Yale University

EliScholar – A Digital Platform for Scholarly Publishing at Yale

Cowles Foundation Discussion Papers | Cowles Foundation

8-1-2015

First Price Auctions with General Information Structures: Implications for Bidding and Revenue

Dirk Bergemann

Benjamin Brooks

Stephen Morris

Follow this and additional works at: https://elischolar.library.yale.edu/cowles-discussion-paper-series

 Part of the Economics Commons

Recommended Citation

Bergemann, Dirk; Brooks, Benjamin; and Morris, Stephen, "First Price Auctions with General Information Structures: Implications for Bidding and Revenue" (2015). *Cowles Foundation Discussion Papers*. 2458.
https://elischolar.library.yale.edu/cowles-discussion-paper-series/2458

This Discussion Paper is brought to you for free and open access by the Cowles Foundation at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Cowles Foundation Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.

FIRST PRICE AUCTIONS WITH GENERAL INFORMATION STRUCTURES: IMPLICATIONS FOR BIDDING AND REVENUE

By

Dirk Bergemann, Benjamin Brooks, and Stephen Morris

August 2015
Revised May 2016

COWLES FOUNDATION DISCUSSION PAPER NO. 2018R2



COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

First-Price Auctions with General Information Structures: Implications for Bidding and Revenue*

Dirk Bergemann Benjamin Brooks Stephen Morris

May 6, 2016

Abstract

We explore the impact of private information in sealed-bid first-price auctions. For a given symmetric and arbitrarily correlated prior distribution over values, we characterize the lowest winning-bid distribution that can arise across all information structures and equilibria. The information and equilibrium attaining this minimum leave bidders indifferent between their equilibrium bids and all higher bids. Our results provide lower bounds for bids and revenue with asymmetric distributions over values.

We report further analytic and computational characterizations of revenue and bidder surplus including upper bounds on revenue. Our work has implications for the identification of value distributions from data on winning bids and for the informationally robust comparison of alternative bidding mechanisms.

KEYWORDS: First-price auction, information structure, Bayes correlated equilibrium, private values, interdependent values, common values, revenue, surplus, welfare bounds, reserve price.

JEL CLASSIFICATION: C72, D44, D82, D83.

*Bergemann: Department of Economics, Yale University, dirk.bergemann@yale.edu; Brooks: Becker Friedman Institute and Department of Economics, University of Chicago, babrooks@uchicago.edu; Morris: Department of Economics, Princeton University, smorris@princeton.edu. We acknowledge financial support through NSF Grant ICES 1215808. We also thank Ben Golub, Jason Hartline, Hanming Fang, Phil Reny, Vasilis Syrgkanis, Satoru Takahashi and Eva Tardos for helpful conversations. Aron Tobias, Elia Sartori, and Kai Hao Yang provided valuable research assistance. We are also grateful to seminar participants at Arizona, Berkeley, Duke, Harvard, Hebrew University, MIT, Northwestern, Queen Mary, Stanford, and Tel Aviv University.

1 Introduction

The first-price auction has been the subject of extensive theoretical study for over fifty years. It is still fair to say, however, that its properties are well understood only in relatively special cases. Under complete information, the first-price auction reduces to classic Bertrand competition. Under incomplete information, most work focuses on the case of one-dimensional type spaces. For example, when bidders know their private values, it is typically assumed that each bidder *only* knows their private value and has no additional source of information. Beyond the private-values case, it is typically assumed that bidders have one-dimensional types that are jointly affiliated and that values are increasing in the profile of types (Milgrom and Weber, 1982). Thus, a strong relationship is assumed between each bidder's belief about his own value and his beliefs about others' information. But in first-price auctions, unlike in second-price auctions, bidders' beliefs about others' information are of central strategic importance, since what others know is informative about what they will bid in equilibrium. In many situations, it is unnatural to impose strong restrictions on the relationship between the conceptually distinct beliefs about one's own value and about others' information.

In this paper, we derive results about equilibrium behavior in the first-price auction that hold across *all* common prior information structures. For a given prior joint distribution over value profiles, we study what can happen for all information structures specifying bidders' information about their own and others' values. Our setting thus incorporates all existing models of information.[1] For any such value distribution, we identify a lower bound on the distribution of winning bids in the sense of first order stochastic dominance. In other words, no matter what the true information structure is, the distribution of winning bids must first-order stochastically dominate the bound that we describe. In addition, when the prior distribution of values is symmetric, we construct an equilibrium and an information structure in which this lower bound is attained. This minimum winning-bid distribution therefore pins down the minimum amount of revenue that can be generated by the auction in expectation. Moreover, the minimum winning-bid distribution is attained in an efficient equilibrium. As a result, this equilibrium also attains an upper bound on the expected surplus of the bidders, which is equal to the maximum feasible surplus minus minimum revenue.[2]

Let us give a brief intuition for how our bounds are obtained. If the distribution of winning bids places too high of a probability on low bids, then some bidder would find that a modest

[1] We will discuss what happens with asymmetric value distributions in the body of the paper. Some results extend as stated to the asymmetric value distributions, and for the others, we will report weaker analogue results.

[2] Where no confusion results, we will write "revenue" for ex-ante expected revenue, "bidder surplus" for ex-ante expected surplus, etc.

increase in their bid would result in a relatively large increase in the probability of winning, so that such a deviation would be attractive. For example, it cannot be that all bidders bid zero with probability one, for then some bidder could increase his bid at arbitrarily small cost, and win all the time. This suggests that the relevant constraints for pinning down minimum bidding are those associated with deviating to higher bids. Indeed, we show that the minimum winning-bid distribution is characterized by bidders being indifferent to *all* upward deviations.

In fact, to characterize the minimum, it turns out to be sufficient to look at a relatively small class of such deviations: For some bid b, we say that a bidder *uniformly deviates up to* b if, whenever he would have bid less than b in equilibrium, he switches to bidding b. It is clearly necessary for equilibrium that the bidders should not want to uniformly deviate upward. Moreover, it turns out that the change in a bidder's surplus from a uniform upward deviation depends only on the distribution of winning bids, and not on losing bids. This motivates a relaxed program in which we minimize the distribution of winning bids, subject only to the uniform upward incentive constraints. The solution to this relaxed program gives us a lower bound on the winning-bid distribution, and with the additional assumption of symmetry of the prior, we subsequently construct an information structure and equilibrium in which this lower bound is attained, thus verifying that it is indeed the minimum. We will further motivate and illustrate this proof strategy in Section 3 with an example in which there are two bidders and a uniformly distributed common value.

We report a number of further results about bidding, revenue, and bidder surplus. A straightforward upper bound on revenue is the efficient surplus, and we show that this bound is in fact tight. We also explore the whole welfare space of possible revenue and bidder-surplus outcomes, including those associated with inefficient equilibria. In the case of two bidders and independent values, we construct a maximally inefficient information structure and equilibrium strategy profile. In the resulting equilibrium, the bidders receive zero surplus, and the seller's revenue is the expectation of the lower of the two of values.

Our primary focus in this paper is on developing insights about how general information structures can affect outcomes in the first-price auction and on the qualitative properties of the information structure that lead to different outcomes. The results we obtain can be used for a variety of applications, e.g., to partially identify the value distribution in settings where the information structure is unknown and to make informationally robust comparisons of mechanisms. We will discuss such applications in the concluding section.

Our work relates to a large literature on first-price auctions. The theoretical study of the first-price auction, going back to Vickrey (1961), has focused almost exclusively on the case where the bidders' information is one-dimensional. A distinctive feature of our analysis

is that we allow for multidimensional information about bidders' values. Fang and Morris (2006) and Azacis and Vida (2015) analyze a model with two bidders and two possible values, where each bidder knows his own value and observes a conditionally independent signal of the other bidder's value.[3] Another distinctive feature of our analysis is that we characterize bidding behavior in all equilibria for all information structures at once. Bergemann and Morris (2013, 2016) show that the range of such behavior can be described using a certain incomplete-information correlated equilibrium that they term *Bayes correlated equilibrium*. Although we do not explicitly use that solution concept in our analysis, a contribution of this paper is to characterize the Bayes correlated equilibria of the first-price auction. Others have also studied the first-price auction under weaker solution concepts than Bayes Nash equilibrium. Motivated by collusion, Lopomo, Marx, and Sun (2011) study communication equilibria of the first-price auction when values are known and independent; they thus impose truth-telling constraints that do not arise in our setting. Battigalli and Siniscalchi (2003) and Dekel and Wolinsky (2003) study rationalizable behavior in the first-price auction with a fixed affiliated information structure: our analysis is both more permissive, in that we consider all information structures, and also more restrictive, in that we use equilibrium rather than rationalizability as the solution concept.

The rest of this paper proceeds as follows. Section 2 presents our model of a first-price auction. Section 3 previews our argument with a two bidder example with a uniformly distributed pure common value. Section 4 contains our main result, a characterization of minimum winning bids, minimum revenue, and maximum bidder surplus. Section 5 describes further results on welfare outlined above, and Section 6 concludes with a discussion of applications to identification and the robust comparison of selling mechanisms. Omitted proofs are contained in the Appendix.

2 Model

We consider the sale of a single unit of a good by a first-price auction. There are N individuals who bid for the good, indexed by $i \in \mathcal{N} = \{1, \ldots, N\}$, each of whom has a value which lies in the compact interval $V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+ = [0, \infty)$. The bidders are assumed to be risk neutral and to have quasilinear preferences over the allocation and payments. Values are jointly distributed according to a symmetric probability measure $\mu \in \Delta\left(V^N\right)$.[4]

[3] In an earlier version of this paper (Bergemann, Brooks, and Morris, 2013), we provide a complete analytic characterization of possible welfare outcomes in the setting of Fang and Morris (2006) and Azacis and Vida (2015).

[4] All sets considered in this paper are regarded as topological spaces with their standard topologies, wherever applicable, and endowed with the Borel σ-algebra. For a topological space X, $\Delta(X)$ denotes the

We allow the symmetric (i.e., excheangable) common prior μ to be a general measure (and not necessarily a density) in order to encompass the pure-common-value case in the same analytic framework as non-common values. We will give a formal definition of symmetry in Section 4 when the hypothesis is used (cf. the discussion before Lemma 3). As we shall see, the distribution of the sum of the $N-1$ lowest values will be essential to our subsequent analysis. For the purpose of analytical convenience we will assume that this distribution is non-atomic, so that its associated cumulative distribution function is continuous. Specifically, we assume that for each $x \in \mathbb{R}$, the event in which the $N-1$ lowest values sum to x has zero probability. In the pure-common-value case, this simply means that the distribution of the common value is non-atomic. This assumption can be dropped without any great conceptual difficulty, and the working paper Bergemann, Brooks, and Morris (2015) solves the present model when values are discrete.

Each individual $i \in \mathcal{N}$ submits a bid $b_i \in B = [0, \overline{v}]$, and the winner is selected randomly from among the high bidders. For a profile of bids $b \in B^N$, let $W(b)$ be the set of high bidders,

$$W(b) = \{i \,|b_i \geq b_j, \, \forall j = 1, \ldots, N\}.$$

Letting $\mathbb{I}_E$ denote the indicator function for an event E, the probability that bidder i receives the good if bids are $b \in B^N$ is

$$q_i(b) = \frac{\mathbb{I}_{i \in W(b)}}{|W(b)|}.$$

Bidders may receive additional information about the profile of values, beyond knowing the prior distribution. This information comes in the form of signals that are correlated with the profile of values. An *information structure* is a collection $\mathcal{S} = \left(\{S_i\}_{i=1}^N, \pi\right)$, where the S_i are measurable spaces[5] and $\pi : V^N \to \Delta(S)$ is a measurable mapping from profiles of values to probability measures over $S = \times_{i=1}^N S_i$. The interpretation is that S_i is the set of bidder i's signals and that π describes the conditional joint distribution of signals given values.

For a fixed information structure $\mathcal{S}$, the first-price auction is a game of incomplete information, in which bidders' strategies are measurable mappings $\sigma_i : S_i \to \Delta(B)$ from signals to probability measures over bids. Let Σ_i denote the set of strategies for agent i. Fixing a profile of strategies $\sigma \in \Sigma = \times_{i=1}^N \Sigma_i$, bidder i's (ex-ante) surplus from the auction is

$$U_i(\mathcal{S}, \sigma) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} (v_i - b_i) \, q_i(b) \, \sigma(db|s) \, \pi(ds|v) \, \mu(dv),$$

set of Borel probability measures on X, endowed with the weak-* topology. For a measure $\mu \in \Delta(X)$ and a measurable function $f : X \to \mathbb{R}$, we denote the integral of f with respect to μ by $\int_{x \in X} f(x) \, \mu(dx)$.

[5]Note that the signal spaces can be arbitrary measurable spaces, and do not need to have any topological structure.

where $\sigma : S \to \Delta\left(B^N\right)$ maps a profile of signals $s \in S$ into the product measure $\sigma_1(s_1) \times \cdots \times \sigma_N(s_N)$. The profile $\sigma \in \Sigma$ is a *Bayes Nash equilibrium*, or *equilibrium* for short, if and only if $U_i(\mathcal{S}, \sigma) \geq U_i(\mathcal{S}, \sigma_i', \sigma_{-i})$ for all i and all $\sigma_i' \in \Sigma_i$. In the event that σ_i is a pure strategy, we will abuse notation slightly by writing $\sigma_i(s_i)$ for the bid that is made with probability one.

We denote the total surplus by

$$T(\mathcal{S}, \sigma) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} v_i q_i(b)\, \sigma(db|s)\, \pi(ds|v)\, \mu(dv).$$

This quantity is bounded above by the efficient total surplus:

$$\overline{T} = \int_{v \in V^N} \max v \mu(dv).$$

We will say that an equilibrium is *efficient* if $T(\mathcal{S}, \sigma) = \overline{T}$.

Throughout our analysis, we restrict attention to strategies in which each bidder does not make weakly dominated bids which he is sure are strictly greater than his value. This requirement is slightly weaker than weak dominance, since we allow bidders to bid their own value with positive probability. Ruling out such bids would lead to non-existence of equilibrium in many complete information specifications.

3 A Pure-Common-Value Example

Before giving our general results, we will illustrate where we are headed with a simple example. There are two bidders who share a common value for the good, which is uniformly distributed between 0 and 1. Since we assume there is no reservation price in the auction, the good is always allocated, regardless of the particular information structure and equilibrium, and as both bidders have the same value, all equilibria are socially efficient and result in a total surplus of 1/2. There may, however, be variation across information structures and equilibria in how this surplus is split between the bidders and the seller.

We allow bidders to observe arbitrary and possibly correlated signals about the common value. Indeed, at one extreme the bidders' signals are perfectly correlated, so that they have exactly the same information about the value. For example, the bidders might have no information beyond the prior, so that they both expect the good to be worth 1/2, or the bidders might both observe the true value of the good, so that they know the good's value exactly. In either case, the bidders will compete the price up to the interim expected value of the good, which results in zero bidder surplus and expected revenue of 1/2. These examples

illustrate our later general result that, unless we make additional assumptions about what the bidders know, a tight upper bound on revenue is the efficient surplus.

When the bidders have private information, however, the distribution of ex-ante surplus can be rather different. An important case has been studied by Engelbrecht-Wiggans, Milgrom, and Weber (1983): bidder 1 observes the true value while the bidder 2 is uninformed and observes nothing. In the case of the uniform distribution, the resulting equilibrium involves bidder 1 bidding $v/2$ and the uninformed bidder randomizing uniformly over the interval $[0, 1/2]$. Let us briefly verify that this is an equilibrium. If bidder 2 bids $b \in [0, 1/2]$, she will win whenever bidder 1 bids less than b, which is when the true value is less than $2b$. The conditional distribution of v in the event that bidder 2 wins is therefore uniform on $[0, 2b]$, so that the expected value of v conditional on bidder 2 winning is exactly b. Thus, any bid of bidder 2 results in at most zero surplus in expectation. On the other hand, conditional on the true value being v, bidder 1 wins with a bid of $b \in [0, 1/2]$ with probability $2b$, resulting in a surplus of $(v - b)\, 2b$, which is maximized at precisely $b = v/2$. This equilibrium results in a surplus of $1/6$ for bidder 1, a surplus of 0 for bidder 2, and revenue of only $1/3$.

But what can we say about the outcome of the auction more generally? A simple lower bound on revenue is zero, but this bound cannot be tight: revenue could be zero only if both bidders bid zero with probability one, in which case they must be each getting a surplus of $1/4$. However, either bidder could deviate up to a bid of $\epsilon > 0$ (independent of his signal) and obtain a surplus of $1/2 - \epsilon$. This suggests a general intuition that there cannot be too many bids too close to zero, lest the probability of winning increase too quickly as bidders deviate to higher bids. More generally, we might expect that the limit of how low bidding can go will be characterized by binding upward incentive constraints, i.e., bidders being indifferent to deviating to higher bids. Note that in the analysis of Engelbrecht-Wiggans et al. (1983), the informed bidder strictly prefers his equilibrium bid over any other bid. This suggests that it might be possible to construct other information structures in which revenue is even lower.

Let us provide one such construction. The two bidders will receive signals $s \in [0, 1]$ that are independent draws from the cumulative distribution $F(s) = \sqrt{s}$, so that the distribution of the maximum signal is standard uniform, the same as the common value. Moreover, we will correlate signals with values so that the maximum signal is always exactly the true common value:

$$v = \max\{s_1, s_2\}. \tag{1}$$

This information structure admits a monotonic pure-strategy equilibrium in which the bidders use the same strategies *as if* their signals were their true value. Under this as-if interpretation, the model is one of independent values (IPV), and it is well-known that the

equilibrium strategy is

$$\sigma(s) = \frac{1}{\sqrt{s}} \int_{x=0}^{s} x \frac{dx}{2\sqrt{x}} = \frac{s}{3},$$

which is the expectation of other bidder's signal, conditional on it being below s (Krishna, 2002). With this bidding function, the winning bid will always be $\max_i s_i/3 = v/3$, so that revenue is 1/6, and as the equilibrium is symmetric, each of the bidders obtains a surplus of 1/6 as well. Thus, this information structure doubles the total rents that the bidders receive relative to the proprietary information structure of Engelbrecht-Wiggans et al. (1983).[6]

Let us now verify that this is an equilibrium. A bidder who deviates by bidding $s'/3$ for some $s' < s$ will only win when their own signal was the highest signal, and therefore the value. Thus, a downward deviator's surplus looks exactly the same as in the as-if IPV setting, and we can immediately conclude that bidders do not want to deviate down. On the other hand, if a bidder deviates up to $s'/3$ with $s' > s$, the bidder continues to win on the event that they had the high signal, and now wins on some events when it was the other bidder who had the high signal, which was the true value. Surplus will be

$$\left(s - \frac{s'}{3}\right)\sqrt{s} + \int_{x=s}^{s'} \left(x - \frac{s'}{3}\right) \frac{1}{2\sqrt{x}} dx = \frac{2}{3} s\sqrt{s}$$

which is independent of s'. In other words, bidders are exactly indifferent to *all* upward deviations!

In fact, it turns out that no matter how one structures the information or the equilibrium strategies, it is impossible for revenue to fall below the level attained in this example, i.e., 1/6 is a tight lower bound on revenue when there are two bidders and there is a pure common value that is standard uniform. Moreover, not only is it impossible for revenue to fall below the level of the example, but the *distribution* of winning bids in any information structure and any equilibrium must always first order stochastically dominate the winning-bid distribution in the equilibrium we just constructed.

The full proof of this result will be established in Theorem 1 below. To develop intuition, we will give a partial proof that revenue cannot fall below 1/6. First, we notice that the equilibria in the no-information, complete-information, and independent-signal constructions all have the feature that the *winning bid* is a deterministic and weakly increasing function of the true value v. We denote the winning bid function by $\beta(v)$ and emphasize that the

[6] In the online appendix we compute the ratio of the revenue in the equilibrium under the information structure (1) and under the proprietary information structure of Engelbrecht-Wiggans et al. (1983) for the class of power distribution functions, $P(v) = v^\alpha$. We show that this ratio increases from 1 to 4 as α increases from 0 to ∞, and thus the uniform case is the special result for $\alpha = 1$ with a revenue ratio of 2.

winning bid $\beta(v)$ is an equilibrium outcome (as is the losing bid) and distinct from the bid strategy of the individual bidder $\sigma(s)$. In the no-information case, the winning bid is always $\beta(v) = 1/2$; under complete information, the winning bid is $\beta(v) = v$; and in the independent-signal construction, $\beta(v) = v/3$. Let us explore more generally what can happen in symmetric equilibria of this form. In other words, we implicitly assume that whatever the information structure and equilibrium strategies are, they induce an outcome in which (i) both players are equally likely to win at any given value v and (ii) the bidder who wins pays an amount that is a deterministic and increasing function $\beta(v)$ of the true value. For simplicity, we will also assume that ties occur with zero probability, and that β is strictly increasing. Notice that we are suppressing a great deal of information about the underlying information structure and equilibrium strategies that induce the winning bid. Nonetheless, the winning bid function β provides sufficient information to calculate revenue

$$R = \int_{v=0}^{1} \beta(v)\, dv$$

and each bidder's expected surplus

$$U_i = \frac{1}{2} \int_{v=0}^{1} (v - \beta(v))\, dv, \tag{2}$$

where the latter follows from symmetry.[7]

Now, an equilibrium in which the winning bid is $\beta(v)$ must deter a large number of deviations, most of which we cannot assess without explicitly modeling the rest of the equilibrium. There is, however, one class of deviations which we can evaluate using only information about winning bids: for some $w \in [0,1]$, bid $\beta(w)$ whenever the equilibrium bid would have been some $b \leq \beta(w)$. We refer to this as a *uniform deviation up to* $\beta(w)$.

We can calculate the surplus of a bidder who uniformly deviates up to $\beta(w)$ as follows. First consider the event in which the true value is less than w, so that the equilibrium winning bid would have been some number less than $\beta(w)$. In this case, the upward deviator *always wins*: the deviator always bids at least $\beta(w)$, which is more than what either the winner or the loser would have bid in equilibrium. Now consider the event in which the true value v is strictly greater than w. In this case, if the deviator were going to win in equilibrium, then he would have bid $\beta(v) > \beta(w)$, so that behavior is unaffected by the deviation and he still wins. On the other hand, if the deviator were going to lose, then the other player would

[7] To be clear, we are *not* making any particular assumption about the dimensionality of the signal space, nor are we assuming that bidders use pure strategies. The hypothesis that the winning bid only depends on the true value only requires that whichever bidder wins the auction be not randomizing given his information, but the bidders could play mixed strategies when they lose.

have bid $\beta(v) > \beta(w)$ and the deviator would have bid $b < \beta(v)$, so that the deviator still loses even after the upward deviation. Thus, the upward deviator's surplus is

$$\int_{v=0}^{w} (v - \beta(w))\, dv + \frac{1}{2}\int_{v=w}^{1} (v - \beta(v))\, dv,$$

and the equilibrium payoff given by (2) deters uniform upward deviations only if

$$\frac{1}{2}\int_{v=0}^{w} (v - \beta(w))\, dv \leq \frac{1}{2}\int_{v=0}^{w} (\beta(w) - \beta(v))\, dv$$

for all w. This condition rearranges to

$$\beta(w) \geq \frac{1}{2w}\int_{v=0}^{w} (v + \beta(v))\, dv. \tag{3}$$

A relaxation of the original problem of minimizing revenue over all information structures and equilibria (of this particular form) is to minimize revenue over all bidding functions that satisfy (3) and the condition that $\beta(w) \geq 0$. The solution to this relaxed program has a simple form. Suppose that at the optimum, the incentive constraint (3) holds as a strict inequality for a positive measure of w. Then we could construct a strictly lower winning bid function $\widetilde{\beta}$ that still deters uniform upward deviations, simply by *defining* $\widetilde{\beta}(w)$ to be equal to the right-hand side of (3). Moreover, inspection of the right-hand side of (3) reveals that decreasing β actually relaxes the constraint even further and makes all uniform upward deviations weakly less attractive. Decreasing β is therefore feasible, and since revenue is increasing in β, the modification must lower revenue.

At the optimum, (3) must therefore hold as an equality for almost all w, and hence it is without loss of generality to assume that it holds as an equality for all v. The unique solution at which this integral inequality holds everywhere as an equality with the initial condition $\beta(0) = 0$ is the minimum winning bid function $\underline{\beta}(v) = v/3$. This bidding function minimizes the distribution of winning bids in the sense of first order stochastic dominance, within the class of equilibria we considered and subject only to the incentive constraints associated with uniform upward deviations. In fact, we will subsequently show in Section 4 that this bound continues to hold even if one allows asymmetric equilibria and equilibria in which the winning bid is stochastic conditional on v. This implies that revenue cannot fall below 1/6 in any equilibrium under any information structure. Thus, the independent-signal information structure and its equilibrium attain a global lower bound on the distribution of winning bids.

4 Minimum Bidding, Minimum Revenue, and Maximum Bidder Surplus

4.1 Preview and Statement of Main Result

We shall see that the characterization of minimum winning bids can be generalized to *any* symmetric prior distribution over values and *any* number of bidders. By minimum, we mean that for any such prior, there is a minimum winning-bid distribution that is first order stochastically dominated by any distribution of equilibrium winning bids that can be induced by some information structure $\mathcal{S}$ and equilibrium σ. We will also construct an $\mathcal{S}$ and σ under which the generalized bound is attained. The qualitative features of the solution, and the methods used to characterize it, closely resemble the arguments in the uniform example of the previous section: the relevant incentive constraints that pin down the minimum are those corresponding to uniform upward deviations, and at the lower bound, the winner's bid will turn out to be a deterministic function of the profile of values.

As a segue to stating our main result, we briefly describe how the example of Section 3 generalizes to progressively broader classes of models. First, consider the case where the common value was drawn from an arbitrary continuous distribution P on the interval $[\underline{v}, \overline{v}]$. In this case, we will see that the minimum distribution over winning bids is characterized by the winning bid being the following function of the true value:[8]

$$\underline{\beta}(w) = \frac{1}{\sqrt{P(w)}} \int_{x=\underline{v}}^{w} x \frac{P(dx)}{2\sqrt{P(x)}}.$$

Thus, the winning bid when the true value is w is equal to the expectation of a random variable drawn from the cumulative distribution $\sqrt{P(x)/P(w)}$ on the range $[\underline{v}, w]$, which is necessarily strictly increasing on the support of P. This induces a minimum winning-bid distribution $\underline{H}(b)$ defined by:

$$\underline{H}(b) = P\left(\underline{\beta}^{-1}(b)\right).$$

where $\underline{\beta}^{-1}(b) = \sup\left\{w | \underline{\beta}(w) \leq b\right\}$. These formulae reduce to $\underline{\beta}(v) = v/3$ and $\underline{H}(b) = 3b$ when P is standard uniform.

Second, consider the case where there are two bidders whose values are drawn from a joint distribution $\mu(dv_1, dv_2)$. For this case, there continues to be a minimum winning bid function that pins down $\underline{H}$, but it is now a function of the *lower of the two values*. Thus, if we write $Q(w) = \mu(\{v | \min\{v_1, v_2\} \leq w\})$ for the cumulative distribution of the lowest

[8] Note that the denominator in the following expression, as in subsequent expressions, vanishes as w approaches its lower bound. A straightforward application of L'Hôpital's rule verifies that $\lim_{w \to \underline{v}} \underline{\beta}(w) = \underline{v}$.

value, then the minimum winning bid function is

$$\underline{\beta}(w) = \frac{1}{\sqrt{Q(w)}} \int_{x=\underline{v}}^{w} x \frac{Q(dx)}{2\sqrt{Q(x)}}, \tag{4}$$

and

$$\underline{H}(b) = Q\left(\underline{\beta}^{-1}(b)\right).$$

The intuition for this structure is as follows. The relevant constraints that pin down $\underline{H}$ continue to be those associated with uniform upward deviations, and when a player uniformly deviates upward, they continue to win whenever they would have won in equilibrium. This means that the change in surplus from such a deviation *only depends on the deviator's values when he loses in equilibrium.* As a result, the minimum winning bid only depends on the value of the player who loses. All else equal, lower losing values result in a weaker incentive to deviate up, so that the distribution of winning bids is minimized at an efficient allocation in which the losing value is the lowest value.

Finally, consider the general case with N bidders whose values are jointly distributed according to the probability measure $\mu(dv_1, \ldots, dv_N)$. $\underline{H}$ continues to be characterized by a winning bid that is a deterministic function of the losing bidders' values. Moreover, for the same reasons as with two bidders, $\underline{H}$ is attained with an efficient allocation. The remaining question is how the winning bid depends on the $N-1$ lowest values that lose the auction in equilibrium.

To develop intuition, let us reason by analogy with the benchmark of complete information, in which all bidders see the entire profile of values. The equilibria in this information structure involve the bidder with the highest value winning the good, and paying a price which is equal to the maximum of the $N-1$ lowest values. Now suppose that instead of knowing the entire profile of values, the bidders only learn (i) whether or not they have the highest value and (ii) the realized $N-1$ lowest values. Importantly, the low-value bidders do not learn which value they have, but only the empirical distribution of values among the $N-1$ lowest. Due to symmetry of μ, low-value bidders believe that they are equally likely to fall anywhere in that distribution, so they expect their value to be the *average* of the $N-1$ lowest:

$$\alpha(v) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \max v\right). \tag{5}$$

The equilibria in this information structure still involve the high-value bidder winning the auction. However, the winner now faces less intense competition from the losers: whereas under complete information the most optimistic losing bidder would compete the price up to

the second-highest value, now losing bidders are only willing to compete the price up to the average of the $N-1$ lowest values given by $\alpha(v)$. Revenue will therefore be substantially lower than under complete information.

Based on this intuition, we can guess that the generalized minimum winning bid will depend only on the average of the losing buyers' values. Let Q denote the distribution of $\alpha(v)$, and write $[\underline{w}, \overline{w}]$ for the convex hull of the support of Q. (Recall that we assumed in Section 2 that the events $\alpha^{-1}(x)$ have zero probability, so that $Q(x)$ is continuous.) Let

$$\underline{\beta}(w) = \frac{1}{Q^{\frac{N-1}{N}}(w)} \int_{x=\underline{w}}^{w} x \frac{N-1}{N} \frac{Q(dx)}{Q^{\frac{1}{N}}(x)}, \tag{6}$$

and let

$$\underline{H}(b) = Q\left(\underline{\beta}^{-1}(b)\right). \tag{7}$$

For a given information structure $\mathcal{S}$ and equilibrium σ, the winning-bid distribution is defined by:

$$H(b; \mathcal{S}, \sigma) = \int_{v \in V^N} \int_{s \in S} \sigma\left([0,b]^N \middle| s\right) \pi(ds|v)\, \mu(dv), \tag{8}$$

where $\sigma\left([0,b]^N \middle| s\right)$ is the conditional probability measure of bids between 0 and b given signal profile s. Our main result is the following:

Theorem 1 (Minimum Winning Bids)**.**

(i) For any information structure $\mathcal{S}$ and equilibrium σ, the distribution of winning bids $H(\mathcal{S}, \sigma)$ first-order stochastically dominates $\underline{H}$, i.e., $H(b; \mathcal{S}, \sigma) \leq \underline{H}(b)$ for all b;

(ii) There exists an information structure $\mathcal{S}$ and an efficient equilibrium σ such that the distribution of winning bids $H(\mathcal{S}, \sigma)$ is exactly equal to $\underline{H}$.

Immediate implications of Theorem 1 are characterizations of minimum revenue and maximum bidder surplus. Let $\underline{R}$ be defined by

$$\underline{R} = \int_{x=\underline{w}}^{\overline{w}} \underline{\beta}(x)\; Q(dx),$$

and similarly define the revenue from a given information structure $\mathcal{S}$ and equilibrium σ as

$$R(\mathcal{S}, \sigma) = \int_{b \in B} bH(db; \mathcal{S}, \sigma).$$

Corollary 1 (Minimum Revenue)**.**
Any equilibrium σ under any information structure $\mathcal{S}$ must result in revenue $R(\mathcal{S}, \sigma) \geq \underline{R}$.

Moreover, there exists an equilibrium σ for some information structure $\mathcal{S}$ in which $R(\mathcal{S},\sigma) = \underline{R}$.

Next, recall that $\overline{T}$ is the total surplus that is generated by the efficient allocation, and let the maximal bidder surplus be:

$$\overline{U} = \overline{T} - \underline{R}.$$

Corollary 2 (Maximum Bidder Surplus)**.**
Any equilibrium in any information structure must result in bidder surplus that is at most $\overline{U}$. Moreover, there exists an equilibrium for some information structure in which bidder surplus is exactly $\overline{U}$.

The rest of this section will be devoted to the proof of Theorem 1. The proof consists of two main pieces. We will first argue that (7) is a lower bound on the distribution of winning bids that can arise in equilibrium. In particular, we will show that (7) is the solution to a relaxed program in which we minimize the distribution of winning bids subject to only the uniform upward incentive constraints. The second piece of the proof is the construction of an equilibrium that exactly attains the solution to the relaxed program. This information structure and equilibrium turn out to be remarkably simple: the bidders receive one-dimensional signals and use a symmetric monotonic pure strategy which is equal to the minimum winning bid function.

4.2 The Relaxed Program

We first describe a relaxed program for minimizing the distribution of winning bids. The choice variables in the relaxed program will be distributions over the identity of the winner and the winning bid, conditional on the true profile of values. For an information structure $\mathcal{S}$ and equilibrium strategies σ, let

$$H_i(b|v;\mathcal{S},\sigma) = \int_{s\in S}\int_{x\in[0,b]^N} q_i(x)\,\sigma(dx|s)\,\pi(ds|v) \tag{9}$$

denote the probability that bidder i wins and the winning bid is less than b when the profile of values is v. Thus,

$$H(b|v;\mathcal{S},\sigma) = \sum_{i=1}^{N} H_i(b|v;\mathcal{S},\sigma)$$

denotes the total probability that the winning bid is less than b when the realized profile of values is v. The aggregate distribution of winning bids can be written as

$$H\left(b;\mathcal{S},\sigma\right)=\int_{v\in V^{N}}H\left(b|v;\mathcal{S},\sigma\right)\mu\left(dv\right).$$

We will hereafter suppress the dependence of these distributions on a particular $(\mathcal{S},\sigma)$ in order to keep our notation compact.

Just as β was sufficient to calculate revenue and bidder surplus in the example of Section 3, so too the winning-bid distributions $\{H_i\left(\cdot|\cdot\right)\}$ are sufficient to calculate welfare outcomes for the general model. In particular, revenue is simply

$$R=\int_{x\in B}xH\left(dx\right)$$

and bidder i's surplus is

$$U_i=\int_{v\in V^{N}}\int_{x\in B}\left(v_i-x\right)H_i\left(dx|v\right)\mu\left(dv\right).$$

Thus, bounds on the distribution of winning bids, in addition to the correlation structure between winning bids and values, immediately imply bounds on revenue and bidder surplus.

The $\{H_i\left(\cdot|\cdot\right)\}$ are a family of marginal distributions that are induced by the equilibrium, and they contain insufficient information to evaluate the merits of all potential deviations. Even so, we can use them to evaluate the class of uniform upward deviations that we introduced previously. In particular, bidders must not want to uniformly deviate up by bidding some fixed b whenever their equilibrium bid would have been some $x\leq b$. Let us in particular consider a b at which $H\left(b\right)$ does not have a mass point. Then by uniformly deviating up to b, the upward deviator will win whenever the equilibrium winning bid would have been a number strictly less than b. When the winning bid would have been greater than b, however, the upward deviator wins if and only if he would have won in equilibrium: if the deviator were going to lose, he would have lost to a bid greater than his equilibrium bid x and greater than b, so that he still loses with a bid of $\max\left\{x,b\right\}$, and if he were going to win, then he would have bid some amount $x>b$, which is unaffected by the deviation. The surplus from uniformly deviating up to b is therefore

$$\int_{v\in V^{N}}\left(\left(v_i-b\right)H\left(b|v\right)+\int_{x=b}^{\overline{v}}\left(v_i-x\right)H_i\left(dx|v\right)\right)\mu\left(dv\right). \tag{10}$$

Thus, the uniform deviation up to b is unattractive only if

$$\int_{v\in V^N} (v_i - b)\, H\,(b|v)\, \mu\,(dv) \leq \int_{v\in V^N} \int_{x=0}^{b} (v_i - x)\, H_i\,(dx|v)\, \mu\,(dv)\,. \tag{11}$$

We summarize the preceding discussion with the following result, whose rigorous proof appears in the Appendix.

Lemma 1 (Uniform upward incentive constraints)**.** *Any equilibrium σ under any information structure $\mathcal{S}$ must induce winning-bid distributions $\{H_i\,(\mathcal{S},\sigma)\}$ that satisfy (11).*

Note that this constraint must be satisfied even if there is a mass point at b, which represents the possibility that the uniform deviation up to b would induce ties with positive probability. For bidders must not want to uniformly deviate up to any $b' > b$, and since H can have only countably many mass points, the infimum surplus over all such $b' > b$ is precisely (10). In the sequel, we will find it useful to have an alternate form of (11) obtained by rearranging and integrating by parts:

$$\int_{v\in V^N} (v_i - b)\,(H\,(b|v) - H_i\,(b|v))\,\mu\,(dv) \leq \int_{v\in V^N} \int_{x=0}^{b} H_i\,(x|v)\,dx\mu\,(dv)\,. \tag{12}$$

To characterize minimum bidding, we will consider a family of relaxed programs in which the objective is, for a given bid b,

$$\max \int_{v\in V^N} \sum_{i=1}^{N} H_i\,(b|v)\,\mu\,(dv)$$

over all mappings $H : V^N \times \mathcal{N} \times \mathbb{R}_+ \to [0,1]$, written $H_i\,(b|v)$, such that: $H_i\,(b|v)$ is measurable with respect to v for every (i,b); $H_i\,(b|v)$ is weakly increasing and right-continuous in b for every (i,v); $\sum_{i=1}^{N} H_i\,(\overline{v}|v) = 1$ for every v; and the incentive constraint (12) is satisfied. The solution will turn out to be independent of the particular b we consider, so that this solution is a lower bound on the distribution of winning bids satisfying (12) in the sense of first-order stochastic dominance, and by Lemma 1, it is also a lower bound on the distribution of winning bids that can arise in equilibrium..

The rest of this subsection will be devoted to solving the relaxed program. The following result, whose proof appears in the Appendix, verifies that a solution exists:

Lemma 2 (Existence)**.**
A solution to the relaxed program exists.

Our next four results show that a solution can be found within a relatively small family of candidate optima that are (i) symmetric, (ii) are associated with efficient allocations, and

(iii) only depend on the average of the losing values, and (iv) correspond to a deterministic and increasing function of that average. Once these results have been established, we will use essentially the same argument as in Section 3 to characterize the optimal winning bid function, with the average of the $N-1$ lowest values playing the role of the common value.

Let us first show that it is without loss of generality to look at solutions that are symmetric. Let Ξ denote the set of permutations of the bidders' identities, i.e., bijective mappings from $\mathcal{N}$ into itself. We associate each $\xi \in \Xi$ with a mapping from V^N into itself, where $v' = \xi(v)$ if $v'_{\xi(i)} = v_i$ for all i. We have assumed that the distribution μ is symmetric, by which we formally mean that $\mu(X) = (\mu \circ \xi)(X) = \mu(\xi(X))$ for all measurable sets $X \subseteq V^N$. In addition, we say that a solution $\{H_i(\cdot|v)\}$ is *symmetric* if for all $\xi \in \Xi$, $v \in V^N$, and $b \in B$, $H_{\xi(i)}(b|\xi(v)) = H_i(b|v)$. In other words, the probability of a bidder winning with bid less than b only depends on (i) the bidder's own value and (ii) the distribution of the values of the other bidders, but it does not depend on how others' values are matched to bidders' individual identities.

Lemma 3 (Symmetry)**.**
For any feasible solution to the relaxed program, there exists a symmetric feasible solution with the same aggregate distribution of winning bids.

The idea behind the proof is that if we had a feasible solution that was asymmetric, it is possible to "symmetrize" the solution by creating new winning-bid distributions that are the average of the winning-bid distributions over all permutations of the bidders' identities:

$$\widetilde{H}_i(b|v) = \frac{1}{N!} \sum_{\xi \in \Xi} H_{\xi(i)}(b|\xi(v)).$$

This new solution $\left\{\widetilde{H}_i(\cdot|\cdot)\right\}$ is symmetric, and since the objective and constraints for the relaxed program are all linear in the H_i, the constraints that were previously satisfied will still be satisfied at the symmetrized solution. In light of Lemma 3, we will henceforth assume that the solution we are working with is symmetric.

Next, we say that a solution to the relaxed program is *efficient* if $H_i(b|v) = 0$ for all b whenever $v_i < \max v$. The second property that we can assume without loss of generality is that the solution is efficient.

Lemma 4 (Efficiency)**.**
For any feasible solution to the relaxed program, there exists an efficient feasible solution with the same aggregate distribution of winning bids.

This result is proven by taking a candidate solution $H_i(b|v)$, and constructing a new

efficient solution according to

$$\widetilde{H}_i(b|v) = \frac{\mathbb{I}_{i\in W(b)}}{|W(b)|} H(b|v).$$

Since only losing values appear in (12) (on the left-hand side), and since this new allocation has weakly lower losing values on average across all bidders, one can show that this new solution relaxes the critical uniform upward incentive constraints, while maintaining the same aggregate distribution of bids.

The observation that only losing values appear in (12) allows us to collapse the relaxed program into a somewhat more compact form. In particular, the fact that the H_i are symmetric implies that the distribution of winning bids will be the same for all permutations of the losing buyers' values. Thus, a bidder who uniformly deviates up to b will win with probability $H(b|v)$ when the profile of values is v, but he will also win with the same probability when the profile of values is $\xi(v)$ for any permutation $\xi \in \Xi$. In short, this bidder believes that he will win with the same probability for all elements of the equivalence class $[v] = \{\xi(v) | \xi \in \Xi\}$. Symmetry of μ implies that the deviator is equally likely to have any of the values in the profile v. Since the good is always won by a bidder with the highest value, the expected value conditional on losing in equilibrium and conditional on $[v]$ must be $\alpha(v)$ as defined by (5), which is the average of the $N-1$ lowest realized values. This must also be the expected surplus that the bidder gains by winning when he would have lost in equilibrium on the event $[v]$.

Since upward deviators only care about $\alpha(v)$, it is without loss of generality to consider solutions to the relaxed program in which the distribution of winning bids only depends on the average of the $N-1$ lowest values. Specifically, we can restrict attention to solutions for which

$$\alpha(v) = \alpha(v') \implies H(\cdot|v) = H(\cdot|v').$$

For such a solution, we can then define the conditional distribution of winning bids given the average losing value as $H : [\underline{w}, \overline{w}] \to \Delta(B)$ by $H(\cdot|w) = H(\cdot|v)$ for any $v \in \alpha^{-1}(w)$. Recalling that Q is the marginal distribution of $\alpha(v)$, (12) can be rewritten as

$$\frac{N-1}{N} \int_{w=\underline{w}}^{\overline{w}} (w-b) H(b|w) Q(dw) \leq \frac{1}{N} \int_{w=\underline{w}}^{\overline{w}} \int_{x=0}^{b} H(x|w) dx Q(dw). \tag{13}$$

The following lemma verifies that this incentive constraint is equivalent to (11).

Lemma 5 (Average Losing Values)**.**
For any feasible solution to the relaxed program, there exists a feasible solution H with the

same aggregate distribution of winning bids such that $H(\cdot|v) = H(\cdot|v')$ *whenever* $\alpha(v) = \alpha(v')$*. Moreover, for such solutions, (13) is equivalent to (12).*

We say that the solution H is *monotonic* if $H(b|w) < 1$ implies that $H(b|w') = 0$ for all $w' > w$. In other words, the supports of the winning-bid distributions are monotonically increasing in the average losing value. Our next result will show that it is without loss of generality to restrict attention to solutions that are monotonic. The reason is the following. Note that the incentive constraint (13) can be rewritten as

$$\frac{N-1}{N}\int_{w=\underline{w}}^{\overline{w}} wH(b|w)\,Q(dw) \leq \frac{N-1}{N}bH(b) + \frac{1}{N}\int_{x=0}^{b} H(x)\,dx. \tag{14}$$

Let us consider how this constraint is affected by varying the solution $H(b|w)$ but while maintaining a fixed aggregate distribution of winning bids $H(b)$. The only piece which depends on the correlation between winning bids b and values w is the left-hand side, which is proportional to the expectation of the average losing value conditional on the winning bid being less than b. On the whole, decreasing this expectation is going to relax the constraint, as lower losing values translate into smaller gains from a uniform upward deviation. Monotonicity essentially says that the lowest losing values should be associated with the lowest winning bids. In fact, this is the structure that will minimize pointwise, for every b, the expectation of $\alpha(v)$ conditional on the winning bid being less than b, and thereby relax the constraint as much as possible while maintaining $H(b)$.

As a result, it is without loss of generality to consider solutions to the relaxed program that correspond to a deterministic winning bid $\beta(w)$ as a function of the average losing value $w = \alpha(v)$, which is defined by

$$\beta(w) = \min\{b|H(b) \geq Q(w)\}. \tag{15}$$

Since $H(b)$ is right-continuous, the set of b such that $H(b) \geq Q(w)$ must be closed. Thus, this minimum always exists and is weakly increasing in w (though β need not be continuous or strictly increasing). We can therefore rewrite (12) one last time as

$$\frac{N-1}{N}\int_{x=\underline{w}}^{w} (x - \beta(w))\,Q(dx) \leq \frac{1}{N}\int_{x=\underline{w}}^{w} (\beta(w) - \beta(x))\,Q(dx). \tag{16}$$

Note that if we define

$$\beta^{-1}(b) = \max\{w|\beta(w) \leq \beta(b)\},$$

then maximizing $H(b) = Q(\beta^{-1}(b))$ is essentially equivalent to minimizing β pointwise.

Lemma 6 (Monotonicity)**.**
For any feasible solution to the relaxed program, there exists a monotonic feasible solution with the same aggregate distribution of winning bids. Moreover, for such solutions, (16) is equivalent to (12).

It is now apparent that our situation is quite close to what we assumed in the example of Section 3. Our object of choice in the relaxed program is a deterministic winning bid as a function of a one-dimensional statistic. In the case of pure common values, this statistic was the true value of the good, and in the general model it is the average of the $N-1$ lowest values.

Lemma 7 (Binding Uniform Upward Constraints)**.**
An optimal solution to the relaxed program must satisfy $\beta(\underline{w}) = \underline{w}$ and solves (16) with equality for all $w \in [\underline{w}, \overline{w}]$.

One can verify that the solution to this formula is given by precisely (7). The solution to the relaxed program is summarized as follows:

Proposition 1 (Solution to the Relaxed Program)**.**
For any bid b, the solution to the relaxed program is $\underline{H}(b)$, where $\underline{H}$ is given by (7). Since this solution is independent of b, $\underline{H}$ must be greater than any feasible solution to the relaxed program, and therefore $\underline{H} \geq H(\mathcal{S}, \sigma)$ for all information structures $\mathcal{S}$ and equilibria σ.

4.3 A Minimum-Bidding Information Structure and Equilibrium

At this point, we have proven the first part of Theorem 1, which is that the mininum winning-bid distribution $\underline{H}$ must be first-order stochastically dominated by any equilibrium winning-bid distribution. To show that the bounds are tight, we will simply construct an efficient equilibrium in which the distribution of winning bids is precisely $\underline{H}$. This construction will generalize the independent-signal information structure of Section 3.

In this winning-bid-minimizing information structure, the bidders will receive signals which are *independent* of one another and drawn from the same distribution $F(s) \triangleq Q^{\frac{1}{N}}(s)$ on the support $S = [\underline{w}, \overline{w}]$. This distribution is chosen so that the highest signal is distributed according to Q. Signals will be correlated with values so that:

(i) the highest signal is always equal to the realized average losing value: $\max s = \alpha(v)$;[9]

[9]In the pure-common-value model, the winning-bid-minimizing information structure coincides with an environment that Bulow and Klemperer (2002) refer to as "The Maximum Game". In the pure-common-value model, the average losing value is simply the common value, which is the maximum of independent signals. Interestingly, they show that in this informational environment, the bidder with the highest signal has the lowest marginal revenue (as described by the virtual utility of the bidder), thus hinting at the low revenue properties of this information structure within the pure-common-value environment.

(ii) it is the bidder with the highest value who receives the highest signal.

We note that we could have alternatively specified the same information structure by first drawing a profile of values v from μ, then giving the highest-value bidder a signal $w = \alpha(v)$, and then giving the losing bidders signals which are independent draws from the conditional distribution $F(s)/F(w)$.

In equilibrium, a bidder with signal s bids $\underline{\beta}(s)$ with probability one. It is a well-known result in auction theory that these strategies are an equilibrium under a different model in which bidders' signals are distributed as above, but each bidder's signal is their value. Specifically, in the independent private-value (IPV) model in which the bidders' values are independent draws from F, there is an equilibrium in monotonic pure strategies in which a bidder with value s_i bids the expected highest signal of others $\max s_{-i}$, conditional on $\max s_{-i}$ being below s_i, which is precisely $\underline{\beta}(s_i)$ (Krishna, 2002).

We can use this connection to prove that $\underline{\beta}$ is an equilibrium under the interdependent values information structure we constructed. Consider a bidder who receives a signal s and bids $\underline{\beta}(w)$ with $w < s$. Such a deviator would only win when the other bidders' signals are less than w, in which case the deviator had the highest signal and expects his value to be the maximum value conditional on $\alpha(v) = s$, which we can denote by $\tilde{v}(s)$. Note that $\tilde{v}(s)$ must be weakly greater than s. As a result, if we write $G(w) = (F(w))^{N-1} = (Q(w))^{(N-1)/N}$ for the probability that $N-1$ of the independent signals are below w, then the deviator's surplus is

$$\left(\tilde{v}(s) - \underline{\beta}(w)\right) G(w) = \left(\widetilde{v}(s) - s\right) G(w) + \left(s - \underline{\beta}(w)\right) G(w).$$

The second piece is exactly the surplus under the IPV interpretation, which must be less than $\left(s - \underline{\beta}(s)\right) G(s)$. In addition, $\left(\widetilde{v}(s) - s\right) G(w) \leq \left(\widetilde{v}(s) - s\right) G(s)$, so that downward deviations are not attractive.

Now consider an upward deviation to $\underline{\beta}(w)$ with $w > s$. Note that a bidder with signal s believes that by following the equilibrium strategy, they win with probability $G(s)$ and have value $\widetilde{v}(s)$. On the other hand, by deviating upwards to some $\underline{\beta}(w)$ with $w > s$, the bidder will also win when the highest signal of others is in $[s, w]$, and their expected value conditional on winning when $\max s_{-i} = x$ is precisely x, since the highest signal of others must equal the average losing value.[10] The surplus from the upward deviation is therefore

[10]Note that for non-common-value models, $\tilde{v}(s)$ is generally strictly greater than s. This means that a bidder's expectation of his own value is non-monotonic in others' signals: when $\max_{j \neq i} s_j < s_i$, bidder i expects his value to be $\tilde{v}(s_i) > s_i$, but when $\max_{j \neq i} s_j > s_i$, bidder i expects his value to be $\max_{j \neq i} s_j$. This non-monotonicity puts our information structure outside the affiliated-values model of Milgrom and Weber (1982), though this violation is not so severe as to disrupt an equilibrium in monotonic pure strategies.

$$\tilde{v}(s) G(s) + \int_{x=s}^{w} xG(dx) - \underline{\beta}(w) G(w) = \int_{x=\underline{w}}^{s} (\widetilde{v}(s) - x) G(dx), \tag{17}$$

due to the fact that

$$\underline{\beta}(w) = \frac{1}{G(w)} \int_{x=\underline{w}}^{w} xG(dx).$$

Thus, (17) is independent of w, and we conclude that bidders are indifferent to upward deviations. We have proved the following:

Proposition 2 (Winning-Bid-Minimizing Equilibrium)**.**
There exists an information structure and efficient equilibrium in which the winning bid is a deterministic function of the average losing value and is given by $\underline{\beta}(w)$.

Proposition 2 completes the proof of Theorem 1. We have constructed an information structure and equilibrium in which the distribution of winning bids is precisely the solution to the relaxed program, so that minimum revenue must be attained. Moreover, it is always a bidder with a high value who receives the good, so that the equilibrium allocation is efficient. We remark that the similarity in strategies between our construction and the "as-if" IPV model leads to the following interpretation of the minimum winning-bid distribution: it is the distribution of winning bids that would arise in an independent private-values model in which the distribution of the highest value is equal to $Q(v)$, the true distribution of the average of the $N-1$ lowest value.

The reader may rightly ask, is there some deeper reason why this information structure is able to attain the bounds? Put differently, what are the essential properties of this construction that make it work? It turns out that while there are some degrees of freedom in how one constructs the winning-bid-minimizing information structure and equilibrium, there are certain necessary features that could be *derived* from the solution to the relaxed program. First, it must work out that the bidder with the highest value wins the good. Second, the bidders must be indifferent to all uniform upward deviations. This second property actually implies that bidders must be indifferent to *all* upward deviations, uniform or otherwise. For if a bidder strictly preferred not to deviate up to some b for a set of bids $X \subset [0, b]$ that arise with positive probability in equilibrium, then indifference to the uniform upward deviation implies that they strictly prefer to deviate up to b when the equilibrium bid is in $[0, b] \setminus X$. Now, the value of an arbitrary upward deviation from a bid x up to $b > x$ depends on the likelihood of losing with a bid of x to a winning bid less than b for each possible value, and it turns out that indifference to all upward deviations exactly pins down those likelihoods, and hence the marginal distribution of each bidder's losing bids. Moreover, it is always possible to normalize signals so that bidders bid b after receiving a signal $\underline{\beta}^{-1}(b)$, in which case the

marginal distribution of losing signals is exactly as in our construction. In the case of $N = 2$ and a pure common value, this describes an information structure that is essentially unique. For $N > 2$, there is some flexibility as to how losing bidders' signals are correlated with one another and with the highest value, neither of which affects incentive constraints. It is always possible, however, to make losing signals independent of one another and of $\max v$ conditional on $\alpha(v)$, as we have done, which is in some sense the simplest construction that would work.

As to a deeper reason why this information structure minimizes the distribution of winning bids, the best explanation we can give is that because the highest signal is $\alpha(v)$ and bidders use monotonic pure strategies, as a bidder deviates up, their value on the marginal event that they win increases in lockstep with their bid. Moreover, independence of the signals means that all bidders face the same tradeoff as they deviate up between additional wins and a higher price for all wins. This makes it possible to have all types of bidders indifferent to deviating up, regardless of the signal they received, thereby filling up all of the upward incentive constraints and pushing bids down as much as possible. It is nonetheless remarkable that all of these properties can be achieved by a single information structure and equilibrium.

Finally, we comment on the maintained assumption that the distribution of values is symmetric. This assumption was used at the critical Lemma 3, in which we proved that it was without loss of generality to restrict attention to symmetric solutions to the relaxed program, and also in the construction of an information structure and equilibrium attaining the bounds. We do not have a tight characterization of minimum bidding for general asymmetric distributions of values. We can, however, use our methods to obtain a lower bound on the winning-bid distribution. Specifically, for an arbitrary distribution μ, we continue to define Q to be the distribution of the average $N-1$ lowest values, and similarly define $\underline{\beta}$ and $\underline{H}$ as in equations (6) and (7). The following proposition is proven in the Online Appendix:

Proposition 3 (Minimum bidding with asymmetric distributions)**.** *If μ is asymmetric, then for any information structure $\mathcal{S}$ and equilibrium σ, the induced distribution of winning bids $H(\mathcal{S}, \sigma)$ must first order stochastically dominate $\underline{H}$.*

Let us briefly describe the logic behind this result. Notice that Lemmas 1 and 2 did not use the hypothesis that μ is symmetric. Thus, the relaxed program is well-defined for asymmetric μ, it has a solution, and for each b, the solution gives an upper bound on the probability that the winning bid is less than b. Now, given a feasible solution for the relaxed program for μ, it turns out that we can construct a feasible solution for the relaxed program for a "symmetrized" prior $\widetilde{\mu}$, which is generated from μ by randomly permuting the identities

of the bidders, and this solution for $\widetilde{\mu}$ induces the same marginal distribution over winning bids. Moreover, the distribution of the average of the $N-1$ lowest values is the same for μ and for $\widetilde{\mu}$. Since we know that the solution to the relaxed program for $\widetilde{\mu}$ is $\underline{H}$, we can conclude that the latter is a lower bound on the distributions of winning bids that can arise under μ.

This concludes our characterization of minimum bidding. In the next section, we will broaden our gaze beyond minimum revenue to consider the whole set of welfare outcomes that could arise under some information structure and equilibrium. By contrast to the opening example of pure common value in Section 3, the leading example in the next section is a model which the bidders' values are independent of one another. With independent values, the efficiency or inefficiency of the equilibrium will become a central issue in the analysis.

5 Further Results on Revenue and Bidder Surplus

5.1 An Independent Value Example

We continue to explore the limits of bidder surplus and revenue. To describe our new results and their relation to Theorem 1, it is useful to report their implications in an example with two bidders whose values are independent draws from the standard uniform distribution. In this case, the efficient surplus is $\overline{T} = 2/3$. The average of the $N-1$ lowest values is simply the lower of the two uniform draws, which has distribution $Q(v) = 1-(1-v)^2$. Thus, the revenue-minimizing bidding function (4) reduces to

$$\underline{\beta}(v) = \frac{1}{\sqrt{1-(1-v)^2}} \int_{x=0}^{v} \frac{x(1-x)}{\sqrt{1-(1-x)^2}} dx.$$

Minimum revenue does not have a simple analytical expression, but it numerically integrates to $\underline{R} \approx 0.096$, so that maximum bidder surplus is $\overline{U} \approx 0.571$.

For comparison, in the independent *private*-values model—when each bidder only knows his own value but maintains the common prior regarding the other bidder's value—the bidders' surplus is $1/3$ and revenue is $1/3$. This is the same outcome as would obtain in the complete information model where both bidders observe both values. Maximum bidder surplus is therefore approximately 1.7 times larger than that predicted by either of those information structures. By contrast, in the no-information environment in which each bidder knows nothing about the values except the prior distribution, the bidders compete the price up to their expected values of $1/2$. As a result, the allocation will be ex-post inefficient, revenue is $1/2$, and bidder surplus is 0.

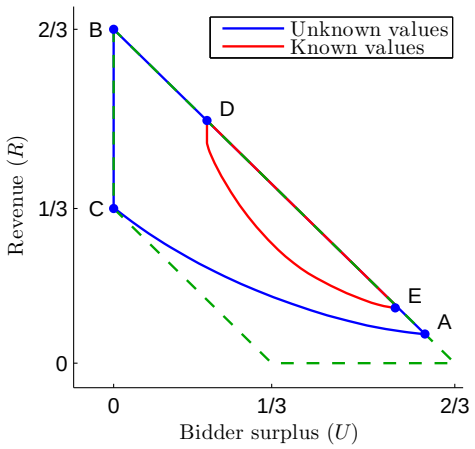


Figure 1: The set of revenue-bidder surplus pairs that can arise in equilibrium for some information structure.

Figure 1 illustrates results for this example. Possible combinations of bidder surplus (on the x-axis) and revenue (on the y-axis) are plotted. As the maximum total surplus is $2/3$, the efficient allocations correspond to the -45 degree line on the right of the picture. The worst case for efficiency would be that the object is always sold to the bidder with the lowest value, which would generate a total surplus of $1/3$. Thus, the green trapezoid represents the surplus pairs that satisfy this range restriction on total surplus and also give non-negative surplus to both the bidders and the seller.

We will now consider the whole range of revenue and bidder surplus across all possible information structures and equilibria. This includes, in particular, maximum revenue, minimum bidder surplus, and minimum total surplus. We start by considering all information structures, including those in which the bidders' signals do not reveal their own values exactly. We refer to this model as one of *unknown values*, to distinguish it from the model we consider next. The set of surplus pairs that can arise in the unknown-values model is the area enclosed by the blue curve in Figure 1. The revenue-minimizing outcome identified in the previous section is attained at point A. Beyond minimum revenue, we can see that there is wide a range of possible welfare outcomes. There are two extreme points which stand out: At point B, the allocation is efficient but bidder surplus is zero, which necessarily attains maximum revenue. At point C bidder surplus is again zero, but the outcome is minimally efficient and the revenue is held down to the minimum feasible surplus. We show that an analogous but somewhat weaker inefficiency result holds for many bidders and independent values.

A natural question to ask is what would happen to our prediction if we assume that the bidders have more information. In the *known-values* model, we assume that the bidders

at least know their own values, and their signals may contain additional information about others' signals. The set of welfare outcomes that can arise under known-values information structures and equilibria is enclosed by the red curve in Figure 1.[11] In contrast to unknown values, known values implies that each bidder can guarantee himself a strictly positive surplus. We will describe a lower bound on bidder surplus and a corresponding upper bound on revenue which turn out to be tight, and in Figure 1 they are attained at point D. The working paper Bergemann, Brooks, and Morris (2015) contains a general proof of this result, and also additional results on minimum revenue in the known-values case, which is attained at point E.

Bergemann and Morris (2016) characterize the set of joint distributions of bids and values that can be induced by some information structure and equilibrium as a particular class of incomplete-information correlated equilibria that they term *Bayes correlated equilibria* (BCE). Loosely speaking, a distribution $\phi \in \left(V^N \times B^N\right)$ is an unknown-values BCE if b_i is a best response to the conditional distribution $\phi\left(v, b_{-i} | b_i\right)$ of players' values and others' bids given that bidder i's bid is b_i. Similarly, ϕ is a known-values BCE if b_i is a best response to $\phi\left(v_{-i}, b_{-i} | b_i, v_i\right)$ for every (v_i, b_i). These incentive constraints are linear, so that for a discretized version of the auction with finitely many values and bids, the set of BCE is a convex polytope, and the problem of maximizing a linear objective, e.g., revenue, over all BCE is a linear program. We used this methodology to compute the range of welfare outcomes for cases in which we do not have an analytical characterization. In particular, while points A–D are derived analytically, as described above, other points are computed numerically for an independent uniform distribution with grids of 10 values and 50 bids between 0 and 1. The axes have been re-scaled to match moments with the continuum limit; for the discretized example, the efficient surplus and minimum surplus are respectively 41/60 and 19/60, as opposed to their limit values of 2/3 and 1/3.

5.2 Maximum Revenue

To maximize revenue and minimize bidder surplus, we would like to generate a highly competitive environment where the highest bid is equal to the highest value. Consider the information structure where each bidder received a signal that was equal to the highest

[11] The fact that the known-values set (in red) is contained within the unknown-values set (in blue) is a reflection of the general observation that providing the bidders with more information decreases the set of outcomes that can be rationalized as an equilibrium with even more information. In other words, the set of Bayes correlated equilibria is decreasing in the minimum information of the players. Bergemann and Morris (2016) formalize the notion of "more information" and give a precise statement of this result.

value, independent of who had the highest value:

$$s_i = \max v, \text{ for all } i.$$

Suppose that there was an efficient tie-breaking rule where the highest-value bidder always wins in case of ties. With efficient tie breaking, there would be an equilibrium where each bidder sets his bid equal to his signal. A bidder would then be indifferent between all bids less than or equal to his signal s_i: if he bid less than the signal, he would lose for sure, and if he bid his signal, he would only win when he has the highest value, in which case he is paying his value and getting a payoff of 0. Moreover, no one would want to deviate up, since this would result in winning for sure and paying a price which is greater than the highest value. Thus, bidding the signal is an equilibrium, and under these strategies the winning bid is the maximum value, so that revenue is the efficient surplus.

This argument relied on the endogenous tie-breaking rule. However, it is possible to achieve approximately the same outcome with the uniform (and thus potentially inefficient) tie-breaking rule by suitably perturbing the information structure:

Theorem 2 (Maximal Revenue and Minimum Bidder Surplus)**.**
For all $\varepsilon > 0$, there exists an information structure and equilibrium such that revenue is at least $\overline{T} - \epsilon$ and bidder surplus is less than ϵ.

To establish the result under the uniform tie-breaking rule, consider the information structure where the bidder with the highest value observes a signal that is a convex combination of the highest and second-highest values, $xv^{(1)} + (1-x)\,v^{(2)}$. The losing bidders observe conditionally independent signals $yv^{(1)} + (1-y)v^{(2)}$, where the weight y is drawn independently across bidders on the interval $[0, x]$ from the distribution

$$y \sim F(y) = \left(\frac{y}{1-y}\frac{1-x}{x}\right)^{\frac{1}{N-1}}.$$

In equilibrium, bidders follow the pure strategy of bidding their signal. As we show in the Online Appendix, for any given x, these strategies form an equilibrium, and for x sufficiently small, the winning bid is arbitrarily close to the highest value of the object among the bidders. In the limit, revenue approaches the efficient surplus, and yet the bidders surplus is arbitrarily close to zero.

Thus, in an environment with unknown values, the private information of each bidder might be sufficiently confounding to induce very aggressive bidding behavior. The bidders are willing to bid a large amount because they think that the bid is less than their value con-

ditional on *winning*, although their value might be quite a bit lower than their bid conditional on *losing*. As a result, the strategy of bidding one's signal is weakly undominated.

We note that the construction of the bid distribution described above exploits symmetry among the bidders, but the argument could be extended to asymmetric distributions of values, assuming only that the asymmetric distribution of values is non-atomic over a symmetric support.

5.3 Minimum Efficiency

Thus far, our analysis has led us to equilibria in which the allocation of the good was efficient, so that the welfare outcome lay on the northeast frontier of Figure 1. As the figure plainly shows, however, there is a large number of possible outcomes in which the allocation of the good is inefficient. That some inefficiency might arise is obvious, for example when the bidders have no information about values except the prior. What is more striking is the extent of this potential inefficiency. In particular, point C attains a maximally inefficient outcome in which the good is always allocated to the buyer with the *lowest* value, all while giving the bidders zero rents.

For the model underlying Figure 1, with two bidders and independent uniform valuations, there is an extremely simple information structure and equilibrium which attains this outcome: each bidder observes the *other* bidder's valuation, and bids half of what they observe. To see that this is an equilibrium, consider a bidder i who has observed a signal s_i. Because of independence, this signal contains no information about the bidder's own value v_i, which has a posterior distribution that is uniform. Now, conditional on bidding some b_i, bidder i wins whenever b_j is less than b_i, which is when s_j is less than $2b_i$. But s_j is equal to v_i, so that the expectation of v_i conditional on winning with a bid of b_i is just the expectation of a uniform random variable conditional it being below $2b_i$, which is precisely b_i! Thus, no matter what bidder i bids in the range $[0, 1/2]$, the expected value conditional on winning equals the bid. Thus, bidders receive zero rents in equilibrium, and since bids are monotonic in the other bidder's value, it is the bidder with the lowest value who wins the auction.

While we have not explored minimum efficiency in its full generality, we report in the Online Appendix a class of information structures and equilibria which generalize this example to the case of many bidders and symmetric independent values drawn from a cumulative distribution $P(v)$. Specifically, each bidder observes the maximum of others' values $s_i = \max v_{-i}$ and bids the expectation of a value drawn from the prior conditional on it being below s_i. In the resulting equilibrium, the one of the $N-1$ low-value bidders win the object with equal probability. When there are two bidders, this construction attains the maximally

inefficient outcome, and more generally, we conjecture that this equilibrium minimizes total surplus subject to the constraint that bidder surplus is zero.

5.4 Minimum and Maximum Revenue with Known Values

In the environment with arbitrary interdependence in values, we have a complete characterization of minimum and maximum revenue and bidder surplus. One might ask how our results would change if we imposed additional restrictions on how much bidders can learn about their values from the outcome of the auction. An extreme assumption, but one which is commonly adopted in independent value models, is that each bidder knows his own value for sure. This is what we call the *known-values* case.

The assumption that bidders know their own values substantially affects the set of possible outcomes. It is no longer the case that bidder surplus can be driven all the way down to zero. In the working paper Bergemann et al. (2015), we derive an elementary lower bound on each bidder's surplus, which we describe here in the context of the two-bidder independent uniform example. As each bidder knows his value for the object, any weakly undominated strategy profile requires that the bidders never bid above their values. Thus, each bidder knows that if they bid b, then they will necessarily win whenever the other bidder's value is less than b. Thus, in the independent uniform example, the surplus from bidding b when the value is v is at least $(v-b)\,b$, and maximizing this quantity over all b implies that the bidder is guaranteed at least $v^2/4$ in surplus. In ex-ante terms, bidder i must receive at least

$$\underline{U}_i = \frac{1}{4}\int_{v=0}^{1} v^2 dv = \frac{1}{12}.$$

In Bergemann et al. (2015) we establish in Theorem 3 that this lower bound is in fact tight: there is an information structure and equilibrium in which bidder i receives *exactly* $1/12$ in surplus. Moreover, bidders can be held to this lower bound while maintaining an efficient allocation. Thus, this equilibrium simultaneously minimizes bidder surplus and maximizes the revenue of the seller at $2/3 - 2\,(1/12) = 1/2$. In fact, this result can be generalized well beyond the two-bidder independent uniform example, to any symmetric or asymmetric joint distribution of values μ.

With regard to minimum revenue, the model with unknown values provides a *lower bound* revenue for the model with known values. In Bergemann et al. (2015), we provide a complete characterization of minimum revenue with known values for the special case where the bidders have values that are either high or low. This characterization employs a similar methodology of (i) formulating and solving a relaxed program for revenue, and (ii)

extending the solution to the relaxed program to an information structure and equilibrium. We also discuss the scope for extending this program beyond binary values, though at the time of writing we do not have a general analytical characterization of minimum revenue in known-values models. This remains an interesting avenue for future research.

To sum up, we have characterized three "corners" of the unknown-values set (for the two-bidder case), and by convexity, we can generate both the western and northeastern flats of the blue region in Figure 1. The remaining feature of Figure 1, hitherto unexplained, is the apparently smooth southwestern frontier that runs from the maximally inefficient equilibrium to the efficient revenue-minimizing equilibrium. In the working paper, Bergemann, Brooks, and Morris (2015) we give a complete description of the class of equilibria that generate this southwestern frontier. They are members of a class of "conditionally-revenue-minimizing" equilibria, which minimize revenue conditional on a fixed allocation of the good. As the allocation ranges from efficient to maximally inefficient, we move smoothly between points A and C.

The known-values surplus set, depicted in red, is significantly smaller than the unknown-values surplus set, and—except for the known-values maximum-revenue result (point D)—is derived from computations. It is notable that inefficiencies that can arise in the known value model are relatively small compared to what can happen with unknown values, as visually expressed by the slimness of the red lens that describes the set of all possible equilibrium surplus realizations. This observation is in line with the result of Syrgkanis and Tardos (2013) and Syrgkanis (2014) who show that the efficiency loss in the independent private-value auction expressed in terms of the ratio between realized surplus and efficient surplus in the first-price auction is bounded below by $1 - 1/e$.

5.5 The Many-Bidder Limit

We conclude this section by relating our model to the classic question of the performance of auctions when the number of bidders becomes large. Consider a sequence of economies indexed by N, each of which is associated with a joint distribution of the potential bidders' values. The preceding analysis tells us that the features of the economy that matter for minimum revenue and maximum bidder surplus are (i) the distribution of the average losing value, which determines the minimum winning-bid distribution, and (ii) the total surplus that is generated by an efficient allocation. Thus, we can analyze behavior in the many-bidder limit by analyzing the behavior of these two objects. Let us suppose that the distribution of the average losing value converges to a limit $Q(w)$ and the limit of total surplus converges

to $\overline{T}$. The mininum winning bid is given by the limit as $N \to \infty$ of (6), i.e.,

$$\underline{\beta}(w) = \frac{1}{Q(w)} \int_{x=\underline{w}}^{w} xQ(dx). \tag{18}$$

Minimum revenue will in turn converge to

$$\underline{R} = \int_{w=\underline{w}}^{\overline{w}} \underline{\beta}(w) Q(dw),$$

and maximum bidder surplus converges to $\overline{U} = \overline{T} - \underline{R}$.

In the pure-common-value case, the distribution of the average losing value is just the distribution of the common value. If we hold this distribution fixed as N grows large, then revenue and bidder surplus converge to the expressions above where Q is the distribution of the common value. Thus, revenue is *bounded away* from the total surplus and bidder surplus is bounded away from zero. This conclusion, while perhaps surprising, is not novel. For example, the same result is obtained by Engelbrecht-Wiggans et al. (1983) when one bidder is informed while $N-1$ bidders are uninformed. In that case, both the informed bidder's strategy and surplus are independent of N, uninformed bidders obtain zero rents, and the uninformed bidders strategies adjust so as to support the informed bidder's behavior. In general, though, maximum bidder surplus is strictly greater than that obtained in the model of Engelbrecht-Wiggans et al.. In fact, the limiting winning bid function (18) is exactly equal to the strategy of the informed bidder, though the informed bidder loses the auction with non-vanishing probability. For example, in the case of a uniform distribution, the informed bidder's surplus is $1/6$, whereas maximum bidder surplus converges to $1/4$ as N grows large.

Another leading case is the one where bidders' values are independent draws from fixed prior $P \in \Delta([0,1])$. When the number of bidders is large, the distribution of the empirical distribution of the $N-1$ lowest values converges weakly to a Dirac measure on P, so that the distribution of the average losing value converges to a Dirac measure on the mean of P:

$$\hat{v} = \int_{v=\underline{v}}^{\overline{v}} vP(dv).$$

In the limit, the winning bid converges almost surely to $\hat{v}$, but the allocation is efficient so total surplus converges to $\overline{v}$ and bidder surplus converges to $\overline{v} - \hat{v}$. Thus, revenue is what would obtain if the bidders had no information beyond the prior, but the allocation is asymptotically efficient and all of the additional surplus goes to the bidders.

The conclusion that minimum revenue in the many-bidder limit is bounded away from the efficient surplus stands in stark contrast with the literature on information aggregation

in large markets (Wilson, 1977; Milgrom, 1979; Bali and Jackson, 2002). This literature considers two distinct but related issues: first, is information aggregated in large markets, in the sense that the winning bid reveals the true maximal value of the good? And second, does this revelation of information induce the bidders to compete away their rents? The positive results in this literature rely upon assumptions about information which our constructions violate. Both Wilson (1977) and Bali and Jackson (2002) assume that there is a uniform lower bound on the quality of bidders' information even as the number of bidders becomes large. In contrast, in the revenue-minimizing information structure, the marginal distribution of each losing bidder's signal converges to a point mass on the lowest possible average losing value, so that signals are asymptotically uninformative about the value. Thus, the sale price is fully revealing about the value without competition forcing the price up to the true value. With non-common-values, however, we have both failure of information aggregation and full surplus extraction in the limit, although enough information is aggregated to ensure an efficient allocation.

6 Discussion

This paper has provided new and general characterizations of equilibrium bidding in the first-price auction. More broadly, this paper contributes to the growing literature on information-free predictions in Bayesian games. We believe that this perspective has value both for theoretical and for applied work. There are many real-world scenarios involving Bayesian games in which practitioners cannot reasonably be expected to know the nature of information held by strategic agents. In such a setting, it would be sensible for econometricians and mechanism designers to base their inference or auction design on the admittedly weaker but also much safer information-free predictions of the present paper. In that spirit, we will conclude by highlighting the positive and normative implications of our results for the theory of applied auctions.

First, there is a large and active literature on auction econometrics that aims to back out payoff relevant fundamentals from observed bidding behavior, and a sizable portion of this literature focuses on the first-price auction (e.g., Laffont, Ossard, and Vuong, 1995, Athey and Haile, 2007, and Somaini, 2015). This work interprets the received model of the first-price auction quite literally, and assumes that bids are generated by an equilibrium under a classical—i.e., affiliated—information structure. Alternatively, one could use our theory to derive bounds on the value distribution that rationalizes an observed distribution of winning bids H. Let us suppose that H was generated by a Bayesian equilibrium under some common prior information structure about a pure common value distributed according

to $P(v)$. Then it must be that the minimum winning-bid distribution $\underline{H}$ associated with P is first order stochastically dominated by H. Loosely speaking, P cannot be too high, or else H will be worse than the worst-case distribution. Since we cannot analytically invert $\underline{\beta}$ to evaluate (7), it is more convenient to express this constraint in terms of percentiles as

$$\underline{\beta}\left(P^{-1}(z)\right) = z^{-\frac{N-1}{N}}\left(z^{\frac{N-1}{N}}P^{-1}(z) - \int_{x=0}^{P^{-1}(z)}\left(P(x)\right)^{\frac{N-1}{N}}dx\right) \leq H^{-1}(z) \tag{19}$$

for all probabilities $z \in [0,1]$, where the expression for $\underline{\beta}\left(P^{-1}(z)\right)$ follows from integrating (6) by parts. Indeed, quantity

$$\Gamma(z;P) = z^{\frac{N-1}{N}}P^{-1}(z) - \int_{x=0}^{P^{-1}(z)}\left(P(x)\right)^{\frac{N-1}{N}}dx$$

denotes the area below the line $y = z^{\frac{N-1}{N}}$ and above the curve $\left(P(w)\right)^{\frac{N-1}{N}}$, which is monotonically increasing in the first-order stochastic dominance ordering and is depicted in Figure 2(a).[12] The consistent distributions are those such that

$$\Gamma(z;P) \leq H^{-1}(z)\, z^{\frac{N-1}{N}}$$

for all $z \in [0,1]$. In sum, we think a promising direction for future research is to further investigate how information-free predictions can be used for partial identification of value distributions from bidding data.

Second, our approach can be used to compare the set of possible welfare outcomes across mechanisms. Myerson (1981) showed that imposing a reserve price can increase revenue for a fixed information structure, and we can also improve our lower bound on revenue by imposing a reserve price. In the Online Appendix, we extend our results on minimum and maximum revenue to the first-price auction with reserve price for pure-common-value environments. Figure 2(b) reports these revenue bounds for the standard-uniform pure-common-value example. With a zero reserve price, the mechanism is exactly the first-price auction we characterized, and maximum revenue $\overline{R}$ and minimum revenue $\underline{R}$ coincide with the theoretical predictions of $1/2$ and $1/6$, respectively. For $r > 0$, $\overline{R}(r)$ and $\underline{R}(r)$ are concave functions, with maximum $\overline{R}(r)$ being attained at any $r < 1/2$ and maximum $\underline{R}(r)$ being attained at $r^* = 1/8$. The latter has the following interesting interpretation: Consider a seller

[12]Thus, if a distribution P satisfies (19) for all z and if $\widetilde{P}$ is first-order stochastically dominated by P, then $\widetilde{P}$ also satisfies (19) for all z. This does not imply, however, that the set of distributions satisfying (19) has a maximal element in the sense of first-order stochastic dominance, nor is it true that every distribution of winning bids H can be rationalized as the minimum winning bid distribution for some distribution of a common value.

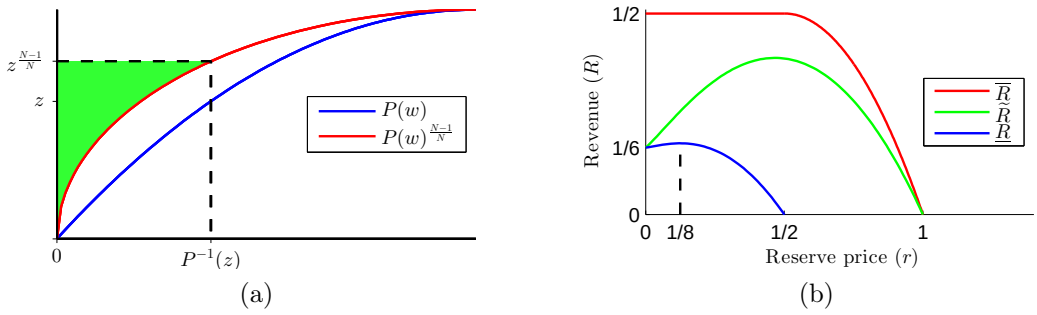


Figure 2: (a) Drawing inference about the distribution of values: for all z, the green shaded area must be less than $z^{\frac{N-1}{N}} H^{-1}(z)$. (b) Maximum revenue $\overline{R}(r)$ and minimum revenue $\underline{R}(r)$ as functions of the reserve price r. $\widetilde{R}(r)$ is revenue at the worst-case information structure for $r = 0$.

who knows that there is a standard-uniform common value, but is ambiguity averse with respect to both the information structure and choice of Bayesian equilibrium, and moreover that this seller has to sell according to a first-price auction with reserve price. Then the optimal such reserve price is precisely 1/8. Note that this reserve price is significantly less than the optimal reserve price for the revenue-minimizing information structure at $r = 0$ which is 1/2.

Stepping back, we have shown that the first-price auction is guaranteed to generate positive revenue, regardless of the information structure and equilibrium. In contrast, other commonly considered auction formats, e.g., the second-price auction, need not admit such a positive lower bound: Even when buyers know their own values, the second-price auction admits weakly dominated equilibria in which revenue is zero, and there are information structures that add small uncertainty about one's values in which revenue is essentially the same but equilibrium strategies are no longer dominated. In our view, an exciting direction for future research is to look for mechanisms that provide even more favorable revenue guarantees than first-price auctions with reserve prices, or more broadly, to look for mechanisms that are guaranteed to perform well according to some metric regardless of the true structure of information and equilibrium.

References

Athey, S. and P. Haile (2007): "Nonparametric Approaches to Auctions," in *Handbook of Econometrics*, ed. by J. Heckman and E. Leamer, Amsterdam: Elsevier, vol. 6A, 3847–3965.

Azacis, H. and P. Vida (2015): "Collusive Communication Schemes in a First-Price Auction," *Economic Theory*, 58, 125–160.

Bali, V. and M. Jackson (2002): "Asymptotic Revenue Equivalence in Auctions," *Journal of Economic Theory*, 106, 161–176.

Battigalli, P. and M. Siniscalchi (2003): "Rationalizable Bidding in First-Price Auctions," *Games and Economic Behavior*, 45, 38–72.

Bergemann, D., B. Brooks, and S. Morris (2013): "Extremal Information Structures in First Price Auctions," 1926 1926, Cowles Foundation for Research in Economics, Yale University.

——— (2015): "First Price Auctions with General Information Structures: Implications for Bidding and Revenue," Tech. Rep. 2018R, Cowles Foundation Discussion Paper.

Bergemann, D. and S. Morris (2013): "Robust Predictions in Games with Incomplete Information," *Econometrica*, 81, 1251–1308.

——— (2016): "Bayes Correlated Equilibrium and the Comparison of Information Structures in Games," *Theoretical Economics*, forthcoming.

Bulow, J. and P. Klemperer (2002): "Prices and the Winner's Curse," *RAND Journal of Economics*, 33, 1–21.

Çinlar, E. (2011): *Probability and Stochastics*, New York: Springer.

Dekel, E. and A. Wolinsky (2003): "Rationalizable Outcomes of Large Private-Value First-Price Discrete Auctions," *Games and Economic Behavior*, 43, 175–188.

Engelbrecht-Wiggans, R., P. Milgrom, and R. Weber (1983): "Competitive Bidding and Proprietary Information," *Journal of Mathematical Economics*, 11, 161–169.

Fang, H. and S. Morris (2006): "Multidimensional Private Value Auctions," *Journal of Economic Theory*, 126, 1–30.

Krishna, V. (2002): *Auction Theory*, San Diego: Academic Press.

Laffont, J.-J., H. Ossard, and Q. Vuong (1995): "Econometrics of First-Price Auctions," *Econometrica*, 953–980.

Lopomo, G., L. Marx, and P. Sun (2011): "Bidder Collusion at First-Price Auctions," *Review of Economic Design*, 15, 177–211.

Milgrom, P. (1979): "A Convergence Theorem for Competitive Bidding with Differential Information," *Econometrica*, 47, 679–88.

Milgrom, P. and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Somaini, P. (2015): "Identification in Auction Models with Interdependent Costs," Tech. rep., Stanford University.

Syrgkanis, V. (2014): "Efficiency of Mechanisms in Complex Markets," Ph.D. thesis, Cornell University.

Syrgkanis, V. and É. Tardos (2013): "Composable and Efficient Mechanisms," in *STOC '13 Proceedings of the 45th Annual ACM Symposium on Theory of Computing*, 211–220.

Vickrey, W. (1961): "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

Wilson, R. (1977): "A Bidding Model of Perfect Competition," *Review of Economic Studies*, 44, 511–518.

A Appendix

Proof of Lemma 1. Fix an information structure $\mathcal{S}$ and strategies σ, and define the uniform deviation up to b by its distribution function $\sigma_i^b([0,x]|s) = \mathbb{I}_{x\geq b}\sigma_i([0,x]|s)$. Now, fix $b \in B$. Observe that the marginal distribution of each bidder's bids

$$\int_{v\in V^N} \sigma_i(db_i|s_i)\,\pi(ds|v)\,\mu(dv)$$

can have only countably many mass points. Thus, it must be possible to find ϵ arbitrarily small such that

$$\int_{v\in V^N} \sigma_i(\{b+\epsilon\}|s_i)\,\pi(ds|v)\,\mu(dv) = 0$$

for all i. Fix such an ϵ. Then for σ to be an equilibrium, it must be that $U_i(\mathcal{S},\sigma) \geq U_i\left(\mathcal{S},\sigma_i^{b+\epsilon},\sigma_{-i}\right)$. The latter utility is

$$U_i\left(\mathcal{S},\sigma_i^{b+\epsilon},\sigma_{-i}\right) = \int_{v\in V^N}\int_{s\in S}\int_{x\in B^N} (v_i - x_i)\,q_i(x)\left(\sigma_i^{b+\epsilon},\sigma_{-i}\right)(dx|s)\,\pi(ds|v)\,\mu(dv).$$

Note that for each i, the joint distribution of bids is unchanged outside of $\{x\in B|x_i \leq b+\epsilon\}$, and bidder i deterministically bids $b+\epsilon$ on this event, so we can rewrite this surplus as

$$\begin{aligned}
&\int_{v\in V^N}\int_{s\in S}\int_{x_{-i}\in B^N} (v_i - b - \epsilon)\,q_i(b+\epsilon,x_{-i})\,\sigma_{-i}(dx_{-i}|s_{-i})\,\sigma_i([0,b+\epsilon)|s_i)\,\pi(ds|v)\,\mu(dv)\\
&\quad + \int_{v\in V^N}\int_{s\in S}\int_{x\in B^N} (v_i - x_i)\,\mathbb{I}_{x_i\geq b+\epsilon}q_i(x)\,\sigma(dx|s)\,\pi(ds|v)\,\mu(dv)\\
&= \int_{v\in V^N}\int_{s\in S} (v_i - b - \epsilon)\,\sigma\left([0,b+\epsilon)^N|s\right)\pi(ds|v)\,\mu(dv)\\
&\quad + \int_{v\in V^N}\int_{s\in S}\int_{x\in B^N} (v_i - x_i)\,\mathbb{I}_{x_i\geq b+\epsilon}q_i(x)\,\sigma(dx|s)\,\pi(ds|v)\,\mu(dv).
\end{aligned}$$

where we have used the facts that each bidder bids $b+\epsilon$ with (ex-ante) probability zero and that bidder i wins with probability one when bids are in $[0,b+\epsilon)^N$. Thus, the incentive constraint is equivalent to

$$\begin{aligned}
&\int_{v\in V^N} (v_i - b - \epsilon)\int_{s\in S}\sigma\left([0,b+\epsilon)^N|s\right)\pi(ds|v)\,\mu(dv)\\
&\leq \int_{v\in V^N}\int_{s\in S}\int_{x\in B^N} (v_i - x_i)\,\mathbb{I}_{x_i<b+\epsilon}q_i(x)\,\sigma(dx|s)\,\pi(ds|v)\,\mu(dv)\\
&= \int_{v\in V^N}\int_{s\in S}\int_{x\in[0,b+\epsilon)^N} (v_i - x_i)\,q_i(x)\,\sigma(dx|s)\,\pi(ds|v)\,\mu(dv)
\end{aligned}$$

where the second equality is coming from the fact that $\mathbb{I}_{x_i<b+\epsilon} q_i(x) > 0$ only if $x \in [0, b+\epsilon)^N$. Now, as $\epsilon \to 0$,

$$\int_{s\in S} \sigma\left([0,b+\epsilon)^N | s\right) \pi(ds|v) \to \int_{s\in S} \sigma\left([0,b]^N | s\right) \pi(ds|v) = H(b|v;\mathcal{S},\sigma),$$

which follows from summing (9) over $i \in \mathcal{N}$ and the fact that $\sum_{i\in\mathcal{N}} q_i(x) = 1$, and invoking countable additivity. Similarly,

$$\int_{s\in S}\int_{x\in[0,b+\epsilon)^N} q_i(x)\,\sigma(dx|s)\,\pi(ds|v) \to \int_{s\in S}\int_{x\in[0,b]^N} q_i(x)\,\sigma(dx|s)\,\pi(ds|v) = H_i(b|v;\mathcal{S},\sigma).$$

Plugging in these expressions yields (11). □

Proof of Lemma 2. We can identify the domain of the relaxed program with the subset D of $\Delta\left(B\times\mathcal{N}\times V^N\right)$ that have $\mu(dv)$ as the marginal over V^N, and satisfy

$$\int_{v\in V^N}\int_{x\in B}\sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b}(v_i - b)\,\phi(dx,j,dv) \le \int_{x\in[0,b]}\left(\int_{v\in V^N}\int_{y\in[0,x]}\phi(dy,i,dv)\right)dx. \quad (20)$$

for every b. For any such distribution ϕ disintegrates into the marginal μ and a probability transition kernel $K : V^N \to \Delta(B\times\mathcal{N})$, for which $H_i(b|v) = K([0,b]\times\{i\}|v)$, and under this transformation, (20) reduces to (12). The objective of the relaxed program is simply

$$\int_{v\in V^N}\int_{x\in B}\sum_{i\in\mathcal{N}} \mathbb{I}_{x\le b}\phi(dx,i,dv),$$

which is upper semi-continuous in ϕ. Thus, we can verify existence of a solution by proving that D is weak-* compact. This is not trivial, however, since for each individual constraint (20), the set of distributions satisfying the constraint is not closed. We will show, however, that the set D as a whole is closed. Note that $\Delta\left(B\times\mathcal{N}\times V^N\right)$ is weak-* compact and completely metrizable, so that compactness of D will follow from the fact that it is closed.

Let $\left\{\phi^k\right\}$ be a sequence D that converges to some ϕ. It is trivial to show that ϕ has μ as a marginal on V^N, and we only need to verify that ϕ satisfies (20). We know that

$$\begin{aligned}\int_{v\in V^N}\int_{x\in B}\sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b}(v_i - b)\,\phi^k(dx,j,dv) &\le \int_{x\in[0,b]}\left(\int_{v\in V^N}\int_{y\in[0,x]}\phi^k(dy,i,dv)\right)dx \\ &= \int_{x\in[0,b]} H_i^k(x)\,dx\end{aligned}$$

for all k, where $H_i^k(x) = \phi^k\left([0,x]\times\{i\}\times V^N\right)$. Because the $H_i^k(x)$ are distribution

functions that are weakly converging to $H_i(x) = \phi\left([0,x] \times \{i\} \times V^N\right)$, it must be that $H_i^k(x) \to H_i(x)$ whenever H_i is continuous at x, which must be true for all but countably many values of x. Thus, the integral on the right-hand side must converge to $\int_{x\in[0,b]} H_i(x)\,dx$ as $k \to \infty$. If we additionally assume that $\phi\left(\{b\} \times \{i\} \times V^N\right) = 0$, then

$$\begin{aligned}\lim_{k\to\infty} \int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b}(v_i - b)\,\phi^k(dx,j,dv) &\ge \lim_{k\to\infty} \int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x<b}(v_i - b)\,\phi^k(dx,j,dv) \\ &\ge \int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x<b}(v_i - b)\,\phi(dx,j,dv).\end{aligned}$$

But of course, the last integral is equal to

$$\int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b}(v_i - b)\,\phi(dx,j,dv)$$

because

$$\left|\int_{v\in V^N} \int_{x\in\{b\}} (v_i - b)\,\phi(dx,j,dv)\right| \le (\overline{v} - b)\,\phi\left(\{b\} \times \{j\} \times V^N\right) = 0.$$

Thus, if ϕ does not have a mass point on $\{b\} \times \mathcal{N} \times V^N$, we conclude that (20) holds.

Now suppose $\phi\left(\{b\} \times \mathcal{N} \times V^N\right) > 0$. Since $H(b) = \phi\left([0,b] \times \mathcal{N} \times V^N\right)$ can have at most countably many mass points, for every $\epsilon > 0$, we can find a $b' \in (b, b+\epsilon)$ such that $\phi\left(\{b'\} \times \mathcal{N} \times V^N\right) = 0$. Fixing such an ϵ and b', we know that

$$\begin{aligned}\int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b}(v_i - b)\,\phi(dx,j,dv) &\le \int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b'}(v_i - b)\,\phi(dx,j,dv) \\ &\le \int_{v\in V^N} \int_{x\in B} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{I}_{x\le b'}(v_i - b')\,\phi(dx,j,dv) + \epsilon \\ &\le \int_{x\in[0,b']} \left(\int_{v\in V^N} \int_{y\in[0,x]} \phi(dy,i,dv)\right) dx + \epsilon \\ &\le \int_{x\in[0,b]} \left(\int_{v\in V^N} \int_{y\in[0,x]} \phi(dy,i,dv)\right) dx + 2\epsilon.\end{aligned}$$

Since this must be true for every $\epsilon > 0$, we conclude that (20) holds at mass points as well, thus completing the proof that D is closed. □

Proof of Lemma 3. Given a feasible solution $\{H_i(\cdot|v)\}$, we can explicitly define a sym-

metrized solution by

$$\widetilde{H}_i(b|v) = \frac{1}{N!}\sum_{\xi\in\Xi} H_{\xi(i)}(b|\xi(v))$$

so that also $\widetilde{H}(b|v) = \sum_{i=1}^{N}\widetilde{H}_i(b|v)$. It is clear that this solution is symmetric, since $\Xi = \{\xi\circ\xi'|\xi\in\Xi\}$ for each $\xi'\in\Xi$, so that

$$\widetilde{H}_{\xi'(i)}(b|\xi'(v)) = \frac{1}{N!}\sum_{\xi\in\Xi} H_{\xi\circ\xi'(i)}(b|\xi\circ\xi'(v)) = \frac{1}{N!}\sum_{\xi\in\Xi} H_{\xi(i)}(b|\xi(v)) = \widetilde{H}_i(b|v).$$

Moreover, $\left\{\widetilde{H}_i\right\}$ will clearly still be increasing and satisfy the probability bounds, since they are just obtained by averaging the H_i. In addition, since μ is symmetric,

$$\begin{aligned}\widetilde{H}(b) &= \int_{v\in V^N}\widetilde{H}(b|v)\,\mu(dv) = \frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N} H(b|\xi(v))\,\mu(dv)\\ &= \frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N} H(b|v)\left(\mu\circ\xi^{-1}\right)(dv) = \frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N} H(b|v)\,\mu(dv)\\ &= \int_{v\in V^N} H(b|v)\,\mu(dv) = H(b)\end{aligned}$$

where $\mu\circ\xi^{-1} = \mu$ by the assumption of exchangeability. Thus, symmetrizing the solution does not change the induced distribution of winning bids.

Finally, we verify that (12) will still be satisfied. Note that if $\xi(i) = j$, then

$$\int_{v\in V^N} H_{\xi(i)}(b|\xi(v))\,\mu(dv) = \int_{v\in V^N} H_{\xi(i)}(b|v)\left(\mu\circ\xi^{-1}\right)(dv) = \int_{v\in V^N} H_j(b|v)\,\mu(dv).$$

Thus,

$$\begin{aligned}&\frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N}(v_i-b)\left(H(b|\xi(v)) - H_{\xi(i)}(b|\xi(v))\right)\mu(dv)\\ &\quad= \frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N}\left(\xi(v)_{\xi(i)} - b\right)\left(H(b|\xi(v)) - H_{\xi(i)}(b|\xi(v))\right)\mu(dv)\\ &\quad= \frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N}\left(v_{\xi(i)} - b\right)\left(H(b|v) - H_{\xi(i)}(b|v)\right)\left(\mu\circ\xi^{-1}\right)(dv)\\ &\quad= \frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N}(v_j-b)\left(H(b|v) - H_j(b|v)\right)\mu(dv),\end{aligned}$$

so that the left-hand side of (11) is simply summed across bidders. By a similar argument,

$$\frac{1}{N!}\sum_{\xi\in\Xi}\int_{v\in V^N} H_{\xi(i)}\left(x|\xi\left(v\right)\right)\mu\left(dv\right)=\frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N} H_j\left(x|v\right)\mu\left(dv\right),$$

so that

$$\begin{aligned}\int_{v\in V^N}\left(v_i-b\right)\left(\widetilde{H}\left(b|v\right)-\widetilde{H}_i\left(b|v\right)\right)\mu\left(dv\right)&=\frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N}\left(v_j-b\right)\left(H\left(b|v\right)-H_j\left(b|v\right)\right)\mu\left(dv\right)\\&\leq\frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N}\int_{x=0}^{b}H_j\left(x|v\right)dx\mu\left(dv\right)=\int_{v\in V^N}\int_{x=0}^{b}\widetilde{H}_i\left(x|v\right)dx\mu\left(dv\right),\end{aligned}$$

as desired. □

Proof of Lemma 4. Suppose that we have an inefficient solution. We can then define an alternative solution

$$\widetilde{H}_i\left(b|v\right)=\frac{\mathbb{I}_{i\in\arg\max v}}{\left|\arg\max v\right|}H\left(b|v\right)$$

It is clear that $\widetilde{H}\left(b|v\right)=H\left(b|v\right)$, so that the aggregate distribution of winning bids is unchanged. We therefore only have to check that $\widetilde{H}$ satisfies (12). Since H_i is symmetric,

$$\int_{v\in V^N}\widetilde{H}_i\left(b|v\right)\mu\left(dv\right)=\frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N}\widetilde{H}_j\left(b|v\right)\mu\left(dv\right)=\frac{1}{N}\int_{v\in V^N}H\left(b|v\right)\mu\left(dv\right),$$

and the right-hand side of (12) is unchanged. Symmetry implies that

$$\begin{aligned}\int_{v\in V^N}v_i\widetilde{H}_i\left(b|v\right)\mu\left(dv\right)&=\frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N}v_j\widetilde{H}_j\left(b|v\right)\mu\left(dv\right)\\=\frac{1}{N}\int_{v\in V^N}\max v\,H\left(b|v\right)\mu\left(dv\right)&=\int_{v\in V^N}\max v\,H_i\left(b|v\right)\mu\left(dv\right).\end{aligned}$$

The left-hand side can be rewritten as

$$\int_{v\in V^N}\left(v_i-b\right)\left(H\left(b|v\right)-H_i\left(b|v\right)\right)\mu\left(dv\right)+\int_{v\in V^N}\left(v_i-b\right)\left(H_i\left(b|v\right)-\widetilde{H}_i\left(b|v\right)\right)\mu\left(dv\right).$$

The second piece reduces to

$$\int_{v\in V^N}\left(v_i-\max v\right)H_i\left(b|v\right)\mu\left(dv\right)\leq 0,$$

so that (12) is satisfied. □

Proof of Lemma 5. Let $\{H(\cdot|v)\}$ be symmetric, efficient, and satisfy (12). Define $\phi \in \Delta\left(V^N \times B\right)$ by

$$\phi(X) = \int_{(v,b)\in X} H(db|v)\,\mu(dv)$$

for measurable sets $X \subseteq V^N \times B$. Since α is measurable, so is the mapping $(\alpha \times \mathbf{I}) : V^N \times B \to [\underline{w}, \overline{w}] \times B$ defined by $(\alpha \times \mathbf{I})(v,b) = (\alpha(v), b)$. Thus, we can define $\psi \in \Delta([\underline{w}, \overline{w}] \times B)$ by $\psi = \phi \circ (\alpha \times \mathbf{I})^{-1}$. Note that the marginal distribution of ψ on B must be H, i.e., $\psi([\underline{w}, \overline{w}] \times [0,b]) = \phi\left(V^N \times [0,b]\right) = H(b)$ for all $b \in B$, and the marginal distribution of ψ on $[\underline{w}, \overline{w}]$ must be Q, i.e., $\psi([\underline{w}, w] \times B) = \phi(\alpha^{-1}([\underline{w}, w]) \times B) = Q(w)$ for all $\underline{w} \leq w \leq \overline{w}$. By the disintegration theorem (Çınlar, 2011, Theorem IV.2.18), there exists a probability transition kernel $K : [\underline{w}, \overline{w}] \to \Delta(B)$ such that $\psi(dw, db) = K(db|w)\,Q(dw)$, and we can define $H(b|w) = K([0,b]\,|w)$.

Now let us argue that $H(\cdot|v)$ satisfies (12) if and only if $H(\cdot|w)$ satisfies (13). From the assumption that H is efficient and symmetric, we can conclude that

$$\int_{v\in V^N} v_i H_i(b|v)\,\mu(dv) = \frac{1}{N}\int_{v\in V^N} \max v\, H(b|v)\,\mu(dv).$$

Moreover, symmetry implies that

$$\int_{v\in V^N} v_i H(b|v)\,\mu(dv) = \int_{v\in V^N} \frac{1}{N}\sum_{j=1}^{N} v_j H(b|v)\,\mu(dv).$$

Letting $\gamma(w,b) = w$, the left-hand side of (13) is simply

$$\begin{aligned}
\int_{v\in V^N} \frac{1}{N}\left(\sum_{i=1}^{N} v_i - \max v\right) H(b|v)\,\mu(dv) &= \frac{N-1}{N}\int_{v\in V^N} \alpha(v)\, H(b|v)\,\mu(dv) \\
&= \frac{N-1}{N}\int_{(v,x)\in V^N\times[0,b]} \gamma\left((\alpha\times\mathbf{I})^{-1}(v,x)\right)\phi(dv,dx) \\
&= \frac{N-1}{N}\int_{(w,x)\in[\underline{w},\overline{w}]\times[0,b]} \gamma(w,x)\,\psi(dw,dx) \\
&= \frac{N-1}{N}\int_{w=\underline{w}}^{\overline{w}} w H(b|w)\,Q(dw)
\end{aligned}$$

where we have used the change of variables formula (cf. Çınlar, 2011, Theorem I.5.2). Thus,

we conclude that

$$\int_{v\in V^N}(v_i-b)\left(H\left(b|v\right)-H_i\left(b|v\right)\right)\mu\left(dv\right)=\frac{N-1}{N}\left(\int_{w=\underline{w}}^{\overline{w}}wH\left(b|w\right)Q\left(dw\right)-bH\left(b\right)\right)$$
$$=\frac{N-1}{N}\int_{w=\underline{w}}^{\overline{w}}\left(w-b\right)H\left(b|w\right)Q\left(dw\right)$$

and similarly

$$\int_{v\in V^N}\int_{x=0}^{b}H_i\left(x|v\right)dx\mu\left(dv\right)=\frac{1}{N}\int_{x=0}^{b}H\left(x\right)dx=\frac{1}{N}\int_{w=\underline{w}}^{\overline{w}}\int_{x=0}^{b}H\left(x|w\right)dxQ\left(dw\right),$$

so that (12) and (13) are equivalent.

Finally, we can always define a new solution

$$\widetilde{H}\left(b|v\right)=H\left(b|\alpha\left(v\right)\right)$$

that induces the same joint distribution of winning bids and $\alpha\left(v\right)$, so that it is feasible for the relaxed program and only depends on $\alpha\left(v\right)$. □

Proof of Lemma 6. Notice that the only piece of the incentive constraint (14) that depends on how b is correlated with $\alpha\left(v\right)$ is through the left-hand side, and all things equal, making $\int_{w=\underline{w}}^{\overline{w}}wH\left(b|w\right)Q\left(dw\right)$ smaller relaxes the incentive constraint and makes uniform upward deviations less attractive. Let β be defined as in (15). Then $\int_{w=\underline{w}}^{\overline{w}}w\widetilde{H}\left(b|w\right)Q\left(dw\right)$ is minimized pointwise and for all $b\in B$ by setting $\widetilde{H}\left(b|w\right)=\mathbb{I}_{b\geq\beta(w)}$. To see this, consider the functions

$$G\left(w\right)=\int_{x=\underline{w}}^{w}H\left(b|x\right)Q\left(dx\right)$$

and

$$\widetilde{G}\left(w\right)=\int_{x=\underline{w}}^{w}\widetilde{H}\left(b|x\right)Q\left(dx\right),$$

where b is in the support of $H\left(db\right)$. Claim: $G\left(w\right)\leq\widetilde{G}\left(w\right)$ for all w, i.e., G first-order stochastically dominates $\widetilde{G}$. For if we let $\hat{w}=\min\left\{w|\beta\left(w\right)\geq b\right\}$ (which exists because the set $\beta^{-1}\left(b\right)$ is non-empty for all b in the support of $H\left(db\right)$), then for $w\leq\hat{w}$, we have $H\left(b|w\right)\leq1=\widetilde{H}\left(b|w\right)$, and for $w>\hat{w}$, we have that $\widetilde{G}\left(w\right)=H\left(b\right)$, and $G\left(w\right)\leq H\left(b\right)$. Thus,

$$\int_{w=\underline{w}}^{\overline{w}}wH\left(b|w\right)Q\left(dw\right)=\int_{w=\underline{w}}^{\overline{w}}wG\left(dw\right)\geq\int_{w=\underline{w}}^{\overline{w}}w\widetilde{G}\left(dw\right)=\int_{w=\underline{w}}^{\overline{w}}w\widetilde{H}\left(b|w\right)Q\left(dw\right).$$

We now prove that (16) is equivalent to (12) and (13). Note that $\beta^{-1}\left(\left[0,\hat{b}\right]\right)$ is simply the set of $w \in [\underline{w},\overline{w}]$ such that $Q(w) \leq H\left(\hat{b}\right)$, so that the equality $Q\left(\beta^{-1}\left(\left[0,\hat{b}\right]\right)\right) = H\left(\hat{b}\right)$ follows immediately from the definition of the cumulative distribution function and the fact that Q is continuous. (If $Q(w) \leq H\left(\hat{b}\right)$, then $\hat{b} \in \{b|H(b) \geq Q(w)\}$, so that $\hat{b} \geq \beta(w)$, and if $Q(w) > H\left(\hat{b}\right)$, then $\hat{b} \notin \{b|H(b) \geq Q(w)\} = [\beta(w), \overline{v}]$, so that $\hat{b} < \beta(w)$.)

Now let us verify that β satisfies (16) for all w. Assuming that $H(b|w)$ is monotonic, we must have

$$\begin{aligned}\int_{x=\underline{w}}^{\overline{w}} (x-\beta(w))\, H(\beta(w)|x)\, Q(dx) &= \int_{x=\underline{w}}^{\overline{w}} (x-\beta(w))\, \mathbb{I}_{\beta(w)\geq\beta(x)} Q(dx) \\ &= \int_{x=\underline{w}}^{w} (x-\beta(w))\, Q(dx).\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{x=\underline{w}}^{\overline{w}} \int_{b=0}^{\beta(w)} H(b|w)\, dbQ(dx) &= \int_{x=\underline{w}}^{\overline{w}} \int_{b=0}^{\beta(w)} \mathbb{I}_{b\geq\beta(x)} dbQ(dx) \\ &= \int_{x=\underline{w}}^{\overline{w}} \mathbb{I}_{\beta(w)\geq\beta(x)} (\beta(w)-\beta(x))\, Q(dx) \\ &= \int_{x=\underline{w}}^{w} (\beta(w)-\beta(x))\, Q(dx).\end{aligned}$$

Going in the other direction, if we have a monotonic function $\beta : [\underline{v},\overline{v}] \to B$, we can simply define $H(b|w) = \mathbb{I}_{b\geq\beta(w)}$. Thus, if we let $\hat{w}$ be the maximum of $\beta^{-1}([0,b])$ (so that $\beta(\hat{w}) \leq b$), then

$$\int_{x=\underline{w}}^{\hat{w}} (x-\beta(\hat{w}))\, Q(dx) \geq \int_{x=\underline{w}}^{\hat{w}} (x-b)\, Q(dx) = \int_{x=\underline{w}}^{\overline{w}} (x-b)\, H(b|x)\, Q(dx),$$

and similarly

$$\int_{x=\underline{w}}^{\hat{w}} (\beta(\hat{w})-\beta(x))\, Q(dx) \leq \int_{x=\underline{w}}^{\hat{w}} (b-\beta(x))\, Q(dx) = \int_{x=\underline{w}}^{\overline{w}} \int_{y=0}^{b} H(y|x)\, dyQ(dx).$$

Combining these inequalities with (16) yields (13). □

Proof of Lemma 7. First, let us argue that any feasible solution must satisfy $\beta(w) \geq \underline{w}$ for all $w \in [\underline{w},\overline{w}]$. This is essentially a consequence of (16). Without loss of generality, we can

assume that $\beta(\underline{w}) = \lim_{w \to \underline{w}} \beta(w)$. Since β is increasing, we know that

$$\frac{1}{N} \int_{x=\underline{w}}^{\underline{w}+\epsilon} (\beta(\underline{w}+\epsilon) - \beta(x)) Q(dx) \leq \frac{1}{N} (\beta(\underline{w}+\epsilon) - \beta(\underline{w})) Q(w+\epsilon)$$

and also that

$$\frac{N-1}{N} \int_{x=\underline{w}}^{\underline{w}+\epsilon} (x - \beta(\underline{w}+\epsilon)) Q(dx) \geq \frac{N-1}{N} (\underline{w} - \beta(\underline{w}+\epsilon)) Q(\underline{w}+\epsilon).$$

These two constraints imply that

$$(N-1)(\underline{w} - \beta(\underline{w}+\epsilon)) \leq (\beta(\underline{w}+\epsilon) - \beta(\underline{w})).$$

Continuity of $\beta(w)$ at $\underline{w}$ in turn implies that $\underline{w} - \beta(\underline{w}) \leq 0$.

Now suppose that (16) holds as a strict inequality at a positive Q-measure of w. We can define a new bidding function $\hat{\beta}$ by

$$\hat{\beta}(w) = \frac{\int_{x=\underline{w}}^{w} \left(\frac{N-1}{N} x + \frac{1}{N} \beta(x)\right) Q(dx)}{Q(w)}.$$

This function is obviously weakly increasing, and one can verify from L'Hôpital's rule that

$$\hat{\beta}(\underline{w}) = \frac{N-1}{N} \underline{w} + \frac{1}{N} \beta(\underline{w}),$$

so as long as $\beta(w) \geq \underline{w}$, $\hat{\beta}(w) \geq \underline{w}$ as well. Moreover, (16) implies that $\hat{\beta}(w)$ must be weakly less than $\beta(w)$ everywhere and strictly less on a positive Q-measure of w, so that $Q\left(\hat{\beta}^{-1}\left(\left[0, \hat{b}\right]\right)\right)$ is weakly greater than $Q\left(\beta^{-1}\left(\left[0, \hat{b}\right]\right)\right)$. □

B Online Appendix

B.1 Pure Common Values

Let us first consider a more general version of the pure-common-values model that we studied in Section 3, in which the bidders have the same value which is distributed according to $P(v)$. Recall the information structure of Engelbrecht-Wiggans et al. (1983), in which one bidder knows the true value and the remaining bidders are uninformed. The corresponding equilibrium has the informed player bid

$$\sigma(v) = \frac{1}{P(v)} \int_{x=\underline{v}}^{v} x\, P(dx),$$

i.e., the expected value of the good conditional on it being below its true value. This bidding function ensures that the uninformed bidders must get zero rents in equilibrium, because no matter what they bid, they must pay the expected value conditional on winning. In equilibrium, the uninformed bidders bid independently of one another and independently of the true value so that the marginal distribution of the highest of the $N-1$ uninformed bids is equal to the marginal distribution of the informed bid.

Let us compare the welfare properties of the equilibrium under their information structure with our bounds for the family of power distributions with support equal to $[0,1]$ and the cumulative distribution

$$P(v) = v^{\alpha},$$

where $\alpha \geq 0$. For this family of distributions, the informed bidder's strategy reduces to a deterministic bid of

$$\sigma(v) = \frac{\alpha}{\alpha+1} v.$$

Given the interpretation of the informed bid, we can immediately conclude that the expected value of the object is

$$\overline{T} = \frac{\alpha}{\alpha+1}.$$

We can think of the highest of the $N-1$ uninformed bids as also being of the same form $\sigma(v)$, but for an independent draw of v from the same prior. Thus, the surplus obtained by the informed bidder is

$$U^{EMW} = \int_{v=0}^{1} \left(v - \frac{\alpha}{1+\alpha} v \right) v^{\alpha} \alpha v^{\alpha-1} dv = \frac{\alpha}{(\alpha+1)(2\alpha+1)}.$$

Given our calculation of total surplus, revenue must be

$$R^{EMW} = \frac{2\alpha^2}{(\alpha+1)(2\alpha+1)}.$$

On the other hand, when $N = 2$, the revenue-minimizing winning bid function we obtained earlier is

$$\underline{\beta}(v) = \frac{\alpha}{\alpha+2}v.$$

Minimum revenue is therefore

$$\underline{R} = \frac{\alpha^2}{(\alpha+2)(\alpha+1)}$$

and maximum bidder surplus is

$$\overline{U} = \frac{2\alpha}{(\alpha+2)(\alpha+1)}.$$

We can now compare the welfare outcome in the equilibrium with the informed bidder with our bounds for the parametrized family of distributions. Note that the ratio of the bidder surplus between these two information structures is

$$\frac{\overline{U}}{U^{EMW}} = 2\left(\frac{2\alpha+1}{\alpha+2}\right).$$

This quantity is 2 when $\alpha = 1$, which corresponds to our earlier observation in the uniform example that the two bidders collectively earn twice as many rents in the bidder-surplus-maximizing equilibrium as does the informed bidder. As $\alpha \to 0$, the ratio converges to 1 so that the informed bidder asymptotically attains the lower bound on bidder surplus (which is zero). As $\alpha \to \infty$, the bidder-surplus ratio converges to 4, meaning that as the distribution of the common value converges weakly to a point mass on $v = 1$, each of the two bidders receives twice as much surplus as the informed bidder in Engelbrecht-Wiggans et al. We will revisit this comparison when we consider the many-bidder limit.

B.2 Proof of Proposition 3

Proof of Proposition 3. Let us construct the symmetrized winning-bid distributions. Let

$$K : V^N \to \Delta(B \times \mathcal{N})$$

denote the transition probability kernel associated with the $H_i(b|v)$, i.e.,

$$K([0,b] \times \{i\}|v) = H_i(b|v).$$

Then we can define a product measure $\phi \in \Delta\left(V^N \times B \times \mathcal{N}\right)$ according to

$$\phi(X) = \int_{(v,b,i)\in X} K(db, di|v)\,\mu(dv)$$

for measurable $X \subseteq V^N \times B \times \mathcal{N}$. Now, let us define the mapping

$$f_\xi : V^N \times B \times \mathcal{N} \to V^N \times B \times \mathcal{N}$$

according to

$$f_\xi(v,b,i) = (\xi(v), b, \xi(i)).$$

Then we can define the symmetrized distribution

$$\widetilde{\phi} = \frac{1}{N!}\sum_{\xi\in\Xi} \phi \circ f_\xi^{-1}.$$

Let us briefly verify that $\widetilde{\phi}$ has the symmetrized prior

$$\widetilde{\mu} = \frac{1}{N!}\sum_{\xi\in\Xi} \mu \circ \xi^{-1}.$$

as a marginal over V^N (which, we remark, is symmetric; cf. the proof of Lemma 3), and that it has the original distribution of winning bids $H(b)$ as its marginal over B. This follows from the observations that for any measurable $X \subseteq V^N$,

$$\phi \circ f_\xi^{-1}(X \times B \times \mathcal{N}) = \mu \circ \xi^{-1}(X),$$

and that for any b,

$$\begin{aligned}\phi \circ f_\xi^{-1}\left(V^N \times [0,b] \times \mathcal{N}\right) &= \sum_{i=1}^{N} \int_{v\in V^N} H_{\xi^{-1}(i)}\left(b|\xi^{-1}(v)\right)\mu \circ \xi^{-1}(dv) \\ &= \int_{v\in V^N} H(b|v)\,\mu(dv) = H(b).\end{aligned}$$

We next observe that

$$\xi \circ \alpha^{-1} = \alpha^{-1}$$

for every permutation ξ, where we recall that α denotes the average of the $N-1$ lowest values. This is because the average is invariant to permutations. As a result,

$$\widetilde{\mu} \circ \alpha^{-1} = \frac{1}{N!} \sum_{\xi \in \Xi} \mu \circ \xi^{-1} \circ \alpha^{-1} = \frac{1}{N!} \sum_{\xi \in \Xi} \mu \circ \alpha^{-1} = \mu \circ \alpha^{-1}.$$

Thus, the distribution of the average of the $N-1$ lowest values associated with μ, i.e., Q, is exactly the same as that associated with $\widetilde{\mu}$.

The final step of the proof is to use $\widetilde{\phi}$ to construct winning-bid distributions that are feasible for the relaxed program for the symmetric prior $\widetilde{\mu}$. We can disintegrate $\widetilde{\phi}$ (Çınlar, 2011, Theorem IV.2.18) to obtain a probability transition kernel $\widetilde{K} : V^N \to \Delta(B \times \mathcal{N})$ such that

$$\widetilde{\phi}(X) = \int_{(v,b,i) \in X} \widetilde{K}(db, di|v) \, \widetilde{\mu}(dv).$$

This kernel induces winning-bid distributions

$$\widetilde{H}_i(b|v) = \widetilde{K}([0,b] \times \{i\} \,|v).$$

Now, consider the left-hand side of (11):

$$\begin{aligned}
\int_{v \in V^N} (v_i - b) \, \widetilde{H}(b|v) \, \widetilde{\mu}(dv) &= \int_{(v,x,j) \in V^N \times [0,b] \times \mathcal{N}} (v_i - b) \, \widetilde{\phi}(dv, dx, dj) \\
&= \frac{1}{N!} \sum_{\xi \in \Xi} \int_{(v,x,j) \in V^N \times [0,b] \times \mathcal{N}} (v_i - b) \, \phi \circ f_\xi^{-1}(dv, dx, dj) \\
&= \frac{1}{N!} \sum_{\xi \in \Xi} \int_{v \in V^N} \left(v_{\xi(i)} - b\right) H(b|v) \, \mu(dv) \\
&= \frac{1}{N} \sum_{j=1}^{N} \int_{v \in V^N} (v_j - b) \, H(b|v) \, \mu(dv).
\end{aligned}$$

By a similar sequence of steps, we conclude that the right-hand side of (11) isv

$$\int_{v \in V^N} \int_{x=0}^{b} (v_i - x) \, \widetilde{H}_i(dx|v) \, \widetilde{\mu}(dv) = \frac{1}{N} \sum_{j=1}^{N} \int_{v \in V^N} \int_{x=0}^{b} (v_j - x) \, H_j(dx|v) \, \mu(dv).$$

Since (11) is satisfied for every $j = 1, \ldots, N$ for the measure μ and winning-bid distributions H_i, we conclude that (11) will also be satisfied for the symmetrized prior $\widetilde{\mu}$ and winning-bid distributions $\widetilde{H}_i$. Since both induce the same distribution of winning bids, we conclude that the solution to the relaxed program for $\widetilde{\mu}$ must be weakly lower than the

solution for μ, and we have argued that the solution for $\widetilde{\mu}$ is the $\underline{H}$ defined relative to the distribution of the average of the $N-1$ lowest values for μ. Finally, since any information structure $\mathcal{S}$ and equilibrium σ (under μ) induce a winning-bid distribution $H(\mathcal{S},\sigma)$ that is feasible for the relaxed program for μ, we conclude that $H(\mathcal{S},\sigma) \leq \underline{H}$. □

B.3 Proof of Theorem 2

Proof of Theorem 2. We construct a sequence of information structures and associated equilibria, indexed by $x \in (0,1)$, such that for every $\epsilon > 0$, there exists a sufficiently large x such that for all $x' \geq x$, revenue is within ϵ of the efficient social surplus.

The information structure is constructed as follows. Fix any $x \in (0,1)$. Given any realization of values, $v_1, ..., v_N$, every bidder i with the highest valuation v_i, and hence $v^{(1)} = v_i$ is told to bid

$$\overline{b} = xv^{(1)} + (1-x)v^{(2)}. \tag{21}$$

The other bidders are given recommendations:

$$\underline{b} = y_j v^{(1)} + (1-y_j)v^{(2)}, \tag{22}$$

where the y_j are independent random variables in $[0,x]$ and a distribution function parametrized by x:

$$y \sim F(y\,|x) = \left(\frac{y}{1-y}\frac{1-x}{x}\right)^{1/(N-1)}. \tag{23}$$

The bid distribution for the losing bidders are determined independently, and thus the highest y among the losing bidders, the first order statistic out of $N-1$ is given by:

$$y^{(1)} \sim F^{(1)}(y\,|x) = \left(\frac{y}{1-y}\frac{1-x}{x}\right).$$

We claim that the bidding strategies given by (21) and (22) are an equilibrium for this information structure for every $x \in (0,1)$. Clearly, conditional on the highest value $v^{(1)}$, the distributions of both winning bids and losing bids are absolutely continuous and have support equal to $\left[v^{(2)}, v^{(1)}\right]$. Thus, bidders can never infer from their bid recommendation that they are bidding more than their own value, and the proposed equilibrium strategy is not weakly dominated.

We now verify that there is no profitable deviation for any bidder. We establish the absence of a profitable deviation pointwise, that is for every realized profile of values, $v_1, ..., v_N$. First, note that if there are several bidders with the highest valuation, then $v^{(1)} = v^{(2)}$, and

by (21) and (22), it follows that $\underline{b} = \overline{b}$, and there are several winning bidders, and each one receives zero bidder surplus; yet clearly there is no profitable deviation for anybody. For the rest of the argument, it is then sufficient to consider the case of $v^{(1)} > v^{(2)}$.

Now, if the bid b is a recommendation for a losing bidder with value v_i, then it is never profitable to deviate to a higher bid since by construction $b > v_i$. Similarly, lowering the bid below b is not profitable either as it will not change the outcome of the auction. Next, if the bid b is a recommendation for a winning bidder i, then $b < v_i = v^{(1)}$ and a bid increase is not profitable as it does not change the outcome but rather leads to higher sale price. It remains to verify that the winning bidder has no incentive to lower his bid. Given the equilibrium bid, the payoff for winning bidder is:

$$v^{(1)} - \overline{b} = v^{(1)} - xv^{(1)} - (1-x)v^{(2)}.$$

By deviating to a lower bid b', the deviator will win whenever the realized y is below a critical level defined by $b' = yv^{(1)} + (1-y)v^{(2)}$. Given the distribution of y as defined by (23), the payoff from such a deviation is:

$$(v^{(1)} - (yv^{(1)} + (1-y)v^{(2)})) \left(\frac{y}{1-y} \frac{1-x}{x} \right) = (v^{(1)} - v^{(2)}) \left(y \frac{1-x}{x} \right),$$

which is increasing in y and at $y = x$ equals:

$$(v^{(1)} - v^{(2)}) (1-x),$$

which is the winning bidder's surplus given x. Thus there is no profitable deviation for the winning bidder either.

Finally, for each x, the expected winning bid is simply a convex combination of the expected highest and the expected second-highest values, with weights x and $1-x$ respectively. As x approaches 1, the expected winning bid converges to the expected highest value, and bidder surplus must therefore converge to zero. □

B.4 Inefficient Equilibria

We argue that the pure strategies given by

$$\sigma(s) = \frac{1}{P(s)} \int_{x=\underline{v}}^{s} xP(dx) \tag{24}$$

constitute an equilibrium. First, consider a bidder i who observes a signal s_i. If bidder i follows the equilibrium strategy and bids $\sigma(s_i)$, then they will win with probability $1/(N-1)$ when they had a high signal, which is when some other bidder had a higher value. But since bidder i's value is independent of the highest of others' values, the posterior distribution of bidder i's value on this event is precisely the truncated prior $P(v_i)/P(s_i)$ with support equal to $[\underline{v}, s_i]$. Thus, the expected valuation conditional on winning is precisely $\sigma(s_i)$, and the bidder obtains zero rents in equilibrium.

Now, consider a bidder i who deviates down to some $\sigma(s')$ with $s' < s_i$. We can separately consider the case of $N = 2$ and $N > 2$. In the latter case, there is more than one bidder who sees a signal equal to the highest value, and therefore in equilibrium there is a tie for the highest bid at $\sigma(s_i)$. Thus, a downward deviator will always lose the auction and obtain zero rents. On the other hand, if $N = 2$, then the bidder wins whenever the other bidder's signal was less than s'. But since the other bidder's signal equal to v_i, the event where bidder i wins is precisely when v_i is in the range $[\underline{v}, s']$, so that the expectation conditional on winning is $\sigma(s')$, and the deviator still obtains zero rents.

Finally, let us consider a bidder i who deviates up to $\sigma(s')$ with $s' > s_i$. This bidder will now win outright whenever s_i was equal to the maximum valuation. Moreover, when s_i was a losing signal, the bidder will now win whenever others' signals were less than s'. But on this event, others' signals are equal to $v_i = \max v$. Thus, the upward deviator will win whenever $v_i \leq s'$, and again the expected value conditional on winning is precisely $\sigma(s')$, so that the deviator's surplus is still zero.

We observe that the realized value among the winning bidders is exactly given by the average value among the $N-1$ bidders with the lowest values, or

$$\alpha(v) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \max v\right).$$

It follows that the revenue of the seller is given exactly by the expectation over the average value among the $N-1$ lowest valuations. We have therefore proven the following:

Theorem 3 (Inefficient Equilibrium)**.**
The strategies (24) are an equilibrium for the information structure in which each bidder observes $s_i = \max v_{-i}$. In this equilibrium, revenue and total surplus are both equal to

$$\int_{\underline{v}}^{\overline{v}} wQ(dw),$$

and bidder surplus is zero.

We note that this equilibrium construction can be extended well beyond the independent values case. In such a generalization, the equilibrium bid would be each bidder's expected value conditional on it being less than the observed maximum of others' values. As long as there is sufficient positive correlation between bidders' values, e.g., affiliation, this bidding function will be strictly increasing, and for this more general class, the upward incentive constraints will be satisfied as strict inequalities.

B.5 Reserve Prices

In this section, we analyze the first-price auction with a minimum bid r in the case of pure common values. Let P denote the distribution of the value on $V = [\underline{v}, \overline{v}]$, and suppose $r \in V$. The auction is as described in Section 2, except that

$$q_i(b) = \frac{\mathbb{I}_{i \in W(b)}}{|W(b)|},$$

where

$$W(b) = \{i | b_i \geq b_j \ \forall j \text{ and } b_i \geq r\}.$$

In other words, a bidder only wins if they bid more than the reserve price, and ties are broken randomly among the winners. Let us construct an equilibrium as follows. Let x_i be i.i.d. draws from $F(s) = (P(s))^{1/N}$, correlated with the value so that $v = \max_i x_i$. Bidder i's signal is $s_i = \max\{\hat{v}, x_i\}$, where $\hat{v}$ solves

$$\frac{1}{P(\hat{v})} \int_{v=\underline{v}}^{\hat{v}} vP(dv) = r. \tag{25}$$

Bidders follow a pure strategy in equilibrium $\underline{\beta}$, defined by $\underline{\beta}(s_i) = 0$ if $s_i = \hat{v}$, and otherwise bid

$$\underline{\beta}(s_i) = \frac{1}{(P(s_i))^{\frac{N-1}{N}}} \left(r + \int_{v=\hat{v}}^{s_i} \frac{N-1}{N} v \frac{P(dv)}{(P(v))^{\frac{1}{N}}} \right), \tag{26}$$

which we note for future reference is the solution to the differential equation

$$\beta'(s_i) = \frac{N-1}{N} (s_i - \beta(s_i)) \frac{P(ds_i)}{P(s_i)}$$

with the boundary condition $\beta(\hat{v}) = r$.

To verify that this is an equilibrium, first observe that a bidder with signal $s_i = \hat{v}$ who deviates up to $b_i = r$ will win if and only if $s_j = \hat{v}$ for all $j \neq i$, which is when $v < \hat{v}$. The

conditional expectation of the value of the good is no more than r, so that the bidder obtains non-positive surplus conditional upon winning. Now consider the surplus from bidding as a type $w > \hat{v}$ when $s_i = \hat{v}$. In this case, surplus is

$$\left(r - \underline{\beta}(w)\right)\left(P(\hat{v})\right)^{\frac{N-1}{N}} + \int_{v=\hat{v}}^{w}\left(v - \underline{\beta}(w)\right)\frac{N-1}{N}\frac{P(dv)}{\left(P(v)\right)^{\frac{1}{N}}}.$$

The marginal change in surplus from an increase in w is therefore

$$\left(w - \underline{\beta}(w)\right)\frac{N-1}{N}\frac{P(dw)}{\left(P(w)\right)^{\frac{1}{N}}} - \underline{\beta}'(w)\left(P(w)\right)^{\frac{N-1}{N}} = 0$$

by definition of the bidding function. Similarly, now consider a bidder with signal $s_i > \hat{v}$ who bids $\underline{\beta}(w)$ for some $w \geq s_i$. Surplus is

$$\left(s_i - \underline{\beta}(w)\right)\left(P(s_i)\right)^{\frac{N-1}{N}} + \int_{v=s_i}^{w}\left(v - \underline{\beta}(w)\right)\frac{N-1}{N}\frac{P(dv)}{\left(P(v)\right)^{\frac{1}{N}}},$$

which also has zero derivative. Finally, let us verify that a bidder with signal $s_i > \hat{v}$ does not want to deviate down to a bid $\hat{\underline{\beta}}(w)$ for $w \leq s_i$. It is straightforward to see that $\underline{\beta}(s_i) \leq s_i$ for all s_i, so that surplus in equilibrium is non-negative. Thus, it is not attractive to deviate for $w = \hat{v}$. Otherwise, surplus from the downward deviation is

$$\left(s_i - \underline{\beta}(w)\right)\left(P(w)\right)^{\frac{N-1}{N}}$$

which has derivative

$$\left(s_i - \underline{\beta}(w)\right)\frac{N-1}{N}\frac{P(dw)}{\left(P(w)\right)^{\frac{1}{N}}} - \underline{\beta}'(w)\left(P(w)\right)^{\frac{N-1}{N}},$$

which, using the formula for $\underline{\beta}'$, becomes

$$\left(s_i - \underline{\beta}(w)\right)\frac{N-1}{N}\frac{P(dw)}{\left(P(w)\right)^{\frac{1}{N}}} - \frac{N-1}{N}\left(w - \underline{\beta}(w)\right)\frac{P(dw)}{P(w)}\left(P(w)\right)^{\frac{N-1}{N}} = (s_i - w)\frac{N-1}{N}\frac{P(dw)}{\left(P(w)\right)^{\frac{1}{N}}},$$

which is positive. This completes the proof that the construction is an equilibrium, and we note that revenue is simply

$$\underline{R} = \int_{v=\hat{v}}^{\overline{v}} \underline{\beta}(v)\, P(dv).$$

Now let us prove that this construction attains a lower bound on revenue. As before, we

set up a relaxed program:

$$\max H\left(\hat{b}\right)$$

subject to

$$\int_{v\in V}(v-b)\,H\left(b|v\right)P\left(dv\right)\leq\int_{v\in V}\int_{x=r}^{b}(v-x)\,H_{i}\left(dx|v\right)P\left(dv\right)$$

for all $b\geq r$, where the H_i are conditional distributions over the high bidder and the highest bid given the true value, and $H\left(b\right)$ is the aggregate distribution of highest bids. Note that we now distinguish between "highest" and "winning", since no one wins the good when the winning bid is less than r, and revenue is only

$$R=\int_{b=r}^{\overline{v}}bH\left(db\right).$$

The constraint of course represents the uniform upward incentive constraint for deviating up to $b\geq r$. It differs from (11) in that the right-hand side only counts the equilibrium surplus from winning with the highest bid recommendation when it is at least r, although the left-hand side still counts surplus from winning whenever the highest recommendation is less than b.

By similar arguments as those provided in Section 4, we can conclude that it is without loss of generality to look at solutions that are symmetric and monotonic. Thus, there is a deterministic and increasing highest bid $\beta\left(v\right)$ as a function of the true value. The incentive constraint thus becomes

$$\int_{v=\underline{v}}^{w}\left(v-\beta\left(w\right)\right)P\left(dv\right)\leq\frac{1}{N}\int_{v=\hat{v}}^{w}\left(v-\beta\left(v\right)\right)P\left(dv\right),$$

where $\hat{v}$ is the critical type at which $\beta\left(v\right)>r$ for all $v>\hat{v}$. Thus, uniform deviations up to $\hat{v}$ are not attractive as long as

$$\int_{v=\underline{v}}^{\hat{v}}\left(v-r\right)P\left(dv\right)\leq0,$$

and since we wish to minimize β (by making it zero for as many types as possible), it is optimal to set $\hat{v}$ as in the construction, so that (25) holds. Thus, the integral inequality

becomes:

$$\begin{aligned}\beta(w) P(w) &\geq \int_{v=\underline{v}}^{w} vP(dv) - \frac{1}{N}\int_{v=\hat{v}}^{w} (v - \beta(v)) P(dv) \\ &= rP(\hat{v}) + \frac{N-1}{N}\int_{v=\hat{v}}^{w} P(dv) + \frac{1}{N}\int_{v=\hat{v}}^{w} \beta(v) P(dv).\end{aligned}$$

It is clear that a minimum must solve this integral inequality as an equality for all w. (26) is precisely the solution to that integral equality. This completes the proof that the construction attains a lower bound on revenue.

We conclude with a description of minimum bidding in the uniform case, in which $V = [0,1]$ and $P(v) = v$. In that case, $\hat{v} = 2r$, and

$$\begin{aligned}\underline{\beta}(w) &= \frac{1}{w^{\frac{N-1}{N}}}\left(r + \frac{N-1}{N}\int_{v=\hat{v}}^{w} v^{\frac{N-1}{N}} dv\right) \\ &= \frac{1}{w^{\frac{N-1}{N}}}\left(r + \frac{N-1}{2N-1}\left(w^{\frac{2N-1}{N}} - (2r)^{\frac{2N-1}{N}}\right)\right) \\ &= \frac{N-1}{2N-1}w + \frac{1}{w^{\frac{N-1}{N}}}\left(r - \frac{N-1}{2N-1}(2r)^{\frac{2N-1}{N}}\right).\end{aligned}$$

Revenue is therefore

$$\begin{aligned}\underline{R} &= \int_{w=2r}^{1}\left(\frac{N-1}{2N-1}w + \frac{1}{w^{\frac{N-1}{N}}}\left(r - \frac{N-1}{2N-1}(2r)^{\frac{2N-1}{N}}\right)\right) dw \\ &= \frac{N-1}{2N-1}\frac{1}{2}\left(1-4r^2\right) + \left(r - \frac{N-1}{2N-1}(2r)^{\frac{2N-1}{N}}\right) N\left(1 - (2r)^{\frac{1}{N}}\right).\end{aligned}$$

For the case of $N = 2$, this reduces to

$$\underline{R} = \frac{1}{6}\left(1-4r^2\right) + 2r\left(1 - \frac{2\sqrt{2}}{3}\sqrt{r}\right)\left(1 - \sqrt{2r}\right).$$

Differentiating with respect to r, we obtain

$$\begin{aligned}-\frac{4}{3}r + 2\left(1 - \frac{2\sqrt{2}}{3}\sqrt{r}\right)\left(1 - \sqrt{2r}\right) - \sqrt{r}\left(1 - \sqrt{2r}\right)\frac{2\sqrt{2}}{3} - \sqrt{2r}\left(1 - \frac{2\sqrt{2}}{3}\sqrt{r}\right) \\ = 2 - 5\sqrt{2}\sqrt{r} + 4r\end{aligned}$$

which has zeros at

$$\sqrt{r^*} = \frac{5\sqrt{2} \pm \sqrt{50-32}}{8} = \frac{(5 \pm 3)\sqrt{2}}{8}.$$

But r must be between 0 and 1, so the critical point must be $\sqrt{r^*} = \sqrt{2}/4$, so that $r^* = 1/8$. We leave it as an exercise for the reader to verify second-order conditions.