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Abstract

In a previous essay, we developed a simple (in)efficiency measure for matrix games. We now address the difficulties encountered in assessing the usefulness and accuracy of such a measure.

JEL Classifications: C63, C72, D61
Keywords: 2×2 matrix games.

1 Introduction

In a prior essay, entitled "Expected Worth for 2×2 Matrix Games with Variable Grid Sizes", we considered both a crude measure of the loss of efficiency in 2×2 matrix games between noncooperative and/or zero-intelligence (entropic) players, and an approach to assess whether this loss is sufficient to justify policing/adjusting the games or player behavior. One important theme was that noncooperative analysis is biased toward individual, local (decentralized), and anonymous behaviors, and therefore a short-run time frame, whereas cooperative analysis is biased toward societal behavior that provides a long-run view. Therefore, the concept of a "game within a game" is as

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natural to social science as is an "organism within an organism" in biology. The present essay stands alone from the previous one in the sense that neither requires the other for comprehension. The first paper's simple (in)efficiency measure was developed without addressing the difficulties encountered in assessing how useful and accurate such a measure might be. In the present paper, we examine many of these difficulties in some detail. In a future essay, we will take these observations into account in proposing and evaluating measures of (in)efficiency and other characteristics of 2×2 matrix games.

1.1 Motivation and Preliminaries

Currently, there are no generally accepted solutions for individuals playing either single-shot or repeated matrix games. These open questions provide the framework to study strategic and other behaviors in a standard, formal setting. Repeated games offer particularly rich opportunities to explore the emergence of player coordination and cooperation over time. One of the key features distinguishing human beings from bacteria, insects, and even more complex animals is that we possess far more intricate language, as well as laws and culture, to police behavior. Although it is likely that many aspects of coordination and cooperation are context-specific and facilitated by the use of language, we tend to abstract from these aspects of decision making in the present paper. Instead, we restrict our model building and analysis to examining a large set of simple but mathematically well-defined games that can serve as the basis for experiments in a gaming laboratory, as well as to two questions of fundamental importance to social psychologists and game theorists: How much does the briefing or "back story" supporting payoff structures influence the nature of play? and how do we measure the impact of such scenarios?

1.2 Clearing the Underbrush

In the development of game theory, many variants of the original formulation of a game have been considered. These include games with: lack of knowledge of the rules; uncertain player types (as suggested by Harsanyi, 1967); asymmetric information; continuous strategy sets; and many other special features. Here we limit ourselves to matrix games with symmetric information among the players. One major distinction made by game theorists is that between games in coalition form, with solutions such as the core (see Shapley and Shubik, 1969), the value (see Shapley, 1953), and the stable set of von Neumann and Morgenstern (1948),[1] and games in strategic form, with solutions such as the von Neumann-Morgenstern minimax or the Nash noncooperative equilibrium. Many have felt that this distinction is unfortunate and that a theory should be constructed that unifies both types. In particular, Harsanyi and Selten (1988) spent considerable time and effort attempting to select a single equilibrium point that would be an appropriate solution for any game. For them, a solution was an algorithm or recipe for selecting a unique equilibrium point for any game in strategic form. This heroic attempt does not appear to have succeeded.

In the last few years, much of the game-theoretic literature has focused on the problems of learning, communication, and coordination in games within a dynamic context. Marimon (1996) notes the possibility that learning may provide a way to select among non-unique equilibrium points, but recognizes the concern that learning may lead almost everywhere: "Does learning theory account for any possible pattern? That is, does it justify 'everything goes'?". With a related goal in mind, we consider a related approach: to seek a satisfactory connections among cooperative, noncooperative, and other behavioral perspectives in terms of games within games. This approach is based upon the proposition that noncooperative theory is best viewed as concerned with independent, local, individual behavior within a societal context that often calls for coordination to prevent inefficient outcomes, whereas cooperative theory emphasizes the longer run, and implicitly assumes society has provided the context and rules for the coordination of otherwise independent agents. The need for coordination and enforced cooperation is captured by setting a game within a game, in which a short-term matrix game may be described verbally, but can be regarded as a component of a long-term game that is slowly changing over time, but which the short-term players take as given. A concrete example is the game of business, embedded in a society whose laws, politicians, judges, and law-enforcement agencies set the context for economic activity. The larger game may contribute resources to (or remove resources from) the subgame through laws, subsidies, taxation, and other mechanisms.

[1] Since these concepts are not used in the present essay, we will not provide precise definitions.

Early in the development of game theory, Nash (1950) realized that the existence of a set of fixed points in the games he studied was insufficient to provide a satisfactory solution concept for individual players in a game of strategy. Although suggesting that players need a means to select a unique equilibrium, he did not consider dynamics explicitly. Subsequently, a virtual sub-industry has sprung up among game theorists, political scientists, social psychologists, economists, and others to (a) address issues of learning, teaching, and signaling; (b) provide explanations of the evolution of cooperation; and (c) deal with problems of plausible threats.

One of the most important open problems in game theory today is to identify a satisfactory solution for games in extensive form. The number of solutions that have been suggested is enormous, as evinced by theoretical work such as that of Mailath and Samuelson (2006), as well as behavioral and experimental critiques, such as those of Smith (2008). We make no attempt to summarize the many solutions proposed, and commentaries offered, in this literature. Instead, we comment on both the structural and behavioral aspects of using general matrix games and within a context or environment linked to specific problems.

In all areas of scientific endeavor, there are ongoing debates regarding the appropriate level of abstraction (and possibly mathematization) of the empirical phenomena under study. One key feature of these debates is that the specific characteristics of the phenomenon under consideration impose considerable influence on a satisfactory resolution. In economics, for example, if the question is "What are the necessary and sufficient conditions for the existence of a static, efficient price system in a finite production and exchange economy without government, externalities, and public goods?", then the contributions of Arrow and Debreu (1954), Debreu (1959), and McKenzie (1981) are masterpieces of parsimonious abstraction. However, the same level of abstraction is inadequate to understand the role of money in an economy.

Given our focus on fundamental problems, we often will consider a high level of abstraction: the 2×2 matrix game, which is the simplest structure reflecting strategic choice between two individuals, denoted by R (the row player) and C (the column player). For purposes of exposition, we further will employ the much-studied Prisoner's Dilemma game of Table 1 as a convenient example.

Table 1: Prisoner's Dilemma

	Left ($j = 1$)	Right ($j = 2$)
Up ($i = 1$)	3,3	1,4
Down ($i = 2$)	4,1	2,2

2 Normative vs. Positive Theories

Over the past 60 years, axiomatic methods have been used extensively in normative game theory. In many cases, this has tended to widen the split between normative approaches – addressing the ideal properties of a solution to a formally described game, and positive approaches – concerned with observed behavior or verifiable claims as to how games are actually played by individuals in practice.

2.1 The Noncooperative/Cooperative Dichotomy

Another major division in game-theoretic research is the convenient, but not airtight, distinction between cooperative and noncooperative games. Cooperative game theory has been primarily normative in its approach, whereas noncooperative game theory, despite its stress on individualistic norms, has been more closely associated with experimental gaming and ad hoc applications. (It should be noted that the cooperative/noncooperative dichotomy has changed considerably over time; see, for example, Simon, 1970.) For studying the evolution of societies, we believe models of games within games, interacting on different time scales, are most appropriate. In this case, the long-term bargaining and discussion of cooperative games can be used to develop, maintain, and enforce the laws or rules for many largely individualistic and anonymously played subgames (whose outcomes may feed back to influence the longer term games).

2.1.1 Some Solution Norms

The following list notes various properties often regarded as desirable for a game's solution:

1. Pareto optimality (or Pareto efficiency). Joint (i.e., all-player) improvement of the outcome is not feasible, even with cooperation.

2. Existence. The solution exists for all games to which it is applied.

3. Symmetry. The solution treats all players in the same way (i.e., names do not matter).

4. Individual rationality. The outcome gives each player at least as much as he/she could have obtained by acting alone. (This condition is easily violated.)

5. Subset or group rationality. For any subset of two or more players, the outcome gives the subset (in aggregate) at least as much as it could have obtained by acting as an independent subgroup. (This condition is not only easily violated, but often infeasible.)[2]

6. Uniqueness. There is only one solution point. (This is rarely the case, but a considerable and highly relevant normative literature on fair division argues for a specific outcome.)

7. Independence from irrelevant alternatives. If the set of feasible outcomes is enlarged to include points not satisfying the solution criteria, the set will remain unchanged. (This is a relatively subtle and sophisticated condition compared to the others in the present list.)

8. Mutually consistent expectations (or noncooperative equilibrium). At the solution point, each player's expectations are consistent with those of all other players. (This is the central tenet of the noncooperative equilibrium solution. How this outcome is achieved is one of the major problems of learning, signaling, and teaching in game dynamics.)

Although researchers have identified other, more technical, desiderata – such as risk dominance and equilibrium perfection (see Harsanyi and Selten, 1988) – we will restrict attention to the above list in the present work. Any subset of these properties, possibly with a hierarchy of importance, can be suggested as criteria for a set of games. However, it is important to note that for some games, certain subsets may be mutually inconsistent, and many proposed solutions empty.

The most general solution concept, which is so broad as to be of little or no practical use, is: "The subset of all feasible outcomes." This highly abstract statement captures the idea that

[2]For group rationality to be feasible, the cooperative solution of the core (see Shapley, 1953 and Shapley and Shubik, 1969) must be nonempty, a condition that frequently fails to hold.

a solution calls for nothing more than narrowing the possibilities. In terms of dynamics, it is an algorithm (i.e., set of instructions) telling one how to get from time t to $t+1$.

Beyond the solution-concept properties discussed above, many more considerations, such as the role of language, the distinction between face-to-face and anonymous communication, and the role of outside enforcement, play crucial roles in the study of games.

2.1.2 Some Behavioral Solutions

Rather than taking an axiomatic approach to constructing solutions for a matrix game, one can evoke a direct behavioral approach by considering several categories of player types. Three common types are given by the following – by no means exhaustive – list:

1. The random (entropic) or "zero-intelligence" player, who typically knows the size of his/her strategy set, but nothing else.

2. The best-response player, who requires information on the initial conditions, including any formal or informal hints from any briefing on context and history (e.g., "The matrix represents a Prisoner's Dilemma game, and in previous plays, the 'Confess' choice was favored 60 percent of the time.").

3. The risk-minimizing (maxmin) player, who must be able to view the matrix as a whole and make maxmin computations over both rows and columns.

For the simple case of a 2-person matrix game, one can allow the game to be played by a pair of identical or different behavioral types, yielding 9 possible permutations and 6 possible combinations.

2.1.3 Behavior and a Single Play

An unadorned matrix game, with no historical record or suggestion of a future course, poses considerable problems for game experimentalists, as well as theorists interested in game dynamics.

Given that an individual's life is embedded in society and time, it is reasonable to question how one can experiment with a matrix game played only once, and what can be learned from doing so. There is no such thing as an experimental game played without context. At the very least and most abstract, a group of students in a classroom or laboratory are presented an abstract payoff matrix – often with a background story (such as the Prisoner's Dilemma) – and basic instructions on how to select a strategy and how they will be rewarded (usually in money) depending on the outcome.

The less context supplied by the experimenter, the more players will supply their own. Human memory and experience (unlike those of a computer) cannot be reset to zero. Since personality and upbringing get in the way, the study of single-period matrix game play lies in the domains of both social psychology and game theory, possibly more in the former than the latter.

A matrix game with supplied context raises concerns about the players' beliefs regarding the relationship between descriptive words and the formal game structure. For example, one could create both a military and business scenario for the same abstract matrix game, and then have students from both a military academy and business school play the two games. Would the students play differently in the two scenarios, and would performance be better or worse in the game in which scenario and training match? Paradoxically, at least one experiment (see Simon, 1970) revealed that military students performed better in the business scenario, and business students in the military scenario. The explanation for this disparity was that professional training in a subject area renders related scenarios too simplistic; in other words, if one does not possess relevant training, then he/she is not hampered by institutional knowledge that cannot be used because of oversimplification. Both groups performed better when the same matrix game was treated as an abstract exercise.

If the players are given several different matrix games with no associated scenarios, then the experimenter may be able to determine whether the structure of the numbers themselves guides behavior in a statistically significant way.

2.2 A Single-Period Game Played by Automata

We now consider the very few behavioral player types that can be considered in a 2×2 matrix-game setting without history.[3] For reasons just discussed, these players are essentially automata, not true human beings. If one adds a single historical period of play, then there still will be only

[3] This subsection is based upon unpublished joint work by Schapia and Shubik.

a manageable few player types. Matters change qualitatively with the addition of more historical periods and/or a future.

2.2.1 Single Information Sets

By dehumanizing the players, we are able to consider simple behavior types for which explicit outcomes may be calculated, and thus provide benchmarks for comparison with human players. We consider the following natural categories of automata. (The numbers in parentheses denote the numbers of player types that fall within the indicated categories.)

1. Constant player (2): One that selects the same strategy from $S = \{1, 2\}$ regardless of the specific game being played (or history of play, if available).

2. Entropic player (1): One that selects a strategy from $S = \{1, 2\}$ uniformly at random.

3. Maxmax player (1): One that selects the strategy from $S = \{1, 2\}$ for which the maximum aggregate payoff (joint maximum) is achieved, acting as if it controls both players' strategic choices.[4]

4. Maxmin player (4): One that assumes the other player wants to inflict the smallest possible payoff on it, and takes defensive action by selecting a pure and/or mixed strategy from $S = \{1, 2\}$. This category has four variants:

   - Maxmin(pure) selects a maxmin pure strategy with respect to its payoff;
   - Maxmin(mix) selects a maxmin pure or mixed strategy with respect to its payoff;
   - MaxminDif(pure) selects a maxmin pure strategy with respect to the difference between its own payoff and that of the other player; and
   - MaxminDif(mix) selects a maxmim pure or mixed strategy with respect to the difference between its own payoff and that of the other player.

[4]If there is more than one joint maximum, then a selection rule is needed. If mixed strategies are involved, then a further rule concerning coordination is required.

(Note this treatment avoids having to provide a detailed justification of the use of mixed strategies.)

5. MinMax player (4): One that assumes the other player wants to obtain the maximum possible payoff, and takes aggressive counter-action by selecting a pure and/or mixed strategy from $S = \{1, 2\}$. Analogously to the Maxmin case, this category has four variants:
   - Minmax(pure) selects a minmax pure strategy with respect to the other player's payoff;
   - Minmax(mix) selects a minmax pure or mixed strategy with respect to the other player's payoff;
   - MinmaxDif(pure) selects a minmax pure strategy with respect to the difference between the other player's payoff and its own; and
   - MinmaxDif(mix) selects a minmax pure or mixed strategy with respect to the difference between the other player's payoff and its own.

6. Introspective regression player (∞): One whose behaviors are based upon assumptions such as "I believe that the other player believes that I believe that the other player believes that ...", which involve a potentially infinite chain of reasoning. This problem has been studied by Aumann and Shapley (1994) and others, and is an epistemological issue in the context of single-period games. However, it does not appear to admit of simple experimentation, and therefore will not be discussed further here.

2.2.2 The Two-Ply Case

As a cautious step towards dynamics, we may consider two successive 2×2 games by retaining a single play of the game, but providing information about how the same game was played in one previous period. This enables us to define more player types. In addition to all the above-mentioned low-information types, we now have:

1. Markovian player (16): One whose current selection from $S = \{1, 2\}$ is a simple function of both players' previous moves. In other words, the player is described by a strategy of the

form "If I and the other player selected i and j, respectively, in the previous game, then I will select k in the current game", where $(i, j, k) \in S^3$.[5]

2. Best-response player (2): One who selects a strategy from $S = \{1, 2\}$ in the current game to maximize its payoff, assuming the other player will select the same strategy chosen in the previous game.

We note that there are 12 $(= 2+1+1+4+4)$ player types in the case of no numerical history, and an additional 16 $(= 18+2)$ types when one considers two-ply games, for 30 in total. Furthermore, we observe that it is possible to divide this collection of player types into two subsets:

1. Syntactic players, whose selection of strategies is independent of the specific game in $G(2 \times 2)$ being played. (These include the constant, entropic, and Markovian players.)

2. Semantic players, whose selection of strategies is dependent of the specific game played. (These include the maxmax, various maxmin/minmax, and best-response players.)

Naturally, one may consider different pairwise combinations of the 30 different types of players listed above (ignoring the introspective regression players). Restricting attention to the case of syntactic players, we immediately see that, in games played between types within this classification, the average payoff to each player across all 144 strategically distinct 2×2 matrix games with strictly ordinal preferences (and payoff values $a_{i,j} \in \{1, 2, 3, 4\}$ and $b_{i,j} \in \{1, 2, 3, 4\}$) is exactly 2.5 – the average of the four possible payoff values. (See Table 2.)

Table 2: Average Payoffs, Syntactic vs. Syntactic Players

	Constant	Entropic	Markovian
Constant	2.5, 2.5		
Entropic	2.5, 2.5	2.5, 2.5	
Markovian	2.5, 2.5	2.5, 2.5	2.5, 2.5

[5] In formal game theory, a 2×2 matrix game played twice can be represented in extensive form by a two-ply game tree in which both players move once simultaneously, then are informed of each others' moves and move again. There are 16 end points in this tree. The same game can be represented in strategic form as a 16×16 matrix. Experimenters who employ a 2×2 matrix game played several times in succession often devise their own notation that is neither the classical extensive nor strategic form of the game. Interestingly, an experiment comparing two 2×2 games played in succession and an equivalent 16x16 matrix played once found that different results were obtained, despite the fact that the two formulations were theoretically equivalent (see Shubik, Wolf, and Poon, 1974).

These simple results arise because the syntactic players are non-learners, and their strategies are averaged over the symmetric set of "all possible worlds". The average outcomes for semantic players would be quite different because they make adjustments when confronted by different payoff structures. For three-ply games, the number of automata behavioral types explodes astronomically, so only the two-ply case is analytically tractable. Considering two players, each with two strategies and a strong ordering on payoffs, yields 144 strategically different structures. Accounting for the 30 different automata behavioral types of two-ply games, we see that the set of behaviorally distinct games contains $62,640 = 44\left(\frac{30!}{2!28!}\right)$ elements. At this level of abstraction, there exists an opportunity to study structure and behavior exhaustively.

2.2.3 Automata, People, and Analogies

We again stress the significance of the gap between the mathematization of automata playing abstract games and the use of verbal briefings and analogy in instructing human beings to play abstract games. The ability of automata to operate with a tabula rasa constitutes a fundamental difference between automata and the human players used in experimental gaming.

It is instructive to consider recent successes in the algorithmic modeling of chess and *Go* players. Because these settings are formally 2-person zero-sum games, they are games of pure opposition with no social component requiring the need for language. All historical information is easily expressed in numerical form, and social context plays no role.

At a far more complex level, an operational war game, such as a tank battle, is much easier to formalize for study than a diplomatic negotiation. This is because the tactical battle is largely dependent on hardware and specific factors of terrain and weather, whereas the diplomatic negotiation has many intangibles involving political and societal feasibility, definition of goals, and the socio-psychological characteristics of the negotiators. Although the tactical-battle analysis must take into account the discipline, training, and morale of the combatants, there are few intangibles, except possibly the valuation of skills at the highest level of command. Naturally, both such games are an order of magnitude more difficult to model and analyze than a 2-person board game.

3 Payoffs and Preferences

In previous work (see Powers, Shubik, and Wang, 2016, and Shubik, 2012), we have studied the set of all 144 2×2 strictly ordinal games, viewing the payoff values as additive (and possibly transferable) monetary units. Extending this analysis further, we observe that there exists a large set of abstract properties that could be attached to payoff values in a game of strategy. We now identify and discuss a few of the properties available to the model builder's toolkit.

Skipping subtler measures and pre-measures with non-transitive relations among outcomes, partial orderings, and other preference relations that require greater discussion, we consider only relatively simple individual-preference properties. In particular, we assume that:

- An ordinal measure of preferences exists. (This may be with or without considering ties.)
- A cardinal measure of preferences exists of the form $\gamma x + \delta$, where γ and δ are arbitrary parameters that may be fixed by specific empirical conditions (e.g., Fahrenheit or Celsius measures of temperature).

3.1 Interpersonal Comparison of Preferences

If individual preferences are ordinal, then it is difficult to find meaning in interpersonal comparisons. However, if the preferences possess linear scales, then the question of comparability becomes operationally meaningful. There are three cases of interest:

1. A linear utility measure without comparable scales, with two free parameters (γ_h, δ_h) for each player $h = \mathrm{R}, \mathrm{C}$.
2. A linear utility measure without comparable scales, but with a natural "zero point" for each $\delta_h = 0$ fixed by the problem, and one free parameter (γ_h) remaining for each player.
3. A linear utility measure with comparable scales, a natural "zero point" for each $\delta_h = 0$ fixed by the problem, and a valuation of the γ_h such that for any payoff pair $(a_{i,j}, b_{i,j})$, the sum $\gamma_\mathrm{R} a_{i,j} + \gamma_\mathrm{C} b_{i,j}$ is a meaningful quantity.

In Powers, Shubik, and Wang (2016), we took the payoff values $a_{i,j} \in \{1, 2, 3, 4\}$ and $b_{i,j} \in \{1, 2, 3, 4\}$ to be units of money of comparable worth to each player, with $\gamma_\mathrm{R} = \gamma_\mathrm{C} = 1$ and $\delta_\mathrm{R} = \delta_\mathrm{C} = 0$. In this case, the joint maximum payoff from any matrix is an element of $\{5, 6, 7, 8\}$.

All three of the above possibilities have been employed in the game-theoretic literature. Analyzing games using money as a crude measure of payment provides a convenient link between cooperative and noncooperative games, because payoffs from a short-term, local noncooperative subgame can be used strategically in a longer-term, global cooperative game. An intuitively simple example is a subgame modeling economic competition for which the economy is embedded in a polity, and the subgame players may use part of their earnings to influence political activity.

3.2 Transfer of Value?

If, as is often the case, an experimental matrix game is run with the payoff values representing the amounts of money that players will receive from the various outcomes, then there are two cases to consider: games with side payments, and games with no side payments. In a side-payment game, physical transfers of wealth are permitted after the game is over. For example, if the payoff pair were $(3, 4)$ and payoffs were permitted, then the players could attain the alternative outcome $(3.5, 3.5)$, if they so desired. In a no-side-payment game, the rules forbid any such ex post transfer of wealth.

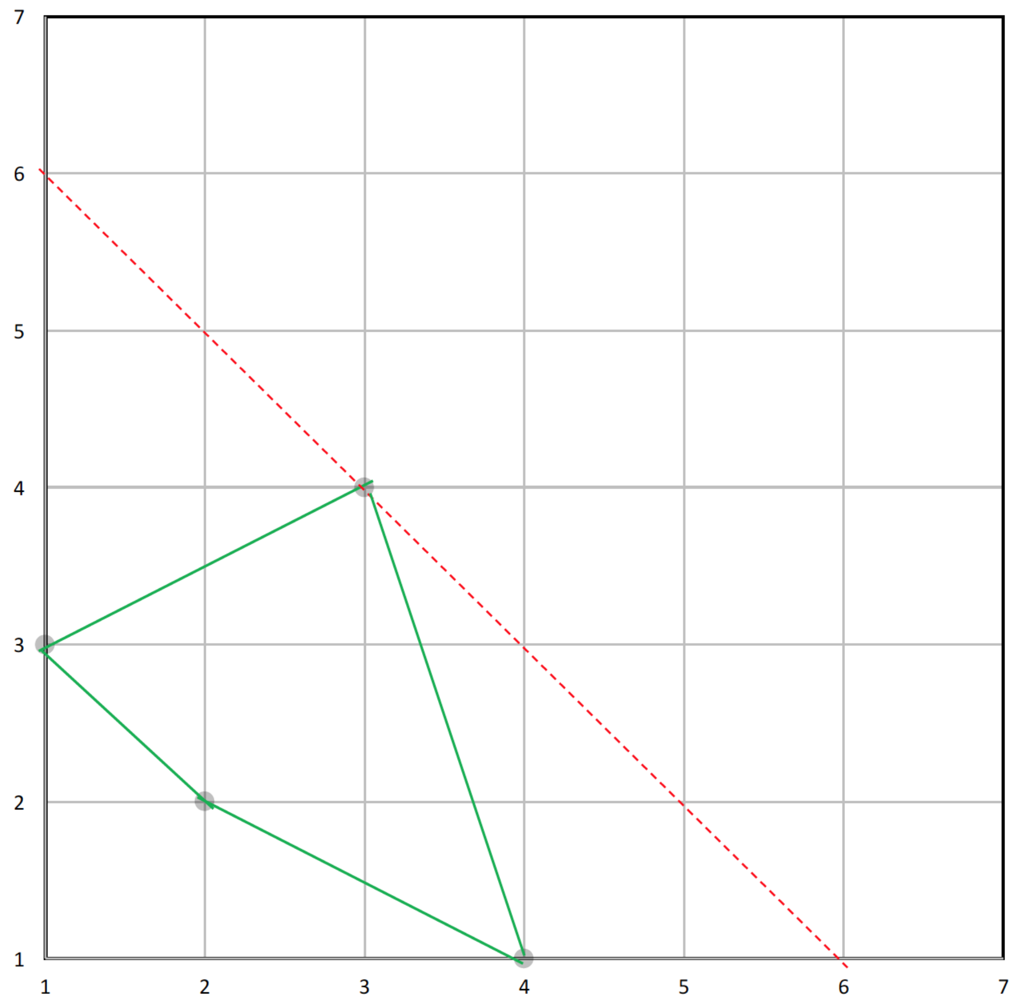
Along with the benefits of helping to attain fair division, the availability of side payments also make extortion and ransom feasible. The asymmetric strictly ordinal game of Table 3 shows how the feasible set of outcomes is affected by the possibility of side payments.

Table 3: An Asymmetric Strictly Ordinal Game

	Left	Right
Up	1,3	3,4
Down	4,1	2,2

For this game, one can draw the payoff set as in Figure 1. We note that the joint maximum occurs in the upper-right cell, with payoffs $(3, 4)$. If side payments are permitted, then the entire range from $(0, 7)$ to $(7, 0)$, including the symmetric joint max of $(3.5, 3.5)$, can be attained. The minimal individually rational payoff is $(2,2)$, achieved when both players employ a maxmin strategy. Finally, we note that there are points within the payoff, set such as $(4,1)$ and $(1,3)$, that are not individually rational for at least one player, but could occur under some behavioral patterns.

Figure 1. 2x2 Payoff Space for Game in Table 3



4 Dynamics and Other Disciplines

The present essay is primarily pre-dynamic because, at the level of abstraction and mathematical description used, one cannot do justice to even elementary dynamics except by analogy and allegory.

We therefore confine our remarks to a few questions associated with dynamics:

- Do all individuals know the rules of the game?
- Do psychological factors matter?
- Does fineness of perception (grid size and just-noticeable difference) matter?
- What scope of memory retention is assumed?
- Are socio-psychological factors relevant?
- Are implicit customs, laws, and social conventions taken into account?
- Is the treatment of language relevant?

It is possible to set aside much, if not all, of the above and see what can be learned from repeated plays of abstract games. This was done in an ingenious questionnaire-experiment by Axelrod (2006).

4.1 Emergence of Cooperation in Repeated Matrix Games

Ever since the Axelrod (2006) experiments, much theory and experimentation has been devoted to considering the emergence of cooperation in repeated (usually 2×2) matrix games.

4.2 The Playable Game and a Simple Universe

4.2.1 Playability

In constructing game-theoretic models of strategic settings, it is useful to consider – and actively seek out – playable-game representations. This approach forces one to be explicit about the level of complexity, as well as to specify both initial and terminal conditions. It also illuminates the relationship between the mathematical model and the physical reality, and exposes relevant problems of interpretation. For these reasons, the process of constructing a playable game provides reality training, helps avoid false generality, and requires "truth in packaging" when linking mathematics and the real world. More specifically, testing the structure of a playable game enables one to address questions concerning the roles of communication, language, player identity/anonymity, and context (e.g., whether it is played in the laboratory, in class, on the Internet, at Las Vegas, or elsewhere).

4.2.2 Cycles in Dynamics

In playing sequential games, it is easy to construct highly nonintuitive examples (see, e.g., Quint, Shubik, and Yan, 1996). For example, the single-period version of the symmetric 3×3 game in Table 4 possesses a mixed-strategy equilibrium of with probability triple $(1/2, 0, 1/2)$ for each player, and also a pure-strategy equilibrium with probability triple $(0, 1, 0)$. However, there exists a best-response 4-cycle alternating around the 4 corner cells.

Table 4: A Symmetric 3×3 Game

	$j = 1$	$j = 2$	$j = 3$
$i = 1$	9,4	0,0	4,9
$i = 2$	0,0	6,6	0,0
$i = 3$	4,9	0,0	9,4

4.2.3 Universality

Whenever gaming exercises are performed on a limited set of games, the resulting deductions must be treated with caution. In particular, we would note that the set of 144 strictly ordinal 2×2 games forms a natural, closed strategic universe that can be utilized as a test bed for investigating many strategic concepts. However, generalizing beyond this set to settings with more players, more strategies, the possibility of ties, etc., may be misleading as important qualitative differences can appear with an enlargement of scope.

5 Cooperative and Noncooperative Solutions

In developing a theory of cooperative n-person games, von Neumann and Morgenstern (1948) selected comparable, linearly transferable utility because they wanted to begin with the simplest assumptions in order to deal with the considerable combinatorics they foresaw for virtually any cooperative solution. More complicated assumptions could (and did) come later. One driving assumption in their models and solution concept was that outcomes should be Pareto optimal. This is consistent with their interest in modeling society as a whole, complete with the implicit use of language and coalitions. In contrast, when Nash (1950) developed the general noncooperative solution some years later, his driving assumption was the mutual consistency of expectations among otherwise isolated, nonsocial, anonymous individuals. To develop his theory, Nash used measurable utilities but left two undetermined parameters for each utility function. In neither the von Neumann-Morgenstern nor Nash approach does the solution concept always select a unique point.[6]

Nash and some other economists viewed the noncooperative equilibrium as an "American" or individual-enterprise solution in keeping with competitive markets and the freedom of individual choice. In contrast, cooperative approaches might be regarded as more collectivist, and possibly "European" in spirit. Alternatively, we suggest a completely different perspective: the noncooperative solution is more a feasible model of short-term, decentralized decision making, whereas many aspects of the various cooperative solutions model longer-term behavior. The noncooperative game in strategic form thus can be regarded as more or less individualistic and economic, in which the rules of the game are derived by a higher-level game that may involve the actions of coalitions and is more political and social in form. This somewhat loosely stated point is noted to provide some context, and is developed more fully below.

At the end of the present essay, we more precisely define the problem of comparing cooperative, noncooperative, and other solutions for the 2-person game. Specifically, the question to be addressed is whether one can estimate the difference in efficiency between cooperatively played games, on the one hand, and the same games played noncooperatively or in some other manner. To this end, we consider noncooperative games with payoffs in money that acts as a linearly transferable utility that permits a comparison of outcomes yielded by different solutions to the same game.

[6]However, for a cooperative game with linearly transferable utility, the Shapley value does provide a single-point solution.

6 2-Person Matrix Games

In this section, we restrict attention to the class of 2-person $k \times k$ matrix games, and focus specifically on the set of 2×2 games described below. An important reason for working with $k \times k$ matrix games – rather than the more general $k \times \ell$ case – is that the former emphasizes the intrinsic symmetry of the roles of the players. Furthermore, it can accommodate the case of asymmetry, without loss of generality, by the simple addition of strategically irrelevant rows or columns.

The 2-person game is the simplest n-person matrix game for which many of the important problems of competition, coordination, and collaboration can be posed. This is because:

- k must be at least 2 for the players to require strategic thinking;
- n must be at least 2 for each player to possess strategic alternatives; and
- the players must select strategies simultaneously if we want symmetric treatment in information.

There is a considerable literature on playing the 2×2 Prisoner's Dilemma, both as a single-period and repeated game, and the utilization of the "Tit for Tat" strategy in the latter case has caught the imagination of many outside the community of professional game theorists. There also are considerable literatures on other specially named 2×2 matrix games, such as the Stag Hunt and Battle of the Sexes. As noted many years ago by the psychologist Miller (in his classic publication, "The Magical Number Seven, Plus or Minus Two", 1956), the range of individual perception of separate items lies between 5 and 9. Consequently, an individual player may easily grasp and keep in mind the 8 payoff numbers needed to describe a 2×2 matrix game, but would have difficulty remembering the 18 numbers in a 3×3 game or the 24 numbers in a $2 \times 2 \times 2$ game. Rapoport, Guyer, and Gordon (1976) were the first to study a complete set of matrix games for the 2×2 case. This would not be feasible, from the perspective of either theory or experimentation, for $n > 2$ and/or $k > 2$.

6.1 The Complete Set of Strictly Ordinal 2×2 Games

We now consider the case of a single-period 2×2 matrix game with strictly ordinal payoffs from the set $\{1, 2, 3, 4\}$ (for which a greater payoff value is preferred to a smaller one). Specifically, the row and column player payoffs are assigned as follows: $a_{i,j} \in \{1, 2, 3, 4\}$ s.t. $a_{i,j} = a_{i',j'}$ iff $i = i'$ and $j = j'$ and $b_{i,j} \in \{1, 2, 3, 4\}$ s.t. $b_{i,j} = b_{i',j'}$ iff $i = i'$ and $j = j'$. It is easy to see that this generates $576 = (4!)^2$ distinct matrices, but this set can be reduced to $144 = 576/4$ by removing all strategically equivalent cases (i.e., games that differ only as the result of interchanging a row or column). The importance of this set of 144 games is that it is the smallest complete, closed universe of strategic structures, and thus provides an overall strategic framework within which to consider the viability of players in all environments. This has been discussed and illustrated in both Powers, Shubik, and Wang (2016) and Shubik (2012).

6.2 Rules and Players

So far, we have addressed game structure, but said nothing about the nature of the player agents. We now confine ourselves to the case of individual human players.

6.2.1 Von Neumann Players

The considerable literature on evolutionary game theory appears to fit insects or microbes better than humans. There also is a sizable literature in which players may be interpreted as organizations of individuals such as firms, nations, or other institutions. Here, however, we consider *homo economicus*, or what one might call a "von Neumann player". This individual is completely rational, and assumed able to carry out any calculation unless specific limitations on his/her abilities are specified. One of the great open challenges in game theory has been to formalize the concept of a capacity-constrained individual agent. Although it may be tempting to try to impose computer science-like limits, such as the bit size of memory, such an approach immediately creates problems in the coding of information. We may, however, place constraints on strategies and outcomes to be considered.

6.2.2 Artificial Agents

"In the kingdom of the blind, the one-eyed man is king." – Old proverb

"In the kingdom of the one-eyed, man is mad." – Counter proverb

In truth, the world does not appear to be populated with von Neumann players. Individuals with limited abilities abound, and it is not always to the advantage of a von Neumann player to confront a large population of individuals with more limited abilities. One way in which we can model a form of limited ability is to assume each agent has a fixed strategy. For example, we may consider a best-response agent who assumes that his/her opponent will retain a previously played strategy, and then optimizes against it; or an entropic (zero-intelligence) agent who simply randomizes over a specified set of strategies. In Subsection 2.2 above, we have listed different player types for low-information games.

6.3 Solution to a Single-Period Matrix Game

It is useful to focus on single-period games because they avoid all complications of learning and teaching. Although pre-game communication and/or language use may appear in many actual one-shot encounters, this is ruled out here unless such communication is formally modeled.

Stated simply, a 2×2-game solution consists of a pair of strategies – $i \in \{1, 2\}$ and $j \in \{1, 2\}$ – the former selected by the row player, and the latter by the column player. We may wish to limit this concept to include extra properties such as: mutually consistent expectations (noncooperative equilibrium); Pareto optimality; the most symmetric cooperative solution (that selects a maxmax over outcome pairs); or a maxmin solution in which each player expects the worst. We also may wish to require that the solution satisfy more than one criterion from this "shopping list". Such an intersection may be empty; or this approach may provide a useful way to identify a unique outcome. The most popular solution for the single-period matrix game is the noncooperative equilibrium.

There have been several attempts to construct a natural taxonomy of the 144 strictly ordinal 2×2 games (see Rapoport, Guyer, and Gordon 1976; Borm, 1987; and Kilgore and Frazer, 1988). The least controversial is to break the set into three categories:

- 36 games of coordination (i.e., games in which a cell contains the joint-maximum payoff pair, $(4,4)$);
- 6 games of pure opposition (i.e., constant-sum games); and
- 102 games in which the coincidence of joint interest or opposition of interest is mixed.

Given that the set of 144 strictly ordinal 2×2 games is the only non-trivial collection of games that can be explored exhaustively, Baranyi, Lee, and Shubik (1992) explored all possible noncooperative equilibria as topological structures. For each game, they constructed the convex hull joining the four payoff pairs $(a_{i,j}, b_{i,j})$, and noted that 24 $(= 4!)$ such figures existed. They further noted that, by rotation and reflection, this set can be reduced to 7 basic shapes. If the payoffs are interpreted as cardinal values, then the convex hulls represent the universe of possible payoffs attainable through the use of mixed strategies. Within the set of 24 structures, there are:

- 6 figures with a 1-point Pareto set;
- 13 figures with a 2-point Pareto set;
- 4 figures with a 3-point Pareto set; and
- 1 figure with a 4-point Pareto set.

The set of 144 strictly ordinal games can be subdivided into:

- 18 games with 0 pure-strategy equilibria;
- 108 games with 1 pure-strategy equilibrium; and
- 18 games with 2 pure-strategy equilibria.

Furthermore, all 36 games without a unique pure-strategy equilibrium possess a single mixed-strategy equilibrium.

Only 1 of the 108 unique pure-strategy equilibria – that associated with the Prisoner's Dilemma – is not Pareto optimal. However, only in the games of pure opposition is a noncooperative equilibrium always Pareto optimal. This latter observation shows that in any situation in which collaboration has no value, independent action determines the shares of a fixed sum. Looking at the set of 144 strictly ordinal games as a whole, we see that the noncooperative equilibria have only the mutually consistent expectations property going for them. They do not always offer uniqueness, fairness, or optimality. In games with a continuum of strategies, noncooperative equilibria generally are not Pareto optimal (see Dubey and Rogawski, 1990).

7 Problems with $k \times k$ Matrix Games

In the present section, we provide a brief overview of some of the most significant modeling problems encountered when the number of strategic choices available to each player is greater than $k = 2$. In Subsection 8.2 below, we will consider the problems arising when the number of players is greater than $n = 2$.

7.1 Coordination and Pure Opposition

As k increases, the proportion of games of coordination drops off as $O\left(1/k^2\right)$, and the proportion of games of pure opposition decreases as $O\left(1/k!\right)$. Thus, for large k, virtually all games are characterized by mixed motives.

7.2 Predominance of Mixed-Strategy Equilibria

A study of 2×2 matrix games might suggest that pure-strategy noncooperative equilibria are quite common. However, for large k, the proportion of $k \times k$ games that possess $m \in \left\{1, 2, \ldots, k^2\right\}$ pure-strategy noncooperative equilibria is given approximately by the Poisson($\lambda = 1$) probability $f\left(m\right) = e^{-1}/m!$ (see Powers, I., 1990). Consequently, the proportion of such games that possess more than 1 noncooperative equilibrium is approximately $1 - 2e^{-1} \approx 0.2642$.

7.3 Purification

If we regard the use of mixed strategies as a means of minimizing the value of one player's learning something about the other's plans, then the presence of outside random events may serve the same purpose. It is an open question whether a mixed strategy is psychologically natural and/or operationally justified. Even assuming it has value, Aumann et al. (1983), Levitan and Shubik (1971), and Skyrms (2004) noted that the presence of exogenous uncertainty removes the need to randomize between strategies because the noise from Nature offers sufficient concealment.[7] Since many reasonable applications entail the presence of exogenous uncertainties, it follows that Nature frequently obviates the need for mixed strategies.

7.4 Strategic Plans

For most human activities, 10 strategies is about as large a set of alternative plans as is ever considered in practice; and at that size, almost all matrix games are characterized by mixed motives. Therefore, even for a 2-person matrix game, the coordination problem is likely to grow in significance unless it is addressed by a special context and/or structure.[8] General $k \times k$ matrix games with $k > 10$ rarely arise in social-science models without such a special structure.

7.5 Ties

We have concentrated on ordinal games without ties, but in fact human preference is often characterized by the inability to perceive distinct valuations for different outcomes. It therefore is reasonable to extend our analysis to consider ordinal games with ties. Even for the 2×2 case, this extension increases the number of strategically different games from 144 to 726. (This is because the domain of the set of games with ties includes $65,536 = \left(4^4\right)^2$ distinct matrices, but this set can be reduced to 726 by removing all strategically equivalent cases.)

Table 5 summarizes the set of all ordinal 2×2 matrix games with ties permitted.

[7] A trivial example is provided by the game of Matching Pennies. Suppose that with probability 1/2, Nature reverses the signs on all payoffs. Then there is no benefit from further randomization.

[8] For example, one might wish to study an economic duopoly model using a 20×20 matrix with a special structure affecting the smoothness of the payoff surfaces.

Table 5: All Ordinal 2×2 Games with Ties Permitted

	4,3,2,1	3,3,2,1	3,2,2,1	3,2,1,1	2,2,2,1	2,2,1,1	2,1,1,1	1,1,1,1	Total
4,3,2,1	78	72	72	72	24	36	24	6	384
3,3,2,1		21	36	36	12	18	12	3	138
3,2,2,1			21	36	12	18	12	3	102
3,2,1,1				21	12	18	12	3	66
2,2,2,1					3	6	4	1	14
2,2,1,1						8	6	3	17
2,1,1,1							3	1	4
1,1,1,1								1	1
Total	78	93	129	165	63	104	73	21	726

This number of cases is too large to study in totality,[9] but a smaller set of games in which the perception of one player is clearly better than the other would illustrate the value of expertise. This could be seen by observing that the more perceptive player is able to throw away his additional information, and thus cannot do worse than if he/she were less perceptive.

7.6 Rough vs. Smooth Surfaces

A 2×2 matrix possesses so few entries that only first (and not second or higher-order) differences among the payoffs can be defined. Consequently, there can be at most two "hills" on the payoff surface. In the 3×3 case, there can be at most five "hills" and four "valleys". Once $k \geq 4$, the complexity of local minima and maxima becomes considerable. Unless there is a specific question with additional structure, it is difficult to link verbal analogy with mathematical structure for any such games in the behavioral sciences.

7.7 Plausible Threats

The problem of how to define a plausible threat is one of the long-standing key problems of game theory. Treating a strategic situation as a matrix game solves this problem by definition, but immediately raises questions about the model itself. In any finite matrix game with numerical payoffs, one can use the greatest lowest bound. For the 2×2 case, this can be found by drawing

[9] The full set of $k \times k$ ordinal games has $\left[(k^2)!\right]^2 / k^2$ distinct members without ties, and $\left[(k^2)^{(k^2)}\right]^2 / k^2$ distinct members with ties.

a two-dimensional figure of the payoff space, and then raising a line with slope -1 from the origin until it touches the feasible set.

A different way of solving for a minimum plausible outcome is to determine that part of the feasible set that contains individually rational outcomes. The least-desirable point in this set is given by the solution (i^*, j^*) to the pair of equations:

$$i^* = \arg\max_{i \in \{1,2\}} \min_{j \in \{1,2\}} U_{\mathrm{R}}(i,j)$$

$$j^* = \arg\max_{j \in \{1,2\}} \min_{i \in \{1,2\}} U_{\mathrm{C}}(i,j),$$

where $U_{\mathrm{R}}(i,j)$ and $U_{\mathrm{C}}(i,j)$ denote the true utilities (as opposed to nominal payoffs) of the row player and column player, respectively.

Yet another way to evaluate a zero point in a plausible-threat game is to consider what happens if each player tries to maximize its relative advantage over the other. With comparable payoffs, this determines an optimal damage-exchange rate; and if there is an interior solution, this rate will be equal for each. More generally, this problem of "ill-fare" economics may be solved as follows:

$$i^* = \arg\max_{i \in \{1,2\}} \min_{j \in \{1,2\}} \left[U_{\mathrm{R}}(i,j) - U_{\mathrm{C}}(i,j)\right]$$

$$j^* = \arg\max_{j \in \{1,2\}} \min_{i \in \{1,2\}} \left[U_{\mathrm{C}}(i,j) - U_{\mathrm{R}}(i,j)\right].$$

Sometimes, a game's zero point or threat point is not defined mathematically, but rather by a physical outcome that serves as a natural boundary condition. For example, in most economic markets, a customer is able to leave the market if he/she sees no better alternative. Moreover, some choices lead to pondering the imponderable, thinking the unthinkable, and/or trying to measure the unmeasurable. How can one evaluate the threat of biological war, nuclear war, and global warming? What units of measurement are available, and what would they mean? Is "mega-death" a measure or simply rhetoric? Such questions remain open; but for meaningful modeling, one must justify any assumptions made.

8 Problems with n-Person Matrix Games

Good modeling must be preceded by a well-specified question and followed by the use of sound approximative and sensitivity-analytic techniques. One always must be prepared to reconsider the model's original intent when considering even simple and mathematically well-defined extensions such as enlarging the numbers of players.

8.1 Cooperative Solutions

The noncooperative equilibrium and other individualistic solutions that assume player behavior does not involve communication are best suited to mass markets in which it is reasonable to view each player as an individual confronting a faceless mechanism (i.e., the aggregation of all other agents). Such 2-person models fit settings such as brief military exercises or experimental games that last no more than a couple of hours.

In contrast, the theory of cooperative games has concentrated not only on the 2-person bargaining problem, but also on what can be said about games with $n \geq 3$ players. Rather than a formulation in matrix or strategic form, cooperative games utilize a different representation. The characteristic function, $\upsilon(S_i)$, provides a value for any coalition $S_i \in N$ of $|S_i|$ players (where N denotes the full coalition of all $n = |N|$ players). Von Neumann and Morgenstern (1948) took a maxmin approach to calculate this value, assuming that the coalition $N - S_i$ acts to damage the coalition S_i as much as it can. Essentially, this is as though they studied an $(n+1)$-player constant-sum game in which the extra player is a strategic dummy, s^*, who obtains the value

$$\upsilon(s^*) = \upsilon(N) - [\upsilon(N - S_i) + \upsilon(S_i)] .$$

In other words, the n-person cooperative game always could be extended to the form of a constant-sum game.

Another method of calculating the characteristic function of an n-person game is to consider how the 2^n values of a coalition S_i are computed under different, less hostile, circumstances. This is discussed further below.

8.2 Different Ways of Going to the Limit

We now return to the strategic form. As we increase the number of players, the expansion of even a game with only two choices per player quickly buries us in an overwhelming number of cases. In particular, the $144 = (4!)^2/4^2$ strategically different cases of the strictly ordinal 2×2 game are replaced by $(4!)^n/4^n$ different cases for the strictly ordinal game with $n > 2$. The formal mathematics of adding players is easy; but the meaning of the problem may have been changed. This is illustrated below by a variant of the Tragedy of the Commons.

8.2.1 Numbers and Context

Suppose there is a newly formed town whose citizens collectively own a cattle-grazing space large enough for optimal grazing by two herds of cattle. We then can represent the strategic encounter between two herd owners through the 2×2 game in Table 6, in which each owner receives a payoff of 0 from not grazing his/her cattle, and a payoff of 4 from grazing. Given that there is sufficient land, the two herders' decisions do not affect each other, and they therefore behave essentially independently.

Table 6: Grazing the Commons

	Do Not Graze	Graze
Do Not Graze	0,0	0,4
Graze	4,0	4,4

Now suppose that the number of herds increases to 3. If the amount of common property remains constant, then the strategic encounter among the three herd owners cannot be represented easily by a $2 \times 2 \times 2$ extension of Table 6. Instead, one can model the case of three herders by a set of three 2×3 matrices of the form shown in Table 7 (with one matrix for each herder).

Table 7: Overgrazing the Commons, One Herder's Perspective

	0 Graze	1 Grazes	2 Graze
Do Not Graze	$0, \bar{0}$	$0, \bar{2}$	$0, \bar{4}$
Graze	$4, \bar{0}$	$4, \bar{2}$	$1, \bar{1}$

For a given herder's matrix, the first number in each cell is the payoff to that herder, and the second number is the average payoff to the other two herders. This allows the herder under consideration to consider what the other herders do and how their payoffs are affected by his actions. In all cells but that of the lower right, the (two or fewer) herders who decide to graze simply share the available 8 payoff units (up to a maximum of 4 units each). In the lower-right corner, however, the three herders fail to use the common ground efficiently, and so each receives a payoff of only 1 unit. From Table 7 it is easy to see that, regardless what the other herders do, a noncooperative herder always will prefer grazing to not grazing, resulting in the inefficient payoff triple $(1, 1, 1)$. If instead they had worked cooperatively, they could have achieved the efficient solution of $(8/3, 8/3, 8/3)$.

Four important lessons may be learned from this example:

1. A change in the number of players (n) requires changes in both the numbers of strategies and payoffs.

2. Changes in the numbers of strategies and payoffs will depend critically on the context of the problem.

3. Conclusions drawn from the case of one value of n (e.g., the 2×2 case) may be misleading for other n, necessitating a change in the mathematical model.

4. The structure of a game with more players and strategies still can be solved using a noncooperative solution, but in some contexts (such as the use of public lands or the generation of atmospheric pollution) the payoff structure will indicate considerable room for gains from cooperation.

8.2.2 Paradox of the Prisoner's Dilemma

It is often the crowning joy of a lecturer using matrix games to teach elementary game theory to present the results of the Prisoner's Dilemma, and note its uniqueness among all 2×2 games. Essentially, this game shows quite nicely how a suboptimal outcome can arise from individual competition. However, it is important to recognize that an outcome that may be suboptimal with two players is not necessarily suboptimal in a game with many more players, even if the additional agents individually do not possess any strategic power.

Within the theory of games, there exists the useful concept of a strategic dummy; that is, a player who has no strategic power, but who may have a payoff function. In the context of competition in economic markets, it is frequently considered reasonable to treat thousands or millions of customers as strategic dummies in the sense that they impart no individual effect on price. Since firms know that these customers will accept the price named, and will adjust the amount they purchase depending only on price, the customers can be regarded as a non-strategic mechanism. Consequently, to model the case of a market with only two firms (i.e., a duopoly), we can use the modified 2×2 game of Table 8 – with three rather than two payoff numbers in each cell. In this example, the third number indicates the gain to consumers collectively.

Table 8: A Duopolistic Market

	High Price	Low Price
High Price	3,3,1	1,4,2
Low Price	4,1,2	2,2,3

Looking at the payoffs to the two duopolists, one can see that they are exactly the same as the corresponding values in the familiar Prisoner's Dilemma of Table 1. Therefore, it follows that: (a) if the two firms were to cooperate (in this case, what might be called "collusion"), then they could achieve the joint-maximum payoff of $(3, 3)$; but (b) if each were to try to grab market share, then they would wind up with the payoff of $(2, 2)$. However, we now see that the latter, "suboptimal," payoff is the most beneficial for customers. In other words, somewhat paradoxically, the "worst" of the 2×2 games actually illustrates the virtues of competition when more players are added.

8.3 Value of Government: Rules, Signals, and Transfers

Much of the richness of strategic decision making can be explored utilizing 2×2 games. In concrete applications, the high level of abstraction may be modified by providing the context of specific problems and making certain adjustments to representation. With the qualifications noted above, one can develop an apparatus to obtain a first approximation to the value of coordinating and policing to agents in a society. This can be achieved for various combinations of the norms discussed above, as well as for a variety of behaviors other than those of purely rational, error-free decision makers. For example, the Tragedy of the Commons example above can easily be modified to show that there are many ways to solve this problem, including establishing a town council to rent or sell the land and then use the proceeds to cover the expenses of enforcement and collections. Many institutional arrangements have been discussed for this problem by Ostrom (1990) and others.

In this essay, we have not been concerned with institutional examples, but rather with the general principle that when one views human activity as a game within a game, spending resources on coordination and policing may be of value to all concerned. In a subsequent essay, we will develop and discuss appropriate measures of value for many of the solutions considered herein.

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