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Michael R. Powers<br>Martin Shubik<br>Wen Wang

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# EXPECTED WORTH FOR $2 \times 2$ MATRIX GAMES WITH VARIABLE GRID SIZES 

By

Michael R. Powers, Martin Shubik, and Wen Wang

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Box 208281
New Haven, Connecticut 06520-8281
http://cowles.yale.edu/

# Expected Worth for $2 \times 2$ Matrix Games with Variable Grid Sizes 

Michael R. Powers, ${ }^{1}$ Martin Shubik, ${ }^{2}$ and Wen Wang ${ }^{3}$

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#### Abstract

We offer a detailed examination of a broad class of $2 \times 2$ matrix games as a first step toward considering measures of resource distribution and efficiency of outcomes. In the present essay, only noncooperative equilibria and entropic outcomes are considered, and a crude measure of efficiency employed. Other solution concepts and the formal construction of an efficiency index will be addressed in a companion paper.


JEL Classifications: C63, C72, D61
Keywords: $2 \times 2$ matrix games, efficiency, coordination, worth of coordination.

## $12 \times 2$ Matrix Games with Cardinal Payoffs

In the folklore and elementary pedagogy of game theory, the $2 \times 2$ matrix game plays a special role.
Several of these games bear well-known names, such as the Prisoner's Dilemma, Stag Hunt, and Battle of the Sexes. Although there are only 144 strategically different $2 \times 2$ games with strictly ordinal preferences, one often is interested in considering related games with cardinal preferences, whose number is unbounded. The present paper is devoted to addressing applications in which it is desirable to examine a large but finite set of $2 \times 2$ games with cardinal preferences.

### 1.1 Outcome Sets

A generic $2 \times 2$ game is described by the matrix shown in Table 1 . Here, the row player has the two strategies, "Up" and "Down" (corresponding to rows $i=1,2$, respectively), whereas the column player

[^0]has "Left" and "Right" (corresponding to columns $j=1,2$, respectively). This yields four possible payoff pairs (outcomes), $\left(a_{i, j} b_{i, j}\right)$, for $i=1,2$ and $j=1,2$.

Table 1: A Generic $2 \times 2$ Matrix Game

$$
\begin{array}{ccc} 
& \text { Left }(j=1) & \text { Right }(j=2) \\
\text { Up }(i=1) & a 1,1, b 1,1 & a 1,2, b 1,2 \\
\text { Down }(i=2) & a 2,1, b 2,1 & a 2,2, b 2,2
\end{array}
$$

The universe of all strictly ordinal games is easily denumerated by noting that each of the payoff vectors $\left[a_{11}, a_{12}, a_{21}, a_{22}\right]$ and $\left[b_{11}, b_{12}, b_{21}, b_{22}\right]$ must be permutations of the ordinal vector [1,2,3,4], yielding a total of $4!\times 4!=576$ different outcomes. This number can be divided by 2 to remove duplications arising from interchanging rows, and by another 2 to account for interchanging columns, leaving the canonical 144 strategically distinct games shown in Appendix 1. Topologically, the outcome sets of these games may be characterized by a smaller set of 24 distinct shapes, 22 of which are two-dimensional (i.e., games of opposition) and the remaining 2 one-dimensional (i.e., games of coordination). These shapes are shown in detail in Appendix 1, where they are associated with the 144 games.

In considering cardinal games, we assume that the payoff pairs, $\left(a_{i, j} b_{i, j}\right)$, may be expressed in welldefined units of money or gold, with a fixed minimal level of fineness that can be perceived and/or traded. ${ }^{4}$ Is there an upper bound on how large an individual payoff can be? Philosophically, one could argue in either direction; but for all practical purposes, one can impose a large enough upper bound that encompasses all possible observations for a given society. We therefore investigate a closed set of $2 \times 2$ matrix games with payoffs given by elements in the set $\left\{1 / 2^{k-1}, 2 / 2^{k-1}, \ldots, 4\right\}$, for $k \in\{1,2, \ldots\}$, with a grid size of $\Delta=1 / 2^{k-1}$. Equivalently, one might choose the payoff set $\left\{1,2, \ldots, 2^{k-1} \times 4\right\}$, with a grid size of $\Delta=1$. Although the latter approach offers the simplicity of an easily comprehensible fixed grid size, the former provides both a bounded maximum payoff size and an intuitively straightforward limiting process to assess the impact of grid size on player behavior.

[^1]In the remainder of the paper, we study various properties of cardinal games arranged into 144 categories associated with their corresponding strictly ordinal games. Our investigation relies on both analytical and simulation methods. In the latter case, we employ a computer program that carries out the following steps for each of games $G=1,2, \ldots, 100,000$, for a given value $k \in\{1,2, \ldots .$.$\} :$

1. For each of the four cells, $(i, j)=(1,1),(1,2),(2,1),(2,2)$, generate two independent random variables, $a_{i, j} \sim$ Uniform $\left\{1 / 2^{k-1}, 2 / 2^{k-1}, \ldots, 4\right\} \quad$ and $b_{i, j} \sim$ Uniform $\left\{1 / 2^{k-1}, 2 / 2^{k-1}, \ldots, 4\right\}$, where the four pairs $\left(a_{i, j} b_{i, j}\right)$ are mutually independent.
2. If either $a_{i, j}=a_{i, j, j}$ or $b_{i, j}=b_{i 0, j}$ for any $(i, j)=\left(i^{0}, j^{0}\right)$, then reject the game and return to step (1).
3. Define the cardinal $2 \times 2$ game $G$ by the four payoff pairs $\left(a_{i j,}, b_{i, j}\right)$.
4. Separately order the four $a_{i, j}$ and four $b_{i, j}$ from lowest to highest, and let ${ }^{\sim} a_{i j}=\operatorname{rank}\left(a_{i, j}\right) \in\{1,2,3,4\}$ and $\tilde{b_{i, j}}=\operatorname{rank}\left(b_{i, j}\right) \in\{1,2,3,4\}$ for all $(i, j)$.
5. Define the ordinal $2 \times 2$ game $G$ by the four payoff pairs $\left(\tilde{a}_{i, j}, \tilde{b}_{i, j}\right)$, and match this game to one of the 144 canonical strictly ordinal games.

By symmetry, we know that the number of ordinal games generated for each of the 144 canonical forms will be approximately the same.

### 1.2 Mass Properties

The generation of a large number of distinct cardinal games, each associated with one of the 144 canonical ordinal games, provides the means to consider the mass properties of several approaches to game play. A solution is the outcome (or set of outcomes) derived by the selection of a strategy by each of the game's players, and may be based upon a wide array of individual player characteristics. For the present, however, we will limit consideration to (1) noncooperative, individually optimizing players, and (2) entropy players selecting each row or column randomly, with probability $1 / 2$.

## 2 Joint Maximum Payoffs

Given a set of cardinal games generated randomly, as above, it is natural to consider the distribution of the joint maximum payoff, $J M_{k}=\max _{i \in\{1,2\}, j \in\{1,2\}}\left\{a_{i, j}+b_{i, j}\right\}$, for a given $k \in\{1,2, \ldots .$.$\} , and easy to see$ that the sample space of $J M_{k}$ is given by the set of values $\left\{5 / 2^{k-1}, 6 / 2^{k-1}, \ldots, 8\right\}$.

### 2.1 Distribution for $k=1$

For the case of $k=1,5$ this sample space is simply the set of integers $\{5,6,7,8\}$, and it is useful to associate these joint maxima with each of three game categories: (1) games of coordination, for which $J M_{1}=8$; (2) mixed-motive games, for which $J M_{1} \in\{7,6\}$; and (3) games of opposition, for which $J M_{1}=$ 5. For this baseline case, one can work out the distribution of the joint maximum as in Table 2, from which it is clear that $J M_{1}$ is negatively skewed, with mean, median, and mode of $6.875,7$, and 7 , respectively.

## Table 2: Distribution of $J M_{1}$

| Value | \# of Games |
| :---: | :---: |
| 8 | 36 |
| 7 | 60 |
| 6 | 42 |
| 5 | 6 |
| Total | 144 |

### 2.2 Distribution for $k>1$

For $k>1$, the distribution of $J M_{k}$ is more complex, but much can be learned simply by considering the limiting case as $k \rightarrow \infty$. Letting $a_{i, j}$ and $b_{i, j}$ be independent and identically distributed (continuous) Uniform(0,4] random variables for all (i,j), one can define $J M_{\infty}=\max _{i \in\{1,2\}, j \in\{1,2\}}\left\{a_{i, j}+b_{i, j}\right\}$, and observe that $J M_{\infty}=\max \left\{X_{1}, X_{2}, X_{3}, X_{4}\right\}$, where $X \sim$ i.i.d. Triangular[0,8]; i.e.,

[^2]\[

$$
\begin{aligned}
& \text { [ } \\
& \text { (T] } \\
& F_{X}(x)=\quad \text { for } x \in[0,4] \\
& \text { [0] }-x^{2} / 32+x / 2-1 \text { for } x \in(4,8]
\end{aligned}
$$
\]

It then follows that

$$
F_{J M_{\infty}}(y)= \begin{cases}\left(y^{2} / 32\right)^{4} & \text { for } y \in[0,4] \\ \left(y^{2} / 32-y / 2+1\right)^{4} & \text { for } y \in[4,8]\end{cases}
$$

and

$$
f_{J M_{\infty}}(y)=\left\{\begin{array}{lr}
\left(y^{2} / 32\right)^{3}(y / & \text { 4) for } y \in[0,4] \\
\left(y^{2} / 32-y / 2+1\right)^{3}(y / 4- & \text { 2) for } y \in(4,8]
\end{array}\right.
$$

This probability density function, plotted in Figure 1, shows that the distribution of $J M_{\infty}$ is negatively skewed, with mean, median, and mode given approximately by $5.6968,5.7436$, and 5.8619, respectively. ${ }^{3}$


Figure 1: Probability Density Function of $J M_{\infty}$
$3^{3}$ These parameter values are calculated as follows: Mean $=\int_{0}^{8}\left[1-F_{J M_{\infty}}(y)\right] d y=$
$\int_{0}^{4}\left[1-\left(y^{2} / 32\right)^{4}\right] d y+\int_{4}^{8}\left[1-\left(y^{2} / 32-y / 2+1\right)^{4}\right] d y=3,589 / 630 \approx 5.6968 ;$ Median $=F_{J M_{\infty}}^{-1}(1 / 2)=$
$\left\{m:\left(m^{2} / 32-m / 2+1\right)^{4}=1 / 2\right\}=8-4 \sqrt{2-\sqrt[4]{8}} \approx 5.7436 ;$ and Mode $={ }_{y \in[0,8]}^{\operatorname{argmax}\left\{f_{M_{\infty} \infty}(y)\right\}=}$
$\substack{\text { root } \\ y \in(4,8]}$
$\left\{\left(y^{2} / 32-y / 2+1\right)^{2}\left(7 y^{2} / 32-7 y / 2+13\right)\right\}=8-8 / \sqrt{14} \approx 5.8619$.

The above analysis reveals that the shape of the distribution of the joint maximum remains negatively skewed for large values of $k$, just as it is for $k=1$. One noteworthy difference, however, is
that games of coordination and games of opposition become less and less probable, approaching sets of measure zero as $k \rightarrow \infty$.

## 3 Noncooperative Equilibrium

The concept of noncooperative equilibrium has existed in the economic literature since the work of Augustin Cournot (1836), but was mathematically fully formalized and generalized by John Nash (1952). We define the outcome of a $2 \times 2$ matrix game as an ordered pair of strategies, $\left(s_{R} s_{C}\right)$, in which the first element denotes the row player's method of selecting a row ( $i \in\{1,2\}$ ), and the second element denotes the column player's method of selecting a column $(j \in\{1,2\}) .{ }^{6}$ We further define a noncooperative equilibrium as an outcome in which each player has no motivation to change his or her strategy, given the indicated strategy of the other player. Restricting attention to pure strategies, in which each player's decision consists of a fixed (as opposed to random) choice of row or column, one can see that $\left(s_{\mathrm{R}}^{*}, s_{\mathrm{C}}^{*}\right)=\left(i^{*}, j^{*}\right)$ constitutes a pure-strategy noncooperative equilibrium (PSNE) if and only if

$$
i^{*}=\underset{i \in\{1,2\}}{\operatorname{argmax}}\left\{a_{i, j}\right\}
$$

and

$$
j^{*}=\underset{j \in\{1,2\}}{\operatorname{argmax}}\left\{b_{i^{*}, j}\right\} .
$$

A simple illustration of PSNE is given by the ordinal Prisoner's Dilemma of Table 3. If both prisoners remain silent, then each will be given only a minor penalty (of 3); however, if one confesses and the other does not, then the former receives a very light penalty (of 4), whereas the latter receives a more severe penalty (of 1) than the penalty if both had confessed (of 2). For this game, it is easy to confirm that the strategy pair (Down,Right) forms a PSNE with payoffs (2,2).

Table 3: Prisoner's Dilemma
Left ("Remain Silent") Right ("Confess")
Up ("Remain Silent") 3,3 1,4
${ }^{6}$ In a matrix or normal-form game, any move is equivalent to a strategy because the players face no contingencies.

One of the most important contributions of Nash (1952) was the extension of noncooperative equilibrium from games with only pure strategies to games allowing each player to select a probability distribution over his or her possible choices. Thus, instead of just the pure-strategy pairs, $\left(s_{\mathrm{R}} S_{C}\right)=(i, j)$, we can consider random-strategy pairs, $\left(s_{\mathrm{R}} S_{C}\right)=(x, y)$, in which $x \in(0,1)$ denotes the row player's (non-trivial) probability of selecting Up $(i=1)$, and $y \in(0,1)$ denotes the column player's (likewise non-trivial) probability of selecting Left ( $j=1$ ). A noncooperative equilibrium that involves random strategies is referred to as a mixed-strategy noncooperative equilibrium (MSNE). One can solve for a game's MSNEs from the two conditions:

$$
\begin{align*}
& x^{*}=\underset{x \in(0,1)}{\operatorname{argmax}}\left\{a_{1,1} x y^{*}+a_{1,2} x\left(1-y^{*}\right)+a_{2,1}(1-x) y^{*}+a_{2,2}(1-x)\left(1-y^{*}\right)\right\}  \tag{4.1}\\
& y^{*}=\underset{y \in(0,1)}{\operatorname{argmax}}\left\{b_{1,1} x^{*} y+b_{1,2} x^{*}(1-y)+b_{2,1}\left(1-x^{*}\right) y+b_{2,2}\left(1-x^{*}\right)(1-y)\right\} . \tag{4.2}
\end{align*}
$$

The simple game of Matching Pennies, shown in Table 4, provides an intuitively reasonable use of mixed strategies. For the mixed-strategy pair $(x, y)=(1 / 2,1 / 2)$, each player receives an expected payoff of 0 . However, if either player selected a pure strategy, then the other player's best response would cause the first player to lose 1 unit with certainty. ${ }^{7}$ Although coin tosses are commonly used by individuals in certain decision-making settings, the use of more complicated mixed strategies appears to depend heavily on player sophistication and problem context.

Table 4: Matching Pennies

|  | Heads | Tails |
| :---: | :---: | :---: |
| Heads | $1,-1$ | $-1,1$ |
| Tails | $-1,1$ | $1,-1$ |

### 3.1 Population of PSNEs

In the study of matrix games, it is useful to separate PSNEs and MSNEs because the former depend only on payoff ordinalities, whereas the latter are sensitive to cardinal differences. For the set of 144 strictly ordinal $2 \times 2$ games, the distribution of the number of PSNEs is given in Table 5 .

[^3]| Table 5: Distribution of PSNEs |  |
| :---: | :---: |
| Value | \# of Games |
| 0 | 18 |
| 1 | 108 |
| 2 | 18 |
| Total | 144 |

### 3.2 Population of MSNEs

From the list in Appendix 2, one can see that the set of 144 canonical games contains 36 games with exactly one MSNE, and no games with more than one MSNE. These 36 games match exactly with the set of games having either zero or two PSNEs, from which it follows that all 144 ordinal games contain at least one noncooperative equilibrium.

### 3.3 Some Easy Calculations

Most features and solutions of $2 \times 2$ matrix games - whether strictly ordinal or cardinal - are easy to calculate. In developing procedures for such calculations, dominant rows and/or columns play an important role.

A row [column] in a matrix game is said to dominate (strictly) another row [column] if and only if each payoff in the first row [column] is greater than the corresponding payoff in the second row [column]. In a $2 \times 2$ game, it is well known that:

- If a game possesses exactly one dominant row or column, then it must possess exactly one PSNE.
- If a game possesses zero dominant rows or columns, then it must possess zero PSNEs (and therefore exactly one MSNE).
- If a game possesses one dominant row and one dominating column, then it may possess either (a) exactly one PSNE or (b) two PSNEs and exactly one MSNE.

The above facts enable us to construct the following algorithm for computing all PSNEs and MSNEs:

1. Check each row and column for the dominance property, and let $d \in\{0,1,2\}$ denote the total number of dominant rows/columns.
2. If $d=1$, then solve for the unique PSNE by identifying the cell in the dominant row [column] that has the greater payoff for the column [row] player.
3. If $d=0$, then solve for the MSNE explicitly as

$$
\left(x^{*}, y^{*}\right)=\left(\frac{a_{2,2}-a_{1,2}}{a_{1,1}-a_{1,2}-a_{2,1}+a_{2,2}}, \frac{b_{2,2}-b_{1,2}}{b_{1,1}-b_{1,2}-b_{2,1}+b_{2,2}}\right) .
$$

4. If $d=2$, then solve for each of the two PSNEs using the method described in step (1) immediately above, and solve for the MSNE as in step (2) immediately above. (The two PSNEs also can be found as corner solutions of the system of equations in (4.1) and (4.2).)

## 4 Populations of Three Special Games

In Appendix 3, we provide the C++ program used to generate cardinal (and strictly ordinal) games as described in steps (1) through (5) of Subsection 1.1. Appendix 4 contains various sample means and variances associated with 100,000 cardinal games generated with a grid size of $\Delta=1 / 256$ (i.e., for $k$ $=9$ ).

In the present section, we consider the populations of three special games discussed widely in the behavioral science literature:

- Prisoner's Dilemma (game 1 of Appendix 2);
- Stag Hunt (game 41 of Appendix 2); and
- Battle of the Sexes (game 10 of Appendix 2).

The common names of these games attach considerable context to the abstract payoff structure, which may or may not be justified. In particular, there is little indication that anyone who does not know these common names would associate them with the specific games involved (see, e.g., I.

Powers and Shubik, 1991). Nevertheless, the structural features of these three settings allow us to illustrate several important aspects of $2 \times 2$ games.

### 4.1 Prisoner's Dilemma

Possibly the most studied of all games is the Prisoner's Dilemma, whose popularity arises at least in part because its ordinal form is the only game within the canonical 144 for which: (a) there is a unique PSNE, (Down,Right), that is strictly dominated by another feasible outcome, (Up,Left); and (b) all other outcomes are Pareto optimal. Figure 22 of Appendix 1 presents the payoff set for this wellknown game.

In exploring the population of cardinal Prisoner's Dilemma games for a given choice of grid size, $\Delta=1 / 2^{k-1}$, it is helpful to think of the entire domain of possible payoffs, from the game with the smallest payoff values, in Table 6, to that with the largest payoff values, in Table 7. Naturally, an ordinal treatment of preferences recognizes no difference between these two games.

Table 6: Prisoner's Dilemma, Smallest Payoffs

$$
\begin{array}{ccc} 
& \text { Left } & \text { Right } \\
\text { Up } & \frac{3}{2^{k-1}}, \frac{3}{2^{k-1}} & \frac{1}{2^{k-1}}, \frac{4}{2^{k-1}} \\
\text { Down } & \frac{4}{2^{k-1}}, \frac{1}{2^{k-1}} & \frac{2}{2^{k-1}}, \frac{2}{2^{k-1}}
\end{array}
$$

Table 7: Prisoner's Dilemma, Largest Payoffs

$$
\begin{array}{ccc} 
& \text { Left } & \text { Right } \\
\text { Up } & 4-\frac{1}{2^{k-1}}, 4-\frac{1}{2^{k-1}} & 4-\frac{3}{2^{k-1}}, 4 \\
\text { Down } & 4,4-\frac{3}{2^{k-1}} & 4-\frac{2}{2^{k-1}}, 4-\frac{2}{2^{k-1}}
\end{array}
$$

As $k$ increases and the grid becomes finer, the population of cardinal Prisoner's Dilemma games covers more and more of the entire square interval $(0,4]^{2}$. In all cases, the game has only the single PSNE (Down,Right), whose payoff pair depends on the particular values of $a_{i, j}$ and $b_{i, j}$ generated in step (1) of Subsection 1.1.

As a baseline, we note that for $k=1$, the PSNE payoff pair is always (2,2), whereas entropy players do better on average, obtaining an expected payoff of (2.5,2.5). Figures 2 and 3 provide scatter
diagrams of payoff pairs from 100,000 games for $k=9$ and $\Delta=1 / 256$. Figure 2 shows the PSNEs, and Figure 3 shows the corresponding entropic outcomes. In each case, the sample-mean payoff pair is indicated by a red diamond near the center of the point cluster.


Figure 2: Prisoner's Dilemma, Noncooperative Equilibrium


Figure 3: Prisoner's Dilemma, Entropy Players

### 4.2 Stag Hunt

The Stag Hunt (Table 8) possesses two PSNEs, one of which, (Up,Left), strictly dominates the other, (Down,Right). However, depending on underlying assumptions, it can be argued that the smaller payoff pair sometimes will be chosen by rational players. In particular, if the row [column] player believes that the column [row] player will choose Right [Down] with probability greater than $1 / 2$, then he or she will be motivated to choose Down [Right].

Table 8: Stag Hunt
Left Right

| Up | 4,4 | 1,3 |
| :---: | :---: | :---: |
| Down | 3,1 | 2,2 |

For $k=1$, the two PSNEs have payoff pairs $(4,4)$ and $(2,2)$, respectively, and the MSNE, $\left(x^{*}, y^{*}\right)=$ $(1 / 2,1 / 2)$, yields expected payoffs of $(2.5,2.5)$. Given the value of $\left(x^{*}, y^{*}\right)$, one can see that the MSNE yields identical strategies and payoffs as the game with entropy players.

Figures 4 and 5 provide scatter diagrams of payoff pairs from 100,000 games for $k=9$ and $\Delta=$ $1 / 256$. Figure 4 includes the PSNEs and MSNE, each with equal frequency, and Figure 5 shows the corresponding entropic outcomes. As before, the sample-mean payoff pairs are indicated by red diamonds near the centers of the point clusters.


Figure 4: Stag Hunt, Noncooperative Equilibrium


Figure 5: Stag Hunt, Entropy Players

### 4.3 Battle of the Sexes

Unlike the Prisoner's Dilemma and Stag Hunt, the Battle of the Sexes (Table 9) is not a symmetric game (i.e., $a_{i, j} 6=b_{j, i}$ for some (i,j)). Like the Stag Hunt, however, it possesses two PSNEs, (Up,Left) and (Down,Right), and 1 MSNE, $\left(x^{*}, y^{*}\right)=(1 / 2,1 / 2)$.

Table 9: Battle of the Sexes

|  | Left | Right |
| :---: | :---: | :---: |
| Up | 4,3 | 2,2 |
| Down | 1,1 | 3,4 |

For $k=1$, the two PSNEs have payoff pairs $(4,3)$ and $(3,4)$, respectively, and the MSNE gives expected payoffs of $(2.5,2.5)$. Thus, as in the case of the Stag Hunt, the game with entropy players yields identical strategies and payoffs as noncooperative players using the MSNE.

Figures 5 and 6 provide scatter diagrams of payoff pairs from 100,000 games for $k=9$ and $\Delta=$ $1 / 256$. Figure 5 includes the PSNEs and MSNE, each with equal frequency, and Figure 6 shows the corresponding entropic outcomes. Once again, the sample-mean payoff pairs are indicated by red diamonds.


Figure 6: Battle of the Sexes, Noncooperative Equilibrium


Figure 7: Battle of the Sexes, Entropy Players

### 4.4 Correlated Strategies and Efficiency

In examining the results of the Stag Hunt and Battle of the Sexes, we observe that the expected payoffs of the MSNE are strictly lower than at least one pair of PSNE payoffs in both games. This suggests a potential problem of ineffective coordination. In other words, the players may arrive at a noncooperative equilibrium for which the individual and/or joint payoffs (i.e., $a_{i^{*}, j^{j},}, b_{i^{*}, j^{*}}$, and/or $a_{i^{*}, j^{*}}+b_{i^{*}, j^{*}}$ ) are less than optimal. Therefore, the value of being able to coordinate strategies may be substantial.

Table 10 offers some quantification of the deficiencies attributable to ineffective coordination for the three games discussed above. For each game, the second column presents the average of the joint payoffs over all noncooperative equilibria, giving equal weight to each PSNE and MSNE. For the

Prisoner's Dilemma, this is $2+2=4$; for the Stag Hunt, it is $[(4+4)+(2+2)+(2.5+2.5)] / 3 \approx$ 5.6667; and for the Battle of the Sexes, it is $[(4+3)+(3+4)+(2.5+2.5)] / 3 \approx 6.3333$. In the third column, we construct a simple efficiency measure by dividing the average joint payoff by the maximum possible joint payoff that can be achieved by either a pure- or mixed-strategy pair imposed by an exogenous agency (custom, law, private intermediation, etc.). Since the agent is exogenous, it can expand the domain of mixed strategies to correlated strategies, for which the players' random selections of Up and Down are statistically dependent. For the Prisoner's Dilemma, this yields $3+3=$ 6; for the Stag Hunt, it yields $4+4=8$; and for the Battle of the Sexes, it yields either $4+3=3+4=7$ or $[(4+3) p+(3+4)(1-p)]=7$, where the latter value comes from any correlated mixed strategy
that chooses (Up,Left) and (Down,Right) with probabilities $p$ and 1-p, respectively. ${ }^{8}$ The fourth and fifth columns present corresponding calculations for games with entropy players.

Table 10: Joint Payoffs for $k=1$

| Game | Avg. of NE Payoffs | NE Efficiency | Entropy Payoffs | Entropy Efficiency |
| :---: | :---: | :---: | :---: | :---: |
| Prisoner's Dilemma | 4.0000 | 0.6667 | 5.0000 | 0.8333 |
| Stag Hunt | 5.6667 | 0.7083 | 5.0000 | 0.6250 |
| Battle of the Sexes | 6.3333 | 0.9048 | 5.0000 | 0.7143 |

## 5 Discussion

### 5.1 Efficiency Analysis of All $2 \times 2$ Games

Table 11 provides efficiency measures - as defined in the previous subsection - for the entire population of cardinal games generated in steps (1) through (5) of Subsection 1.1 for $k=1$ and $k=9$. This table addresses the nature of the optimality of individual behavior within all possible $2 \times 2$-game structures, subdivided by the values of $J M_{1}$ in the associated canonical ordinal game. Thus, for clarity, we would note that: (a) Prisoner's Dilemma games are included in the category of $J M_{1}=6$; (b) Stag Hunt games are included in $J M_{1}=7$; and (c) Battle of the Sexes games are included in $J M_{1}=8$.

Table 11: Efficiencies of All Games

|  | $k=1$ | $k=1$ | $k=9$ | $k=9$ |
| :--- | :---: | :---: | :---: | :---: |
|  | NE | Entropy | NE | Entropy |
| $J M_{1}=8$ | 0.9410 | 0.6250 | 0.7532 | 0.5000 |
| $J M_{1}=7$ | 0.9127 | 0.7143 | 0.7305 | 0.5716 |
| $J M_{1}=6$ | 0.9352 | 0.8333 | 0.7491 | 0.6669 |
| $J M_{1}=5$ | NA | NA | NA | NA |
| Average | 0.9300 | 0.7386 | 0.7445 | 0.5910 |

As is often true in game theory, behavioral paradoxes abound. We purposely indicate that efficiency measures are "not applicable" for the case of $J M_{1}=5$, because these games of opposition are qualitatively different from all others. Formally, the efficiency measures could be defined as 1.0 , but

[^4]such calculations would be misleading because these games do not reflect the characteristics of a society. Specifically, there is no room for cooperation, coordination, or any form of discourse, and whatever one individual gains the other loses. As noted before, such structural dystopias become increasingly rare in the "Flatland" (see Abbott, 1952 [1884]) of matrix games that arises for large values of $k$.

By definition, the category of $J M_{1}=8$ comprises games of coordination. These games always include an outcome with payoff pair $(4,4)$, and such outcomes must be PSNEs. In this case, the reason why efficiency is not exactly 1.0 is that some games have two PSNEs and 1 MSNE, and these lower the average.

Somewhat surprisingly, the fall-off in the efficiency of noncooperative equilibria as $k$ increases from 1 to 9 is rather large for all game categories. For $k=1$, the efficiency loss in games of coordination $\left(J M_{1}=8\right)$ and mixed-motive games $\left(J M_{1}=7,6\right)$ is between 6 and 9 percent. For $k=9$, this grows to between 25 and 27 percent, and results for $k=10$ indicate that $k=9$ is very close to the limit, with differences in efficiency of less than 0.01 percent. (See Table 12.) Part of the decrease in efficiency is attributable to the fact that $E\left[J M_{k}\right] \rightarrow E\left[J M_{\infty}\right] \approx 5.6968$ for large $k$
, which is substantially less than $E\left[J M_{1}\right]=6.875$.

| Table 12: Efficiencies for $k=9$ and $k=10$ |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | $k=9$ | $k=9$ | $k=10$ | $k=10$ |
|  | NE | Entropy | NE | Entropy |
| $J M_{1}=8$ | 0.7532 | 0.5000 | 0.7528 | 0.5004 |
| $J M_{1}=7$ | 0.7305 | 0.5716 | 0.7310 | 0.5717 |
| $J M_{1}=6$ | 0.7491 | 0.6669 | 0.7489 | 0.6672 |
| $J M_{1}=5$ | NA | NA | NA | NA |
| Average | 0.7445 | 0.5910 | 0.7447 | 0.5914 |

We also consider the completely different behavior of entropy players, who may be viewed as "know-nothing" or "zero-intelligence" decision makers. Although it is easy to construct games (e.g., the Prisoner's Dilemma) in which the entropy players' expected payoffs of $(2.5,2.5)$ are greater than those of noncooperative players, Table 11 shows that for both $k=1$ and $k=9$, entropy players perform worse on average than noncooperative players. This is because they are unable to take advantage of certain game structures, such as row and/or column dominance, that assist coordination. Even in this
highly constrained environment, differences in the cardinal measures of payoffs yield far greater variability and inequality when $k$ is large than when $k=1$. The meaning of this change is that as the variety of outcomes grows, the worth of coordination or collaboration grows as well.

### 5.2 Why the $2 \times 2$ Case Is So Important

There are many reasons why $2 \times 2$ games are crucial both to the study of game theory specifically, and behavioral science more generally. These include:

1. They offer a highly useful starting point for illustrating and contrasting many problems andparadoxes in strategic analysis.
2. They are widely used by introductory instructors of game theory in the behavioral sciences.(Is this pedagogical use justified? We would argue that it is, with appropriate qualifications.)
3. They greatly facilitate analogy generation and storytelling in connecting specific realworldproblems to abstract models. Hence, they provide valuable exercises in tying the physical world to mathematics.
4. They offer minimal repeated-game models for the dynamics of learning, signaling, and othercomplex human behaviors.
5. Through a handful of special cases (e.g., the Prisoner's Dilemma, Stag Hunt, and Battle ofthe Sexes), they successfully illustrate fundamental problems in strategy and society.

Items (1), (2), and (4) are highly related for both teaching and research, especially if we believe that dyadic relations are of considerable importance in describing human and other animal behavior. It is thus for good reason that elementary textbooks in the behavioral sciences abound with $2 \times 2$-game examples.

## $6 \quad$ For $n>3$, a Basic Change in the Paradigm

On the whole, we would argue that the study of $2 \times 2$ games is important because so much of human activity is well modeled by interactions between two individuals or between one individual and an
institution, with relatively few choices for each party in the short term. This perspective, however, can blind us against the enormous complexities that arise when increasing matrix size. For example, in the case of a $3 \times 3$ matrix, the number of distinct strategic cases rises to $(9!)^{2} /(3!)^{2} \approx$ $3.6578 \times 10^{9}$; and in the case of 3 players with 2 strategies each, we have a $2 \times 2 \times 2$ matrix with (8! $)^{3}$ $/(2!)^{3} \approx 8.1935 \times 10^{12}$ strategically different possibilities.

In the case of $3 \times 3$ games, the simple example of Table 13 is sufficient to destroy the hopes of those interested in developing plausible dynamic strategic solutions. One natural candidate for simple dynamics is optimal response; that is, players consider where they have been in a previous play of the game, and use that as the basis for their current optimization. However, a brief glance at Table 13 shows that if the row and column players begin with strategies $\left(s_{C}, s_{\mathrm{R}}\right)=(1,1)$, then the column player will move to $s_{\mathrm{C}}=3$, and a 4 -cycle will emerge that never converges to the joint maximum PSNE at $\left(s_{G}, s_{\mathrm{R}}\right)=(2,2)$. (Note that if the payoff at $\left(s_{G} s_{\mathrm{R}}\right)=(2,2)$ were $(1,1)$ instead of $(9,9)$, that particular outcome would still be a PSNE.) Quint, Shubik, and Yan (1995) demonstrated the extensive potential for cycling in a large class of $n \times n$ games involving both sequential and simultaneous moves.

Table 13: A Simple $3 \times 3$ Matrix Game

|  | $s_{\mathrm{C}}=1$ | $s_{\mathrm{C}}=2$ | $s_{\mathrm{C}}=3$ |
| :---: | :---: | :---: | :---: |
| $s_{\mathrm{R}}=1$ | 4,1 | 0,0 | 1,4 |
| $s_{\mathrm{R}}=2$ | 0,0 | 9,9 | 0,0 |
| $s_{\mathrm{R}}=3$ | 1,4 | 0,0 | 4,1 |

### 6.1 Smooth and Rough Games

In any $2 \times 2$ game, it is possible to compute 4 first differences between cell payoffs. In the $3 \times 3$ case, one can compute 12 first differences and 4 second differences, and the payoff sets still have only few hills and valleys. However, for matrices of $4 \times 4$ and above, the potential roughness of the payoff structure increases rapidly. Although the mathematical structure is clearly defined, the analysis of applied decision problems becomes extremely difficult unless some appropriate smoothing mechanism is imposed on the payoff surfaces. In corporate, military, and political planning, the adjustment, evaluation, refinement, and discarding of many features of a strategic process tend to
narrow the final choices to a set that includes certainly less than ten, and often not more than two or three, alternatives.

These somewhat terse remarks will be enlarged in further work.

### 6.2 Coordination, Opposition, and Noncooperative Equilibria

As we increase the number of strategies, or players, or both, the relative numbers of games of coordination and opposition become vanishingly small.

If we consider $m \times n$ matrix games with $m, n \geq 3$, the relative number of PSNEs drops, and MSNEs proliferate. There is a literature illustrating this, which includes limiting formulas for the probability of encountering a PSNE in a large, random matrix game. (See Goldberg, Goldman, and Newman, 1968, Dresher, 1970, and I. Powers, 1990.)

### 6.3 Life Is a Set of Measure Zero

A guiding principle for exploring the enormous universe of matrix games is to select appropriate limiting processes to obtain robust sets of games addressing important questions of interest. We do that here in taking the limit of all cardinal $2 \times 2$ games with a minimal grid size.

In general, many problems of interest require such specification that they can be regarded as sets of measure zero with respect to far larger abstract categories to which they belong. An important example (to which we will return in a subsequent paper) is the presence of ties. In many contexts of human activity, ties in perceived valuation are present. Furthermore, when one individual has finer perceptions than another, the increased precision almost always works to his or her advantage. In the present essay, we have ruled out ties for simplicity and manageability. If we had permitted them, the number of canonical ordinal games would have increased from 144 to 726 (NEED REFERENCE HERE).

## 7 Concluding Remarks

The principal purpose of this paper was to find a reasonably natural way to consider all cardinal $2 \times 2$ games within a given finite grid. The essentially combinatoric aspects of the investigation called for simulation to evaluate structural aspects that are difficult to see using only analytic methods. Our study of this structure provided a sufficiently rich background for examining in detail both noncooperative equilibria and entropy-player solutions. In a subsequent paper, we will consider other solution concepts that enable one to investigate the influence of structure on behavior with various intents.

The amount of "fat left in the system" depends on the solution used. In many cases, a referee, government, or other outside agency could be used to guide the system to a superior outcome, while consuming fewer resources than it adds. The gap between the current solution and the joint maximum is the maximum worth of the coordinator. Given the basic analysis of the present paper, we now are in a position to consider developing improved measures of efficiency and symmetry for any outcome in a matrix game. We plan to discuss this topic as well in a future paper.

In short, this first essay was aimed at providing a simple idea of the worth of government, with a quick and crude estimate. A second essay will be devoted to the many problems of structure and behavior arising from noncooperative equilibria in $2 \times 2$ games. Finally, a third essay will address the development of more sophisticated efficiency measures based upon the analyses of the prior work.

## References

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## Appendix 1

$$
\left[\begin{array}{cc}
(4,4 & 3,3 \\
2,2 & 1,1
\end{array}\right]\left[\begin{array}{cc}
(4,4 & 3,3 \\
1,1 & 2,2
\end{array}\right]\left[\begin{array}{cc}
4,4 & 2,2 \\
1,1 & 3,3
\end{array}\right]
$$



20,92,124

FIGURE 1


19,3,93,122
$\left[\begin{array}{cc}3,4 & 4,3 \\ 2,2 & 1,1\end{array}\right]\left[\begin{array}{|cc}3,4 & 4,3 \\ 1,1 & 2,2\end{array}\right]\left[\begin{array}{cc}2,2 & 3,4 \\ 4,3 & 1,1\end{array}\right]\left[\begin{array}{cc}2,2 & 4,3 \\ 3,4 & 1,1\end{array}\right]$


FIGURE 2

$$
\text { (3,4 } 4,2 \mathrm{oa} 3,44,2 \mathrm{oa} 2,34,2 \mathrm{oa} 2,3 \text { 3,40a 3,4 } 1,1 \circ
$$



FIGURE 3
FIGURE 4


FIGURE 5


FIGURE 6


1


FIGURE 7

$$
\begin{aligned}
& \mathrm{a} 4,4 \quad 1,2^{\circ \circ} 4,4 \quad 3,1^{\circ} \text { ค } 4,4 \quad 2,3^{\circ} \\
& \text { « } \neg 3,12,3 »<1 / 4 \neg 1,22,3 »<1 / 4 \neg 3,11,2^{1 / 4} \text { » } \\
& 4,1 »<1 / 4 \neg 1,2 \quad 3,4 » 1 / 4
\end{aligned}
$$




FIGURE 8

21

$$
\begin{aligned}
& \text { 《 } \neg 2,31,2 »<1 / 4 \neg 2,34,1 »<1 / 4 \neg 1,22,3 »<1 / 4 \neg 1,2
\end{aligned}
$$



1
3
4




FIGURE 9


2,46
FIGURE 11

$$
\begin{aligned}
& \begin{array}{lllll}
\mathrm{a} 4,4 & 3,2^{\mathrm{o}}(4,4 & 3,2^{\circ} \times 4,4 & 2,1 \circ \times 4,4 & 2,1^{\circ}
\end{array} \\
& \text { 《 } \neg 1,32,1 »<1 / 4 \neg 2,11,3 »<1 / 4 \neg 3,21,31 / 4 \neg><1,3 \\
& 1,3 » 1 / 4 \\
& 1,3 » 1 / 4
\end{aligned}
$$



FIGURE 10
$\left[\begin{array}{ll}3,3 & 4,1 \\ 1,2 & 2,4\end{array}\right]\left[\begin{array}{ll}2,4 & 4,1 \\ 1,2 & 3,3\end{array}\right]\left[\begin{array}{ll}2,4 & 3,3 \\ 1,2 & 4,1\end{array}\right]$


84,108,41
FIGURE 12

22

$3,2 » 1 / 4 \quad \neg<1,3 \quad 2,1 »<1 / 4 \neg 1,34,2$ »<1/4$\neg 2,11,3 »<1 / 4 \neg 4,2$

1
3
4


FIGURE 13


FIGURE 14


1
3
4


FIGURE 15


FIGURE 16

23

1
3
4


FIGURE 17

$$
\begin{gathered}
M \\
\text { a3,2 } 2,1^{\circ} \\
« \neg 4,31,4 » 1 / 4
\end{gathered}
$$



$114,45,6,88$
FIGURE 18

$$
\begin{aligned}
& \text { (3,3 } 4,2^{\circ} \sqrt{c} \sqrt{3,3} \quad 4,2^{\text {oad }} 3,3 \quad 2,1^{\circ} \\
& \text { 《 } \neg 1,42,1 \text { »《 } 1 / 4 \neg 2,11,4 » \ll 1 / 4 \neg 4,21,4 » 1 / 4
\end{aligned}
$$

3
4


FIGURE 19


FIGURE 21

FIGURE 20

24


FIGURE 22


118,144


FIGURE 24

## Appendix 2

WhatisaSolutiontoaMatrixGame?

| Game\# | Payoff Matrix |  | Shape | $\begin{aligned} & \text { Join } \\ & \text { Max } \end{aligned}$ | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Row |  |  |  | Col. |  |  |  |
| 1 | $(1,4)$ | $(3,3)$ |  | 22 | 6 | 1 | Sym | 2 | 2 | 2 | 3 | NA |
|  | $(2,2)$ | $(4,1)$ |  |  |  |  |  |  |  |  |  |  |
|  | $(1,2)$ | $(3,1)$ | 11 | 7 | 1 |  | 2 | 4 | 2 | 2 | 115 |  |
| 2 | $(2,4)$ | $(4,3)$ |  |  |  |  |  |  |  |  |  |  |
|  | $(1,1)$ | $(3,2)$ | 3 | 8 | 1 | Sym | 4 | 4 | 2 | 1 | NA |  |
| 3 | $(2,3)$ | $(4,4)$ |  |  |  |  |  |  |  |  |  |  |
|  | $(1,4)$ | $(3,3)$ | 20 | 6 | 1 |  | 4 | 2 | 1 | 3 | 108 |  |
|  | $(4,2)$ | $(2,1)$ |  |  |  |  |  |  |  |  |  |  |


| 5 | $(1,3)$ | $(3,4)$ |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |




| 34 | $(1,1)$ $(4,3)$ | $\begin{array}{\|l\|} \hline(2,2) \\ \hline(3,4) \end{array}$ | 2 | 7 | 1 |  | 3 | 4 | 2 | 2 | 104 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 35 | $(1,1)$ | $(2,4)$ | 6 | 6 | 1 | Sym | 3 | 3 | 2 | 3 | NA |
|  | $(4,2)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |
| 36 | $(1,1)$ | $(2,4)$ | 5 | 7 | 1 |  | 4 | 3 | 1 | 2 | 125 |
|  | $(4,3)$ | $(3,2)$ |  |  |  |  |  |  |  |  |  |
|  | $(1,4)$ | $(4,3)$ | 21 | 7 | 1 |  | 2 | 2 | 1 | 2 | 116 |
| 37 | $(2,2)$ | $(3,1)$ |  |  |  |  |  |  |  |  |  |
|  | $(1,4)$ | $(4,2)$ | 23 | 6 | 1 |  | 2 | 3 | 1 | 3 | 88 |
| 38 | $(2,3)$ | $(3,1)$ |  |  |  |  |  |  |  |  |  |
|  | $(1,4)$ | $(4,1)$ | 22 | 6 | 0 |  | 2.5 | 2.5 | 0 | 3 | 138 |
|  | $(2,2)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |
| 40 | $(1,4)$ | $(4,1)$ | 24 | 5 | 1 |  | 2 | 3 | 1 | 4 | 100 |
|  | $(2,3)$ | $(3,2)$ |  |  |  |  |  |  |  |  |  |
| 41 | $(1,3)$ | $(4,4)$ | 15 | 8 | 2 | Sym | 4 | 4 | 0 | 1 | NA |
|  | $(2,2)$ | $(3,1)$ |  |  |  |  | 2 | 2 |  |  |  |
| 42 | $(1,3)$ | $(4,4)$ | 13 | 8 | 1 |  | 4 | 4 | 1 | 1 | 106 |
|  | $(2,1)$ | $(3,2)$ |  |  |  |  |  |  |  |  |  |
| 43 | $(1,3)$ | $(4,2)$ | 17 | 6 | 1 |  | 2 | 4 | 1 | 2 | 97 |
|  | $(2,4)$ | $(3,1)$ |  |  |  |  |  |  |  |  |  |
|  | $(1,3)$ | $(4,2)$ | 14 | 7 | 0 |  | 2.5 | 2.5 | 0 | 2 | 126 |
| 44 | $(2,1)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |

30

| Game\# | Payoff Matrix | Shape | Joint Max | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  | Row | Col. |  |  |  |


| 45 | $(1,3)$ $(2,4)$ | $(4,1)$ $(3,2)$ | 18 | 6 | 1 |  | 2 | 4 | 1 | 3 | 91 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 46 | $(1,2)$ | $(4,3)$ | 11 | 7 | 2 |  | 4 | 3 | 0 | 2 | 133 |
|  | $(2,4)$ | $(3,1)$ |  |  |  |  | 2 | 4 |  |  |  |
| 47 | $(1,2)$ | $(4,3)$ | 8 | 7 | 1 |  | 4 | 3 | 1 | 2 | 94 |
|  | $(2,1)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |
| 48 | $(1,2)$ | $(4,4)$ | 7 | 8 | 1 |  | 4 | 4 | 1 | 1 | 82 |
|  | $(2,1)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |
| 49 | $(1,2)$ | $(4,1)$ | 12 | 6 | 1 |  | 2 | 4 | 1 | 3 | 121 |
|  | $(2,4)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |
| 50 | $(1,2)$ | $(4,1)$ | 10 | 7 | 0 |  | 2.5 | 2.5 | 0 | 2 | 143 |
|  | $(2,3)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |
| 51 | $(1,1)$ | $(4,3)$ | 2 | 7 | 1 |  | 4 | 3 | 1 | 2 | 79 |
|  | $(2,2)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |
| 52 | $(1,1)$ | $(4,2)$ | 4 | 7 | 1 |  | 4 | 2 | 1 | 2 | 101 |
|  | $(2,3)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |
| 53 | $(1,1)$ | $(4,2)$ | 6 | 6 | 2 | Sym | 4 | 2 | 0 | 3 | NA |
|  | $(2,4)$ | $(3,3)$ |  |  |  |  | 2 | 4 |  |  |  |
| 54 | $(1,1)$ | $(4,4)$ | 1 | 8 | 1 |  | 4 | 4 | 1 | 1 | 92 |
|  | $(2,2)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |
| 55 | $(1,1)$ | $(4,4)$ | 3 | 8 | 2 |  | 4 | 4 | 0 | 1 | 122 |
|  | $(2,3)$ | $(3,2)$ |  |  |  |  | 2 | 3 |  |  |  |
|  | $(1,4)$ | $(4,3)$ | 19 | 7 | 1 |  | 3 | 2 | 1 | 2 | 110 |
| 56 | $(3,2)$ | $(2,1)$ |  |  |  |  |  |  |  |  |  |


| 57 | $(1,4)$ <br> $(3,1)$ | $\begin{array}{\|l\|} \hline(4,3) \\ \hline(2,2) \\ \hline \end{array}$ | 21 | 7 | 0 | 2.5 | 2.5 | 0 | 2 | 141 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 58 | $(1,4)$ <br> $(3,3)$ | $\begin{array}{\|l\|} \hline(4,2) \\ \hline(2,1) \\ \hline \end{array}$ | 20 | 6 | 1 | 3 | 3 | 1 | 3 | 84 |
| 59 | $(1,4)$ <br> $(3,2)$ | $\begin{aligned} & (4,1) \\ & \hline(2,3) \end{aligned}$ | 24 | 5 | 0 | 2.5 | 2.5 | 0 | 4 | 136 |
| 60 | $(1,4)$ <br> $(3,3)$ | $(4,1)$ $(2,2)$ | 22 | 6 | 1 | 3 | 3 | 1 | 3 | 99 |
| 61 | $(1,3)$ <br> $(3,2)$ | $(4,4)$ $(2,1)$ | 13 | 8 | 2 | 4 3 | $\begin{aligned} & 4 \\ & 2 \end{aligned}$ | 0 | 1 | 140 |
| 62 | $(1,3)$ <br> $(3,1)$ | $\begin{array}{\|l\|} \hline(4,4) \\ \hline(2,2) \end{array}$ | 15 | 8 | 1 | 4 | 4 | 1 | 1 | 111 |
| 63 | $(1,3)$ <br> $(3,4)$ | $(4,2)$ $(2,1)$ | 14 | 7 | 1 | 3 | 4 | 1 | 2 | 95 |
| 64 | $(1,3)$ <br> $(3,1)$ | $(4,2)$ $(2,4)$ | 17 | 6 | 0 | 2.5 | 2.5 | 0 | 2 | 130 |
| 65 | $(1,3)$ <br> $(3,2)$ | $(4,1)$ $(2,4)$ | 18 | 6 | 0 | 2.5 | 2.5 | 0 | 3 | 144 |
| 66 | $(1,3)$ <br> $(3,4)$ | $(4,1)$ <br> $(2,2)$ | 16 | 7 | 1 | 3 | 4 | 1 | 2 | 90 |

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| Game\# | Payoff Matrix |  | Shape | Joint <br> Max | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Row |  |  |  | Col. |  |  |  |
| 67 | $(1,2)$ | $(4,3)$ |  | 8 | 7 | 2 |  | 4 | 3 | 0 | 2 | 129 |
|  | $(3,4)$ | $(2,1)$ |  |  |  |  | 3 | 4 |  |  |  |  |


| 68 | $(1,2)$ $(3,1)$ | $(4,3)$ $(2,4)$ | 11 | 7 | 1 |  | 4 | 3 | 1 | 2 | 96 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 69 | $(1,2)$ | $(4,4)$ | 7 | 8 | 2 | Sym | 4 | 4 | 0 | 1 | NA |
|  | $(3,3)$ | $(2,1)$ |  |  |  |  | 3 | 3 |  |  |  |
| 70 | $(1,2)$ | $(4,4)$ | 9 | 8 | 1 |  | 4 | 4 | 1 | 1 | 85 |
|  | $(3,1)$ | $(2,3)$ |  |  |  |  |  |  |  |  |  |
| 71 | $(1,2)$ | $(4,1)$ | 10 | 7 | 1 |  | 3 | 4 | 1 | 2 | 119 |
|  | $(3,4)$ | $(2,3)$ |  |  |  |  |  |  |  |  |  |
| 72 | $(1,2)$ | $(4,1)$ | 12 | 6 | 0 |  | 2.5 | 2.5 | 0 | 3 | 142 |
|  | $(3,3)$ | $(2,4)$ |  |  |  |  |  |  |  |  |  |
| 73 | $(1,1)$ | $(4,3)$ | 5 | 7 | 1 |  | 4 | 3 | 1 | 2 | 80 |
|  | $(3,2)$ | $(2,4)$ |  |  |  |  |  |  |  |  |  |
| 74 | $(1,1)$ | $(4,3)$ | 2 | 7 | 2 | Sym | 4 | 3 | 0 | 2 | NA |
|  | $(3,4)$ | $(2,2)$ |  |  |  |  | 3 | 4 |  |  |  |
|  | $(1,1)$ | $(4,2)$ | 6 | 6 | 1 |  | 4 | 2 | 1 | 3 | 103 |
|  | $(3,3)$ | $(2,4)$ |  |  |  |  |  |  |  |  |  |
| 76 | $(1,1)$ | $(4,2)$ | 4 | 7 | 2 |  | 3 | 4 | 0 | 2 | 139 |
|  | $(3,4)$ | $(2,3)$ |  |  |  |  | 4 | 2 |  |  |  |
| 77 | $(1,1)$ | $(4,4)$ | 3 | 8 | 1 |  | 4 | 4 | 1 | 1 | 93 |
|  | $(3,2)$ | $(2,3)$ |  |  |  |  |  |  |  |  |  |
| 78 | $(1,1)$ | $(4,4)$ | 1 | 8 | 2 |  | 4 | 4 | 0 | 1 | 124 |
|  | $(3,3)$ | $(2,2)$ |  |  |  |  | 3 | 3 |  |  |  |
|  | $(1,1)$ | $(2,2)$ | 2 | 7 | 1 |  | 3 | 4 | 1 | 2 | 51 |
| 79 | $(3,4)$ | $(4,3)$ |  |  |  |  |  |  |  |  |  |



32

| Game\# | Payoff <br> Matrix |  | Shape | Joint Max | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Row |  |  |  | Col. |  |  |  |
| 89 | $(1,4)$ | $(2,1)$ |  | 20 | 6 | 1 |  | 3 | 3 | 2 | 3 | 31 |
|  | $(3,3)$ | $(4,2)$ |  |  |  |  |  |  |  |  |  |
|  | $(1,4)$ | $(2,2)$ | 21 | 7 | 1 |  | 4 | 3 | 1 | 2 | 66 |  |
| 90 | $(3,1)$ | $(4,3)$ |  |  |  |  |  |  |  |  |  |  |


| 91 | $(1,4)$ $(3,1)$ | $\begin{array}{\|l\|} \hline(2,3) \\ \hline(4,2) \\ \hline \end{array}$ | 23 | 6 | 1 | 4 | 2 | 1 | 3 | 45 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 92 | $(1,1)$ $(4,4)$ | $\begin{array}{\|l\|} \hline(2,2) \\ \hline(3,3) \\ \hline \end{array}$ | 1 | 8 | 1 | 4 | 4 | 1 | 1 | 54 |
| 93 | $(1,1)$ $(4,4)$ | $\begin{array}{\|l\|} \hline(2,3) \\ \hline(3,2) \\ \hline \end{array}$ | 3 | 8 | 1 | 4 | 4 | 1 | 1 | 77 |
| 94 | $(1,2)$ $(4,3)$ | $\begin{array}{\|l\|} \hline(2,1) \\ \hline(3,4) \\ \hline \end{array}$ | 6 | 7 | 1 | 3 | 4 | 1 | 2 | 47 |
| 95 | $(1,2)$ $(4,3)$ | $\begin{array}{\|l\|} \hline(2,4) \\ \hline(3,1) \\ \hline \end{array}$ | 11 | 7 | 1 | 4 | 3 | 1 | 2 | 63 |
| 96 | $(1,3)$ $(4,2)$ | $\begin{array}{\|l\|} \hline(2,1) \\ \hline(3,4) \\ \hline \end{array}$ | 14 | 7 | 1 | 3 | 4 | 1 | 2 | 68 |
| 97 | $(1,3)$ $(4,2)$ | $\begin{array}{\|l\|} \hline(2,4) \\ \hline(3,1) \\ \hline \end{array}$ | 17 | 6 | 1 | 4 | 2 | 1 | 2 | 43 |
| 98 | $(1,4)$ <br> $(4,3)$ | $\begin{array}{\|l\|} \hline(2,1) \\ \hline(3,2) \end{array}$ | 19 | 7 | 1 | 4 | 3 | 2 | 2 | 30 |
| 99 | $(1,4)$ $(4,1)$ | $\begin{array}{\|l\|} \hline(2,2) \\ \hline(3,3) \\ \hline \end{array}$ | 22 | 6 | 1 | 3 | 3 | 1 | 3 | 60 |
| 100 | $(1,4)$ <br> $(4,1)$ | $\begin{array}{\|l\|} \hline(2,3) \\ \hline(3,2) \\ \hline \end{array}$ | 24 | 5 | 1 | 3 | 2 | 1 | 4 | 40 |
| 101 | $(1,1)$ <br> $(2,4)$ | $(3,2)$ $(4,3)$ | 5 | 7 | 1 | 2 | 4 | 1 | 2 | 52 |
| 102 | $(1,1)$ <br> $(2,2)$ | $\begin{array}{\|l\|} \hline(3,3) \\ \hline(4,4) \\ \hline \end{array}$ | 1 | 8 | 1 | 4 | 4 | 2 | 1 | 20 |



33

| Game\# | Payoff <br> Matrix |  | Shape | Joint Max | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Row |  |  |  | Col. |  |  |  |
| 111 | $(1,3)$ | $(3,1)$ |  | 15 | 8 | 1 |  | 4 | 4 | 1 | 1 | 62 |
|  | $(2,2)$ | $(4,4)$ |  |  |  |  |  |  |  |  |  |
| 112 | $(1,3)$ | $(3,1)$ | 17 | 6 | 1 |  | 2 | 4 | 2 | 2 | 14 |  |
|  | $(2,4)$ | $(4,2)$ |  |  |  |  |  |  |  |  |  |  |
| 113 | $(1,3)$ | $(3,2)$ | 13 | 8 | 1 |  | 4 | 4 | 1 | 1 | 8 |  |
|  | $(2,1)$ | $(4,4)$ |  |  |  |  |  |  |  |  |  |  |



| 126 | $(1,2)$ $(4,3)$ | $\begin{array}{\|l\|} \hline(3,1) \\ \hline(2,4) \\ \hline \end{array}$ | 11 | 7 | 0 | 2.5 | 2.5 | 0 | 2 | 44 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 127 | $(1,2)$ | $(3,3)$ | 12 | 6 | 1 | 3 | 3 | 1 | 3 | 24 |
|  | $(4,1)$ | $(2,4)$ |  |  |  |  |  |  |  |  |
| 128 | $(1,2)$ | $(3,4)$ | 10 | 7 | 1 | 3 | 4 | 1 | 2 | 13 |
|  | $(4,1)$ | $(2,3)$ |  |  |  |  |  |  |  |  |
| 129 | $(1,2)$ | $(3,4)$ | 8 | 7 | 2 | 4 | 3 | 0 | 2 | 67 |
|  | $(4,3)$ | $(2,1)$ |  |  |  | 3 | 4 |  |  |  |
|  | $(1,3)$ | $(3,1)$ | 17 | 6 | 0 | 2.5 | 2.5 | 0 | 2 | 64 |
| 130 | $(4,2)$ | $(2,4)$ |  |  |  |  |  |  |  |  |
| 131 | $(1,3)$ | $(3,1)$ | 15 | 8 | 1 | 4 | 4 | 1 | 1 | 15 |
|  | $(4,4)$ | $(2,2)$ |  |  |  |  |  |  |  |  |
|  | $(1,3)$ | $(3,2)$ | 13 | 8 | 1 | 4 | 4 | 1 | 1 | 29 |
| 132 | $(4,4)$ | $(2,1)$ |  |  |  |  |  |  |  |  |


| Game\# | Payoff <br> Matrix |  | Shape | Joint Max | PSNEs | Symmetric | NashPayoff |  | Dom. | Pareto Optima | Transpose |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Row |  |  |  | Col. |  |  |  |
| 133 | $(1,3)$ | $(3,4)$ |  | 14 | 7 | 2 |  | 3 | 4 | 0 | 2 | 46 |
|  | $(4,2)$ | $(2,1)$ |  |  |  |  | 4 | 2 |  |  |  |  |
| 134 | $(1,4)$ | $(3,1)$ | 23 | 6 | 0 |  | 2.5 | 2.5 | 0 | 3 | 6 |  |
|  | $(4,2)$ | $(2,3)$ |  |  |  |  |  |  |  |  |  |  |
| 135 | $(1,4)$ | $(3,1)$ | 21 | 7 | 1 |  | 4 | 3 | 1 | 2 | 27 |  |
|  | $(4,3)$ | $(2,2)$ |  |  |  |  |  |  |  |  |  |  |
| 136 | $(1,4)$ | $(3,2)$ | 24 | 5 | 0 |  | 2.5 | 2.5 | 0 | 4 | 59 |  |
|  | $(4,1)$ | $(2,3)$ |  |  |  |  |  |  |  |  |  |  |
| 137 | $(1,4)$ | $(3,2)$ | 19 | 7 | 1 |  | 4 | 3 | 1 | 2 | 16 |  |
|  | $(4,3)$ | $(2,1)$ |  |  |  |  |  |  |  |  |  |  |
| 138 | $(1,4)$ | $(3,3)$ | 22 | 6 | 0 |  | 2.5 | 2.5 | 0 | 3 | 39 |  |
|  | $(4,1)$ | $(2,2)$ |  |  |  |  |  |  |  |  |  |  |
| 139 | $(1,1)$ | $(4,3)$ | 5 | 7 | 2 |  | 4 | 3 | 0 | 2 | 76 |  |
|  | $(2,4)$ | $(3,2)$ |  |  |  |  | 2 | 4 |  |  |  |  |
| 140 | $(1,2)$ | $(4,4)$ | 9 | 8 | 2 |  | 4 | 4 | 0 | 1 | 61 |  |
|  | $(2,3)$ | $(3,1)$ |  |  |  |  | 2 | 3 |  |  |  |  |
| 141 | $(1,3)$ | $(4,1)$ | 16 | 7 | 0 |  | 2.5 | 2.5 | 0 | 2 | 57 |  |
|  | $(2,2)$ | $(3,4)$ |  |  |  |  |  |  |  |  |  |  |
| 142 | $(1,4)$ | $(4,2)$ | 20 | 6 | 0 |  | 2.5 | 2.5 | 0 | 3 | 72 |  |
|  | $(2,1)$ | $(3,3)$ |  |  |  |  |  |  |  |  |  |  |
| 143 | $(1,4)$ | $(4,3)$ | 19 | 7 | 0 |  | 2.5 | 2.5 | 0 | 2 | 50 |  |
|  | $(2,1)$ | $(3,2)$ |  |  |  |  |  |  |  |  |  |  |



## Appendix 3

\#include <iostream>
\#include <fstream>
\#include <stdio.h>
\#include <time.h>
\#include <stdlib.h>
\#include <iomanip>
using namespace
std;
const int maxnum=1<<11;
const double di=1<<9;
const int loop=300000;
class cmatr

```
{ public: int aul,bul,aur,bur,adl,bdl,adr,bdr; public: int
        d1,d2,d3,d4,d5,d6,d7,d8; public: int nash; public: double
        na1,na2,na3,na4; public: void setnum(int a1,int a2,int a3,int a4,int
        a5,int a6,int a7,int a8)
    {
        aul=a1;bul=a2;aur=a3;bur=a4;
        adl=a5;bdl=a6;adr=a7;bdr=a8;
        d1=a1;d2=a2;d3=a3;d4=a4;
        d5=a5;d6=a6;d7=a7;d8=a8;
    }
    public: void print()
    {
    cout<<"cm"<<endl<<"aul "<<aul<<" bul "<<bul<<" aur "<<aur<<" bur
    "<<bur<<endl; cout<<"ad| "<<ad|<<" bdl "<<bd|<<" adr "<<adr<<" bdr
    "<<bdr<<endl; cout<<d1<<<' '<<d2<<<' '<<d3<<'' '<<d4<<endl<<d5<<<' '<<d6<<'
    '<<d7<<' '<<d8<<endl;
    }
    public: bool eql(cmatr a)
    {
        for (int i=1;i<=4;i++)
        {
            if ((aul==a.aul)&&(bul==a.bul)&&(aur==a.aur)&&(bur==a.bur)
                &&(adl==a.adl)&&(bdl==a.bdl)&&(adr==a.adr)&&(bdr==a.bdr)) return true;
            a.setnum(a.adr,a.bdr,a.aul,a.bul,a.aur,a.bur,a.adl,a.bdl)
        ;}
        return false;
    }
    public: void rep()
    {
        aul=1;aur=1;bul=1;bur=1;adl=1;adr=1;bdl=1;bdr=1;
                    36
    if (d1>d3) aul++; if (d1>d5) aul++; if (d1>d7) aul++;
    if (d3>d1) aur++; if (d3>d5) aur++; if (d3>d7) aur++;
    if (d5>d1) adl++; if (d5>d3) adl++; if (d5>d7) adl++;
    if (d7>d1) adr++; if (d7>d3) adr++; if (d7>d5) adr++;
    if (d2>d4) bul++; if (d2>d6) bul++; if (d2>d8) bul++;
    if (d4>d2) bur++; if (d4>d6) bur++; if (d4>d8)
    bur++; if (d6>d2) bdl++; if (d6>d4) bdl++; if (d6>d8)
    bdl++; if (d8>d2) bdr++; if (d8>d4) bdr++; if (d8>d6)
    bdr++;
    }
};
class matr
{ public: int ul,ur,dl,dr;
    public: int
    d1,d2,d3,d4; public:
    void randnum()
    {
```

```
    ul=rand()%maxnum+1;d1=ul;
    ur=rand()%maxnum+1;d2=ur;
    dl=rand()%maxnum+1;d3=dl;
    dr=rand()%maxnum+1;d4=dr;
}
public: bool eql()
{
    if ((ul==ur)| |(ul==dl)| |(ul==dr)) return
    true; if ((ur==dl)| |(ur==dr)) return true; if
    (dl==dr) return true; return false;
}
public: void rep()
{
    int a1=1,a2=1,a3=1,a4=1; if (ul>ur) a1++; if
    (ul>dl) a1++; if (ul>dr) a1++; if (ur>ul) a2++; if
    (ur>dl) a2++; if (ur>dr) a2++; if (dl>ul) a3++;
    if (dl>ur) a3++; if (dl>dr) a3++; if (dr>ul)
    a4++; if (dr>ur) a4++; if (dr>dl) a4++;
    ul=a1;ur=a2;dl=a3;dr=a4;
}
public: cmatr combine(matr b)
{
    cmatr c;
    c.setnum(ul,b.ul,ur,b.ur,dl,b.dl,dr,b.dr);
    return c;
}
public: void print()
{
```

```
        cout<<ul<<' '<<ur<<' '<<d|<<' '<<dr<<endl;
    }
};
    double a[500000],b[500000],a1[500000],b1[500000]; double
    oa[150][4000],ob[150][4000],oa1[150][4000],ob1[150][4000]; int
    soa[150],sob[150],soa1[150],sob1[150]; int ty[500000]; char
    nu[4]=""; string s="out"; string s2="solution"; string s1; int m=0;
int main()
{ srand((int)time(0)); int aul,bul,aur,bur,adl,bdl,adr,bdr; int i,j,k,t,tp; int res[150]; double
    mr[150],mc[150],vr[150],vc[150],sr[150],sc[150],ms[150],vs[150],ss[150]; double
    mr1[150],mc1[150],vr1[150],vc1[150],sr1[150],sc1[150],ms1[150],vs1[150],ss1[150]; double
    mmr,mmc,vvr,vvc,ssr,ssc,mms,vvs,sss; double
    mmr1,mmc1,vvr1,vvc1,ssr1,ssc1,mms1,vvs1,sss1; int rc[150],rc1[150]; cmatr samp[150];
    cmatr c,d; cmatr *p; matr pa,pb;
    FILE * fp=NULL;
    fp=fopen("www.txt","r");
    for (i=1;i<=144;i++)
    {
    fscanf(fp,"%d%d%d%d%d%d%d%d",&aul,&bul,&aur,&bur,&adl,&bdl,&adr,&bdr
    ); samp[i].setnum(aul,bul,aur,bur,adl,bdl,adr,bdr); res[i]=0;
    }
    for (i=1;i<=144;i++)
    {
    fscanf(fp,"%d",&samp[i].nash); if
    ((samp[i].nash==1)||(samp[i].nash==0))
    fscanf(fp,"%lf%lf",&samp[i].na1,&samp[i].na2);
    if (samp[i].nash==2)
        fscanf(fp,"%lf%lf%lf%lf",&samp[i].na1,&samp[i].na2,&samp[i].na3,&samp[i].na4);
    } fclose(fp);
    fp=NULL; for
    (i=1;i<=loop;i++)
    {
    pa.randnum(); while (pa.eql())
    pa.randnum(); pb.randnum(); while
    (pb.eql()) pb.randnum();
    c=pa.combine(pb); c.rep();
    a1[i]=(double)
    (c.d1+c.d3+c.d5+c.d7)/4;
    b1[i]=(double)
    (c.d2+c.d4+c.d6+c.d8)/4; for
    (k=1;k<=144;k++)
    {
        if (samp[k].eql(c))
```

```
{ res[k]++; ty[i]=k; a[i]=0;
    b[i]=0; if
    (samp[k].nash==0)
    {
        a[i]=c.d1+c.d3+c.d5+c.d7;
        a[i]=a[i]/4;
        b[i]=c.d2+c.d4+c.d6+c.d8;
        b[i]=b[i]/4;
    }else if (samp[k].nash==1)
    {
        if ((c.aul==samp[k].na1)&&(c.bul==samp[k].na2))
        {a[i]=c.d1;b[i]=c.d2;
        };
        if ((c.aur==samp[k].na1)&&(c.bur==samp[k].na2))
        { a[i]=c.d3;b[i]=c.d4;
        };
        if ((c.adl==samp[k].na1)&&(c.bdl==samp[k].na2))
        { a[i]=c.d5;b[i]=c.d6;
        };
        if ((c.adr==samp[k].na1)&&(c.bdr==samp[k].na2))
        {a[i]=c.d7;b[i]=c.d8;
        };
    }else if (samp[k].nash==2)
    {
        if ((c.aul==samp[k].na1)&&(c.bul==samp[k].na2))
        { a[i]=c.d1;b[i]=c.d2;
        };
        if ((c.aur==samp[k].na1)&&(c.bur==samp[k].na2))
        {a[i]=c.d3;b[i]=c.d4;
        };
        if ((c.adl==samp[k].na1)&&(c.bdl==samp[k].na2))
        {a[i]=c.d5;b[i]=c.d6;
        };
        if ((c.adr==samp[k].na1)&&(c.bdr==samp[k].na2))
        { a[i]=c.d7;b[i]=c.d8;
        };
        if ((c.aul==samp[k].na3)&&(c.bul==samp[k].na4))
        { a[i]+=c.d1;b[i]+=c.d2;
        };
        if ((c.aur==samp[k].na3)&&(c.bur==samp[k].na4))
        {a[i]+=c.d3;b[i]+=c.d4;
        };
        if ((c.adl==samp[k].na3)&&(c.bdl==samp[k].na4))
        { a[i]+=c.d5;b[i]+=c.d6;
```

```
                    };
            if ((c.adr==samp[k].na3)&&(c.bdr==samp[k].na4))
            { a[i]+=c.d7;b[i]+=c.d8;
            };
            a[i]+=(double) (c.d1+c.d3+c.d5+c.d7)/4;
            b[i]+=(double)
            (c.d2+c.d4+c.d6+c.d8)/4;
            a[i]=(double)a[i]/3; b[i]=(double)b[i]/3;
                };
                break;
            }
    }
}
for (i=1;i<=144;i++)
{ mr[i]=0;mc[i]=0;vr[i]=0;vc[i]=0;sr[i]=0;sc[i]=0;rc[i]=0;
    mr1[i]=0;mc1[i]=0;vr1[i]=0;vc1[i]=0;sr1[i]=0;sc1[i]=0
    ; rc1[i]=0; soa[i]=0;sob[i]=0; soa1[i]=0;sob1[i]=0;
}
mmr=0;mmc=0;vvr=0;vvc=0;ssr=0;ssc=0;mms=0;vvs=0;sss=0;
mmr1=0;mmc1=0;vvr1=0;vvc1=0;ssr1=0;ssc1=0;mms1=0;vvs1=0;sss1=0;
for (i=1;i<=loop;i++)
{ a[i]=a[i]/di;
    b[i]=b[i]/di;
    a1[i]=a1[i]/di;
    b1[i]=b1[i]/di
    ;
}
for (i=1;i<=loop;i++)
{tp=ty[i]; rc[tp]++; rc1[tp]++;
    sr[tp]+=a[i];sr1[tp]+=a1[i];
    sc[tp]+=b[i];sc1[tp]+=b1[i];
    ss[tp]+=a[i]+b[i];ss1[tp]+=a1[i]+b1[i];
    ssr+=a[i];ssc+=b[i];sss+=a[i]+b[i];
    ssr1+=a1[i];ssc1+=b1[i];sss1+=a1[i]+b1[i]
    ; soa[tp]++; oa[tp][soa[tp]]=a[i];
    sob[tp]++; ob[tp][sob[tp]]=b[i];
    soa1[tp]++; oa1[tp][soa1[tp]]=a1[i];
    sob1[tp]++; ob1[tp][sob1[tp]]=b1[i];
}
for (i=1;i<=144;i++)
{mr[i]=sr[i]/rc[i];mr1[i]=sr1[i]/rc1[i];
    mc[i]=sc[i]/rc[i];mc1[i]=sc1[i]/rc1[i];
    ms[i]=ss[i]/rc[i];ms1[i]=ss1[i]/rc1[i];
```

```
        sr[i]=0;sc[i]=0;ss[i]=0;sr1[i]=0;sc1[i]=0;ss1[i]=0
        ;
    }
    mmr=ssr/loop;mmc=ssc/loop;mms=sss/loop;
    mmr1=ssr1/loop;mmc1=ssc1/loop;mms1=sss1/loop
    ; ssr=0;ssc=0;sss=0; ssr1=0;ssc1=0;sss1=0; for
    (i=1;i<=loop;i++)
    { tp=ty[i]; sr[tp]+=(a[i]-mr[tp])*(a[i]-mr[tp]);sr1[tp]+=(a1[i]-mr1[tp])*(a1[i]-
        mr1[tp]); sc[tp]+=(b[i]-mc[tp])*(b[i]-mc[tp]);sc1[tp]+=(b1[i]-
        mc1[tp])*(b1[i]-mc1[tp]);
ss[tp]+=(a[i]+b[i]-ms[tp])*(a[i]+b[i]-ms[tp]);ss1[tp]+=(a1[i]+b1[i]-ms1[tp])*(a1[i]+b1[i]-ms1[tp]);
        ssr+=(a[i]-mmr)*(a[i]-mmr);ssr1+=(a1[i]-mmr1)*(a1[i]-mmr1); ssc+=(b[i]-mmc)*(b[i]-
        mmc);ssc1+=(b1[i]-mmc1)*(b1[i]-mmc1); sss+=(a[i]+b[i]-mms)*(a[i]+b[i]-
        mms);sss1+=(a1[i]+b1[i]-mms1)*(a1[i]+b1[i]-mms1);
    }
    vvr=ssr/loop;vvc=ssc/loop;vvs=sss/loop;
    vvr1=ssr1/loop;vvc1=ssc1/loop;vvs1=sss1/loop;
    for (i=1;i<=144;i++)
    { vr[i]=sr[i]/rc[i];vr1[i]=sr1[i]/rc1[i];
        vc[i]=sc[i]/rc[i];vc1[i]=sc1[i]/rc1[i]
        ;
        vs[i]=ss[i]/rc[i];vs1[i]=ss1[i]/rc1[i];
    } cout.setf(ios::fixed);
    fp=fopen("out.txt","w")
    ;
    fprintf(fp,"r mean %.3lf c mean %.3If sum mean %.3If r var %.3If c var %.3lf sum var
%.3If\n",mmr,mmc,mms,vvr,vvc,vvs);
    fprintf(fp,"r mean %.3lf c mean %.3lf sum mean %.3lf r var %.3lf c var %.3lf sum var
%.3If\n",mmr1,mmc1,mms1,vvr1,vvc1,vvs1); for (i=1;i<=144;i++)
    {
    //cout<<setprecision(3)<<"BOX "<<i<<":"<<"r mean:"<<mr[i]<<" c mean"<<mc[i]<<"
sum mean:"<<ms[i]<<" r var"<<vr[i]<<" c var"<<vc[i]<<" sum var:"<<vs[i]<<endl; fprintf(fp,"BOX
%d:r mean %.3lf c mean %.3If sum mean %.3lf r var %.3If c var %.3lf
sum var %.3If\n",i,mr[i],mc[i],ms[i],vr[i],vc[i],vs[i]);
fprintf(fp,"BOX %d:r mean %.3lf c mean %.3lf sum mean %.3lf r var %.3If c var %.3lf
sum var %.3lf\n",i,mr1[i],mc1[i],ms1[i],vr1[i],vc1[i],vs1[i]);
    } fclose(fp); for
    (i=1;i<=144;i++)
    {
        m++;
        itoa(m,nu,10); s1=s+nu+".txt";
        fp=fopen(s1.c_str(),"w"); fprintf(fp,"BOX
```

```
    %d\n",i); for (k=1;k<=soa[i];k++)
    fprintf(fp,"%.3lf %.3If\n",oa[i][k],ob[i][k]);
    fclose(fp);
    }
    m=0;
    for (i=1;i<=144;i++)
    {
    m++;
    itoa(m,nu,10); s1=s2+nu+".txt";
    fp=fopen(s1.c_str(),"w"); fprintf(fp,"BOX %d\n",i);
    for (k=1;k<=soa1[i];k++) fprintf(fp,"%.3lf
            %.3If\n",oa1[i][k],ob1[i][k]);
        fclose(fp);
}
    return 0;
}
```

| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | NE Entropy | $\begin{aligned} & 1.579 \\ & 1.989 \end{aligned}$ | $\begin{aligned} & 1.577 \\ & 1.995 \end{aligned}$ | $\begin{aligned} & 3.156 \\ & 3.984 \end{aligned}$ | $\begin{aligned} & 0.658 \\ & 0.350 \end{aligned}$ | $\begin{aligned} & 0.618 \\ & 0.315 \end{aligned}$ | $\begin{aligned} & 1.297 \\ & 0.684 \end{aligned}$ |
| 2 | NE <br> Entropy | $\begin{aligned} & 1.609 \\ & 1.996 \end{aligned}$ | $\begin{aligned} & 3.202 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 4.811 \\ & 3.990 \end{aligned}$ | $\begin{aligned} & 0.623 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.428 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 1.062 \\ & 0.662 \end{aligned}$ |
| 3 | NE <br> Entropy | $\begin{aligned} & 3.197 \\ & 1.994 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.228 \\ & 2.011 \\ & \hline \end{aligned}$ | $\begin{aligned} & 6.425 \\ & 4.005 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.429 \\ & 0.330 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.403 \\ & 0.325 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.820 \\ & 0.667 \\ & \hline \end{aligned}$ |
| 4 | NE <br> Entropy | $\begin{aligned} & 3.198 \\ & 1.992 \end{aligned}$ | $\begin{aligned} & 1.585 \\ & 1.996 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.784 \\ & 3.987 \end{aligned}$ | $\begin{aligned} & 0.415 \\ & 0.319 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.629 \\ & 0.329 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.044 \\ & 0.647 \end{aligned}$ |
| 5 | NE <br> Entropy | $\begin{aligned} & 2.377 \\ & 1.986 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.184 \\ & 2.002 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.561 \\ & 3.987 \end{aligned}$ | $\begin{aligned} & 0.639 \\ & 0.321 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.437 \\ & 0.341 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.039 \\ & 0.654 \\ & \hline \end{aligned}$ |
| 6 | NE <br> Entropy | $\begin{aligned} & 1.995 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 2.001 \\ & 1.999 \end{aligned}$ | $\begin{aligned} & 3.996 \\ & 3.993 \end{aligned}$ | $\begin{aligned} & 0.336 \\ & 0.336 \end{aligned}$ | $\begin{aligned} & 0.347 \\ & 0.347 \end{aligned}$ | $\begin{aligned} & 0.669 \\ & 0.669 \end{aligned}$ |
| 7 | NE <br> Entropy | $\begin{aligned} & 2.518 \\ & 1.983 \end{aligned}$ | $\begin{aligned} & 2.536 \\ & 1.993 \end{aligned}$ | $\begin{aligned} & 5.053 \\ & 3.976 \end{aligned}$ | $\begin{aligned} & 0.370 \\ & 0.327 \end{aligned}$ | $\begin{aligned} & 0.373 \\ & 0.318 \end{aligned}$ | $\begin{aligned} & 0.741 \\ & 0.645 \end{aligned}$ |
| 8 | NE Entropy | $\begin{aligned} & 3.217 \\ & 2.023 \end{aligned}$ | $\begin{aligned} & 3.224 \\ & 2.011 \end{aligned}$ | $\begin{aligned} & 6.441 \\ & 4.034 \end{aligned}$ | $\begin{aligned} & 0.422 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 0.405 \\ & 0.332 \end{aligned}$ | $\begin{aligned} & 0.848 \\ & 0.678 \end{aligned}$ |
| 9 | NE Entropy | $\begin{aligned} & 2.391 \\ & 1.998 \end{aligned}$ | $\begin{aligned} & 1.603 \\ & 2.011 \end{aligned}$ | $\begin{aligned} & 3.994 \\ & 4.009 \end{aligned}$ | $\begin{aligned} & 0.658 \\ & 0.347 \end{aligned}$ | $\begin{aligned} & 0.613 \\ & 0.314 \end{aligned}$ | $\begin{aligned} & 1.244 \\ & 0.641 \end{aligned}$ |
| 10 | NE <br> Entropy | $\begin{aligned} & 2.543 \\ & 1.999 \end{aligned}$ | $\begin{aligned} & 2.538 \\ & 2.001 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.081 \\ & 4.001 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.377 \\ & 0.327 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.366 \\ & 0.320 \end{aligned}$ | $\begin{aligned} & 0.732 \\ & 0.630 \end{aligned}$ |
| 11 | NE <br> Entropy | $\begin{aligned} & 2.398 \\ & 2.004 \end{aligned}$ | $\begin{aligned} & 1.605 \\ & 2.000 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.003 \\ & 4.004 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.631 \\ & 0.335 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.644 \\ & 0.328 \end{aligned}$ | $\begin{aligned} & 1.260 \\ & 0.666 \\ & \hline \end{aligned}$ |
| 12 | NE <br> Entropy | $\begin{aligned} & 2.409 \\ & 2.004 \\ & \hline \end{aligned}$ | $\begin{array}{r} 2.399 \\ 1.994 \\ \hline \end{array}$ | $\begin{aligned} & 4.808 \\ & 3.999 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.647 \\ & 0.340 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.639 \\ & 0.338 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.293 \\ & 0.673 \\ & \hline \end{aligned}$ |
| 13 | NE <br> Entropy | $\begin{aligned} & 3.192 \\ & 2.007 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.410 \\ & 2.012 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.601 \\ & 4.019 \end{aligned}$ | $\begin{aligned} & 0.415 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.634 \\ & 0.336 \end{aligned}$ | $\begin{aligned} & 1.051 \\ & 0.669 \end{aligned}$ |
| 14 | NE Entropy | $\begin{aligned} & 3.200 \\ & 1.998 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.606 \\ & 1.995 \end{aligned}$ | $\begin{aligned} & 4.807 \\ & 3.993 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.426 \\ & 0.329 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.648 \\ & 0.328 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.058 \\ & 0.668 \end{aligned}$ |
| 15 | NE <br> Entropy | $\begin{aligned} & 3.192 \\ & 2.010 \end{aligned}$ | $\begin{aligned} & 3.201 \\ & 1.986 \end{aligned}$ | $\begin{aligned} & 6.393 \\ & 3.996 \end{aligned}$ | $\begin{aligned} & 0.430 \\ & 0.342 \end{aligned}$ | $\begin{aligned} & 0.420 \\ & 0.319 \end{aligned}$ | $\begin{aligned} & 0.866 \\ & 0.666 \end{aligned}$ |
| 16 | NE Entropy | $\begin{aligned} & 2.390 \\ & 1.989 \end{aligned}$ | $\begin{aligned} & 3.234 \\ & 2.014 \end{aligned}$ | $\begin{aligned} & 5.624 \\ & 4.003 \end{aligned}$ | $\begin{aligned} & 0.655 \\ & 0.335 \end{aligned}$ | $\begin{aligned} & 0.414 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 1.034 \\ & 0.664 \end{aligned}$ |
| 17 | $\begin{gathered} \text { NE } \\ \text { Entropy } \end{gathered}$ | $\begin{aligned} & 3.206 \\ & 2.014 \end{aligned}$ | $\begin{aligned} & 2.420 \\ & 2.015 \end{aligned}$ | $\begin{aligned} & 5.625 \\ & 4.029 \end{aligned}$ | $\begin{aligned} & 0.435 \\ & 0.341 \end{aligned}$ | $\begin{aligned} & 0.607 \\ & 0.320 \end{aligned}$ | $\begin{aligned} & 1.048 \\ & 0.660 \end{aligned}$ |
| 18 | NE Entropy | $\begin{aligned} & 2.378 \\ & 1.986 \end{aligned}$ | $\begin{aligned} & 3.217 \\ & 2.013 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.595 \\ & 3.998 \end{aligned}$ | $\begin{aligned} & 0.641 \\ & 0.341 \end{aligned}$ | $\begin{aligned} & 0.424 \\ & 0.334 \end{aligned}$ | $\begin{aligned} & 1.036 \\ & 0.663 \end{aligned}$ |


| 19 | NE | 3.170 | 3.199 | 6.369 | 0.440 | 0.432 | 0.868 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Entropy | 1.970 | 1.999 | 3.969 | 0.325 | 0.327 | 0.643 |
| 20 | NE | 3.209 | 3.196 | 6.405 | 0.435 | 0.416 | 0.841 |
|  | Entropy | 2.005 | 2.008 | 4.013 | 0.332 | 0.312 | 0.638 |
| 21 | NE | 2.404 | 2.392 | 4.796 | 0.620 | 0.641 | 1.303 |
|  | Entropy | 2.004 | 2.004 | 44 | 4.008 | 0.337 | 0.341 |
| 22 | NE | 3.199 | 1.599 | 4.798 | 0.419 | 0.645 | 1.050 |
|  | Entropy | 1.990 | 1.995 | 3.985 | 0.326 | 0.342 | 0.652 |
| 23 | NE | 3.236 | 2.389 | 5.625 | 0.434 | 0.649 | 1.090 |
|  | Entropy | 2.025 | 1.985 | 4.010 | 0.329 | 0.328 | 0.653 |

Table 14: The output for $\mathrm{k}=9$ for the 144 games

| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 26 | NE Entropy | $\begin{aligned} & 3.192 \\ & 1.995 \end{aligned}$ | $\begin{aligned} & 3.184 \\ & 1.989 \end{aligned}$ | $\begin{aligned} & 6.375 \\ & 3.984 \end{aligned}$ | $\begin{aligned} & 0.443 \\ & 0.335 \end{aligned}$ | $\begin{aligned} & 0.430 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 0.848 \\ & 0.675 \end{aligned}$ |
| 27 | NE Entropy | $\begin{aligned} & 2.395 \\ & 1.988 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.204 \\ & 2.000 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.599 \\ & 3.988 \end{aligned}$ | $\begin{aligned} & 0.648 \\ & 0.329 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.443 \\ & 0.356 \end{aligned}$ | $\begin{aligned} & 1.101 \\ & 0.673 \end{aligned}$ |
| 28 | NE Entropy | $\begin{aligned} & 3.208 \\ & 1.999 \end{aligned}$ | $\begin{aligned} & 3.223 \\ & 2.017 \end{aligned}$ | $\begin{aligned} & 6.431 \\ & 4.016 \end{aligned}$ | $\begin{aligned} & 0.425 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.424 \\ & 0.343 \end{aligned}$ | $\begin{aligned} & 0.840 \\ & 0.692 \end{aligned}$ |
| 29 | NE <br> Entropy | $\begin{aligned} & 3.215 \\ & 2.010 \end{aligned}$ | $\begin{aligned} & 3.203 \\ & 2.002 \end{aligned}$ | $\begin{aligned} & 6.417 \\ & 4.012 \end{aligned}$ | $\begin{aligned} & 0.418 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 0.420 \\ & 0.328 \end{aligned}$ | $\begin{aligned} & 0.836 \\ & 0.634 \end{aligned}$ |
| 30 | NE Entropy | $\begin{aligned} & 2.379 \\ & 1.987 \end{aligned}$ | $\begin{aligned} & 3.207 \\ & 2.006 \end{aligned}$ | $\begin{aligned} & 5.586 \\ & 3.993 \end{aligned}$ | $\begin{aligned} & 0.644 \\ & 0.337 \end{aligned}$ | $\begin{aligned} & 0.407 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.055 \\ & 0.666 \end{aligned}$ |
| 31 | NE Entropy | $\begin{aligned} & 2.412 \\ & 2.010 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.376 \\ & 1.988 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.788 \\ & 3.997 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.643 \\ & 0.331 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.630 \\ & 0.326 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.302 \\ & 0.674 \end{aligned}$ |
| 32 | NE <br> Entropy | $\begin{aligned} & 3.192 \\ & 1.984 \end{aligned}$ | $\begin{aligned} & 3.198 \\ & 1.991 \end{aligned}$ | $\begin{aligned} & 6.390 \\ & 3.975 \end{aligned}$ | $\begin{aligned} & 0.420 \\ & 0.319 \end{aligned}$ | $\begin{aligned} & 0.417 \\ & 0.332 \end{aligned}$ | $\begin{aligned} & 0.822 \\ & 0.637 \end{aligned}$ |
| 33 | NE Entropy | $\begin{aligned} & 2.386 \\ & 1.991 \end{aligned}$ | $\begin{aligned} & 3.206 \\ & 1.995 \end{aligned}$ | $\begin{aligned} & 5.592 \\ & 3.987 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.629 \\ & 0.326 \end{aligned}$ | $\begin{aligned} & 0.420 \\ & 0.303 \end{aligned}$ | $\begin{aligned} & 1.112 \\ & 0.636 \end{aligned}$ |
| 34 | NE Entropy | $\begin{aligned} & 2.413 \\ & 2.015 \end{aligned}$ | $\begin{aligned} & 3.209 \\ & 2.010 \end{aligned}$ | $\begin{aligned} & 5.621 \\ & 4.024 \end{aligned}$ | $\begin{aligned} & 0.628 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 0.405 \\ & 0.321 \end{aligned}$ | $\begin{aligned} & 1.042 \\ & 0.630 \end{aligned}$ |
| 35 | NE Entropy | $\begin{aligned} & 2.402 \\ & 1.989 \end{aligned}$ | $\begin{aligned} & 2.430 \\ & 2.011 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.832 \\ & 4.000 \end{aligned}$ | $\begin{aligned} & 0.618 \\ & 0.320 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.648 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 1.283 \\ & 0.642 \end{aligned}$ |
| 36 | NE <br> Entropy | $\begin{aligned} & 3.193 \\ & 1.993 \end{aligned}$ | $\begin{aligned} & 2.418 \\ & 2.003 \end{aligned}$ | $\begin{aligned} & 5.611 \\ & 3.996 \end{aligned}$ | $\begin{aligned} & 0.439 \\ & 0.351 \end{aligned}$ | $\begin{aligned} & 0.644 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 1.085 \\ & 0.701 \end{aligned}$ |
| 37 | NE <br> Entropy | $\begin{aligned} & 1.593 \\ & 1.997 \end{aligned}$ | $\begin{aligned} & 1.616 \\ & 2.017 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.209 \\ & 4.015 \end{aligned}$ | $\begin{aligned} & 0.620 \\ & 0.326 \end{aligned}$ | $\begin{aligned} & 0.646 \\ & 0.323 \end{aligned}$ | $\begin{aligned} & 1.262 \\ & 0.661 \end{aligned}$ |
| 38 | NE Entropy | $\begin{aligned} & 1.580 \\ & 1.988 \end{aligned}$ | $\begin{aligned} & 2.406 \\ & 2.004 \end{aligned}$ | $\begin{aligned} & 3.986 \\ & 3.992 \end{aligned}$ | $\begin{aligned} & 0.623 \\ & 0.318 \end{aligned}$ | $\begin{aligned} & 0.639 \\ & 0.325 \end{aligned}$ | $\begin{aligned} & 1.218 \\ & 0.619 \end{aligned}$ |
| 39 | NE Entropy | $\begin{aligned} & 1.999 \\ & 1.997 \end{aligned}$ | $\begin{aligned} & 2.020 \\ & 2.019 \end{aligned}$ | $\begin{aligned} & 4.019 \\ & 4.016 \end{aligned}$ | $\begin{aligned} & 0.326 \\ & 0.326 \end{aligned}$ | $\begin{aligned} & 0.334 \\ & 0.334 \end{aligned}$ | $\begin{aligned} & 0.663 \\ & 0.663 \end{aligned}$ |
| 40 | NE Entropy | $\begin{aligned} & 1.604 \\ & 2.000 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.412 \\ & 2.006 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.016 \\ & 4.005 \end{aligned}$ | $\begin{aligned} & 0.663 \\ & 0.347 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.647 \\ & 0.336 \end{aligned}$ | $\begin{aligned} & 1.309 \\ & 0.672 \end{aligned}$ |
| 41 | NE <br> Entropy | $\begin{aligned} & 2.278 \\ & 2.008 \end{aligned}$ | $\begin{aligned} & 2.268 \\ & 1.998 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.546 \\ & 4.005 \end{aligned}$ | $\begin{aligned} & 0.363 \\ & 0.346 \end{aligned}$ | $\begin{aligned} & 0.332 \\ & 0.315 \end{aligned}$ | $\begin{aligned} & 0.695 \\ & 0.655 \\ & \hline \end{aligned}$ |
| 42 | NE Entropy | $\begin{aligned} & 3.215 \\ & 2.011 \end{aligned}$ | $\begin{aligned} & 3.213 \\ & 1.988 \end{aligned}$ | $\begin{aligned} & 6.428 \\ & 4.000 \end{aligned}$ | $\begin{aligned} & 0.413 \\ & 0.326 \end{aligned}$ | $\begin{aligned} & 0.414 \\ & 0.319 \end{aligned}$ | $\begin{aligned} & 0.853 \\ & 0.668 \end{aligned}$ |
| 43 | NE <br> Entropy | $\begin{aligned} & 1.620 \\ & 2.005 \end{aligned}$ | $\begin{aligned} & 3.204 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 4.824 \\ & 3.999 \end{aligned}$ | $\begin{aligned} & 0.612 \\ & 0.334 \end{aligned}$ | $\begin{aligned} & 0.429 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.063 \\ & 0.670 \end{aligned}$ |
| 44 | NE Entropy | $\begin{aligned} & 2.029 \\ & 2.028 \end{aligned}$ | $\begin{aligned} & 2.017 \\ & 2.016 \end{aligned}$ | $\begin{aligned} & 4.046 \\ & 4.044 \end{aligned}$ | $\begin{aligned} & 0.327 \\ & 0.327 \end{aligned}$ | $\begin{aligned} & 0.331 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 0.645 \\ & 0.645 \end{aligned}$ |
| 45 | NE <br> Entropy | $\begin{aligned} & 1.587 \\ & 1.995 \end{aligned}$ | $\begin{aligned} & 3.218 \\ & 2.013 \end{aligned}$ | $\begin{aligned} & 4.805 \\ & 4.008 \end{aligned}$ | $\begin{aligned} & 0.634 \\ & 0.328 \end{aligned}$ | $\begin{aligned} & 0.406 \\ & 0.314 \end{aligned}$ | $\begin{aligned} & 1.045 \\ & 0.645 \end{aligned}$ |


| 46 | NE | 2.269 | 2.556 | 4.825 | 0.349 | 0.372 | 0.715 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Entropy | 2.003 | 2.023 | 4.025 | 0.331 | 0.339 | 0.669 |
| 47 | NE | 3.202 | 2.416 | 5.618 | 0.445 | 0.605 | 1.040 |
|  | Entropy | 1.998 | 1.991 | 3.989 | 0.339 | 0.313 | 0.624 |
| 48 | NE | 3.196 | 3.193 | 6.389 | 0.417 | 0.418 | 0.809 |
|  | Entropy | 1.983 | 2.007 | 45 | 3.989 | 0.327 | 0.325 |
| 49 | NE | 1.592 | 3.193 | 4.785 | 0.638 | 0.440 | 1.070 |
|  | Entropy | 1.992 | 1.979 | 3.971 | 0.338 | 0.331 | 0.663 |
| 50 | NE | 1.989 | 2.004 | 3.992 | 0.332 | 0.335 | 0.672 |
|  | Entropy | 1.987 | 2.002 | 3.990 | 0.332 | 0.335 | 0.672 |


| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 51 | NE Entropy | $\begin{aligned} & 3.207 \\ & 2.000 \end{aligned}$ | $\begin{aligned} & 2.373 \\ & 1.991 \end{aligned}$ | $\begin{aligned} & 5.579 \\ & 3.991 \end{aligned}$ | $\begin{aligned} & 0.406 \\ & 0.317 \end{aligned}$ | $\begin{aligned} & 0.626 \\ & 0.328 \end{aligned}$ | $\begin{aligned} & 1.082 \\ & 0.657 \end{aligned}$ |
| 52 | NE <br> Entropy | $\begin{aligned} & 3.201 \\ & 2.011 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.580 \\ & 1.995 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.781 \\ & 4.005 \end{aligned}$ | $\begin{aligned} & 0.419 \\ & 0.333 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.638 \\ & 0.332 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.090 \\ & 0.674 \end{aligned}$ |
| 53 | NE Entropy | $\begin{aligned} & 2.278 \\ & 2.009 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.250 \\ & 1.982 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.528 \\ & 3.991 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.340 \\ & 0.325 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.344 \\ & 0.323 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.663 \\ & 0.635 \\ & \hline \end{aligned}$ |
| 54 | NE <br> Entropy | $\begin{aligned} & 3.202 \\ & 2.007 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.177 \\ & 2.006 \\ & \hline \end{aligned}$ | $\begin{aligned} & 6.380 \\ & 4.013 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.437 \\ & 0.331 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.439 \\ & 0.339 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.889 \\ & 0.664 \end{aligned}$ |
| 55 | NE <br> Entropy | $\begin{aligned} & 2.253 \\ & 1.985 \end{aligned}$ | $\begin{aligned} & 2.521 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 4.774 \\ & 3.980 \end{aligned}$ | $\begin{aligned} & 0.365 \\ & 0.345 \end{aligned}$ | $\begin{aligned} & 0.397 \\ & 0.347 \end{aligned}$ | $\begin{aligned} & 0.739 \\ & 0.670 \end{aligned}$ |
| 56 | NE Entropy | $\begin{aligned} & 2.433 \\ & 2.024 \end{aligned}$ | $\begin{aligned} & 1.600 \\ & 2.002 \end{aligned}$ | $\begin{aligned} & 4.034 \\ & 4.026 \end{aligned}$ | $\begin{aligned} & 0.623 \\ & 0.342 \end{aligned}$ | $\begin{aligned} & 0.633 \\ & 0.325 \end{aligned}$ | $\begin{aligned} & 1.308 \\ & 0.696 \end{aligned}$ |
| 57 | NE Entropy | $\begin{aligned} & 1.989 \\ & 1.988 \end{aligned}$ | $\begin{aligned} & 2.009 \\ & 2.008 \end{aligned}$ | $\begin{aligned} & 3.999 \\ & 3.996 \end{aligned}$ | $\begin{aligned} & 0.363 \\ & 0.363 \end{aligned}$ | $\begin{aligned} & 0.330 \\ & 0.330 \end{aligned}$ | $\begin{aligned} & 0.681 \\ & 0.681 \end{aligned}$ |
| 58 | NE Entropy | $\begin{aligned} & 2.411 \\ & 2.002 \end{aligned}$ | $\begin{aligned} & 2.431 \\ & 2.024 \end{aligned}$ | $\begin{aligned} & 4.841 \\ & 4.026 \end{aligned}$ | $\begin{aligned} & 0.620 \\ & 0.310 \end{aligned}$ | $\begin{aligned} & 0.655 \\ & 0.346 \end{aligned}$ | $\begin{aligned} & 1.315 \\ & 0.668 \end{aligned}$ |
| 59 | NE Entropy | $\begin{aligned} & 2.008 \\ & 2.006 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.015 \\ & 2.014 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.023 \\ & 4.020 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.313 \\ & 0.313 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.329 \\ & 0.329 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.656 \\ & 0.656 \\ & \hline \end{aligned}$ |
| 60 | NE <br> Entropy | $\begin{aligned} & 2.378 \\ & 1.992 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.410 \\ & 1.995 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.788 \\ & 3.988 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.639 \\ & 0.328 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.661 \\ & 0.344 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.277 \\ & 0.677 \\ & \hline \end{aligned}$ |
| 61 | NE <br> Entropy | $\begin{aligned} & 2.527 \\ & 1.991 \end{aligned}$ | $\begin{aligned} & 2.279 \\ & 2.010 \\ & \hline \end{aligned}$ | $\begin{aligned} & 4.806 \\ & 4.001 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.369 \\ & 0.330 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.363 \\ & 0.341 \end{aligned}$ | $\begin{aligned} & 0.724 \\ & 0.659 \end{aligned}$ |
| 62 | NE Entropy | $\begin{aligned} & 3.187 \\ & 1.981 \end{aligned}$ | $\begin{aligned} & 3.210 \\ & 1.992 \end{aligned}$ | $\begin{aligned} & 6.397 \\ & 3.973 \end{aligned}$ | $\begin{aligned} & 0.432 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.414 \\ & 0.323 \end{aligned}$ | $\begin{aligned} & 0.832 \\ & 0.646 \end{aligned}$ |
| 63 | NE Entropy | $\begin{aligned} & 2.359 \\ & 1.972 \end{aligned}$ | $\begin{aligned} & 3.210 \\ & 2.006 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.569 \\ & 3.978 \end{aligned}$ | $\begin{aligned} & 0.633 \\ & 0.328 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.405 \\ & 0.330 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.028 \\ & 0.660 \end{aligned}$ |
| 64 | NE <br> Entropy | $\begin{aligned} & 2.022 \\ & 2.020 \end{aligned}$ | $\begin{aligned} & 2.007 \\ & 2.005 \end{aligned}$ | $\begin{aligned} & 4.028 \\ & 4.025 \end{aligned}$ | $\begin{aligned} & 0.340 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 0.344 \\ & 0.344 \end{aligned}$ | $\begin{aligned} & 0.669 \\ & 0.669 \end{aligned}$ |
| 65 | NE Entropy | $\begin{aligned} & 1.994 \\ & 1.992 \end{aligned}$ | $\begin{aligned} & 1.986 \\ & 1.985 \end{aligned}$ | $\begin{aligned} & 3.980 \\ & 3.977 \end{aligned}$ | $\begin{aligned} & 0.315 \\ & 0.315 \end{aligned}$ | $\begin{aligned} & 0.354 \\ & 0.354 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.643 \\ & 0.643 \end{aligned}$ |
| 66 | NE Entropy | $\begin{aligned} & 2.415 \\ & 2.016 \end{aligned}$ | $\begin{aligned} & 3.178 \\ & 1.982 \end{aligned}$ | $\begin{aligned} & 5.594 \\ & 3.998 \end{aligned}$ | $\begin{aligned} & 0.644 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 0.435 \\ & 0.338 \end{aligned}$ | $\begin{aligned} & 1.073 \\ & 0.692 \end{aligned}$ |
| 67 | NE Entropy | $\begin{aligned} & 2.557 \\ & 2.015 \end{aligned}$ | $\begin{aligned} & 2.529 \\ & 1.999 \end{aligned}$ | $\begin{aligned} & 5.086 \\ & 4.014 \end{aligned}$ | $\begin{aligned} & 0.371 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 0.379 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 0.763 \\ & 0.685 \end{aligned}$ |
| 68 | NE <br> Entropy | $\begin{aligned} & 3.193 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 2.420 \\ & 2.008 \end{aligned}$ | $\begin{aligned} & 5.614 \\ & 4.001 \end{aligned}$ | $\begin{aligned} & 0.434 \\ & 0.335 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.604 \\ & 0.316 \end{aligned}$ | $\begin{aligned} & 1.057 \\ & 0.655 \\ & \hline \end{aligned}$ |
| 69 | NE <br> Entropy | $\begin{aligned} & 2.522 \\ & 1.984 \end{aligned}$ | $\begin{aligned} & 2.562 \\ & 2.021 \end{aligned}$ | $\begin{aligned} & 5.084 \\ & 4.005 \end{aligned}$ | $\begin{aligned} & 0.379 \\ & 0.329 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.356 \\ & 0.317 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.734 \\ & 0.657 \end{aligned}$ |
| 70 | NE Entropy | $\begin{aligned} & 3.195 \\ & 1.989 \end{aligned}$ | $\begin{aligned} & 3.214 \\ & 2.000 \end{aligned}$ | $\begin{aligned} & 6.409 \\ & 3.989 \end{aligned}$ | $\begin{aligned} & 0.435 \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 0.425 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.885 \\ & 0.658 \end{aligned}$ |


| 71 | NE | 2.427 | 3.204 | 5.631 | 0.633 | 0.400 | 1.079 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Entropy | 2.018 | 2.000 | 4.018 | 0.336 | 0.313 | 0.656 |
| 72 | NE | 2.007 | 1.997 | 4.004 | 0.340 | 0.323 | 0.638 |
|  | Entropy | 2.006 | 1.996 | 4.002 | 0.340 | 0.323 | 0.639 |
| 73 | NE | 3.196 | 2.382 | 5.578 | 0.424 | 0.659 | 1.057 |
|  | Entropy | 1.996 | 1.989 | 46 | 3.984 | 0.330 | 0.342 |
| 74 | NE | 2.536 | 2.550 | 5.086 | 0.373 | 0.361 | 0.731 |
|  | Entropy | 2.003 | 2.015 | 4.017 | 0.329 | 0.328 | 0.666 |
| 75 | NE | 3.189 | 1.618 | 4.807 | 0.439 | 0.625 | 1.036 |
|  | Entropy | 1.983 | 2.007 | 3.990 | 0.337 | 0.316 | 0.640 |


| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 76 | NE Entropy | $\begin{aligned} & 2.532 \\ & 2.001 \end{aligned}$ | $\begin{aligned} & 2.277 \\ & 2.005 \end{aligned}$ | $\begin{aligned} & 4.809 \\ & 4.006 \end{aligned}$ | $\begin{aligned} & 0.380 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 0.340 \\ & 0.321 \end{aligned}$ | $\begin{aligned} & 0.723 \\ & 0.658 \end{aligned}$ |
| 77 | NE Entropy | $\begin{aligned} & 3.212 \\ & 2.002 \\ & \hline \end{aligned}$ | $\begin{aligned} & 3.197 \\ & 1.995 \\ & \hline \end{aligned}$ | $\begin{aligned} & 6.409 \\ & 3.996 \end{aligned}$ | $\begin{aligned} & 0.429 \\ & 0.332 \end{aligned}$ | $\begin{aligned} & 0.426 \\ & 0.332 \end{aligned}$ | $\begin{aligned} & 0.869 \\ & 0.657 \end{aligned}$ |
| 78 | NE Entropy | $\begin{aligned} & 2.504 \\ & 1.968 \end{aligned}$ | $\begin{aligned} & 2.537 \\ & 2.001 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.041 \\ & 3.969 \end{aligned}$ | $\begin{aligned} & 0.379 \\ & 0.334 \end{aligned}$ | $\begin{aligned} & 0.378 \\ & 0.332 \end{aligned}$ | $\begin{aligned} & 0.756 \\ & 0.669 \end{aligned}$ |
| 79 | NE <br> Entropy | $\begin{aligned} & 2.427 \\ & 2.011 \end{aligned}$ | $\begin{aligned} & 3.198 \\ & 2.003 \end{aligned}$ | $\begin{aligned} & 5.625 \\ & 4.014 \end{aligned}$ | $\begin{aligned} & 0.637 \\ & 0.325 \end{aligned}$ | $\begin{aligned} & 0.413 \\ & 0.319 \end{aligned}$ | $\begin{aligned} & 1.054 \\ & 0.649 \end{aligned}$ |
| 80 | NE Entropy | $\begin{aligned} & 2.412 \\ & 2.025 \end{aligned}$ | $\begin{aligned} & 3.201 \\ & 1.999 \end{aligned}$ | $\begin{aligned} & 5.613 \\ & 4.024 \end{aligned}$ | $\begin{aligned} & 0.633 \\ & 0.328 \end{aligned}$ | $\begin{aligned} & 0.422 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.050 \\ & 0.651 \end{aligned}$ |
| 81 | NE Entropy | $\begin{aligned} & 3.196 \\ & 1.986 \end{aligned}$ | $\begin{aligned} & 2.408 \\ & 2.006 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.605 \\ & 3.992 \end{aligned}$ | $\begin{aligned} & 0.433 \\ & 0.332 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.637 \\ & 0.337 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.074 \\ & 0.676 \end{aligned}$ |
| 82 | NE <br> Entropy | $\begin{aligned} & 3.190 \\ & 2.009 \end{aligned}$ | $\begin{aligned} & 3.178 \\ & 1.990 \end{aligned}$ | $\begin{aligned} & 6.368 \\ & 3.999 \end{aligned}$ | $\begin{aligned} & 0.433 \\ & 0.345 \end{aligned}$ | $\begin{aligned} & 0.454 \\ & 0.344 \end{aligned}$ | $\begin{aligned} & 0.877 \\ & 0.684 \end{aligned}$ |
| 83 | NE Entropy | $\begin{aligned} & 3.213 \\ & 2.021 \end{aligned}$ | $\begin{aligned} & 3.205 \\ & 1.992 \end{aligned}$ | $\begin{aligned} & 6.419 \\ & 4.013 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.449 \\ & 0.349 \end{aligned}$ | $\begin{aligned} & 0.426 \\ & 0.326 \end{aligned}$ | $\begin{aligned} & 0.867 \\ & 0.678 \end{aligned}$ |
| 84 | NE Entropy | $\begin{aligned} & 2.398 \\ & 2.001 \end{aligned}$ | $\begin{aligned} & 2.383 \\ & 1.988 \end{aligned}$ | $\begin{aligned} & 4.780 \\ & 3.988 \end{aligned}$ | $\begin{aligned} & 0.633 \\ & 0.329 \end{aligned}$ | $\begin{aligned} & 0.642 \\ & 0.339 \end{aligned}$ | $\begin{aligned} & 1.307 \\ & 0.686 \end{aligned}$ |
| 85 | NE Entropy | $\begin{aligned} & 3.188 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 3.198 \\ & 2.000 \\ & \hline \end{aligned}$ | $\begin{aligned} & 6.386 \\ & 3.994 \end{aligned}$ | $\begin{aligned} & 0.430 \\ & 0.326 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.433 \\ & 0.327 \end{aligned}$ | $\begin{aligned} & 0.885 \\ & 0.653 \end{aligned}$ |
| 86 | NE <br> Entropy | $\begin{aligned} & 2.437 \\ & 2.022 \end{aligned}$ | $\begin{aligned} & 3.180 \\ & 1.974 \end{aligned}$ | $\begin{aligned} & 5.617 \\ & 3.997 \end{aligned}$ | $\begin{aligned} & 0.649 \\ & 0.336 \end{aligned}$ | $\begin{aligned} & 0.450 \\ & 0.337 \end{aligned}$ | $\begin{aligned} & 1.057 \\ & 0.664 \end{aligned}$ |
| 87 | NE <br> Entropy | $\begin{aligned} & 2.399 \\ & 2.003 \end{aligned}$ | $\begin{aligned} & 3.202 \\ & 2.002 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.602 \\ & 4.005 \end{aligned}$ | $\begin{aligned} & 0.610 \\ & 0.327 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.424 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.015 \\ & 0.646 \\ & \hline \end{aligned}$ |
| 88 | NE Entropy | $\begin{aligned} & 2.387 \\ & 1.994 \end{aligned}$ | $\begin{aligned} & 1.621 \\ & 2.012 \end{aligned}$ | $\begin{aligned} & 4.008 \\ & 4.006 \end{aligned}$ | $\begin{aligned} & 0.632 \\ & 0.324 \end{aligned}$ | $\begin{aligned} & 0.636 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.257 \\ & 0.661 \end{aligned}$ |
| 89 | NE Entropy | $\begin{aligned} & 2.417 \\ & 2.011 \end{aligned}$ | $\begin{aligned} & 2.420 \\ & 2.007 \end{aligned}$ | $\begin{aligned} & 4.837 \\ & 4.019 \end{aligned}$ | $\begin{aligned} & 0.630 \\ & 0.325 \end{aligned}$ | $\begin{aligned} & 0.649 \\ & 0.340 \end{aligned}$ | $\begin{aligned} & 1.335 \\ & 0.688 \end{aligned}$ |
| 90 | NE Entropy | $\begin{aligned} & 3.194 \\ & 1.988 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.388 \\ & 2.000 \\ & \hline \end{aligned}$ | $\begin{aligned} & 5.582 \\ & 3.988 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.443 \\ & 0.335 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.633 \\ & 0.338 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.120 \\ & 0.696 \\ & \hline \end{aligned}$ |
| 91 | NE <br> Entropy | $\begin{aligned} & 3.211 \\ & 2.003 \end{aligned}$ | $\begin{aligned} & 1.624 \\ & 2.020 \end{aligned}$ | $\begin{aligned} & 4.835 \\ & 4.023 \end{aligned}$ | $\begin{aligned} & 0.424 \\ & 0.321 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.644 \\ & 0.333 \end{aligned}$ | $\begin{aligned} & 1.092 \\ & 0.666 \\ & \hline \end{aligned}$ |
| 92 | NE Entropy | $\begin{aligned} & 3.178 \\ & 1.979 \end{aligned}$ | $\begin{aligned} & 3.194 \\ & 1.990 \end{aligned}$ | $\begin{aligned} & 6.372 \\ & 3.969 \end{aligned}$ | $\begin{aligned} & 0.460 \\ & 0.339 \end{aligned}$ | $\begin{aligned} & 0.419 \\ & 0.322 \end{aligned}$ | $\begin{aligned} & 0.905 \\ & 0.664 \end{aligned}$ |
| 93 | NE <br> Entropy | $\begin{aligned} & 3.211 \\ & 2.000 \end{aligned}$ | $\begin{aligned} & 3.217 \\ & 2.028 \end{aligned}$ | $\begin{aligned} & 6.428 \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 0.400 \\ & 0.318 \end{aligned}$ | $\begin{aligned} & 0.427 \\ & 0.335 \end{aligned}$ | $\begin{aligned} & 0.832 \\ & 0.660 \end{aligned}$ |
| 94 | NE Entropy | $\begin{aligned} & 2.400 \\ & 1.993 \end{aligned}$ | $\begin{aligned} & 3.198 \\ & 2.004 \end{aligned}$ | $\begin{aligned} & 5.598 \\ & 3.997 \end{aligned}$ | $\begin{aligned} & 0.647 \\ & 0.335 \end{aligned}$ | $\begin{aligned} & 0.425 \\ & 0.341 \end{aligned}$ | $\begin{aligned} & 1.077 \\ & 0.683 \end{aligned}$ |
| 95 | NE <br> Entropy | $\begin{aligned} & 3.186 \\ & 1.980 \end{aligned}$ | $\begin{aligned} & 2.417 \\ & 2.010 \end{aligned}$ | $\begin{aligned} & 5.602 \\ & 3.991 \end{aligned}$ | $\begin{aligned} & 0.422 \\ & 0.329 \end{aligned}$ | $\begin{aligned} & 0.651 \\ & 0.342 \end{aligned}$ | $\begin{aligned} & 1.058 \\ & 0.654 \end{aligned}$ |


| 96 | NE | 2.393 | 3.168 | 5.561 | 0.624 | 0.443 | 1.067 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Entropy | 1.985 | 1.970 | 3.956 | 0.330 | 0.341 | 0.646 |
| 97 | NE | 3.209 | 1.581 | 4.790 | 0.416 | 0.622 | 1.052 |
|  | Entropy | 2.007 | 1.980 | 3.988 | 0.328 | 0.322 | 0.656 |
| 98 | NE | 3.193 | 2.404 | 5.597 | 0.422 | 0.612 | 1.024 |
|  | Entropy | 2.002 | 2.007 | 47 | 4.009 | 0.336 | 0.324 |
| 99 | NE | 2.391 | 2.423 | 4.814 | 0.651 | 0.623 | 1.260 |
|  | Entropy | 1.994 | 2.018 | 4.012 | 0.337 | 0.328 | 0.660 |
| 100 | NE | 2.372 | 1.602 | 3.974 | 0.663 | 0.651 | 1.308 |
|  | Entropy | 1.984 | 1.993 | 3.977 | 0.344 | 0.337 | 0.677 |


| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 101 | NE | 1.594 | 3.206 | 4.800 | 0.633 | 0.422 | 1.038 |
|  | Entropy | 1.997 | 1.995 | 3.992 | 0.331 | 0.330 | 0.676 |
| 102 | NE | 3.209 | 3.201 | 6.411 | 0.451 | 0.437 | 0.896 |
|  | Entropy | 2.025 | 2.011 | 4.036 | 0.350 | 0.344 | 0.685 |
| 103 | NE | 1.603 | 3.176 | 4.779 | 0.658 | 0.419 | 1.084 |
|  | Entropy | 2.002 | 1.998 | 4.000 | 0.346 | 0.323 | 0.661 |
| 104 | NE | 3.206 | 2.372 | 5.578 | 0.425 | 0.660 | 1.116 |
|  | Entropy | 2.010 | 1.989 | 3.999 | 0.327 | 0.345 | 0.677 |
| 105 | NE | 1.568 | 2.401 | 3.969 | 0.617 | 0.628 | 1.285 |
|  | Entropy | 1.974 | 2.022 | 3.997 | 0.320 | 0.329 | 0.660 |
| 106 | NE | 3.214 | 3.215 | 6.430 | 0.422 | 0.431 | 0.853 |
|  | Entropy | 2.015 | 2.009 | 4.024 | 0.338 | 0.337 | 0.681 |
| 107 | NE | 3.186 | 3.228 | 6.414 | 0.431 | 0.426 | 0.860 |
|  | Entropy | 1.990 | 2.019 | 4.009 | 0.329 | 0.335 | 0.696 |
| 108 | NE | 1.618 | 3.206 | 4.824 | 0.636 | 0.411 | 1.031 |
|  | Entropy | 2.013 | 1.993 | 4.006 | 0.338 | 0.316 | 0.665 |
| 109 | NE | 3.205 | 2.405 | 5.610 | 0.427 | 0.651 | 1.053 |
|  | Entropy | 2.003 | 2.012 | 4.016 | 0.332 | 0.339 | 0.649 |
| 110 | NE | 1.571 | 2.403 | 3.974 | 0.671 | 0.645 | 1.268 |
|  | Entropy | 1.989 | 1.991 | 3.980 | 0.351 | 0.327 | 0.652 |
| 111 | NE | 3.226 | 3.205 | 6.431 | 0.410 | 0.430 | 0.834 |
|  | Entropy | 2.019 | 1.987 | 4.006 | 0.325 | 0.333 | 0.656 |
| 112 | NE | 1.612 | 3.215 | 4.827 | 0.633 | 0.420 | 0.995 |
|  | Entropy | 2.005 | 2.021 | 4.026 | 0.333 | 0.333 | 0.637 |
| 113 | NE | 3.192 | 3.206 | 6.398 | 0.436 | 0.423 | 0.827 |
|  | Entropy | 1.990 | 1.994 | 3.984 | 0.335 | 0.335 | 0.664 |
| 114 | NE | 1.611 | 3.209 | 4.820 | 0.641 | 0.411 | 1.040 |
|  | Entropy | 1.996 | 2.006 | 4.002 | 0.349 | 0.334 | 0.677 |
| 115 | NE | 3.209 | 1.630 | 4.839 | 0.420 | 0.623 | 1.041 |
|  | Entropy | 2.001 | 2.027 | 4.028 | 0.325 | 0.332 | 0.654 |
| 116 | NE | 1.600 | 1.597 | 3.197 | 0.651 | 0.637 | 1.252 |
|  | Entropy | 1.998 | 1.991 | 3.989 | 0.336 | 0.340 | 0.664 |
| 117 | NE | 3.212 | 2.388 | 5.600 | 0.400 | 0.652 | 1.089 |
|  | Entropy | 1.999 | 2.000 | 3.998 | 0.316 | 0.342 | 0.676 |
| 118 | NE | 1.628 | 2.409 | 4.037 | 0.639 | 0.640 | 1.278 |
|  | Entropy | 2.017 | 2.002 | 4.018 | 0.344 | 0.338 | 0.700 |
| 119 | NE | 3.181 | 2.389 | 5.570 | 0.452 | 0.649 | 1.115 |
|  | Entropy | 1.975 | 1.996 | 3.971 | 0.344 | 0.345 | 0.688 |
| 120 | NE | 1.617 | 2.405 | 4.022 | 0.673 | 0.622 | 1.301 |
|  | Entropy | 2.004 | 2.007 | 4.012 | 0.341 | 0.325 | 0.664 |


| 121 | NE | 3.195 | 1.553 | 4.748 | 0.434 | 0.599 | 1.078 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Entropy | 1.993 | 1.968 | 3.961 | 0.323 | 0.327 | 0.662 |
| 122 | NE | 2.552 | 2.254 | 4.806 | 0.382 | 0.350 | 0.719 |
|  | Entropy | 2.014 | 1.990 | 4.004 | 0.344 | 0.331 | 0.658 |
| 123 | NE | 2.359 | 2.395 | 4.754 | 0.631 | 0.649 | 1.291 |
|  | Entropy | 1.969 | 1.996 | 48 | 3.965 | 0.325 | 0.336 |
| 124 | NE | 2.532 | 2.539 | 5.071 | 0.369 | 0.378 | 0.754 |
|  | Entropy | 2.002 | 2.009 | 4.011 | 0.332 | 0.335 | 0.680 |
| 125 | NE | 2.384 | 3.195 | 5.579 | 0.624 | 0.422 | 1.032 |
|  | Entropy | 1.987 | 2.007 | 3.994 | 0.320 | 0.335 | 0.634 |


| Game\# | Type | Row mean | Column mean | Sum mean | Row var | Column var | Sum var |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 126 | NE | 2.003 | 2.000 | 4.003 | 0.339 | 0.335 | 0.688 |
|  | Entropy | 2.002 | 1.999 | 4.000 | 0.339 | 0.335 | 0.688 |
| 127 | NE | 2.408 | 2.426 | 4.834 | 0.654 | 0.641 | 1.284 |
|  | Entropy | 2.011 | 2.016 | 4.027 | 0.342 | 0.332 | 0.666 |
| 128 | NE | 2.383 | 3.202 | 5.585 | 0.672 | 0.410 | 1.032 |
|  | Entropy | 1.970 | 2.008 | 3.978 | 0.342 | 0.330 | 0.656 |
| 129 | NE | 2.537 | 2.551 | 5.088 | 0.382 | 0.375 | 0.774 |
|  | Entropy | 2.001 | 2.022 | 4.023 | 0.345 | 0.335 | 0.695 |
| 130 | NE | 2.002 | 2.020 | 4.022 | 0.333 | 0.337 | 0.672 |
|  | Entropy | 2.000 | 2.019 | 4.019 | 0.333 | 0.337 | 0.672 |
| 131 | NE | 3.218 | 3.205 | 6.423 | 0.414 | 0.426 | 0.817 |
|  | Entropy | 2.017 | 1.991 | 4.008 | 0.335 | 0.325 | 0.671 |
| 132 | NE | 3.189 | 3.200 | 6.389 | 0.435 | 0.428 | 0.870 |
|  | Entropy | 1.999 | 1.997 | 3.996 | 0.338 | 0.332 | 0.679 |
| 133 | NE | 2.545 | 2.287 | 4.832 | 0.354 | 0.348 | 0.713 |
|  | Entropy | 2.007 | 2.020 | 4.027 | 0.322 | 0.327 | 0.658 |
| 134 | NE | 2.021 | 1.996 | 4.018 | 0.333 | 0.328 | 0.648 |
|  | Entropy | 2.020 | 1.995 | 4.015 | 0.333 | 0.328 | 0.649 |
| 135 | NE | 3.225 | 2.396 | 5.621 | 0.419 | 0.642 | 1.034 |
|  | Entropy | 2.014 | 1.997 | 4.011 | 0.327 | 0.337 | 0.634 |
| 136 | NE | 1.999 | 1.981 | 3.980 | 0.346 | 0.321 | 0.679 |
|  | Entropy | 1.998 | 1.979 | 3.977 | 0.346 | 0.321 | 0.679 |
| 137 | NE | 3.212 | 2.384 | 5.596 | 0.416 | 0.645 | 1.070 |
|  | Entropy | 2.002 | 1.994 | 3.995 | 0.320 | 0.343 | 0.657 |
| 138 | NE | 2.004 | 1.999 | 4.002 | 0.340 | 0.334 | 0.703 |
|  | Entropy | 2.002 | 1.997 | 3.999 | 0.340 | 0.334 | 0.703 |
| 139 | NE | 2.271 | 2.543 | 4.814 | 0.350 | 0.376 | 0.737 |
|  | Entropy | 2.003 | 1.998 | 4.001 | 0.327 | 0.333 | 0.667 |
| 140 | NE | 2.285 | 2.521 | 4.805 | 0.345 | 0.372 | 0.710 |
|  | Entropy | 2.017 | 1.998 | 4.015 | 0.331 | 0.330 | 0.651 |
| 141 | NE | 1.977 | 1.991 | 3.968 | 0.320 | 0.344 | 0.674 |
|  | Entropy | 1.975 | 1.990 | 3.965 | 0.320 | 0.344 | 0.674 |
| 142 | NE | 2.007 | 2.002 | 4.009 | 0.348 | 0.323 | 0.679 |
|  | Entropy | 2.005 | 2.001 | 4.006 | 0.348 | 0.323 | 0.679 |
| 143 | NE | 2.011 | 2.009 | 4.020 | 0.328 | 0.327 | 0.645 |
|  | Entropy | 2.009 | 2.008 | 4.017 | 0.328 | 0.327 | 0.645 |
| 144 | NE | 2.014 | 2.017 | 4.031 | 0.319 | 0.329 | 0.641 |
|  | Entropy | 2.013 | 2.016 | 4.028 | 0.319 | 0.329 | 0.642 |


[^0]:    ${ }^{1}$ Zurich Group Professor of Risk Mathematics, School of Business and Management, and Professor, Schwarzman Scholars Program, Tsinghua University.
    ${ }^{2}$ Seymour Knox Professor of Mathematical Institutional Economics (Emeritus), Yale University, and External Faculty, Santa Fe Institute.
    ${ }^{3}$ International Business School, Nankai University.

[^1]:    ${ }^{4}$ We do not concern ourselves with individual preferences directly, but allow for the possibility that each amount of money/gold is mapped onto individual preferences in some risk-averse manner.

[^2]:    ${ }^{5}$ We note that for $k=1$, the sample space of the random cardinal games is identical to the set of 144 canonical ordinal games.

[^3]:    ${ }^{7}$ Alternatively, one could consider an ordinal version of this game in which the payoffs -1 and 1 are replaced by the payoff ranks 1 and 2, respectively. In the ordinal game, using the mixed strategy $(x, y)=(1 / 2,1 / 2)$ gives each player the same opportunity to receive 1 or 2 .

[^4]:    ${ }^{8}$ Within the conventional Battle of the Sexes storyline, the man and woman who are trying to decide which movie to see could choose between "his" movie and "her" movie by tossing a coin (for which $p=1 / 2$ ).

