive in** r_s **within** $First_s(r)$ if $i', i'' \in First_s(r)$ and there is no $i''' \in First_s(r)$ such that $r_{i's} < r_{i'''s} < r_{i''s}$ or $r_{i's} > r_{i'''s} > r_{i''s}$. For each r, consider a permutation $\hat{\sigma}_s(r) \neq \sigma^*_s(r)$ of r_s that switches only two students i' and i'' who are consecutive in r_s within $First_s(r)$. If there are no such i' and i'', let $\hat{\sigma}_s(r) = r_s$. Let $\hat{\sigma}(r)$ be the following. If φ uses MTB, let $\hat{\sigma}(r) = (\hat{\sigma}_s(r), r_{-s})$. If φ uses STB, let $\hat{\sigma}(r) = \times_{|S|}\hat{\sigma}_s(r)$.

Lemma 2. *(A breakdown of the Fisher property discussed in Section 3.3) There exits lottery number profile r consistent with gDA mechanism φ's lottery structure (STB or MTB) such that $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$ or $D_{is}(\hat{\sigma}(r)) \neq D_{is}(r)$ for some student $i \in First_s(r)$.*

Proof of Lemma 2. It is enough to show that if there is no r such that there exists $i \in First_s(r)$ such that $D_{is}(\hat{\sigma}(r)) \neq D_{is}(r)$, then there exits r such that $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$. Suppose to the contrary that for all r, it is not the case $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$.

Step 2.A. For all lottery number profile r with $i_0(r) \neq \emptyset$ and $i_1(r) \neq \emptyset$, $D_{i_0(r)s}(\sigma^*(r)) = 1, D_{i_1(r)s}(\sigma^*(r)) = 0$, and $D_{is}(\sigma^*(r)) = D_{is}(r)$ for all student $i \in First_s(r)$ with $i \neq i_0(r)$ and $i \neq i_1(r)$.

Proof of Step 2.A. By the above assumption that for all r, it is not the case $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$, it is enough to show that for all such r, $D_{i_1(r)s}(\sigma^*(r)) = 0$ and $D_{is}(\sigma^*(r)) = D_{is}(r)$ for all $i \in First_s(r)$ with $i \neq i_0(r)$ and $i \neq i_1(r)$. Let me consider $\texttt{STAGE1}(r)$ and $\texttt{STAGE1}(\sigma^*(r))$. Since everything except $i_0(r)$ and $i_1(r)$'s lottery numbers is the same between r and $\sigma^*(r)$, both $\texttt{STAGE1}(r)$ and $\texttt{STAGE1}(\sigma^*(r))$ produce the same tentative assignment $\mu_{st-1}(r) = \mu_{st-1}(\sigma^*(r)) \equiv \mu_{st-1}$. Now start $\texttt{STAGE2}(r)$ and $\texttt{STAGE2}(\sigma^*(r))$. Under r, McVitie and Wilson (1970)'s order irrelevance result implies s rejects $i_0(r)$ and tentatively accepts $i_1(r)$, which implies $\rho^{\varphi}_{i_0(r)s} + r_{i_0(r)s} > c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in \mu_{st-1} \cup i_0(r) \cup i_1(r)})$ where $c_s\text{-th}(\cdot)$ is the c_s-th order statistic in the input set. Under $\sigma^*(r)$, by definition of $\sigma^*(r)$, $\rho^{\varphi}_{i_1(r)s} + \sigma^*_{i_1(r)s}(r) = \rho^{\varphi}_{i_0(r)s} + r_{i_0(r)s} > c_s\text{-th}(\{\rho^{\varphi}_{is} + r_{is}\}_{i \in \mu_{st-1} \cup i_0(r) \cup i_1(r)})$, resulting in s's rejecting $i_1(r)$. Since any rejected student is never be accepted in the DA algorithm, this implies $D_{i_1(r)s}(\sigma^*(r)) = 0$. $D_{is}(\sigma^*(r)) = D_{is}(r)$ for all $i \in First_s(r)$ with $i \neq i_0(r)$ and $i \neq i_1(r)$ by the following reason. Suppose not. There exists $i \in \mu_{st-1} \cap First_s(r) \setminus \{i_0(r), i_1(r)\}$ for whom, without loss of generality, $D_{is}(\sigma^*(r)) = 0$ and $D_{is}(r) = 1$. This implies that $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$ since $\rho^{\varphi}_{is} + r_{is} < \rho^{\varphi}_{i_1(r)s} + r_{i_1(r)s} < \rho^{\varphi}_{i_0(r)s} + r_{i_0(r)s}$ (the first inequality is by definition of $i_1(r)$) and so $\rho^{\varphi}_{is} + \sigma^*_{is}(r) < \rho^{\varphi}_{i_0(r)s} + \sigma^*_{i_0(r)s}(r) < \rho^{\varphi}_{i_1(r)s} + \sigma^*_{i_1(r)s}(r)$. This is a contradiction to the assumption that for all r, it is not the case $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$. □

Step 2.B. For all lottery number profile r with $i_0(r) = i_1(r) = \emptyset$, it holds $\varphi(\sigma^*(r)) = \varphi(r)$. For each r and each permutation $\hat{\sigma}(r) \neq \sigma^*(r)$ defined right before Lemma 2, $D_{is}(\hat{\sigma}(r)) = D_{is}(r)$ for all $i \in First_s(r)$.

Proof of Step 2.B. For all r with $i_0(r) = i_1(r) = \emptyset$, $\sigma^*(r) = r$ and it is trivial that $\varphi(\sigma^*(r)) = \varphi(r)$. The second part is by assumption. □

Step 2.C. For each lottery number profile r, define $o_s(r) \equiv |\{i \in First_s(r) | D_{is}(r) = 1\}|$. For each r and each permutation $\sigma_s(r)$ of r_s that permutes lottery numbers only among members of $First_s(r)$, let $\sigma(r)$ be the following. If φ uses MTB, $\sigma(r) = (\sigma_s(r), r_{-s})$. If φ uses STB,

$\sigma(r) = \times_{|S|}\sigma_s(r)$. *Then the following is true for all* r.

(1) $First_s(\sigma(r)) = First_s(r)$.

(2) $o_s(\sigma(r)) = o_s(r)$.

(3) *For all* i *and* i' *in* $First_s(\sigma(r))$, *if* $D_{is}(\sigma(r)) > D_{i's}(\sigma(r))$, *then* $\sigma_{is}(r) < \sigma_{i's}(r)$.

Proof of Step 2.C. Since any permutation can be expressed as a composition of contrapositions (permutations switching consecutive two elements), I can express any σ as a composition of σ^* and $\hat{\sigma}$'s defined right before Lemma 2. Steps 2.A and 2.B imply (1) and (2). (3) follows from the fact that for all r and all $i, i' \in First_s(r)$, $\rho^{\varphi}_{is} = \rho^{\varphi}_{i's}$ and another well-known property of the DA algorithm that for applicants who rank s first and share ρ^{φ}_{is}, D_{is} is monotonically decreasing in r_{is} (Balinski and Sönmez, 1999). □

Let $\mathcal{R}$ be the set of all possible values of r. Partition $\mathcal{R}$ into $\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_N$ such that within each $\mathcal{R}_n$, for all $r, r' \in \mathcal{R}_n$, r can be obtained from r' by permutation $r' = \sigma(r)$, where $\sigma(r)$ is a permutation of r defined in Step 2.C. This partition is well-defined by Step 2.C(1): Since Step 2.C(1) guarantees $First_s(r) = First_s(r')$, $r' = \sigma(r)$ for such a permutation σ if and only if $r = \sigma(r')$ for such a (possibly different) permutation σ. Let r_n be a generic element of $\mathcal{R}_n$. Note that $First_s(r_n)$ and $o_s(r_n)$ are the same for all $r_n \in \mathcal{R}_n$ by Step 2.C(1) and 2.C(2), respectively. Step 2.C guarantees that conditional on each $\mathcal{R}_n$, $D_{is}(R)$ is independent of i's type for all $i \in First_s(r_n)$, i.e., for all n, and θ,

$$P(D_{is}(R) = 1 | i \in First_s(r_n), R \in \mathcal{R}_n, \theta_i = \theta) = \frac{o_s(r_n)}{|First_s(r_n)|},$$

which is independent of $\theta_i = \theta$. The equality holds since $First_s(r_n)$ and $o_s(r_n)$ stay constant across all $r_n \in \mathcal{R}_n$ and under each r_n, students with the $o_s(r_n)$-best lottery numbers have $D_{is}(r_n) = 1$. Therefore, for all $j \in First_s(r_0)$,

$$\begin{aligned}
&P(D_{is}(R) = 1 | i \in First_s(r_0), R \in \mathcal{R}_n, \theta_i = \theta) \\
&= \begin{cases} \dfrac{o_s(r_n)}{|First_s(r_n)|} & \text{if } \rho^{\varphi}_{is} = \rho^{\varphi}_{js} \text{ for any } j \in First_s(r_n) \\ 1 & \text{if } \rho^{\varphi}_{is} < \rho^{\varphi}_{js} \text{ for any } j \in First_s(r_n) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \dfrac{o_s(r_n)}{|First_s(r_n)|} & \text{if } First_s(r_0) = First_s(r_n) \\ 1 & \text{if } \rho^{\varphi}_{is} < \rho^{\varphi}_{js} \text{ for any } j \in First_s(r_n) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$\equiv p_n$,

where the second last equality holds because $\rho^{\varphi}_{is} = \rho^{\varphi}_{js}$ for any $j \in First_s(r_n)$ if and only if $First_s(r_0) = First_s(r_n)$. p_n is independent of θ_j. This implies that for all $j \in First_s(r_0)$

$$
\begin{aligned}
&P(D_{is}(R) = 1 | i \in First_s(r_0), \theta_i = \theta) \\
&= \Sigma_{n=1}^{N} P(D_{is}(R) = 1 | i \in First_s(r_0), R \in \mathcal{R}_n, \theta_i = \theta) \times P(R \in \mathcal{R}_n | i \in First_s(r_0), \theta_i = \theta) \\
&\quad \text{(by the law of total probability)} \\
&= \Sigma_{n=1}^{N} p_n \times P(R \in \mathcal{R}_n),
\end{aligned}
$$

which is again independent of $\theta_i = \theta$.Thus the first-choice research design extracts a random assignment, a contradiction. This completes the proof of Lemma 2. □

Remark 1. Under the alternative definition of a random assignment in Section 5.1, this part simplifies to the following. No other part of the proof depends on which random assignment definition I use.

$$
\begin{aligned}
&P(D_{is}(R) = 1 | i \in First_s(R), \theta_i = \theta) \\
&= \Sigma_{n=1}^{N} P(D_{is}(R) = 1 | R \in \mathcal{R}_n, i \in First_s(R), \theta_i = \theta) \times P(R \in \mathcal{R}_n | i \in First_s(R), \theta_i = \theta) \\
&= \Sigma_{n=1}^{N} \frac{o_s(r_n)}{|First_s(r_n)|} \times \frac{|\{r \in \mathcal{R}_n | i \in First_s(r)\}|}{|\{r \in \mathcal{R} | i \in First_s(r)\}|},
\end{aligned}
$$

which is independent of $\theta_i = \theta$ conditional on $i \in First_s(R)$.

Case 1: There exists lottery number profile r consistent with gDA mechanism φ's lottery structure (STB or MTB) such that $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$. For each r, define r^*_s as the following permutation of r_s. If $i_0(r) = i_1(r) = \emptyset$, let $r^*_s = r_s$. Otherwise, let

- $r^*_{is} = r_{is}$ for all i with $\rho^{\varphi}_{is} \neq \rho^{\varphi}_{i_0(r)s} = \rho^{\varphi}_{i_1(r)s}$
- $r^*_{i_1(r)s} = \max\{r_{is} | \rho^{\varphi}_{is} = \rho^{\varphi}_{i_0(r)s} = \rho^{\varphi}_{i_1(r)s} \text{ and } D_{is}(r) = 1\}$
- $r^*_{i_0(r)s} > r^*_{i_1(r)s}$ and there is no such i that $\rho^{\varphi}_{is} = \rho^{\varphi}_{i_0(r)s} = \rho^{\varphi}_{i_1(r)s}$ and $r^*_{i_0(r)s} > r^*_{is} > r^*_{i_1(r)s}$
- $r^*_{is} > r^*_{js}$ if and only if $r_{is} > r_{js}$ for all i, j such that $\rho^{\varphi}_{is} = \rho^{\varphi}_{js} = \rho^{\varphi}_{i_0(r)s} = \rho^{\varphi}_{i_1(r)s}$ and $i \neq i_0(r)$, $i \neq i_1(r)$, $j \neq i_0(r)$, and $j \neq i_1(r)$.

For each lottery number profile r and each school $s' \neq s$, define $\tilde{\sigma}_{s'}(r)$ as the following permutation of $r_{s'}$. (Note that $\tilde{\sigma}_{s'}(r)$ implicitly depends on whole r.) If $i_0(r) = i_1(r) = \emptyset$ or MTB is used by φ, then $\tilde{\sigma}_{s'}(r) = r_{s'}$. Otherwise, $\tilde{\sigma}_{s'}(r)$ is obtained by moving $i_1(r)$ to right above $i_0(r)$, i.e.,

- $\tilde{\sigma}_{i_1(r)s'}(r) = \max\{r_{is'} | r_{is'} < r_{i_0(r)s'}\}$
- $\tilde{\sigma}_{is'}(r) > \tilde{\sigma}_{js'}(r)$ if and only if $r_{is'} > r_{js'}$ for all students i, j such that $i, j \neq i_1(r)$.

Lemma 3. *(Outcome-equivalence between lottery number profiles r and $(r^*_s, \tilde{\sigma}_{-s}(r))$) For all lottery number profile r, $\varphi(r) = \varphi(r^*_s, \tilde{\sigma}_{-s}(r))$. Therefore, $D_{i_0(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 0 (= D_{i_0(r)s}(r))$ and $D_{i_1(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 1 (= D_{i_1(r)s}(r))$.*

Proof of Lemma 3. The following Steps 3.A and 3.B imply Lemma 3. □

Step 3.A. For all lottery number profile r, $\varphi(r) = \varphi(r^*_s, r_{-s})$.

Proof of Step 3.A. If $i_0(r) = i_1(r) = \emptyset$ and so $r^*_s = r_s$, the above equality is trivial. Otherwise, r^*_s is obtained from r_s through a composition of two permutations. The first one permutes lottery numbers only among students in some set I_1 such that $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$ for all $i', i'' \in I_1$ and there exists i with $D_{is}(r) = 1$ such that $\max_{i' \in I_1}\{\rho^{\varphi}_{i's} + r_{i's}\} \leq \rho^{\varphi}_{is} + r_{is}$. The second permutation permutes lottery numbers only among students in some set I_0 such that $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$ for all $i', i'' \in I_0$ and $\min_{i' \in I_0}\{\rho^{\varphi}_{i's} + r_{i's}\} > \rho^{\varphi}_{is} + r_{is}$ for all i with $D_{is}(r) = 1$. The first permutation is a composition of special permutations δ satisfying the conditions in Lemma 1.b. If φ is not strategy-proof for schools, then the proof of Theorem 1 is complete. If φ is strategy-proof for schools, the second permutation is a composition of special permutations δ satisfying the conditions in Lemma 1.a. Therefore Lemma 1 implies Step 2.A. □

Step 3.B. For all lottery number profile r, $\varphi(r^*_s, r_{-s}) = \varphi(r^*_s, \tilde{\sigma}_{-s}(r))$.

Proof of Step 3.B. If $i_0(r) = i_1(r) = \emptyset$ and so $\tilde{\sigma}_{-s}(r) = r_{-s}$, the above inequality is trivial. Otherwise, at the first step of the DA algorithm constituting φ, students apply for schools in the same way both under (r^*_s, r_{-s}) and $(r^*_s, \tilde{\sigma}_{-s}(r))$. In particular, $i_1(r)$ applies for s since $i_1(r) \in First_s(r)$. Schools also tentatively accept students in the same way both under (r^*_s, r_{-s}) and $(r^*_s, \tilde{\sigma}_{-s}(r))$: s does so since s has the same strict priority $\succ^{\varphi}_{r^*_s}$ both under (r^*_s, r_{-s}) and $(r^*_s, \tilde{\sigma}_{-s}(r))$. The other schools also do so since the only possible difference between $\succ^{\varphi}_{r_{s'}}$ and $\succ^{\varphi}_{\tilde{\sigma}_{s'}(r)}$ is the position of $i_1(r)$, who applies for s. As a result, since Step 2.A implies $D_{i_1(r)s}(r^*_s, r_{-s}) = 1$, s tentatively accepts $i_1(r)$ at the first step of the DA algorithm both under (r^*_s, r_{-s}) and $(r^*_s, \tilde{\sigma}_{-s}(r))$. Since (a) s has the same preference $\succ^{\varphi}_{r^*_s}$ both under (r^*_s, r_{-s}) and $(r^*_s, \tilde{\sigma}_{-s}(r))$, (b) the only possible difference between $\succ^{\varphi}_{r_{s'}}$ and $\succ^{\varphi}_{\tilde{\sigma}_{s'}(r)}$ is the position of $i_1(r)$, and (c) $i_1(r)$ is tentatively kept by s and is never be rejected by s under (r^*_s, r_{-s}), the DA algorithm operates in the same way for the remaining steps, producing the same matching. □

Lemma 4. *(Partial outcome-equivalence between $\sigma^*(r)$ and $(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$) For all lottery number profile r with $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$, it is the case $D_{i_0(r)s}(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r)) = D_{i_1(r)s}(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r)) = 0$.*

Proof of Lemma 4. If $i_0(r) = i_1(r) = \emptyset$ and so $\sigma^*(r) = r = (\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$, Lemma 4 is immediate. Otherwise, I first prove the following result.

Step 4.A. For all lottery number profile r with $D_{i_0(r)s}(\sigma^*(r)) = D_{i_1(r)s}(\sigma^*(r)) = 0$, it is the case $\varphi(\sigma^*(r)) = \varphi(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$.

Proof of Step 4.A. Note that

a) $\sigma^*_{i_0(r)s}(r) = r_{i_1(r)s} \leq \min\{r_{i_0(r)s}, r^*_{i_0(r)s}, r^*_{i_1(r)s}\} = \min\{\sigma^*_{i_1(r)s}(r), \sigma^*_{i_1(r)s}(r_s^*, r_{-s}), \sigma^*_{i_0(r)s}(r_s^*, r_{-s})\}$, where the last equality follows from $i_0(r_s^*, r_{-s}) = i_0(r)$ and $i_1(r_s^*, r_{-s}) = i_1(r)$ by Step 2.A.

b) $\rho^\varphi_{js} + \sigma^*_{js}(r) > \rho^\varphi_{is} + \sigma^*_{is}(r)$ for all j with $\rho^\varphi_{js} = \rho^\varphi_{i_0(r)s}$ and $\sigma^*_{js}(r) > \sigma^*_{i_0(r)s}(r)$ and all i with $D_{is}(\sigma^*(r)) = 1$ since $D_{i_0(r)s}(\sigma^*(r)) = 0$ by assumption and $i_0(r) \in First_s(r)$ and so $i_0(r)$ ranks s first.

(a) and (b) imply that starting from $\sigma^*(r)$, $\sigma_s^*(r_s^*, r_{-s})$ is obtained from $\sigma_s^*(r)$ through a permutation that permutes lottery numbers only among students in some set I_0 such that $\rho^\varphi_{i's} = \rho^\varphi_{i''s}$ for all $i', i'' \in I_0$ and $\min_{i' \in I_0}\{\rho^\varphi_{i's} + \sigma^*_{i's}(r)\} > \rho^\varphi_{is} + \sigma^*_{is}(r)$ for all i with $D_{is}(\sigma^*(r)) = 1$. This permutation is a composition of special permutations δ's that satisfy the conditions in Lemma 1.a. Therefore Lemma 1 implies Step 4.A. □

Now let me compare $\varphi(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$ and $\varphi(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$. At the first step of the DA algorithm constituting φ, students apply for schools in the same way both under $(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$ and $(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$. In particular, $i_0(r)$ and $i_1(r)$ apply for s since $i_0(r), i_1(r) \in First_s(r)$. Schools also tentatively accept students in the same way both under $(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$ and $(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$: s does so since s has the same strict priority $\succ^\varphi_{\sigma_s^*(r_s^*, r_{-s})}$ both under $(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$ and $(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$. The other schools also do so since the only possible differences between $\succ^\varphi_{\sigma^*_{s'}(r)}$ and $\succ^\varphi_{\tilde{\sigma}_{s'}(r)}$ are the positions of $i_0(r)$ and $i_1(r)$, both of whom apply for s.

If s rejects both $i_0(r)$ and $i_1(r)$ at the first step, the proof is complete. Otherwise, s tentatively accepts at least $i_0(r)$ since $\rho^\varphi_{i_0(r)s} + \sigma^*_{i_0(r)s}(r_s^*, r_{-s}) < \rho^\varphi_{i_1(r)s} + \sigma^*_{i_1(r)s}(r_s^*, r_{-s})$. Since $\succ^\varphi_{\sigma^*_{s'}(r)}$ and $\succ^\varphi_{\tilde{\sigma}_{s'}(r)}$ are equivalent over $I \setminus \{i_0(r)\}$, the remaining steps of the DA algorithm operate in the same way both under $(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$ and $(\sigma_s^*(r_s^*, r_{-s}), \tilde{\sigma}_{-s}(r))$ until the point where s rejects $i_0(r)$. s finally rejects $i_0(r)$ since it does so under $(\sigma_s^*(r_s^*, r_{-s}), \sigma_{-s}^*(r))$

by Step 4.A and s has the same preference $\succ^{\varphi}_{\sigma^*_s(r^*_s, r_{-s})}$ both under $(\sigma^*_s(r^*_s, r_{-s}), \sigma^*_{-s}(r))$ and $(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r))$. This implies Lemma 4. □

Lemma 5. *(Existence of a profitable preference manipulation) There exist (ρ^*, r^*), school s's preference $\succ_s$ responsive with respect to (c_s, ρ^*_s, r^*_s), and (ρ'_s, r'_s) such that $\varphi_s((\rho'_s, r'_s), (\rho^*_{-s}, r^*_{-s}))$ $\succ_s \varphi_s(\rho^*, r^*)$.*

Proof of Lemma 5. Lemmas 2, 3, and 4 imply that there exists r such that

- $D_{i_0(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 0$
- $D_{i_1(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 1$
- $D_{i_0(r)s}(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = D_{i_1(r)s}(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = 0$.

Step 5.A. $\varphi_s(r^*_s, \tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$ where $\mu_{st}(\cdot)$ is s's tentative assignment at the end of step t in 2STAGES(r).

Proof of Step 5.A. Execute STAGE1$(r^*_s, \tilde{\sigma}_{-s}(r))$ and start STAGE2$(r^*_s, \tilde{\sigma}_{-s}(r))$. s rejects $i_0(r)$ and tentatively keeps $i_1(r)$ since $D_{i_0(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 0$ and $D_{i_1(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 1$. Suppose to the contrary $\varphi_s(r^*_s, \tilde{\sigma}_{-s}(r)) \neq \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$. Since $|\varphi_s(r^*_s, \tilde{\sigma}_{-s}(r))| = |\mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))| = c_s$ (because s rejects $i_0(r)$ when choosing $\mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$ at step t), this implies there exists a student $i_2 \in \varphi_s(r^*_s, \tilde{\sigma}_{-s}(r))$ such that $i_2 \notin \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$. In addition, $i_2 \notin \mu_{st-1}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_1(r)$ has to be the case since otherwise (i.e., if $i_2 \notin \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$ but $i_2 \in \mu_{st-1}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_1(r)$ and so i_2 applies for s at step $t' < t$ in STAGE1$(r^*_s, \tilde{\sigma}_{-s}(r))$), s rejects i_2 at step t and so $i_2 \notin \varphi_s(r^*_s, \tilde{\sigma}_{-s}(r))$, a contradiction. This means i_2 applies for s at a step $t' > t$ and is tentatively kept by s. This requires that s rejects $i_1(r)$ before or at step t' since by definition $i \succ^{\varphi}_{r^*_s} i_1(r)$ for any $i \in \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \setminus i_1(r)$, which is because s rejects $i_0(r)$ at step t and there is no such i that $\rho^{\varphi}_{is} = \rho^{\varphi}_{i_0(r)s} = \rho^{\varphi}_{i_1(r)s}$ and $r^*_{i_0(r)s} > r^*_{is} > r^*_{i_1(r)s}$. This is a contradiction to the above fact that $D_{i_1(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 1$. □

Step 5.B. There exists a step $t' > t$ in 2STAGES$(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r))$ such that $\mu_{st'}(\sigma^*_s(r^*_s, r_{-s}),$ $\tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$ where i_2 is a student with $i_2 \succ^{\varphi}_{r^*_s} i_1(r)$.

Proof of Step 5.B. Execute STAGE1$(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r))$ and start STAGE2$(\sigma^*_s(r^*_s, r_{-s}),$ $\tilde{\sigma}_{-s}(r))$. School s rejects $i_1(r)$ and tentatively keeps $i_0(r)$ at step t since (1) μ_{st-1} is the same between $(r^*_s, \tilde{\sigma}_{-s}(r))$ and $(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r))$, (2) $\rho^{\varphi}_{i_1(r)s} + \sigma^*_{i_1(r)s}(r^*_s, r_{-s}) = \rho^{\varphi}_{i_0(r)s} + r^*_{i_0(r)s}$, and (3) $D_{i_1(r)s}(r^*_s, \tilde{\sigma}_{-s}(r)) = 1$. By $D_{i_0(r)s}(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = 0$, school

s rejects $i_0(r)$ at a later step $t' > t$ and tentatively keeps i_2 with $i_2 \succ^{\varphi}_{\sigma^*_s(r^*_s, r_{-s})} i_0(r)$, which implies $i_2 \succ^{\varphi}_{r^*_s} i_1(r)$ since $\rho^{\varphi}_{i_0(r)s} + \sigma^*_{i_0(r)s}(r^*_s, r_{-s}) = \rho^{\varphi}_{i_1(r)s} + r^*_{i_1(r)s}$ and $\sigma^*_{i_2 s}(r^*_s, r_{-s}) = r^*_{i_2 s}$. Therefore, $\mu_{st'}(\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$ where i_2 is a student with $i_2 \succ^{\varphi}_{r^*_s} i_1(r)$. □

I am ready to construct a profitable preference manipulation for s. Let $\succ_s$ be any preference for s that is responsive with respect to (c_s, ρ_s, r^*_s). Let ρ'_s be a coarse priority order for s such that $\rho'_{ks} > \rho'_{js}$ for all $k \notin \cup^t_{t_0=1} \mu_{st_0}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$ and $j \in \cup^t_{t_0=1} \mu_{st_0}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$ while $\rho'_{ks} = \rho'_{js}$ if and only if $\rho_{ks} = \rho_{js}$ for all $j, k \notin \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$ or $j, k \in \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$. By Steps 5.A and 5.B, $\varphi_s(r^*_s, \tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r))$ while $\varphi_s((\rho'_s, \rho_{-s}), (\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r)$.

Also, $i_2 \succ^{\varphi}_{r^*_s} i_1(r)$ established in Step 5.B implies $i_2 \succ_{r^*_s} i_1(r)$ as follows:

$$
\begin{aligned}
& i_2 \succ^{\varphi}_{r^*_s} i_1(r) \\
\Leftrightarrow\ & f^{\varphi}(\rho_{i_2 s}) + g^{\varphi}(rank_{i_2 s}) + r^*_{i_2 s} < f^{\varphi}(\rho_{i_1(r)s}) + g^{\varphi}(rank_{i_1(r)s}) + r^*_{i_1(r)s} \\
\Rightarrow\ & f^{\varphi}(\rho_{i_2 s}) + r^*_{i_2 s} < f^{\varphi}(\rho_{i_1(r)s}) + r^*_{i_1(r)s} \\
& \quad \text{(since } rank_{i_1(r)s} = 1 \leq rank_{i_2 s} \text{ and } g^{\varphi}(\cdot) \text{ is weakly increasing)} \\
\Leftrightarrow\ & \rho_{i_2 s} + r^*_{i_2 s} < \rho_{i_1(r)s} + r^*_{i_1(r)s} \\
& \quad \text{(since } f^{\varphi}(\cdot) \text{ is strictly increasing)} \\
\Leftrightarrow\ & i_2 \succ_{r^*_s} i_1(r).
\end{aligned}
$$

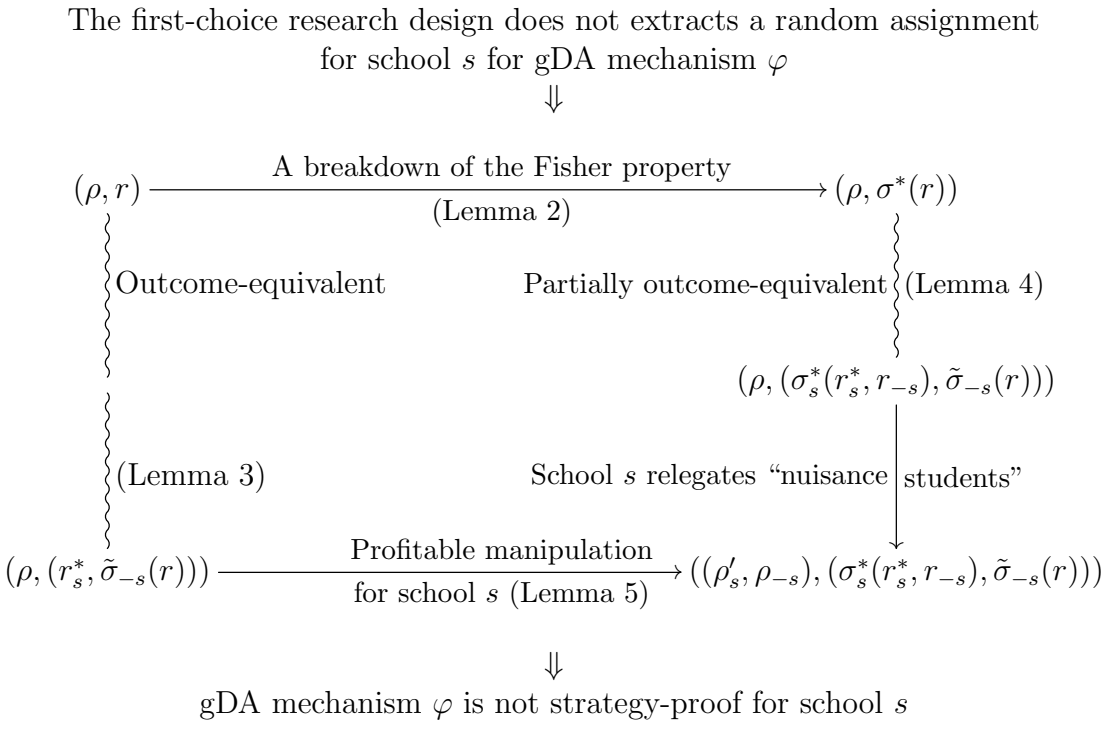
Thus $\varphi_s((\rho'_s, \rho_{-s}), (\sigma^*_s(r^*_s, r_{-s}), \tilde{\sigma}_{-s}(r)) = \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) \cup i_2 \setminus i_1(r) \succ_s \mu_{st}(r^*_s, \tilde{\sigma}_{-s}(r)) = \varphi_s(\rho, (r^*_s, \tilde{\sigma}_{-s}(r)))$ since $\succ_s$ is responsive with respect to (c_s, ρ_s, r^*_s), showing that when (ρ_s, r^*_s) is s's true private information, $(\rho'_s, \sigma^*_s(r^*_s, r_{-s}))$ is a profitable manipulation for s with respect to any $\succ_s$ responsive with respect to (c_s, ρ_s, r^*_s); therefore φ is not strategy-proof for s at X. Figure 3 summarizes the structure of the above proof.

Case 2: There exist r consistent with φ's lottery structure (STB or MTB) and $i \in First_s(r)$ such that $D_{is}(\hat{\sigma}(r)) \neq D_{is}(r)$ where $\hat{\sigma}(r)$ is a permutation defined right before Lemma 2, i.e., a permutation that switches only two students i' and i'' who are consecutive in r_s within $First_s(r)$. By the definition of $\hat{\sigma}(r)$, it is the case $D_{i's}(r) = D_{i''s}(r)$. There are two cases to consider.

Case 2.a: $D_{i's}(r) = D_{i''s}(r) = 1$. Then there exists j with $D_{is}(r) = 1$ such that $\max\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} \leq \rho^{\varphi}_{js} + r_{js}$. If φ uses MTB, $\hat{\sigma}(r)$ satisfies the conditions in Lemma 1.b, implying $\varphi(\hat{\sigma}(r)) = \varphi(r)$ by Lemma 1.b. This is a contradiction to $D_{is}(\hat{\sigma}(r)) \neq D_{is}(r)$.

If φ uses STB, suppose to the contrary that $D_{i's}(r) = D_{i''s}(r) = 1$. At the first step of the

Figure 3: Structure of the proof (Case 1)



DA algorithm, students apply for schools in the same way both under r and $\hat{\sigma}(r)$. In particular, i' and i'' apply for s since $i', i'' \in First_s(r)$ and both of them rank s first. Schools also tentatively accept students in the same way both under r and $\hat{\sigma}(r)$: The other schools than s do so since the only possible differences between $\succ^{\varphi}_{r_{s'}}$ and $\succ^{\varphi}_{\hat{\sigma}_{s'}(r)}$ are the positions of i' and i'', both of whom apply for s. s accepts the same students including i' and i'' since $D_{i's}(r) = D_{i''s}(r) = 1$ and $\{\rho^{\varphi}_{js} + r_{js} | j \text{ applies for } s \text{ at the first step of the DA algorithm under } r\} = \{\rho^{\varphi}_{js} + \hat{\sigma}_{js}(r) | j \text{ applies for } s \text{ at the first step of the DA algorithm under } \hat{\sigma}(r)\}$, which is because the same students apply for s both under r and $\hat{\sigma}(r)$, $\rho^{\varphi}_{js} + r_{js} = \rho^{\varphi}_{js} + \hat{\sigma}_{js}(r)$ for all $j \neq i', i''$, $\rho^{\varphi}_{i's} + r_{i's} = \rho^{\varphi}_{i''s} + \hat{\sigma}_{i''s}(r)$, and $\rho^{\varphi}_{i''s} + r_{i''s} = \rho^{\varphi}_{i's} + \hat{\sigma}_{i's}(r)$. Since (a) the only possible differences between $\succ^{\varphi}_{r_{s'}}$ and $\succ^{\varphi}_{\hat{\sigma}_{s'}(r)}$ are the positions of i' and i'', and (b) i' and i'' are tentatively kept by s and is never be rejected by s under r, the DA algorithm operates in the same way for the remaining steps, producing the same matching. This implies $D_{is}(\hat{\sigma}(r)) = D_{is}(r)$ for all $i \in First_s(r)$, a contradiction.

Case 2.b: $D_{i's}(r) = D_{i''s}(r) = 0$. Without loss of generality, assume that there exist r and $i \in First_s(r)$ such that $D_{is}(\hat{\sigma}(r)) = 0 \neq 1 = D_{is}(r)$. Let i^* be the student with $i^* \in First_s(r)$, $D_{i^*s}(r) = 1$, and $r_{i^*s} \geq r_{js}$ for all $j \in First_s(r)$ with $D_{js}(r) = 1$. Until the end of Case 2.b, change i^*'s preference $\succ_{i^*}$ to $\succ'_{i^*}$ such that $s \succ'_{i^*} \emptyset \succ'_{i^*} s'$ for all $s' \neq s$.

This does not change $D_{i^*s}(r) = 1$ or $D_{i's}(r) = D_{i''s}(r) = 0$. Note that $D_{i^*s}(\hat{\sigma}(r)) = 0 \neq 1 = D_{i^*s}(r)$ since $D_{is}(\hat{\sigma}(r)) = 0$ and $\rho^{\varphi}_{i^*s} + \hat{\sigma}_{i^*s}(r) = \rho^{\varphi}_{i^*s} + r_{i^*s} \geq \rho^{\varphi}_{is} + r_{is} = \rho^{\varphi}_{is} + \hat{\sigma}_{is}(r)$ for any $i \in First_s(r)$ with $D_{is}(\hat{\sigma}(r)) = 0 \neq 1 = D_{is}(r)$. Without loss of generality, assume $r_{i's} < r_{i''s}$ so that $\hat{\sigma}_{i''s}(r) < \hat{\sigma}_{i's}(r)$. Let $\hat{\sigma}^{\#}_s(r)$ be the further permutation of $\hat{\sigma}_s(r)$ such that $\hat{\sigma}^{\#}_{i''s}(r) = \min\{\hat{\sigma}_{js}(r) | j \in First_s(r), D_{js}(r) = 0\}$, $\hat{\sigma}^{\#}_{i's}(r) = \min\{\hat{\sigma}_{js}(r) \neq \hat{\sigma}^{\#}_{i''s}(r) | j \in First_s(r), D_{js}(r) = 0\}$, and $\hat{\sigma}^{\#}_{js}(r) > \hat{\sigma}^{\#}_{ks}(r)$ if and only if $\hat{\sigma}_{js}(r) > \hat{\sigma}_{ks}(r)$ for all $j, k \in I \setminus \{i', i''\}$.

Lemma 6. *$D_{i^*s}(\hat{\sigma}^{\#}_s(r), \hat{\sigma}_{-s}(r)) = D_{i''s}(\hat{\sigma}^{\#}_s(r), \hat{\sigma}_{-s}(r)) = 0$.*

Proof of Lemma 6. Note that $\min\{\rho^{\varphi}_{i's} + \hat{\sigma}_{i's}(r), \rho^{\varphi}_{i''s} + \hat{\sigma}_{i''s}(r)\} = \min\{\rho^{\varphi}_{i's} + r_{i's}, \rho^{\varphi}_{i''s} + r_{i''s}\} > \rho^{\varphi}_{i^*s} + r_{i^*s} = \rho^{\varphi}_{i^*s} + \hat{\sigma}_{i^*s}(r)$, where the first and last equalities are by $\rho^{\varphi}_{i's} = \rho^{\varphi}_{i''s}$ (implied by $i', i'' \in First_s(r)$) and the definition of $\hat{\sigma}_s(r)$ while the middle inequality is by $D_{i^*s}(r) = 1$ and $D_{i's}(r) = D_{i''s}(r) = 0$. Thus, $D_{i''s}(\hat{\sigma}(r)) = D_{i's}(\hat{\sigma}(r)) = 0$. If φ is not strategy-proof for schools, then the proof of Theorem 1 is complete. If φ is strategy-proof for schools, by Lemma 1.a and the definition of $\hat{\sigma}^{\#}_s(r)$, it holds $\varphi(\hat{\sigma}^{\#}_s(r), \hat{\sigma}_{-s}(r)) = \varphi(\hat{\sigma}(r))$, which implies Lemma 6. □

Let $\hat{\sigma}^{\#\#}_s(r)$ be the permutation of $\hat{\sigma}^{\#}_s(r)$ that switches i^* and i'', who are consecutive within $First_s(r)$ under $\hat{\sigma}^{\#}_s(r)$.

Lemma 7. *$D_{i^*s}(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r)) = 0$ and $D_{i''s}(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r)) = 1$.*

Proof of Lemma 7. Note that $D_{i^*s}(\hat{\sigma}^{\#}_s(r), r_{-s}) = 1$ and $D_{i''s}(\hat{\sigma}^{\#}_s(r), r_{-s}) = 0$ by Lemma 1.a and the definition of $\hat{\sigma}^{\#}_s(r)$. Since the only differences between $(\hat{\sigma}^{\#}_s(r), r_{-s})$ and $(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r))$ are the positions of i^* and i'' in the priority order at s and the positions of i' and i'' in the priority order at $s' \neq s$, both under $(\hat{\sigma}^{\#}_s(r), r_{-s})$ and $(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r))$, the DA algorithm operates in the same way until i' is rejected by s. School s rejects i' in both scenarios since $D_{i''s}(\hat{\sigma}^{\#}_s(r), \hat{\sigma}_{-s}(r)) = 0$ (as shown at the start of this proof) and $\rho^{\varphi}_{i's} + \hat{\sigma}^{\#\#}_{i's}(r) = \rho^{\varphi}_{i's} + \hat{\sigma}^{\#}_{i's}(r) > \rho^{\varphi}_{i''s} + \hat{\sigma}^{\#}_{i''s}(r) = \rho^{\varphi}_{i^*s} + \hat{\sigma}^{\#\#}_{i^*s}(r)$.

Since i' has a weakly worse lottery number under $\hat{\sigma}_{s'}(r)$ than under $r_{s'}$ for all $s' \neq s$, i is less likely to crowd other applicants out from other schools than s and the chain reactions of new rejections and applications caused by s's rejection of i' are less likely to go back to s under $(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r))$ than under $(\hat{\sigma}^{\#}_s(r), r_{-s})$. Also, since the only other difference between $(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r))$ and $(\hat{\sigma}^{\#}_s(r), r_{-s})$ is the school-s lottery numbers of i'' and i^* , i.e., $\hat{\sigma}^{\#}_{i^*s}(r) = \hat{\sigma}^{\#\#}_{i''s}(r) \neq \hat{\sigma}^{\#}_{i''s}(r) = \hat{\sigma}^{\#\#}_{i^*s}(r)$, and $\rho^{\varphi}_{i''s} = \rho^{\varphi}_{i^*s}$, when i'' is rejected by s under $(\hat{\sigma}^{\#}_s(r), r_{-s})$, i^* may be rejected by s under $(\hat{\sigma}^{\#\#}_s(r), \hat{\sigma}_{-s}(r))$. But i^* ranks only s in $\succ'_{i^*}$ and causes no additional rejections at other schools while i'' may rank other schools than s and may cause additional rejections at other schools.

By these two factors, the set of rejections made by schools other than s is weakly larger in the set inclusion sense under $(\hat{\sigma}_s^{\#}(r), r_{-s})$ than under $(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r))$, i.e., $\{(j, s')|D_{js''}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 0$ for all $s'' \succeq_j s'\} \subseteq \{(j, s')|D_{js''}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 0$ for all $s'' \succeq_j s'\}$. Therefore, the set of applicants for s is weakly larger in the set inclusion sense under $(\hat{\sigma}_s^{\#}(r), r_{-s})$ than under $(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r))$, i.e., $\{j|D_{js'}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 0$ for all $s' \succ_j s\} \subseteq \{j|D_{js'}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 0$ for all $s' \succ_j s\}$. As a result it has to be the case that the cutoff at s is smaller (more strict) under $(\hat{\sigma}_s^{\#}(r), r_{-s})$ than under $(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r))$, i.e.,

$$
\begin{aligned}
&\max\{\rho_{js}^{\varphi} + \hat{\sigma}_{js}^{\#\#}(r)|D_{js}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 1\} \\
&= c_s\text{-th}\{\rho_{js}^{\varphi} + \hat{\sigma}_{js}^{\#\#}(r)|D_{js'}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 0 \text{ for all } s' \succ_j s\} \\
&\geq c_s\text{-th}\{\rho_{js}^{\varphi} + \hat{\sigma}_{js}^{\#}(r)|D_{js'}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 0 \text{ for all } s' \succ_j s\} \\
&= \max\{\rho_{js}^{\varphi} + \hat{\sigma}_{js}^{\#}(r)|D_{js}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 1\} \\
&\geq \rho_{i^*s}^{\varphi} + r_{i^*s} \\
&= \rho_{i''s}^{\varphi} + \hat{\sigma}_{i''s}^{\#\#}(r),
\end{aligned}
$$

where the first inequality is by $\{j|D_{js'}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 0$ for all $s' \succ_j s\} \subseteq \{j|D_{js'}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 0$ for all $s' \succ_j s\}$ (shown above), the second inequality is by $D_{i^*s}(\hat{\sigma}_s^{\#}(r), r_{-s}) = 1$ (shown at the start of this proof), and the last equality is by the definition of $\hat{\sigma}_{i''s}^{\#\#}(r)$. c_s-th$\{\cdot\}$ is the c_s-th order statistic. This implies $D_{i''s}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 1$ (by $i'' \in First_s(r)$). This also implies $D_{i^*s}(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r)) = 0$ since otherwise $D_{i^*s}(\hat{\sigma}_s^{\#}(r), \hat{\sigma}_{-s}(r)) = D_{i''s}(\hat{\sigma}_s^{\#}(r), \hat{\sigma}_{-s}(r))$ $= 1$ by Lemma 1.b, a contradiction to $D_{i^*s}(\hat{\sigma}_s^{\#}(r), \hat{\sigma}_{-s}(r)) = 0$ in Lemma 6. □

Lemmas 6 and 7 imply gDA mechanism φ is not strategy-proof for schools by the same argument as Case 1 where students i'' and i^* perform the roles of students $i_1(r)$ and $i_0(r)$, respectively, in Case 1 while the permutation from lottery number profile $(\hat{\sigma}_s^{\#\#}(r), \hat{\sigma}_{-s}(r))$ to $(\hat{\sigma}_s^{\#}(r), \hat{\sigma}_{-s}(r))$ performs the role of the permutation from lottery number profile r to $\sigma^*(r)$ in Case 1.

A.2 Proof of Corollary 1.b

Proof. Consider a special case of the proof of Theorem 1 where I suppose that the first-choice research design does not extract a random assignment for the DA mechanism with STB when there are no priorities, i.e., $\rho_{is} = \rho_{js}$ for all i, j, and s. By the STB lottery structure, $r_{is'} = r_{is''}$ for all i, s', and s'', and the order of r_{is} is the same as the order of ρ_{is}^{φ} for any s. In this case, Case 2 never happens and only Case 1 is relevant. In Case 1, by the no-priority and STB assumptions, $r_s^* = \tilde{\sigma}_{s'}(r)$ for all $s' \neq s$. This implies that under the preferences induced by $(\rho, (r_s^*, \tilde{\sigma}_{-s}(r)))$, all schools share the same preference as s's $\succ_{r_s^*}$.

The proof of Theorem 1 implies that when all the other other schools than s commonly report (ρ_s, r_s^*), reporting $(\rho_s', \sigma_s^*(r_s^*, r_{-s}))$ is a profitable preference manipulation for s with respect to $\succ_{r_s^*}$. This contradicts the fact that for the DA mechanism, truth-telling is optimal for s when all the other schools report the same preference as s's true preference. (For a formal proof of this well-known fact, see Hatfield et al. (2016) Proposition 4 and Lemma 1.) Therefore, for problems with no priorities, the first-choice research design must extract a random assignment for the DA mechanism with STB. □

A.3 Proof of Proposition 2

Proof. Take any three schools and label them as A, B, and C. Consider any student preference profile that can be written as follows for some $k \neq l$ and $k, l \geq 1$:

$\succ_1$: $B, A, \emptyset$
$\succ_2, ..., \succ_k$: $B, \emptyset$
$\succ_{k+1}$: $C, A, \emptyset$
$\succ_{k+2}, ..., \succ_{k+l}$: $C, \emptyset$
$\rho_A, \rho_B, \rho_C : \{1, 2, 3, 4, 5\}$.

Without loss of generality, assume $l > k$. If $k = 1$, set $\succ_k$: $B, A, \emptyset$. The capacity of each school is 1 while the treatment school is A. Since both students 1 and $k+1$ rank A second and have the same priority at A, for any gDA mechanism φ, we have

$$\rho_{1A}^{\varphi} \equiv f^{\varphi}(\rho_{1A}) + g^{\varphi}(rank_{1A}) = f^{\varphi}(\rho_{k+1,A}) + g^{\varphi}(rank_{k+1,A}) \equiv \rho_{k+1,A}^{\varphi},$$

which I denote by ρ. Nevertheless, it turns out that for any gDA mechanism with any lottery structure,

$$P(Z_{iA}(R) = 1 | \rho_{iA}^{\varphi} = \rho, \theta_i = \theta_1) < P(Z_{iA}(R) = 1 | \rho_{iA}^{\varphi} = \rho, \theta_i = \theta_{k+1}).$$

To see this, note that for any gDA mechanism, student 1 is assigned to B with probability $\frac{1}{k}$ since only students 1 to k rank B, and all of them rank B first so that for all $i, j \in \{1, ..., k\}$,

$$\rho_{iB}^{\varphi} \equiv f^{\varphi}(\rho_{iB}) + g^{\varphi}(rank_{iB}) = f^{\varphi}(\rho_{jB}) + g^{\varphi}(rank_{jB}) \equiv \rho_{jB}^{\varphi}.$$

Likewise, for any gDA mechanism, student $k+1$ is assigned to C with probability $\frac{1}{l}$. Based

on these facts, I first analyze any gDA mechanism with STB by considering the following cases.

Case i: Neither student 1 nor $k+1$ applies for A, i.e., 1 and $k+1$ are assigned B and C, respectively. This case happens with probability $\frac{1}{k} \times \frac{1}{l}$ since student 1 is assigned to B with probability $\frac{1}{k}$, student $k+1$ is assigned to C with probability $\frac{1}{l}$, and these two events are independent since there is no overlap between applicants for B and those for C so that $\{R_{iB} |\ i \text{ ranks } B\}$ and $\{R_{iC} |\ i \text{ ranks } C\}$ are independent. In this case, no student applies for A, and A is undersubscribed. Recall that I define $Z_{iA}(r) = 1$ for all i if there is no j with $D_{jA}(r) = 1$. Both students 1 and $k+1$ are therefore qualified for A, i.e., $Z_{1A}(r) = Z_{k+1,A}(r) = 1$.

Case ii: Only student $k+1$ applies for A. This case happens with probability $\frac{1}{k} \times \frac{l-1}{l}$. In this case, student $k+1$ is always assigned to A and qualified for A. By $\rho^{\varphi}_{1A} = \rho^{\varphi}_{k+1,A}$ shown above and the fact that student $k+1$ gets the single seat at A, student 1 is qualified for A (i.e., $Z_{1A}(R) = 1$) if and only if student 1 has a better lottery number than student $k+1$ at A (i.e., $R_{1A} < R_{k+1,A}$). Let $U[a,b]$ be a random variable drawn from the uniform distribution over $[a,b]$, $Beta(\alpha,\beta)$ be a random variable drawn from the beta distribution with parameters (α,β), $f(x;\alpha,\beta)$ and $F(x;\alpha,\beta)$ be the pdf and cdf, respectively, of $Beta(\alpha,\beta)$, and $\Gamma(\cdot)$ be the Gamma function. Conditional on Case *ii*, student 1 has a better lottery number than student $k+1$ at A and so qualified there ($Z_{1A}(R) = 1$) with the following probability.

$$
\begin{aligned}
&\Pr(R_{1A} < R_{k+1,A} | D_{1B}(R) = 1, D_{k+1,C}(R) = 0) \\
&= \Pr(\min\{R_{1B}, ..., R_{kB}\} < U[\min\{R_{k+2,C}, ..., R_{k+l,C}\}, 1]) \\
&= \Pr(\min\{R_{1A}, ..., R_{kA}\} < U[\min\{R_{k+2,A}, ..., R_{k+l,A}\}, 1]) \\
&= \int_0^1 \Pr(Beta(1,k) < U[x,1]) \times f(x;1,l-1)dx \\
&= \int_0^1 \int_x^1 F(y;1,k) \times \frac{1}{1-x} dy f(x;1,l-1)dx \\
&= \int_0^1 \int_x^1 \frac{\Gamma(1+k)}{\Gamma(1)\Gamma(k)} \int_0^y (1-t)^{k-1} dt \frac{1}{1-x} dy \frac{\Gamma(l)}{\Gamma(1)\Gamma(l-1)} (1-x)^{l-2} dx \\
&= \frac{\Gamma(1+k)\Gamma(l)}{\{\Gamma(1)\}^2\Gamma(k)\Gamma(l-1)} \int_0^1 \int_x^1 (-\frac{(1-y)^k - 1}{k}) dy (1-x)^{l-3} dx
\end{aligned}
$$

$$= \frac{\Gamma(1+k)\Gamma(l)}{\{\Gamma(1)\}^2\Gamma(k)\Gamma(l-1)} \int_0^1 (-\frac{\frac{(1-x)^{k+1}}{k+1}+x-1}{k})(1-x)^{l-3}dx$$

$$= \frac{\Gamma(1+k)\Gamma(l)}{\{\Gamma(1)\}^2\Gamma(k)\Gamma(l-1)} \times \frac{k+l}{(1+k)(l-1)(k+l-1)}$$

$$= \frac{k(k+l)}{\{\Gamma(1)\}^2(1+k)(k+l-1)}$$

$$\equiv p_1(k,l),$$

where the second equality is by the STB lottery structure while the first and third equalities use the following facts, respectively:

- If $X \sim U[0,1]$, then the distribution of X conditional on $X \geq x_0$ is $U[x_0,1]$ where x_0 is any constant on $[0,1]$.
- R_{iA}'s are i.i.d. samples from $U[0,1]$ while the k-th order statistic of n i.i.d. samples from $U[0,1]$ is distributed according to $Beta(k, n+1-k)$ (Casella and Berger (2002), p.230).

Case iii: Only student 1 applies for A. This case happens with probability $\frac{k-1}{k} \times \frac{1}{l}$ by the same reason as in Case ii. In this case, 1 is always assigned A and qualified for A. Since $\rho^{\varphi}_{1A} = \rho^{\varphi}_{k+1,A}$ and student 1 gets the single seat at A, student $k+1$ is qualified for A if and only if student $k+1$ has a better lottery number than 1 at A. By the same reasoning as in Case *ii*, conditional on Case *iii*, student $k+1$ has a better lottery number than 1 at A and so qualified there ($Z_{k+1,A}(R)=1$) with probability $p_1(l,k)$.

Case iv: Both students 1 and $k+1$ apply for A. This case happens with probability $\frac{k-1}{k} \times \frac{l-1}{l}$ by the same reason as in Cases iii and iv. In this case, again by $\rho^{\varphi}_{1A} = \rho^{\varphi}_{k+1,A}$, only one of students 1 and $k+1$ with a better lottery number is assigned to A and qualified for A. Conditional on Case *iv*, student 1 has a better lottery number than $k+1$ at A and so qualified there ($Z_{1A}(R)=1$) with the following probability.

$$\begin{aligned}
&\underline{p} \\
&\equiv \Pr(R_{1A} < R_{k+1,A} | D_{1B}(R)=0, D_{k+1,C}(R)=0) \\
&= \Pr(U[\min\{R_{2,B},...,R_{k,B}\},1] < U[\min\{R_{k+2,C},...,R_{k+l,C}\},1]) \\
&= \Pr(U[\min\{R_{2,A},...,R_{k,A}\},1] < U[\min\{R_{k+2,A},...,R_{k+l,A}\},1])
\end{aligned}$$

$$
\begin{aligned}
&= \Pr(U[Beta(1,k-1),1] < U[Beta(1,l-1),1]) \\
&= \int_0^1 \int_0^1 \Pr(U[x,1] < U[y,1]) \times f(x;1,k-1) \times f(y;1,l-1) dxdy \\
&= \int_0^1 \int_0^1 \int_y^1 \frac{\max\{t-x,0\}}{1-x} \frac{1}{1-y} dt \frac{\Gamma(k)}{\Gamma(1)\Gamma(k-1)} (1-x)^{k-2} \times \frac{\Gamma(l)}{\Gamma(1)\Gamma(l-1)} (1-y)^{l-2} dxdy \\
&= \int_0^1 \int_0^1 \int_y^1 \max\{t-x,0\} dt \frac{\Gamma(k)}{\Gamma(1)\Gamma(k-1)} (1-x)^{k-3} \times \frac{\Gamma(l)}{\Gamma(1)\Gamma(l-1)} (1-y)^{l-3} dxdy,
\end{aligned}
$$

where the third equality uses the STB lottery regime while the fifth equality uses the fact that $(R_{2A}, ..., R_{kA})$ and $(R_{k+2,A}, ..., R_{k+l,A})$ are independent. Letting

$$
\begin{aligned}
&\bar{p} \\
&\equiv \Pr(R_{1A} > R_{k+1,A} | D_{1B}(R) = 0, D_{k+1,C}(R) = 0) \\
&= \int_0^1 \int_0^1 \int_x^1 \max\{t-y,0\} dt \frac{\Gamma(k)}{\Gamma(1)\Gamma(k-1)} (1-x)^{k-3} \times \frac{\Gamma(l)}{\Gamma(1)\Gamma(l-1)} (1-y)^{l-3} dxdy,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\underline{p}/\bar{p} \\
&= \frac{\int_0^1 \int_0^1 \int_y^1 \max\{t-x,0\} dt (1-x)^{k-3} \times (1-y)^{l-3} dxdy}{\int_0^1 \int_0^1 \int_x^1 \max\{t-y,0\} dt (1-x)^{k-3} \times (1-y)^{l-3} dxdy} \\
&\leq 1,
\end{aligned}
$$

where the last inequality is because of $l > k$. Therefore, since $\underline{p} + \bar{p} = 1$, we have $\underline{p} \leq \frac{1}{2}$ and $\bar{p} \geq \frac{1}{2}$.

To sum up all cases, students 1 and $k+1$'s qualification probabilities at A are different as follows:

$$
\begin{aligned}
&\Pr(Z_{iA}(R) = 1 | \rho_{iA}^{\varphi} = \rho, \theta_i = \theta_1) \\
&= \Sigma_{x=i,ii,iii,iv} \Pr(\text{Case } x) \times \Pr(Z_{iA}(R) = 1 | \rho_{iA}^{\varphi} = \rho, \theta_i = \theta_1, \text{Case } x) \\
&= \frac{1}{k} \times \frac{1}{l} + \frac{1}{k} \times \frac{l-1}{l} \times p_1(k,l) + \frac{k-1}{k} \times \frac{1}{l} + \frac{k-1}{k} \times \frac{l-1}{l} \times \underline{p}
\end{aligned}
$$

$$= \underbrace{\frac{1}{k} \times \frac{1}{l} + \frac{1}{k} \times \frac{l-1}{l} + \frac{k-1}{k} \times \frac{1}{l}}_{\equiv p_2(k,l)} + \underbrace{\frac{1}{k} \times \frac{l-1}{l} \times (p_1(k,l)-1)}_{\equiv p_3(k,l)} + \underbrace{\frac{k-1}{k} \times \frac{l-1}{l} \times \underline{p}}_{\equiv p_4(k,l)}$$

$$< p_2(k,l) + \underbrace{\frac{k-1}{k} \times \frac{1}{l} \times (p_1(l,k)-1)}_{> p_3(k,l)} + \underbrace{\frac{k-1}{k} \times \frac{l-1}{l} \times \bar{p}}_{\geq p_4(k,l) \text{ (by } \bar{p} \geq \underline{p})}$$

$$= \frac{1}{k} \times \frac{1}{l} + \frac{1}{k} \times \frac{l-1}{l} + \frac{k-1}{k} \times \frac{1}{l} \times p_1(l,k) + \frac{k-1}{k} \times \frac{l-1}{l} \times \bar{p}$$

$$= \Sigma_{x=i,ii,iii,iv} \Pr(\text{Case } x) \times \Pr(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_{k+1}, \text{Case } x)$$

$$= P(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_{k+1}),$$

where the key inequality $\frac{k-1}{k} \times \frac{1}{l} \times (p_1(l,k)-1) > p_3(k,l)$ comes from the following facts:

- $\frac{1}{k} \times \frac{l-1}{l} > \frac{k-1}{k} \times \frac{1}{l} > 0$ (by $l > k \geq 2$).
- $p_1(k,l) - 1 < p_1(l,k) - 1 < 0$ (the first inequality is by $l > k$ while the second inequality is because both $p_1(k,l)$ and $p_1(k,l)$ are nondegenerate conditional probabilities).

This proves that there is no gDA mechanism with the STB lottery structure for which the qualification IV research design extracts a random assignment.

For any gDA mechanism with MTB, the argument is simplified as follows.

Case i: Neither student 1 nor $k+1$ applies for A, i.e., 1 and $k+1$ are assigned B and C, respectively. This case happens with probability $\frac{1}{k} \times \frac{1}{l}$. In this case, no student applies for A, and A is undersubscribed. Both students 1 and $k+1$ are therefore qualified for A.

Case ii: Only student $k+1$ applies for A. This case happens with probability $\frac{1}{k} \times \frac{l-1}{l}$. In this case, student $k+1$ is always assigned A and qualified for A. Student 1 is qualified for A if and only if student 1 has a better lottery number than student $k+1$ at A. By $\rho^{\varphi}_{1A} = \rho^{\varphi}_{k+1,A}$, this happens with probability 1/2 by the MTB lottery structure, where R_{1A} and $R_{k+1,A}$ are i.i.d. even conditional on Case *ii* ($D_{1B}(R) = 1$ and $D_{k+1,C}(R) = 0$).

Case iii: Only student 1 applies for A. This case happens with probability $\frac{k-1}{k} \times \frac{1}{l}$. In this case, student 1 is always assigned A and qualified for A. Student $k+1$ is qualified for A if and only if student $k+1$ has a better lottery number than 1 at A. By the same reasoning as in Case *ii*, this happens with probability 1/2.

Case iv: Both students 1 and $k+1$ apply for A. This case happens with probability $\frac{k-1}{k} \times \frac{l-1}{l}$. In this case, only one of students 1 and $k+1$ with a better lottery number is assigned A and qualified for A. Conditional on Case iv, by $\rho^{\varphi}_{1A} = \rho^{\varphi}_{k+1,A}$, student 1 has a better lottery number than $k+1$ at A with probability 1/2.

To sum up all cases, students 1 and $k+1$'s qualification probabilities at A are different as follows:

$$
\begin{aligned}
&\Pr(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_1) \\
&= \Sigma_{x=i,ii,iii,iv} \Pr(\text{Case } x) \times \Pr(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_1, \text{Case } x) \\
&= \frac{1}{k} \times \frac{1}{l} + \frac{1}{k} \times \frac{l-1}{l} \times \frac{1}{2} + \frac{k-1}{k} \times \frac{1}{l} + \frac{k-1}{k} \times \frac{l-1}{l} \times \frac{1}{2} \\
&< \frac{1}{k} \times \frac{1}{l} + \frac{1}{k} \times \frac{l-1}{l} + \frac{k-1}{k} \times \frac{1}{l} \times \frac{1}{2} + \frac{k-1}{k} \times \frac{l-1}{l} \times \frac{1}{2} \\
&= \Sigma_{x=i,ii,iii,iv} \Pr(\text{Case } x) \times \Pr(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_{k+1}, \text{Case } x) \\
&= P(Z_{iA}(R) = 1 | \rho^{\varphi}_{iA} = \rho, \theta_i = \theta_{k+1}),
\end{aligned}
$$

where the key inequality is by $l > k$. Therefore, at the above problem, there is no gDA mechanism with any lottery structure for which the qualification IV research design extracts a random assignment. □

B Additional Discussion

B.1 General Definition of a Random Assignment

Definitions 2 and 4 in the main body define a random assignment under the first-choice and qualification instrumental variable (IV) research designs, respectively. This section explains these definitions are special cases of a unified definition of a random assignment under general empirical research designs. It is therefore legitimate to compare the first-choice and qualification IV research designs based on Definitions 2 and 4.

Given any dataset from any assignment problem, I consider a class of empirical research designs that try to identify causal effects of being assigned to any given treatment school s. Each research design in this class tries to achieve the goal by instrumenting for the treatment assignment ($D_{is}(r_0)$) with some instrumental variable $Z^*_{is}(r_0)$, where r_0 is the realized lottery number profile in the data. For simplicity, I consider only binary instrumental variables, i.e., $Z^*_{is}(\cdot) : \mathcal{R} \to \{0, 1\}$ where $\mathcal{R}$ is the set of all possible lottery number profiles. Define Θ as

the set of all possible student types $\theta_i = (\succ_i, (\rho_{is})_{s \in S})$. Let $(\Theta_1, ..., \Theta_m)$ be a partition of Θ with $\cup_{k=1}^{m} \Theta_k = \Theta$. I allow the research design to do instrumenting conditional on which type partition cell contains each student's type (i.e., conditional on $(1\{\theta_i \in \Theta_k\})_{k=1,...,m}$) and within a restricted sample $I_s^*(r_0) \subset I$, where $I_s^*(\cdot) : \mathcal{R} \to \mathcal{I}$ where $\mathcal{I}$ is the set of all subsets of the set of students I.

For outcome Y_i of interest, the research design measures the effect of treatment assignment $D_{is}(r_0)$ on Y_i by estimating the following Two Stage Least Square regression model or a similar IV model within the restricted sample $I_s^*(r_0)$:

$$Y_i = \alpha_2 + \beta_2 D_{is}(r_0) + \Sigma_{k=1}^{m} \gamma_2^k 1\{\theta_i \in \Theta_k\}) + \epsilon_{2i} \text{ (second stage regression)}$$
$$D_{is}(r_0) = \alpha_1 + \beta_1 Z_{is}^*(r_0) + \Sigma_{k=1}^{m} \gamma_1^k 1\{\theta_i \in \Theta_k\}) + \epsilon_{1i} \text{ (first stage regression)}$$

The above class of research designs is parametrized by what IV to use (Z_{is}^*), which aspects of student type to control for $(\Theta_1, ..., \Theta_m)$, and what sample restriction to impose (I_s^*). I allow these objects to change depending on different gDA mechanisms. For any research design in the class, I introduce the following definition of extracting a random assignment.

Definition 7. An empirical research design with instrumental variable Z_{is}^*, conditioning $(\Theta_1, ..., \Theta_m)$, and sample restriction I_s^* **extracts a random assignment** for a gDA mechanism φ if at any assignment problem X and for any school s,

$$P(Z_{is}^*(R) = 1 | i \in I_s^*(r), (1\{\theta_i \in \Theta_k\})_{k=1,...,m} = v, \theta_i = \theta)$$
$$= P(Z_{is}^*(R) = 1 | i \in I_s^*(r), (1\{\theta_i \in \Theta_k\})_{k=1,...,m} = v),$$

for any potential lottery realizations r, any vector $v \in \{0,1\}^m$, and any student type θ for which these conditional probabilities are well-defined.

Only under this conditionally random assignment does the IV Z_{is}^* generate an exogenous or random variation in assignment treatment D_{is} (Heckman and Vytlacil (2007) chapter 4, Manski (2008) chapter 3, Angrist and Pischke (2009) chapter 4). Whether a research design extracts a random assignment depends on which gDA mechanism generates the data since different mechanisms produce different Z_{is}^*, $(\Theta_1, ..., \Theta_m)$, and I_s^*.

The first-choice and qualification IV research designs are two members of this research design class. The first-choice design corresponds to a research design with the treatment assignment as the instrumental variable $Z_{is}^*(r) = D_{is}(r)$, no conditioning $(\Theta_1, ..., \Theta_m) = \Theta$, and sample restriction $I_s^*(r) = First_s(r)$. The qualification IV design corresponds to a research design with the qualification instrumental variable $Z_{is}^*(r) = Z_{is}(r)$, modified priority

conditioning $\Theta_k = \{\theta \in \Theta | \rho_{is}^{\varphi} = k\}$, and no sample restriction $I_s^*(r) = I$. Substituting these corresponding objects shows that Definition 7 nests as special cases Definitions 2 and 4 for the first-choice and qualification IV research designs, respectively. Definitions 2 and 4 are therefore comparable, making it legitimate to use these definitions to compare the first-choice and qualification IV research designs.

B.2 Strategy-proofness for Schools is not Exactly Necessary

Section 3.4 shows that the first-choice research design does not extract a random assignment for the DA, Charlotte, and top trading cycles mechanisms, which are not strategy-proof for schools. This suggests strategy-proofness for schools is almost necessary for the first-choice research design to extract a random assignment. On the other hand, strategy-proofness for schools turns out to be not exactly necessary. To see this, consider the following mechanism. Given an assignment problem and realized lottery numbers, the **partially deferred acceptance mechanism** is defined through the following algorithm.[32]

- Step 1: Each student i applies to her most preferred acceptable school (if any). Each school accepts its highest-priority (with respect to $\rho_{is} + r_{is}$) students up to its capacity and rejects every other student. Finalize these acceptances and subtract the number of each school's acceptances from that school's capacity.

- Step 2: Each student who has not been accepted by any school applies to her most preferred acceptable school that has not rejected her (if any). Each school tentatively keeps the highest-ranking students up to its remaining capacity (after the subtraction at step 1), and rejects every other student.

In general, for any step $t \geq 3$,

- Step t: Each student i who was not tentatively assigned to any school in Step $t-1$ applies to her most preferred acceptable school that has not rejected her (if any). Each school tentatively keeps the highest-ranking students up to its remaining capacity (after the subtraction at step 1) from the set of students tentatively assigned to this school in previous step $t-1$ and the students newly applying, and rejects every other student.

The algorithm terminates at the first step at which no student applies to any school. Each student tentatively kept by a school at that step or accepted by that school at step 1 is allocated a seat in that school, resulting in an assignment.

[32] Agarwal and Somaini (2015) call this mechanism the "first preferences first" mechanism while the same name is used by others to mean a different mechanism. I use a different name to avoid confusion.

The partially deferred acceptance mechanism is a mix of the Boston mechanism and the DA mechanism in that the first step is process as in the Boston mechanism while the remaining steps are processed as in the DA mechanism. The partially deferred acceptance mechanism can be interpreted as modifying priorities so that each school prioritizes students ranking it first over students ranking it lower, and the partially deferred acceptance mechanism is a gDA mechanism with $(f^{\varphi}(m) = m, g^{\varphi}(n) = 1\{n \neq 1\}(K+1))$.

The partially deferred acceptance mechanism is not strategy-proof for schools by a similar reason for the DA mechanism. However, the first-choice research design extracts a random assignment for the partially deferred acceptance mechanism with any lottery regime. The reason is that given any assignment problem and lottery realization, the treatment assignments of students ranking the treatment school first are finalized at the first step of the algorithm, and their treatment assignments (whether each of them is assigned to the first-choice treatment school) are the same as those produced by the Boston mechanism with the same lottery realization. Corollary 1(a) therefore implies that the first-choice design extracts a random assignment under the partially deferred acceptance mechanism with any lottery regime. Hence, the first-choice research design may extract a random assignment even for a mechanism that is not strategy-proof for schools.

Nevertheless, I am not aware of any empirical study that uses data from the partially deferred acceptance mechanism. As long as more widely-used and widely-discussed mechanisms such as the Boston, DA, Charlotte, and top trading cycles mechanisms are concerned, strategy-proof for schools is necessary, as summarized in Proposition 1.