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By

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Dynamic Moral Hazard without Commitment*

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Abstract. We study a discrete-time model of repeated moral hazard without commitment. In every period, a principal finances a project, choosing the scale of the project and a contingent payment plan for an agent, who has the opportunity to appropriate the returns of a successful project unbeknownst the principal. The absence of commitment is reflected both in the solution concept (perfect Bayesian equilibrium) and in the ability of the principal to freely revise the project's scale from one period to the next. We show that removing commitment from the equilibrium concept is relatively innocuous—if the players are sufficiently patient, there are equilibria with payoffs low enough to effectively endow the players with the requisite commitment, within the confines of perfect Bayesian equilibrium. In contrast, the frictionless choice of scale has a significant effect on the project's dynamics. Starting from the principal's favorite equilibrium, the optimal contract eventually converges to the repetition of the stage-game Nash equilibrium, operating the project at maximum scale and compensating the agent (only) via immediate payments.

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1 Introduction

1.1 Dynamic Moral Hazard and Commitment

The objective of this paper is to understand the role of commitment in dynamic moral hazard problems. A growing literature (discussed in Section 1.2) has developed a theory of firm dynamics under financial constraints and asymmetric information. This literature typically assumes that imperfections in financial markets force the agent to rely on external financing from a principal to achieve the requisite liquidity to operate a profitable project, but the ability of the principal and the agent to commit to a long-term contract suffers no such imperfection. This might be a reasonable approximation for financial contracts enforceable by courts, but we believe that this commitment is problematic in a number of applications. These include lending between sovereign nations, where there may be no applicable courts, as well as cases in which contracts cannot be written so precisely as to admit legal enforcement.

Commitment is usually embedded in dynamic moral hazard models in two ways. The first is the solution concept. Rather than solving for the best (perfect Bayesian or subgame-perfect) equilibrium, the literature typically models the problem as solving for the best dynamic mechanism from the principal's point of view, assuming that the principal will adhere to the resulting contract after all contingencies, regardless of whether it will be in her best interest to do so. Instead of giving the principal such unlimited commitment power, we explicitly model the long-term relationship as a two-player repeated extensive-form game, with no commitment power on either side. The second source of commitment is the extensive-form stage game itself. By assuming that it is costly to adjust the relationship's size, existing models endow the principal with some commitment, as (for instance) scaling down the project cannot be easily reversed in the future. In particular, Biais, Mariotti, Rochet and Villeneuve [3] assume that this adjustment is asymmetric: while downsizing is unconstrained, the pace of expansion is limited. Instead, we assume that the project's size can be costlessly revised (up or down) from one period to the next.

Our goal is twofold. First, we characterize the the behavior underlying equilibria giving payoffs on the frontier of equilibrium payoffs. Second, we characterize the long-run dynamics of the project. We show that (the lack of) commitment has important implications. As is usual in this class of models, the principal has three instruments with which to create incentives for the agent. In each period, the principal chooses the scale of the project, makes contingent payments to the agent, and makes promises of continuation utility to the agent. The latter reflects promises involving the scale of the project and payments in future periods, and is of course constrained by equilibrium considerations. We show that equilibria giving frontier payoffs combine these instruments in one of three ways, depending upon the agent's equilibrium payoff. If the agent's payoff is sufficiently small, then no payments are made and incentives for the agent arise entirely out of promised continuation utilities. Depending on the performance of the agent, this continuation utility increases or decreases. The scale of the project dynamically adjusts accordingly, increasing or decreasing as does the agent's promised utility. Nonetheless, both the scale and the promised utility drift upward on average. This process lasts a random length of time but with probability one the agent's payoff eventually climbs into a region in which again no payments are made, but the project is always operated at maximum scale. Incentives are created by promises of continuation utility that increase or decrease in response to the agent's performance. This process again lasts a random amount of time, and may result in the agent's payoff slipping back into the region in which the scale of the contract varies, but with probability one the agent's payoff eventually climbs to a third and absorbing level. Here, the project is operated at maximum scale and continuation utilities are constant, with incentives created entirely via immediate payments.

This contrasts with the dynamics found in the literature. When adjustment costs are positive, there is a positive probability that the scale of the project declines to zero and the project is abandoned. While an extraordinary string of bad luck may push our project to an arbitrarily small size, we show that on average it grows, and it eventually (almost surely) reaches a plateau on which it is thereafter operated at maximum scale.[1]

Our first step toward this characterization of behavior is to show that of the two ways we relax the standard commitment assumptions, it is relatively innocuous to model examine perfect Bayesian equilibria. We show that for sufficiently high discount factors, equilibria exist that are arbitrarily severe for the principal, yielding a zero payoff. Using these equilibria as threats in case the principal deviates from equilibrium behavior, we construct equilibria in which the principal effectively commits to future courses of action. It is of course a familiar result that patient players can achieve outcomes in repeated games that cannot be implemented under high discounting. However, our results are not limiting results for arbitrarily patient players, but are built on the (less obvious) demonstration that the set

[1]Another difference with the literature is that we do not need to assume that the principal and agent have different discount factors.

of commitment outcomes is realized for a fixed discount rate. In contrast, the frictionless adjustment in the size plays an important role, and is in particular critical in showing that the project is never abandoned.

1.2 Related Literature

Our paper is related to two bodies of literature. The first is concerned with financial contracting. The paper in this area most directly related to ours, and the motivation for this paper, is Biais, Mariotti, Rochet and Villeneuve [3]. Biais, Mariotti, Rochet and Villeneuve [3] consider a quite similar problem, but solve for the optimal contract for the principal under the assumption that the principal commits to her future course of action. We do not allow the principal the luxury of commitment to future contract terms, but as we have noted, sufficient patience allows us to construct equilibria in which both players receive zero payoff. This in turn can be used to punish the principal for deviations from a putative equilibrium path, effectively recovering the ability to commit. More importantly, however, Biais, Mariotti, Rochet and Villeneuve [3] work with a model in which the principal can at any time implement a downward jump in the scale of the project, but cannot choose upward jumps. Instead, the principal can expand the scale of the project only gradually, by choosing the rate at which the project grows, with the set of possible rates constrained to a bounded interval. This forcibly commits the principal to not let the project grow too rapidly. By allowing the principal to choose any project scale, we work without this commitment possibility.

Related papers include Clementi and Hopenhayn [4], DeMarzo and Fishman [5, 6] and Quadrini [10]. Just as we do, DeMarzo and Fishman [5] work in a discrete-time framework, but with a finite horizon which allows them to solve their model by backward induction. In contrast, our environment is stationary, and the infinite horizon allows us to investigate the long-run properties of the model. In related models in which the moral hazard problem is one in which the firm's manager privately observes the firm's cash flow and can divert it to himself, as in ours, but which differ from our analysis in assuming that size adjustments are costly, Quadrini [10] and DeMarzo and Fishman [6] contrast the commitment solution to the renegotiation-proof contract. We discuss the relationship between our notion of no-commitment with renegotiation-proofness in Section 4.1.

An early contribution to the study of repeated principal-agent problems which developed many of the technical and conceptual ideas in this literature is Thomas and Worrall [16]. They show that the marginal payoff of the principal is a martingale, a property that plays a key role in our analysis as well, and they investigate conditions under which the agent's utility converges to zero in the long-run ("immiseration"), a property that does not arise in our model.

The second strand of related literature considers repeated games, disavowing commmitment. Radner [11], Rubinstein [12] and Rubinstein and Yaari [13] show how review strategies can be used to achieve efficient outcomes in repeated principal-agent problems when the players are sufficiently patient. In contrast, efficient outcomes can be achieved as equilibria of the stage game in our setting, though the principal's best payoff in the repeated game hinges upon inefficiency. Wen [17] (see also Mailath and Samuelson [9, Section 9.6]) establishes a folk theorem for repeated extensive form games that (unlike) ours involve no moves by nature. In contrast to this literature, we are concerned not with characterizing the limiting set of equilibrium payoffs as players become arbitrarily patient but with characterizing equilibrium behavior and payoffs for fixed discount factors.

2 The Game

The horizon is discrete and infinite, with rounds indexed by $n = 0, 1 \ldots$

2.1 The Stage Game

In each round n, the following three-stage game unfolds. There are two players, the principal and the agent. First, the principal chooses a scale for the project, which is a scalar $q_n \in [0, 1]$, as well as two (conditional) payments $z_n = (z_n^F, z_n^S) \in \mathbf{R}_+^2$. This choice is publicly observed. Second, Nature determines the outcome of a binary random variable $\omega_n \in \{s, f\}$, which takes the value s ("success") with probability $p \in (0, 1)$, f ("failure") occurring with complementary probability. This outcome is drawn independently across rounds, and independently of all choices of the players. It is privately observed by the agent. Finally, the agent then sends a public message, $m_n \in \{S, F\}$. However, the message set is a function of the realization of ω_n: in the event $\{\omega_n = f\}$, only the message F is available. In the event $\{\omega_n = s\}$, both messages are available. Hence, the message S reveals that the state is s, but the message F does not reveal the state. We interpret S as disclosure of as success, and F as non-disclosure.[2]

The principal has access to a public randomization device, on whose realization she can condition her actions. As is customary, this randomization device is dropped from the

[2]Hence, messages aren't "cheap." Adding messages by the agent to this binary disclosure decision would not affect the analysis: the agent would have to be indifferent over messages, and from the principal's point of view such messages wouldn't achieve anything she cannot achieve with the public randomization device.

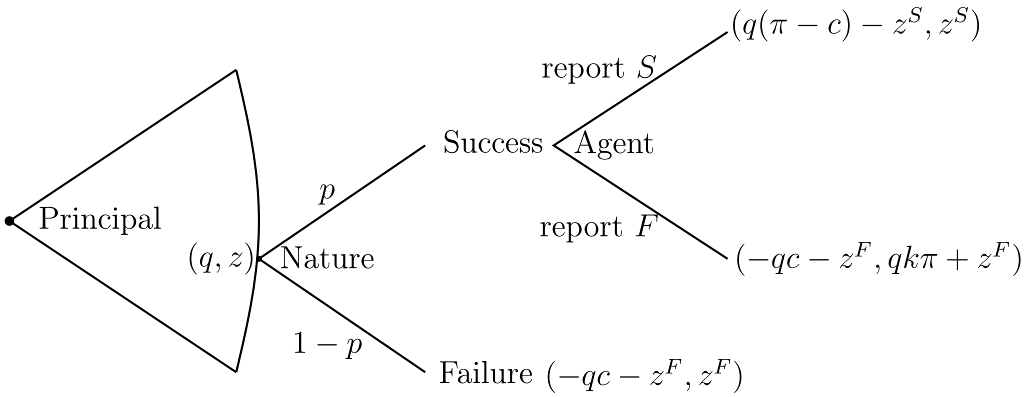

Figure 1: Extensive form (with rewards).

notation.

Realized rewards in round n are as follows. If $(\omega_n, m_n) = (f, F)$, the agent's utility is $u_n(q, z, f, F) = z^F$; if $(\omega_n, m_n) = (s, F)$, then $u_n(q, z, s, F) = qk\pi + z^F$, where $k \in (0, 1)$. We interpret k as the inefficiency in the agent's appropriating the success of the project for his own use, where $\pi > 0$ is the per-unit value of the project. If a success is reported, then the state must be s, and the agent's reward is $u_n(q, z, s, S) = z^S$.

As for the principal, she must always pay a cost $c > 0$ (per unit) for the project. If it is a success that is reported (and hence, a success actually obtained), her reward is $v_n(q, z, s, S) = q(\pi - c) - z^S$; if a failure is reported, then $v_n(q, z, \omega, F) = -qc - z^F$, independently of the state. Hence, q scales the project's costs and returns.

Figure 1 illustrates the stage game.

2.2 The Repeated Game

Players discount rewards with the common factor $\delta \in [0, 1)$. Hence, the principal's and agent's realized (average) payoffs are, respectively,

$$(1-\delta)\sum_{n=0}^{\infty} \delta^n v_n, \quad (1-\delta)\sum_{n=0}^{\infty} \delta^n u_n.$$

A history up to round N for the principal is a sequence $h_P^N := \{(q_n, z_n, m_n)\}_{n=0}^{N-1}$, with $(q_n, z_n, m_n) \in [0,1] \times \mathbf{R}_+^2 \times \{S, F\}$. The set of such histories is denoted by H_P^N (where we set $H_P^0 = \{\emptyset\}$, with singleton element h_P^0). A history up to round N for the agent is a sequence $h_A^N := \{(q_n, z_n, \omega_n, m_n)\}_{n=0}^{N-1}$ and a triple (q_N, z_N, ω_N), with $(q_n, z_n, \omega_n, m_n) \in$

$[0,1] \times \mathbf{R}^2_+ \times \{s, f\} \times \{S, F\}$, and the obvious restriction that $\omega_n = f \Rightarrow m_n = F$. The set of such histories is denoted by H^N_A (where we set $H^0_A = [0,1] \times \mathbf{R}^2_+ \times \{s, f\}$, with generic element $h^0_A = (q_0, z_0, \omega_0)$). Notice that realized rewards are not publicly observed, so that if the agent reports a failure in period n, the principal never learns whether Nature drew a success or failure in that period.

A (behavior) strategy σ_P for the principal is a sequence $\{\sigma^n_P\}$, where σ^n_P is a probability transition from H^n_P into $[0,1] \times \mathbf{R}^2_+$.[3] A (behavior) strategy σ_A for the agent is a sequence $\{\sigma^n_A\}$, where σ^n_A is a probability transition from H^n_A into $\{S, F\}$, with the obvious restriction on feasible messages.

A strategy profile $\sigma = (\sigma_P, \sigma_A)$ defines a distribution over infinite histories in the usual way, and players maximize their expected payoff. We write

$$W(\sigma) = (1-\delta)\mathbf{E}_\sigma\left[\sum_{n=0}^{\infty} \delta^n v_n\right], \quad V(\sigma) = (1-\delta)\mathbf{E}_\sigma\left[\sum_{n=0}^{\infty} \delta^n u_n\right],$$

although we often drop the argument σ when the strategy profile is understood. We are also interested in the total surplus,

$$S = W + V.$$

The solution concept is perfect Bayesian equilibrium. Hence, no commitment is assumed on either side, although within a round the principal commits to the payments z that she announces.

We note that information is imperfect, but has a product structure, because the principal does not observe the state, while the agent does. Hence, any perfect Bayesian equilibrium in which the agent's strategy in a given round n depends on private information acquired during previous rounds, conditional on the public information, is outcome-equivalent to an equilibrium in which it does not. Hence, without loss, we assume that σ^n_A only depends on the public information h^n_P up to the current round, as well as the triple (q_n, z_n, ω_n) relevant to the current round. This ensures that, given h^n_P, continuation payoffs are common knowledge (so that dynamic programming can be used), and with an abuse of notation, we write $W(h^n_P), V(h^n_P)$ for continuation. The principal has no private information, and even if the agent makes an observable deviation, the serial independence of the process $\{\omega_n\}_{n=0}^\infty$ makes it irrelevant for updating about future moves by Nature, and hence perfect Bayesian equilibrium leaves no ambiguity about how to revise beliefs.

[3]That is, for each $h^n_P \in H^n_P$, $\sigma^n_P(h^n_P)$ is a probability distribution over $[0,1] \times \mathbf{R}^2_+$, and the probability $\sigma^n_P(\cdot)[A]$ assigned to any Borel set $A \subset [0,1] \times \mathbf{R}^2_+$ is a measurable function of h^n_P, and similarly for σ^n_A.

2.3 Stage-Game Equilibrium

We assume throughout that

$$p\pi > c,$$

so that operating the project is efficient. In the absence of an agency problem, the efficient outcome is thus to operate the project at scale $q = 1$. We refer to $p\pi - c$ as the efficient surplus.

Here, we examine subgame-perfect equilibria of the extensive-form stage game (played once), suppressing the time subscript.[4] In the event of a success, the agent will report

$$\begin{array}{lll} S & \text{if} & z^S > qk\pi + z^F, \\ F & \text{if} & z^S < qk\pi + z^F. \end{array}$$

The agent is indifferent in the event that $z^S = qk\pi + z^F$, but equilibrium will require that this indifference be broken in favor of reporting a success.

The principal can induce the agent to report a success, given that Nature has drawn a success, if (given that an indifferent agent reports a success) and only if the payoff differential $z^S - z^F$ is at least as large as the value the agent receives from appropriating the success, *i.e.*, if and only if $z^S - z^F \geq qk\pi$. It is then immediate that the principal will set $z^F = 0$, and the principal's payoffs are then

$$\begin{array}{rll} -qc & \text{if} & z^S < qk\pi, \\ p(q\pi - z^S) - qc & \text{if} & z^S \geq qk\pi. \end{array}$$

The principal will operate the project on either the largest ($q = 1$) or smallest ($q = 0$) scale, setting $z^S = qk\pi$ in the first case and with z^S arbitrary in the second, depending on whether operating the project on a positive scale is profitable ($p(1-k)\pi - c > 0$) or unprofitable ($p(1-k)\pi - c < 0$). We can summarize with the following immediate result:

Proposition 1

[1.1] Suppose $p(1-k)\pi - c > 0$. Then in the unique equilibrium outcome of the one-shot game the principal sets $q = 1$ and $z = k\pi$. Payoffs to the principal and the agent are:

$$W^* = pk\pi, \tag{1}$$

$$V^* = p(1-k)\pi - c. \tag{2}$$

[4] The stage game is a game of imperfect information, since the principal does not observe Nature's draw, but nonetheless subgame perfection implies sequential rationality, and hence a subgame-perfect equilibrium is also (part of) a perfect Bayesian equilibrium.

[1.2] Suppose $p(1-k)\pi - c < 0$. Then in the unique equilibrium outcome the principal sets $q = 0$ and payoffs are $(0, 0)$.

In the first case ($p(1-k)\pi - c > 0$), the stage game exhibits a unique subgame-perfect equilibrium outcome (notice that the agent's action is arbitrary in the out-of-equilibrium event that $q = 0$ and $\omega = s$) in which the total payoff equals the efficient surplus. In the second case ($p(1-k)\pi - c < 0$), even though the total surplus is positive, inducing the agent to reveal a success is too expensive for the principal to induce. There is again a unique subgame-perfect equilibrium outcome, in which payoffs are $(0, 0)$.

3 Equilibria in the Dynamic Moral Hazard Game

Our objective is to characterize the set of equilibrium payoff vectors $(W, V) \in \mathbf{R}^2_+$ in the repeated game, which we denote by $\mathcal{V}$. In particular, we characterize the boundary of this set, and identify the equilibrium strategies that give rise to these boundary payoffs.

The set $\mathcal{V}$ of equilibrium payoffs in the repeated game is a subset of the set of feasible and individually rational payoffs $\mathcal{F} := \{(W, V) \in \mathbf{R}^2_+ : W + V \leq p\pi - c\}$. We continue to maintain the assumption that $p\pi - c > 0$, so that this set is not only nonempty but contains some strictly positive payoffs. We say that an equilibrium with payoffs (W, V) achieves the efficient surplus if $W + V = p\pi - c$.

Note that ours is a repeated extensive-form game, which as is well known raises some subtleties regarding incentives and the lowest equilibrium payoffs that differ from those in standard repeated games. Nonetheless, we will adopt standard terminology from repeated games, such as self-generation (see Abreu, Pearce and Stacchetti [1]).

3.1 Constrained Efficient Equilibria

We say that an equilibrium is constrained efficient if it maximizes, over the set of equilibria, the sum of the principal's and the agent's payoff. Section 5.1 proves the following lemma, which characterizes the constrained efficient equilibria of the repeated game:

Lemma 1

[1.1] If $p(1-k)\pi - c > 0$, then in any constrained efficient equilibrium, we have $q = 1$ in each period on path, with a sum of payoffs equal to $W^* + V^* = p\pi - c$.

[1.2] If $p(1-k)\pi - c > 0$, then in any constrained efficient equilibrium outcome, the agent's payoff satisfies $W \geq W^*$.

[1.3] If $p(1-k)\pi - c < 0$, *then the repeated game has a unique equilibrium payoff,* $(0,0)$.

The first result is straightforward. If $p(1-k)\pi - c > 0$, then the stage game has a subgame-perfect equilibrium that achieves the efficient surplus, and one can then also achieve the efficient surplus in the repeated game by playing this stage-game equilibrium in every period. The second result is achieved by showing that whenever the project is operated, generating surplus $p\pi - c$, at least $pk\pi$ of this surplus must go to the agent in order to satisfy the agent's incentive constraint. This is enough to ensure that the agent's payoff is at least W^*. The final result follows from noting that if $p(1-k)\pi - c < 0$, then allocating payoff W^* to the agent relegates the principal to a negative payoff. The only equilibrium must then be trivial, giving payoff $(0,0)$.

The inequality $p(1-k)\pi - c < 0$ suffices to ensure that in the stage game, there is a unique equilibrium payoff, namely $(0,0)$. It is a familiar result that repeated games allow equilibrium payoffs that cannot be obtained in the stage game. Recalling that $p\pi - c > 0$, so that the efficient surplus is positive, one might have thought that the repeated game would allow positive payoffs even if the stage game does not. However, the final part of this lemma indicates that when $p(1-k)\pi - c < 0$, repeating the stage game allows no new opportunities for positive payoffs.

If $p(1-k)\pi - c > 0$, the repeated game has equilibria that achieve the efficient surplus. There may be other equilibria that attain the efficient surplus, but all must give the agent a higher payoff. If the repeated game is to bring the principal a higher payoff than does the stage game, this must come at the cost of inefficiency.

In light of this result we restrict attention to the interesting case in which $p(1-k)\pi - c > 0$, since both the stage game and the repeated game are trivial if this strict inequality is reversed.

3.2 A Self-Generating Set

Lemma 1 shows that, under our maintained assumption of $p(1-k)\pi - c > 0$, so that the game is not trivial, the repeated game admits an equilibrium that achieves the efficient surplus, with payoffs (W^*, V^*). But these payoffs are already available as the unique equilibrium payoffs in the stage game. What does repeating the game add to the set of possible equilibria?

To ensure that the set of repeated-game equilibria is sufficiently rich, we impose a lower bound on the discount factor by maintaining throughout the following assumption:

$$\frac{1-\delta}{\delta} k\pi < p(1-k)\pi - c < \frac{p\delta}{1-\delta}(p\pi - c). \tag{3}$$

We explain as we proceed where we use these conditions, and explore in Section 4.2 what happens if this assumption fails. Notice that a necessary condition for the left inequality is the familiar condition $p(1-k)\pi - c > 0$. Given this inequality, both inequalities in (3) will hold as long as the players are sufficiently patient. Neither of the conditions in (3) implies the other.

We then establish the following:

Lemma 2 *There exists a self-generating set of equilibrium payoffs corresponding to two equilibria with the following properties:*

- *The first equilibrium yields payoffs* $(0,0)$. *The principal sets* $q = 0$ *in each period. Should the principal set* $q > 0$ *in any period, play immediately switches to that specified by the second equilibrium, which specifies the agent's response.*
- *The second equilibrium yields payoffs* $(p\pi - c, 0)$. *The principal sets* $q = 1$, $z^F = 0$, *and* $z^S = \pi - c/p$ *in every period, while the agent reports any successes. Should the principal ever choose a different triple* (q, z^S, z^F) *with the property that a reported success would give the principal a positive payoff, play switches immediately to that specified by the first equilibrium.*

The second equilibrium in this self-generating pair achieves the efficient surplus, but offers all of this surplus to the agent. The first equilibrium realizes none of the surplus. Each equilibrium exhibits a stationary outcome path, with incentives created by the threat of switching to the other equilibrium. The equilibrium with payoffs $(0,0)$ establishes that the minmax values can be simultaneously achieved as equilibrium outcomes. We refer to this as the minmax equilibrium.

The intuition behind this construction is that the principal's payoffs are zero in either equilibrium. Deviations on the part of the principal simply prompt a change to the other equilibrium, ensuring the principal's payoffs are zero after every history, and hence that the principal's behavior is optimal. Incentives for the agent are constructed by specifying that prescribed behavior leads to the continuation payoff $p\pi - c$, while proscribed behavior leads to continuation payoff 0.

Proof. Consider the minmax equilibrium. If the principal sets $q = 0$, the agent's actions have no effect on payoffs, and hence any action is a best response. In response to any (q, z^S, z^F) with $qk\pi + z^F \geq z^S$, the agent conceals any success, and continuation play exhibits payoffs $(0,0)$. This is a best response for the agent and is at least weakly suboptimal for the principal. Should the principal offer a triple (q, z^S, z^F) with $qk\pi + z^F < z^S$ and $q(p\pi - c) - pz^S - (1-p)z^F \leq 0$, the agent accepts, and continuation payoffs are given by $(0,0)$. Once again this is a best response for the agent and at least weakly suboptimal for the principal. Should the principal offer a triple (q, z^S, z^F) with $qk\pi + z^F < z^S$ and $q(p\pi - c) - pz^S - (1-p)z^F > 0$, the agent rejects and continuation payoffs are given by $(p\pi - c, 0)$. This is again weakly suboptimal for the principal, with the agent's incentives to be verified.

Consider the equilibrium with payoffs $(p\pi - c, 0)$. In equilibrium, the principal offers $q = 1$, $z^F = 0$, and $z^S = \pi - c/p$ in each period, while the agent reveals any successes. The value of z^S is calculated to push the principal's payoff to zero. Suppose the principal offers some other triple (q, z^S, z^F). If this triple satisfies $q(p\pi - c) - pz^S - (1-p)z^F \leq 0$, then the agent reports a success if $z^S \geq qk\pi + z^F$ and conceals it otherwise, with play in either case continuing with the equilibrium giving payoffs $(p\pi - c, 0)$. The agent's actions in this case affect payoffs only in the current period, and are obviously optimal. Because $q(p\pi - c) - pz^S - (1-p)z^F \leq 0$, such a deviation cannot increase the principal's payoffs, and hence it is optimal for the principal to undertake no such deviation.

If the principal offers a triple (q, z^S, z^F) with $q(p\pi - c) - pz^S - (1-p)z^F > 0$ and the agent conceals a success, then continuation play proceeds with the equilibrium giving payoffs $(p\pi - c, 0)$. If the principal offers a triple (q, z^S, z^F) with $q(p\pi - c) - pz^S - (1-p)z^F > 0$ and the agent reveals a success, then continuation play proceeds with the equilibrium giving payoffs $(0, 0)$. There is clearly no gain to the principal from making an offer that induces the agent to conceal a success. We must then establish that it is impossible to get the agent to reveal a success when the strategies call for concealing the success, which is identical to the missing incentive constraint for the agent in the minmax equilibrium. We need to show that there is no triple (q, z^S, z^F) satisfying

$$\begin{aligned} q(p\pi - c) - pz^S - (1-p)z^F &> 0, \\ (1-\delta)z^S &> (1-\delta)[qk\pi + z^F] + \delta(p\pi - c). \end{aligned}$$

It is immediate that the best case for making these inequalities hold is to set $z^F = 0$. We can then rearrange these inequalities as

$$\frac{p\delta}{1-\delta}(p\pi - c) < z^S < q[p(1-k)\pi - c].$$

The right inequality in (3) ensures that there exists no (q, z^S) solving these inequalities. ■

We thus already have three equilibria, with payoffs

$$
\begin{gathered}
(0, 0) \\
(W^*, V^*) \\
(p\pi - c, 0).
\end{gathered}
$$

Moreover, we can support all of the payoffs on the line segment connecting (W^*, V^*) and $(p\pi - c, 0)$ by equilibria featuring stationary outcome paths, with $q = 1$, $z^F = 0$, and the value of z^S adjusting from $k\pi$ for payoff W^* to $\pi - \frac{c}{p}$ for payoff $p\pi - c$. Deviations from equilibrium play prompt a switch to the minmax equilibrium. The calculations confirming that the proposed strategies are part of an equilibrium are immediate, since they involve the same punishments but less profitable deviations than the self-generating pair $\{(0, 0), (p\pi - c)\}$.

Feasibility ensures that there can be no equilibrium giving a sum of payoffs larger than $p\pi - c$. Lemma 1 shows that there are no equilibria that realize the efficient surplus, $W + V = p\pi - c$, and give $W < W^*$. We thus have a complete characterization of payoffs and behavior, on the boundary of the equilibrium payoffs, for values $W > W^*$, as well as an indication that equilibria with $W < W^*$ are inefficient. The remaining task is then to characterize equilibria for values of $W < W^*$.

Remark 1 The equilibria described in Lemma 2 remain equilibria under complete information, *i.e.*, when both players can observe the outcome of Nature's draw. This ensures that under complete information, the set of efficient payoffs is given by $\{(W, V) \in \mathbb{R}^2_+ : W + V = p\pi - c\}$. There is thus an equilibrium that achieves the efficient surplus and splits it between the two players, for any arbitrary such split. The outcome in this equilibrium is stationary, with $q = 1$ and with the value of z chosen so as to achieve the appropriate division of the surplus, and with any deviations prompting a switch to the minmax equilibrium. The key in this construction is that under full information a deviation on the part of the agent to conceal a success can be observed, undermining the argument leading to the lower bound on the agent's payoff (for equilibria that achieve the efficient surplus) that arises when the principal does not observe Nature's draw.

3.3 An Example

Lemma 1 ensures that equilibria for which $W < W^*$ are necessarily inefficient. This section presents an example showing that such equilibria may nonetheless give the principal a payoff larger than V^*, the largest principal payoff from the set of constrained efficient equilibria.

The principal sets $z^F = 0$ throughout. The principal initially offers a value $q < 1$ (to be determined) and $z^S = 0$. A failure causes this path of play to begin anew, while a success prompts a switch to the subsequent repeated play of the stage-game Nash equilibrium. Once the latter switch has been made, of course, we need no longer worry about incentives.

We first solve for the agent's equilibrium payoff W and q. We will ensure that the agent's incentive constraint for revealing a success in the first period binds, so that the agent's payoff is given by the value of reporting a failure, or

$$\begin{aligned} W &= \delta p W^* + \delta(1-p)W \\ &= \frac{\delta p^2 k\pi}{1-\delta(1-p)} \\ &< pk\pi. \end{aligned}$$

We ensure the incentive constraint binds by choosing the value of q to satisfy

$$(1-\delta)qk\pi + \delta W = \delta pk\pi,$$

or

$$(1-\delta)qk\pi = \delta\left[pk\pi - \frac{\delta p}{1-\delta(1-p)}pk\pi\right],$$

and hence

$$q = \frac{\delta p}{1-\delta(1-p)}.$$

We then need confirm the incentives for the principal. We assume that any deviation on the principal's part prompts an immediate switch to the repetition of the stage-game Nash equilibrium, with payoffs (W^*, V^*). Hence, the principal's behavior will be optimal as long as $V \geq V^*$. The equilibrium we are constructing features an initial sequence of identical actions, followed by a switch to the perpetual play of the efficient stage-game equilibrium. This will be an equilibrium if the principal's payoff in these initial periods exceeds that of the efficient equilibrium, or $q(p\pi - c) \geq (p(1-k)\pi - c)$, for which it suffices that

$$p(1-k)\pi - c \leq \frac{p\delta}{1-\delta(1-p)}(p\pi - c),$$

a slight strengthening of (3).

If it exists, this equilibrium gives the principal a payoff larger than V^*. But this equilibrium does not maximize the principal's payoff it turns out; it is not even best for the principal given the agent's payoff. Indeed, all of the excess payoff for the principal comes in the first period, and hence as δ gets large the payoffs from this equilibrium converge to (W^*, V^*). Our attention now turns to characterizing the entire payoff frontier. We will confirm that the equilibrium we have just constructed does not yield a payoff on the boundary of the payoff set, and that no strengthening of (3) will be required to improve the principal's payoff above V^*.

3.4 The Payoff Frontier

We characterize the payoff frontier and the attendant optimal strategies for values of $W \leq W^*$. We concentrate on the function $S(W)$, giving the maximum surplus as a function of the agent's payoff W, defined for $W \in [0, p\pi - c]$ by

$$\begin{aligned} S(W) &= \max\{(1-\delta)q(p\pi - c) + \delta p S(W^S) + \delta(1-p)S(W^F)\} \\ &\quad \text{s.t. } (1-\delta)z^S + \delta W^S \geq (1-\delta)(qk\pi + z^F) + \delta W^F \\ &\qquad W = (1-\delta)[pz^S + (1-p)z^F] + \delta p W^S + \delta(1-p)W^F, \end{aligned}$$

and subject to $q \in [0,1]$, $z^S, z^F \geq 0$, and that the continuation payoffs be equilibrium payoffs. The first constraint above is the agent's *incentive (compatibility)* constraint and the second is the *bookkeeping* constraint that we provide the agent with the appropriate payoff. In the sequel, these expressions refer to the two constraints above. Lemma 1 has established that $S(W) = p\pi - c$ for $W \geq W^*$.

3.4.1 Preliminaries

This section collects some intuitive results that are helpful in focussing the subsequent development. We begin with some characteristics of the function S.

Lemma 3

[3.1] $S(0) = 0$ *and* $S(W) = p\pi - c$ *for* $W \in [W^*, p\pi - c]$.

[3.2] $S(W)$ *is concave, and hence continuous.*

[3.3] $S(W)$ *is increasing.*

Proof. [3.1] That $S(0) = 0$ follows from the observation that a positive payoff can be generated only if the principal at some point sets $q > 0$, which the principal in turn will do only if at some point the agent reports a success, which the agent can be induced to do only with the prospect of a positive payoff. As noted above, Lemma 1 has established that $S(W) = p\pi - c$ for $W \geq W^*$.

[3.2] To establish concavity, suppose that σ and σ' are two equilibrium strategies, corresponding to payoffs (W, V) and (W', V'). Then consider the strategy σ'' that, after any history, sets $q'' = \lambda q + (1 - \lambda)q'$ and sets z'' so that $q''z'' = \lambda qz + (1 - \lambda)q'z'$. Then this new strategy satisfies the incentive constraints and gives $W'' = \lambda W + (1 - \lambda)W'$ and $V'' = \lambda V + (1 - \lambda)V'$, giving concavity.

[3.3] To show that S is increasing, we suppose that $W < W^*$, and then note that increasing W relaxes the bookkeeping constraint, allowing a simultaneous increase in (q, z^S, z^F, W^S, W^F) that preserves the incentive constraint and that increases at least one of these variables in addition to z^S and z^F, ensuring that the objective increases. ■

The function $S(W)$ thus begins at the origin, reaches V^* as W reaches W^*, and is constant above W^*. The function is concave and increasing. Notice that its rate of increase must slow down as W approaches W^*. In particular, we know from Section 3.3 that we have $V > V^*$ for some values of $W < W^*$, so that V is decreasing in W for values of W close to W^*, even as S continues to increase.

We can immediately establish some characteristics of the behavior behind these payoffs:[5]

Lemma 4 *Let $W \leq W^*$ and consider an equilibrium giving surplus $S(W)$. Then without loss,*

[4.1] $z^F = 0$.

[4.2] $W^S \geq W^F$.

[4.3] The incentive constraint binds.

[4.4] $z^S = 0$.

Proof. [4.1] If $z^F > 0$, reducing z^F while increasing z^S to preserve the bookkeeping constraint leaves the objective unaffected while relaxing the incentive constraint.

[4.2] The fact that $W^S \geq W^F$ is an implication of the concavity of S. If $W^S < W^F$, then an expected-value-preserving contraction in these values preserves the bookkeeping constraint while relaxing the incentive constraint and increasing the objective.

[4.3] Suppose the incentive constraint does not bind. First, suppose also that $W^S \neq W^F$ (and hence $W^S > W^F$). Then an expected-value-preserving reduction in W^S and increase in W^F preserves the bookkeeping constraint and (at least weakly) increases the (concave)

[5] It is here that we use the left inequality in (3). If this inequality fails, we will sometimes have $z^S > 0$.

objective. Let this continue until either the incentive constraint binds (in which case we have the desired result) or $W^S = W^F$.

Hence, we may assume the incentive constraint does not bind and $W^S = W^F = \tilde{W}$. Then the incentive and bookkeeping constraints are

$$\begin{aligned} z^S &> qk\pi \\ W &= (1-\delta)pz^S + \delta\tilde{W}. \end{aligned}$$

Then increase q, preserving the bookkeeping constraint and again increasing the objective, until either the incentive constraint binds (in which case we again have the result) or $q = 1$. In the latter case, we can then decrease z^S and increase $\tilde{W}$ to preserve the bookkeeping constraint and increase the objective, until either the incentive constraint binds (finishing the argument) or $\tilde{W} = p\pi - c$ with slack remaining in the incentive constraint. However, the latter event yields

$$W \geq (1-\delta)pk\pi + \delta(p\pi - c),$$

contradicting our assumption that $W \leq W^* = pk\pi$.

[4.4] Finally, notice that the variable z^S does not appear in the objective. The variables z^S and W^S enter the two constraints in identical proportions. Hence, one can increase the objective by decreasing z^S and increasing W^S so as to preserve the constraints, until either z^S hits its lower bound of 0 or W^S hits its upper bound of $p\pi - c$. The first constraint to bind will be $z^S = 0$, and hence we will have $z^S = 0$ for all $W \leq W^*$, if and only if we can satisfy

$$W = \delta p W^S + \delta(1-p)W^F, \tag{4}$$

for values of W as large as $pk\pi = W^*$. From the incentive constraint, we have $(1-\delta)k\pi = \delta(W^S - W^F)$. Using this to eliminate W^F from (4) and inserting $W^* = pk\pi$ on the left, we need

$$pk\pi \leq \delta p W^S + (1-p)[pW^S - (1-\delta)k\pi],$$

for some value of $W^S \leq p\pi - c$. Hence, we need

$$pk\pi \leq \delta(p\pi - c) - (1-p)(1-\delta)k\pi,$$

which is implied by (and motivates) the left inequality in (3). It is expected that some lower bound on the discount factor is needed for this result. If $\delta = 0$, then future payoffs are utterly ineffective in creating current incentives, and the agent can be induced to report a success only if $z^S > 0$. ■

We now know a great deal about the principal's behavior. The principal either uses current payments or continuation values to create incentives, but never uses both. For values $W < W^*$, the agent is rewarded for reporting a success by receiving a higher continuation payoff than if the agent reports a failure, but the agent receives no payment flow. Should the agent's continuation payoff reach W^*, then this continuation payoff remains constant, with subsequent incentives created entirely by the flow of payoffs.

We can characterize the dynamics of continuation values in the region $W \in [0, W^*]$. We can write the incentive and bookkeeping constraints as

$$\begin{aligned} W^S - W^F &= \frac{1-\delta}{\delta} qk\pi, \\ \frac{W}{\delta} &= W^F + p(W^S - W^F), \end{aligned}$$

which we can solve for

$$W^F = \frac{W}{\delta} - p\frac{1-\delta}{\delta} qk\pi, \tag{5}$$

$$W^S = \frac{W}{\delta} + (1-p)\frac{1-\delta}{\delta} qk\pi. \tag{6}$$

To check that W remains constant once the agent's payoff reaches W^*, we need only verify that $W = pk\pi$ and $q = 1$ imply $W^F = W$. The latter equality is then

$$W = \frac{W}{\delta} - p\frac{1-\delta}{\delta} k\pi,$$

which is solved by $W = pk\pi$.

Our task is then one of tracking the dynamics of the agent's payoff W. Assuming that we begin with some value in the interval $(0, W^*)$, then (as we establish below) the agent's continuation payoff increases after a success and decreases after a failure, with the agent receiving no flow payoffs in the meantime. This process continues until either the agent's payoff is absorbed at 0, terminating the project (an event we show below does not occur) or is absorbed at a level at least W^*, at which point the payoff thereafter remains constant and incentives are created by a flow of payoffs to the agent.

3.4.2 The Dynamics of Continuation Payoffs

To characterize the dynamics of continuation payoffs, it is helpful to rewrite the surplus maximization problem in terms of the project scale q and the agent's payoffs, giving, for

$W \in [0, W^*]$:

$$S(W) = \max_{q\in[0,1]} \left\{ (1-\delta)q(p\pi - c) + \delta p S\left(\frac{W + (1-\delta)(1-p)qk\pi}{\delta}\right) + \delta(1-p)S\left(\frac{W - (1-\delta)pqk\pi}{\delta}\right) \right\},$$

where we used Lemma 4 to eliminate all other variables, and where we set $S(W) = p\pi - c$ for $W \in [W^*, p\pi - c]$.

It is clear that $W < W^* \Rightarrow S(W) < p\pi - c$. If not, we note that $S(W) = p\pi - c$ requires $q = 1$ and $S(W^F) = p\pi - c$ while $W < W^*$ and $q = 1$ implies $W^F < W$, allowing us to obtain a contradiction by considering the infimum over W for which $S(W) = p\pi - c$. We also note that this immediately implies (alongside weak concavity) that S is *strictly* increasing on $[0, W^*]$.

We next note from the definition of S that, provided it is differentiable, the process $\{S'(W_n)\}_{n=0}^{\infty}$, viewed as a stochastic process with W_{n+1} taking values W^F and W^S (given $W_n = W$), is a martingale (differentiating with respect to W, we immediately get $S'(W_n) = \mathbf{E}[S'(W_{n+1})]$). We use this martingale property in many of the arguments.

Section 5.2 proves:

Lemma 5

[5.1] The function S is differentiable, with an infinite right derivative at $W = 0$ and a zero derivative at $W = W^*$.

[5.2] For any $W \in (0, W^*)$, we have $S'(W^F) \neq S'(W^S)$.

[5.3] For any $W \in (0, W^*)$, $q > W/W^*$.

[5.4] The function S is strictly increasing and strictly concave on $[0, W^*]$.

[5.5] For any $W \in (0, W^*)$, $W^F < W < W^S$.

The fact that S has an infinite right derivative at $W = 0$ allows us to conclude that the principal never lets the project lie idle, which is to say that $q > 0$ for all $W \in (0, W^*)$. Suppose this is not the case, *i.e.*, there exists $W \in (0, W^*)$ such that $q(W) = 0$. Then plugging in q in the functional equation, we obtain

$$\frac{S(W)}{\delta} = S\left(\frac{W}{\delta}\right),$$

which is inconsistent with weak concavity and the infinite right derivative of $S(W)$ at $W = 0$.

An implication of the differentiability of S is that $W > 0 \Rightarrow W^F > 0$. Hence, the principal will never choose a scale q for the project that causes a failure to terminate the relationship. To see this, we note that because S is differentiable, the optimal q must either solve the first-order condition with equality (if interior, in which case $W^F > 0$) or must be positive (if constrained by $W^F = 0$). But the fact that $S'(0) = \infty$ precludes the latter possibility.

The second statement in the lemma indicates that not only is $W^S > W^F$ (as we have seen), but also that for any $W \in (0, W^*)$, we have $W^S > W > W^F$ as well as

$$S'(W^F) \neq S'(W^S).$$

This in turn implies, because $S'(W) = pS'(W^S) + (1-p)S'(W^F)$, that $S'(W)$ is distinct from both $S'(W^F)$ and $S'(W^S)$. This in turn serves as an important input into the demonstration that s is strictly concave. It also allows us to show not only that the principal never lets the project sit idle, but puts a lower bound, $q > W/W^*$, on the scale at which the principal operates the object.

3.4.3 Equilibrium Behavior

We can now characterize the limiting behavior of the relationship. The following is an immediate combination of our preceding results.

Proposition 2 *With probability 1, the agent's continuation value reaches a value $W \geq W^*$.*

Given any initial condition W_0, the process $\{W_n : n \in \mathbf{N}\}$ defined by the "best" equilibrium given W_0 converges almost surely to the set $\{W \geq W^*\}$. This is because $\{S'(W_n) : n \in \mathbf{N}\}$ is a non-negative martingale, and hence converges. Given the strict concavity of S, this implies that the process $\{W_n\}$ must almost surely exit the interval $(0, W^*)$ (since $W_{n+1} \in \{W_n^S, W_n^F\}$). Given that $\lim_{W \downarrow 0} S'(W) = +\infty$, it follows from the martingale property that $\mathbf{P}[\lim_n W_n = 0] = 0$. Hence the result.

Section 5.3 proves (and further discusses):

Proposition 3 *The optimal scale of the project $q(W)$ increases in W, with some $\hat{W} \in (0, W^*)$ such that $q = 1$ if and only if $W \geq \hat{W}$.*

3.5 Summary

These results give us a complete characterization of the set of equilibrium strategies and equilibrium payoffs that we summarize here.

Figure 2 shows the set of equilibrium payoffs. The payoff (W^*, V^*) features stationary behavior, both on and off the equilibrium path, and is achieved by repeating the subgame-perfect equilibrium of the stage game in each period. Continuation payoffs thus never move, and incentives for the agent to report successes are created by offering the appropriate flow payoffs for such reports. This equilibrium captures all of the possible surplus.

Equilibria on the segment connecting (W^*, V^*) with $(p\pi - c, 0)$ also capture all of the available surplus, but allocate more of the surplus to the agent. The equilibrium path is stationary, differing from the equilibrium giving (W^*, V^*) in that the agent receives a higher payoff in the event of a success. These equilibria deter deviations by switching to the continuation equilibrium exhibiting payoffs $(0, 0)$. It is not immediately obvious that the latter is an equilibrium payoff, but we have shown that the set $\{(0, 0), (p\pi - c, 0)\}$ is a self-generating pair. Figure 2 also displays the equilibrium payoff of the example from Section 3.3, making it plain that this equilibrium is not constrained efficient, despite improving upon V^* for the principal. This should come as no surprise, since any equilibrium achieving the boundary payoff (for values $W < W^*$) requires $W^F < W$, and hence cannot be reconciled with the construction of this equilibrium.

Equilibria achieving payoffs for the agent below W^* are necessarily inefficient. The principal receives a higher payoff from these equilibria than from any of the efficient equilibria as long as W is not too far below W^*, with payoffs collapsing to $(0, 0)$ as W approaches 0. The strategies behind these equilibria have a simple structure. The principal sets the flow payoff $z^F = 0$ throughout, counting on continuation payoffs to create incentives. The principal sets a scale q for the project that is increasing in W. A report of a success leads to a higher continuation value and a report of a failure leads to a lower continuation value, with the spread between these two values increasing in q. As W approaches 0, the attendant value of q declines sufficiently rapidly as to ensure that the adverse continuation payoff W^F never hits 0, and hence the project is never forced to terminate. As W approaches W^*, we enter a region in which $q = 1$, but in which there are still no flow payoffs. Figure 3 illustrates the policy function $q(W)$ for two values of the discount factor δ. The policy function appears to be nearly linear (though it is not) for values of $W \leq \hat{W}$. Figure 4 illustrates the corresponding surplus function. Eventually (with probability 1), the value of W hits W^*. At this point, continuation values cease to move, and the continuation equilibrium is stationary. Incentives for the agent are then created entirely via flow payoffs.

As Figure 2 indicates, the equilibrium that maximizes the principal's payoff does not achieve the efficient surplus. This result is general. Example 3.3 ensures that there are equilibria in which the principal earns more than V^*, while Lemma 1 ensures that any such equilibrium fails to achieve the efficient surplus. Figures 2 and 3 suggest that in the principal's preferred equilibrium, the project starts a full scale, *i.e.*, $q = 1$. This latter result is also general. In particular, let W be the agent's payoff in the equilibrium that maximizes the principal's payoff. Then we can use (5) to write the inequality $q \leq 1$ as

$$W^F - \left(\frac{W - (1-\delta)pk\pi}{\delta} \right) \geq 0.$$

Given W, Let us add this as a constraint, with multiplier $\lambda \geq 0$, to the surplus maximization problem, and then use the substitutions $q = (W - \delta W^F)/(1-\delta)pk\pi$ (from the incentive constraint) and $W^S = W/(p\delta) - (1-p)W^F/p$ (from the bookkeeping constraint) to obtain

$$\begin{aligned} S(W) &= \max_{W^F \geq 0} \left\{ (W - \delta W^F) \frac{p\pi - c}{pk\pi} + \delta \left[pS\left(\frac{W}{p\delta} - \frac{1-p}{p} W^F \right) + (1-p)S(W^F) \right] \right. \\ &+ \left. \lambda \left(W^F - \frac{W - (1-\delta)pk\pi}{\delta} \right) \right\}. \end{aligned}$$

Then the envelope theorem gives

$$S'(W) = \frac{p\pi - c}{pk\pi} + S'(W^S) - \frac{\lambda}{\delta}.$$

For the choice of W_0 that corresponds to the equilibrium that maximizes the principal's payoff, we must have $S'(W) = 1$. However, since $\frac{p\pi - c}{pk\pi} > 1$ and $S'(W^S) \geq 0$, we can have $S'(W) = 1$ only if $\lambda > 0$, which implies $q = 1$.

Remark 2 Consider an equilibrium whose value to the agent satisfies $W < W^*$. Then the equilibrium generates a sequence of values of W, as the agent's continuation payoff moves upward after a success and downward after a failure. Similarly, we have a sequence of values of q, as the scale of the project moves upward after a success and downward after a failure. Until the value of q first hits one, the set of values of q has a simple ladder structure, as it turns out, containing a countable number of values with the property that a success moves the equilibrium up to the next value, while a failure moves the equilibrium down one value. Hence, given an initial value W^0 and given that the project has not yet been operated at full scale, one need know only the total number of successes and failures along a history to identify the continuation payoff and the optimal value of q.

3.6 The Comparative Statics of Patience

This section characterizes payoffs and behavior in the limit as the discount factor δ approaches one. Figure 4 suggests that payoffs are increasing in δ, and we confirm that in the

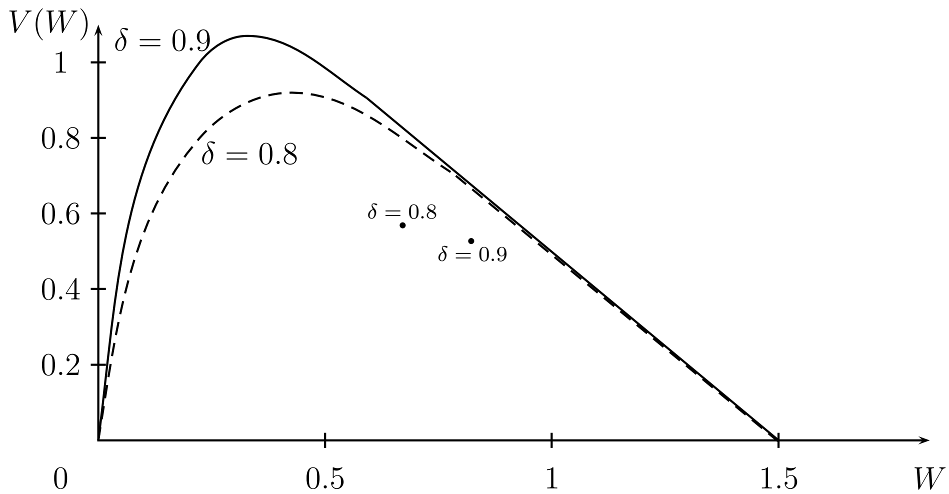

Figure 2: Equilibrium Payoffs. The figure shows the function $V(W)$, identifying the largest equilibrium payoff V for the principal for each equilibrium payoff W for the agent, for $\delta = 0.9$ (solid line) and $\delta = 0.8$ (dashed line). In each case, $(W^*, V^*) = (1, 1/2)$. The dots identify the payoffs for the equilibrium constructed in Section 3.3. Parameter values in Figures 2–4 are $\pi = 4$ and $c = p = k = 1/2$.

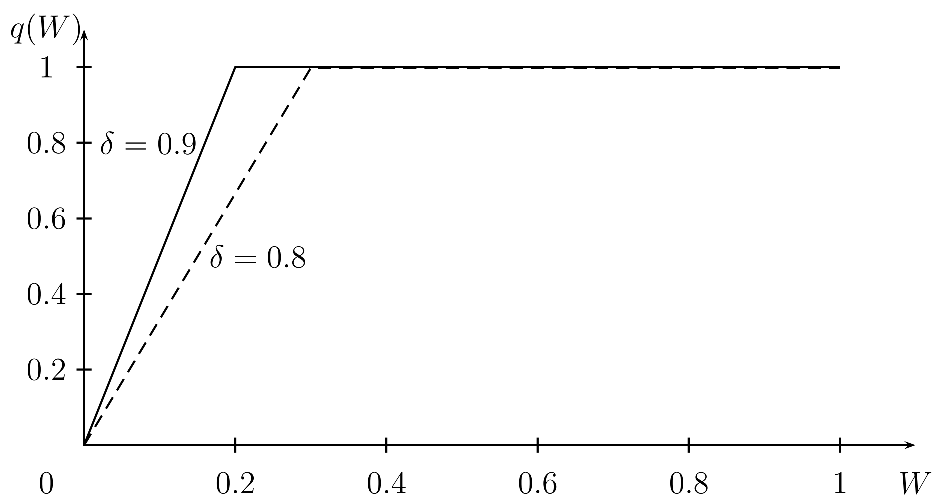

Figure 3: Policy function, $q(W)$, for $\delta = .8$ and $\delta = .9$. The optimal scale of the project $q(W)$ is less than one and strictly increasing for small values of W, and equals one for values of W that are larger but still less than $W^* = 1$. The accompanying flow payoff z^S is positive only for values $W \geq W^*$.

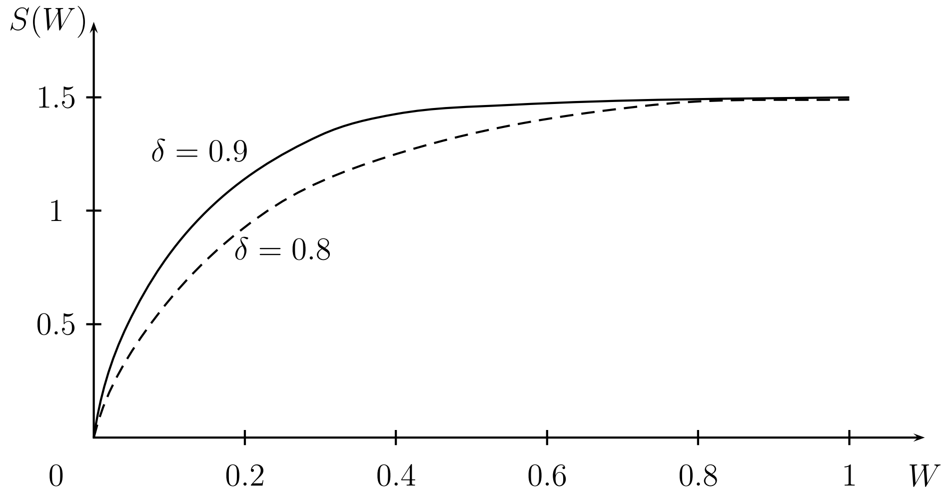

Figure 4: (Surplus) Value function, for $\delta = .8$ and $\delta = .9$. The function $S(W)$ is strictly increasing on the interval $[0, W^*]$.

limit, any division of the efficient surplus can be approximately achieved as an equilibrium payoff. Notice that we keep the other parameters fixed, most notably p, the probability of a success within a given period, so this limit must be interpreted as a change in patience rather than a shortening of the length of a period.

Section 5.5 proves:

Proposition 4

[4.1] As δ increases, the surplus function $S(W)$ increases, converging pointwise to $p\pi - c$ in the limit as $\delta \to 1$.

[4.2] The function $q(W)$ converges pointwise to 1 in the limit as $\delta \to 1$.

As expected, the first result gives us a version of the folk theorem for our game—as the players become arbitrarily patient, the frontier of equilibrium payoffs approaches the efficient frontier, and the set of equilibrium payoffs approaches the set of feasible, individually-rational payoffs. The second result is expected. The function $q(W)$ describes behavior in a constrained efficient equilibrium. As the players become increasingly patient, this payoff approaches but never achieves the efficient surplus, which can be captured only by consistently setting $q = 1$.

4 Discussion

4.1 Renegotiation

Several of the papers concerned with dynamic moral hazard include an analysis of another kind of limited commitment, namely renegotiation-proofness (see, e.g., Quadrini [10] and DeMarzo and Fishman [6]). As is well known, there are several competing definitions for renegotiation-proofness in infinite-horizon games, and we discuss here weak renegotiation-proofness, as defined by Farrell and Maskin [7]. As the name suggests, this is a weak notion, yet as we argue it is already more restrictive than what perfect Bayesian equilibrium implies. Recall that an equilibrium is weakly renegotiation-proof if there do not exist two continuation equilibria such that one would strictly Pareto-dominate the other.[6]

Plainly, the repetition of the stage-game Nash equilibrium yielding payoffs (W^*, V^*) is renegotiation-proof. In fact, so are the payoff vectors $(0, 0)$ and $(p\pi - c, 0)$, as neither strictly Pareto-dominates the other (though stronger notions of renegotiation-proofness would eliminate at least $(0, 0)$). Hence, we get again the entire line segment connecting $(0, 0)$ to (W^*, V^*) while maintaining weak renegotiation-proofness.

The more interesting question is whether the principal can obtain a payoff larger than V^* in a renegotiation-proof equilibrium. We note that the equilibria achieving the upper boundary payoffs (W, V) for $W \leq W^*$ required W to decline after a reported failure, so that, after a long enough string of such reports, the continuation payoff would necessarily be Pareto-dominated by the initial equilibrium achieving (W, V)—a contradiction. Hence, under renegotiation-proofness those boundary points (W, V) for which $W \in (0, W^*)$ must yield the principal a payoff strictly lower than those characterized in Section 3.4.

Nonetheless, whenever it exists, the equilibrium constructed in Section 3.3 is renegotiation-proof.[7] This equilibrium only involves two continuation strategies, corresponding to two payoff vectors, including (W^*, V^*) and the achieved equilibrium payoff (W, V), with $W < W^*$, $V > V^*$. This equilibrium has the property that the scale of the project is precisely calibrated so that the report of a failure leads to an unchanged continuation payoff vector, while one success report takes us to (W^*, V^*).

While this equilibrium payoff tends to (W^*, V^*) as $\delta \to 1$, there is an entire class of payoffs

[6]While renegotiation-proofness has been introduced for games with complete information, and so refers to subgame-perfect equilibrium, we note that the perfect Bayesian equilibria we consider are recursive as well, given that the agent's strategy does not depend on his past private information. Hence, continuation equilibria are well-defined.

[7]Recall that it exists if we assume $p(1-k)\pi - c \leq \frac{\delta p}{1-\delta(1-p)}(p\pi - c)$.

that can be obtained with equilibria of the same kind, with a very simple ladder structure: one success moves the strategies up one step, as long as (W^*, V^*) is not yet reached, whereas one failure leads to an unchanged continuation. This class is parameterized by the number of successes required to get to the payoff (W^*, V^*), starting at the "bottom" of the ladder. With the notation and normalization of Section 3.4.2, we require $q = W/W^*$ (where W stands for the agent's payoff at a given step) so that a failure leads to $(W - (1-\delta)W^* q)/\delta = W$, and we get the functional equation (where, with abuse of notation, $S(\cdot)$ refers to the resulting surplus)

$$S(W) = (1-\delta)(p\pi - c)\frac{W}{pk\pi} + \delta p S\left(\frac{W + (1-\delta)W(1-p)/p}{\delta}\right) + \delta(1-p)S(W),$$

with boundary condition $S(W^*) = p\pi - c$. The solution is

$$\frac{S(W)}{p\pi - c} = \frac{W}{W^*} + \frac{(1-\delta)(W/W^*)\ln(W/W^*)}{(1-\delta(1-p))\ln\left(1 + \frac{1-\delta}{\delta p}\right)},$$

the second term being the (normalized) payoff of the principal. The size of the ladder is constrained by the requirement that each step upward (leading to an improvement in the agent's payoff) leads to a lower payoff to the principal. Ignoring integer issues, it suffices to make sure that the second term be decreasing in W, which upon differentiation is equivalent to $W/W^* \geq e^{-1}$. As $\delta \to 1$, this gives us a curve $S(W) = (W/W^*)(1 - \ln(W/W^*))(p\pi - c)$, for $W \in [W^* e^{-1}, W^*]$. To be sure, this is a lower bound on the highest renegotiation-proof equilibrium payoffs for a given W in this range—we suspect more sophisticated constructions could further improve the principal's payoff without running afoul the renegotiation constraint.

The upshot of this discussion is that, within our model, renegotiation-proofness is strictly more demanding than perfect Bayesian equilibrium. Nonetheless, the principal can do strictly better than in the repetition of the stage-game Nash equilibrium, in equilibria that bear some resemblance with those that achieve the extreme payoffs in the case of perfect Bayesian equilibrium.

4.2 Less Patient Players

In this section we examine the case in which the second inequality in (3) fails, so that

$$p(1-k)\pi - c > \frac{p\delta}{1-\delta}(p\pi - c),$$

which will be the case if the players are not sufficiently patient. However, we maintain the left inequality in (3).

We used the second inequality in (3) to ensure that there existed an equilibrium with payoffs $(0,0)$. This equilibrium allows us to simultaneously achieve the minimum equilibrium payoff for both players.

Our first goal is to characterize the minimum equilibrium payoffs. Let $\underline{W}$ and $\underline{V}$ be (respectively) the lowest equilibrium payoffs for the agent and for the principal. A necessary condition for the existence of an equilibrium delivering value $\underline{V}$ to the principal is that there exist no (q, z^S) satisfying

$$\begin{aligned}(1-\delta)[q(p\pi - c) - pz^S] + \delta V^S &> \underline{V} \\ (1-\delta)z^S + \delta W^S &> (1-\delta)qk\pi + \delta W^F.\end{aligned}$$

This is the statement that there is no offer the principal can make that will increase the principal's payoff if the agent reports a success, and that can also induce the agent to report a success. To find the smallest candidate value for $\underline{V}$, we make these inequalities most difficult to satisfy by setting $V^S = \underline{V}$, $W^S = \underline{W}$ and $W^F = p\pi - c - \underline{V}$, where $p\pi - c - \underline{V}$ is the maximum possible equilibrium payoff for the agent. Hence, a necessary condition for $\underline{V}$ to be the principal's lowest equilibrium payoff is that there be no (q, z) satisfying

$$qk\pi + \frac{\delta}{1-\delta}(p\pi - c - \underline{V} - \underline{W}) < z^S < \frac{q}{p}(p\pi - c) - \frac{\underline{V}}{p},$$

or equivalently, we need it to be the case that for all q,

$$\frac{p\delta}{1-\delta}(p\pi - c - \underline{V} - \underline{W}) + \underline{V} \geq q[p(1-k)\pi - c].$$

Noting that this inequality is most difficult to satisfy for $q = 1$, the boundary on values of $(\underline{W}, \underline{V})$ satisfying our necessary condition for $\underline{V}$ to be the principal's lowest equilibrium payoff value is then given by

$$\underline{V} = \frac{p(1-k)\pi - c - \frac{p\delta}{1-\delta}(p\pi - c - \underline{W})}{1 - \frac{p\delta}{1-\delta}} \equiv h(\underline{W}).$$

The point (W^*, V^*) always satisfies this equation. This equation defines a linear relationship between $\underline{V}$ and $\underline{W}$ whose slope is nonnegative. When $\delta = 0$, the slope is zero, giving a horizontal line passing through (W^*, V^*). In this case we know the lowest equilibrium payoffs $(\underline{W}, \underline{V}) = (W^*, V^*)$ are drawn from h. At the other extreme if δ is just large enough to satisfy the second inequality in (3) with equality, then the slope of h is given by $(p(1-k)\pi - c)/(pk\pi)$, and then h passes through the origin. In this case, the minmax values $(0,0)$ also lie on h. For intermediate values of δ, the function h continues to pass through (W^*, V^*) but has a positive intersection with the vertical axis, indicating that $\underline{V} > 0$. This confirms that when the second inequality in (3) is violated, $(0,0)$ is no longer an equilibrium payoff profile, and in particular $\underline{V} > 0$.

Section 5.6 establishes the following:

- There exists an equilibrium that simultaneously achieves the lowest equilibrium payoffs $(\underline{W}, \underline{V})$, satisfying $\underline{V} = h(\underline{W})$.
- There exists $\underline{\delta} > 0$ such that for all $\delta < \underline{\delta}$, $(\underline{W}, \underline{V})$ equals (W^*, V^*).

For sufficiently patient players, for which (3) is satisfied, the lowest equilibrium payoffs payoffs are $(0,0)$. For sufficiently impatient players, the only equilibrium payoff in the repeated game is (W^*, V^*). In between, there is a range of discount factors for which lowest equilibrium payoffs lie between $(0,0)$ and (W^*, V^*), and are drawn from the graph of the function h. See Section 5.6 for a discussion.

What is then the equilibrium payoff set? It is a (convex) set with lower boundary the horizontal line segment $[(\underline{W}, h(W)), (p\pi - c - h(W), h(W))]$; the upper boundary consists of a curve, for values of $W \in (\underline{W}, W^*)$, along with an upper line segment $[(W^*, V^*), (p\pi - c - h(W), h(W))]$ for $W \geq W^*$. The curve is the solution to a functional equation, as in the patient case. Namely, in terms of surplus, consider the functional equation

$$S(W) = \sup_{q \in [0,1]} \left\{ (1-\delta)q(p\pi - c) + \delta p S\left(\frac{W + (1-\delta)(1-p)qk\pi}{\delta}\right) + \delta(1-p) S\left(\frac{W - (1-\delta)pqk\pi}{\delta}\right) \right\},$$

subject to $W^F = (W - (1-\delta)pqk\pi)/\delta \geq \inf\{W : S(W) \geq h(W) + W\}$.[8,9]

Note that this is the same equation as in Section 3.4.2, except that values of W (in particular, the lower one that follows a failure) must exceed $\underline{W}$, or equivalently correspond to values for the principal above $h(\underline{W})$ –in terms of surplus, $S(W) \geq h(W) + W$. Note that this functional equation is actually defined for all $W \geq 0$, and it is precisely the lowest value for which $S(W) > -\infty$ (if it exists) that defines $\underline{W}$. The curve that defines the upper

[8] Uniqueness of the solution of this equation follows from value iteration, since it is a contraction on the domain of functions for which the constraint is satisfied.

[9] $\underline{W} = W^*$ if and only if the constraint cannot be satisfied for any $W < W^*$.

locus for $W \leq W^*$, then, is precisely $S(W) - W$ wherever this (principal's) payoff is finite (and hence above $h(W)$). By construction, $V = S(W) - W$ is above $h(W)$, and so satisfies the principal's incentive constraint, and by construction also the continuation payoffs are chosen so that the agent's incentive constraint is satisfied. It is easy to see that (whenever $\underline{W} < W^*$), this locus is differentiable at $W = W^*$, as in the patient case.

The dynamics, then, are similar (though not identical) to the patient case: as in the patient case, payoffs on this boundary (in particular, the one maximizing the principal's payoff) are implemented via continuation promises that go up or down according to whether a success or failure is reported. The continuation payoff, however, bounces back from $\underline{W}$ after a success, so that –as in the patient case– the agent's payoff is eventually absorbed at $W \geq W^*$ and the project is operated efficiently.

4.3 Conclusion

Dynamic moral hazard problems provide a convenient setting for examining repeated relationships with incomplete information. The literature has typically focussed on the case in which the principal can commit to a mechanism. This yields a relatively tractable model that may be appropriate in many circumstances. Our goal has been to ask what happens when the players are constrained by sequential rationality.

We find a simple characterization of optimal contracts. These contracts share some features of the commitment case (such as the fact that continuation payoffs are used to create rewards when the agent's payoff is relatively low, while current payoffs are used to create incentives when the agent's payoff is relatively high), but also some marked differences (such as the fact that with probability one, the project in our case is never abandoned and indeed grows to full size).

There is always an equilibrium that captures the efficient surplus, and which has the simple form of repeating the subgame-perfect equilibrium of the stage game in every period. Why is that not the end of the story? There are other equilibria that give the principal a higher payoff, even though they do so at the cost of introducing inefficiency into the relationship. These equilibria do not have stationary outcomes, and to examine them we must characterize the boundary of the equilibrium payoff set, identifying the maximum payoff available payoff to the principal for any agent payoff, no matter how small.

These equilibria have a simple structure. For small agent's payoffs, the principal makes no payments to the agent, creating incentives by scaling down the size of the project and the agent's payoff after every failure, and scaling them up after every success. Repeated failures may push the agent's payoff arbitrarily close to zero, but this payoff (almost surely) never hits this lower absorbing boundary. Instead, there is on average an upward drift in the agent's payoff, which with probability one eventually escapes into an region in which still no payments are made, with incentives created by adjusting continuation payoffs, but with the project operated at maximum scale. Too many failures will push the equilibrium back into the first region, but again with probability one the equilibrium will enter a higher, absorbing region in which continuation payoffs are constant and incentives are created solely through making payments to the agent.

5 Appendix: Proofs

5.1 Proof of Lemma 1

[1.1] If $p(1-k)\pi - c > 0$, then the repeated game has an equilibrium, which repeats the stage-game Nash equilibrium in each period, that achieves the efficient surplus (cf. Proposition 1.1). It is then immediate that every constrained efficient equilibrium gives this sum of payoffs, which can only be achieved by setting $q = 1$ in every period.

[1.2] Let (q, z^F, z^S) identify the scale of the project and the payments to the agent (in the event of a reported failure and success) in the first period. Let (W^S, V^S) and (W^F, V^F) denote continuation payoffs in the event of a (reported) success and failure. A constrained efficient equilibrium outcome solves the maximization problem

$$\max_{q,\mathbf{1}_S,z^F,z^S,W^S,V^S,W^F,V^F} W + V \quad \text{such that} \tag{7}$$

$$W = (1-\delta)(p\mathbf{1}_S z^S + p(1-\mathbf{1}_S)(z^F + qk\pi) + (1-p)z^F) + \delta(p\mathbf{1}_S W^S + (1-p\mathbf{1}_S)W^F) \tag{8}$$

$$V = (1-\delta)(p\mathbf{1}_S(q\pi - z^S) - (1-p\mathbf{1}_S)z^F - qc) + \delta(p(\mathbf{1}_S V^S + (1-p\mathbf{1}_S)V^F) \tag{9}$$

$$\mathbf{1}_S = 1 \Longleftarrow (1-\delta)z^S + \delta W^S > (1-\delta)(z^F + qk\pi) + \delta W^F \tag{10}$$

$$\mathbf{1}_S = 0 \Longleftarrow (1-\delta)z^S + \delta W^S < (1-\delta)(z^F + qk\pi) + \delta W^F, \tag{11}$$

subject to the constraints that W^S, V^S, W^F, and V^F are equilibrium payoffs and that W and V are nonnegative, where $\mathbf{1}_S$ is an indicator for the event that the agent reports a success.

Because $p(1-k)\pi - c > 0$, we know that this problem has a positive solution, with a value equal to $p\pi - c$. This value in turn can be generated only if $q = 1$ and $\mathbf{1}_S = 1$, so that we can rewrite the problem solved by a constrained efficient equilibrium outcome as

$$\max_{z^F,z^S,W^S,V^S,W^F,V^F} W+V$$

$$\begin{aligned} s.t. \qquad & W=(1-\delta)(pz^S+(1-p)z^F)+\delta(pW^S+(1-p)W^F) \\ & V=(1-\delta)(p(q\pi-z^S)-(1-p)z^F-c)+\delta(pV^S+(1-p)V^F) \\ & (1-\delta)z^S+\delta W^S \geq (1-\delta)(z^F+qk\pi)+\delta W^F, \end{aligned}$$

subject to the constraints that W^S, V^S, W^F, and V^F are equilibrium payoffs and that W and V are nonnegative. Now note that it has no effect on the objective, and preserves the constraints, and hence sacrifices no generality, to assume that $z^F=0$. Hence, a constrained efficient equilibrium outcome must solve the maximization problem

$$\max_{z^S,W^S,V^S,W^F,V^F} W+V$$

$$\begin{aligned} s.t. \qquad & W=(1-\delta)pz^S+\delta(pW^S+(1-p)W^F) \\ & V=(1-\delta)(p\pi-c)-pz^S+\delta(pV^S+(1-p)V^F) \\ & (1-\delta)z^S+\delta W^S \geq (1-\delta)k\pi+\delta W^F, \end{aligned}$$

subject to the constraints that W^S, V^S, W^F, and V^F are equilibrium payoffs and that W and V are nonnegative. We can rearrange the incentive constraint to give $(1-\delta)pz^S \geq (1-\delta)pk\pi+\delta p(W^F-W^S)$, and inserting into the expression for W,

$$W \geq (1-\delta)pk\pi+\delta W^F.$$

Now let $\underline{W}$ be the lowest constrained-efficient payoff for the agent. Then $W^F \geq \underline{W}$, and hence

$$W \geq (1-\delta)pk\pi+\delta\underline{W}$$

holds for every agent payoff consistent with a constrained-efficient equilibrium, including $\underline{W}$, which gives $\underline{W} \geq (1-\delta)pk\pi+\delta\underline{W}$ and hence $\underline{W} \geq pk\pi$.

[1.3] We again consider the maximization problem given by (7)–(11), subject to the constraints that W^F, V^S, W^F, and V^F are equilibrium payoffs and that W and V are nonnegative. Assume this problem has a positive solution, with value $\hat{S}$.

First suppose $q=0$. Then the sum of continuation payoffs is positive (since the problem has a positive solution), and hence a higher sum of payoffs can be achieved by simply beginning in the first period with an equilibrium whose payoffs are the maximum of $W^F + V^F$ and $W^S + V^S$. We can thus assume $q > 0$.

Now suppose that $q > 0$ and $\mathbf{1}_S = 0$. Then it has no effect on continuation payoffs and preserves the constraints, and hence sacrifices no generality, to assume that $W^S = W^F$ (simply choose (W^S, V^S) and (W^F, V^F) to both equal whichever of these pairs has the larger sum) and $z^S = z^F = 0$. But then the sum of payoffs can be increased (by $p(1-k)\pi$) while preserving continuation values and increasing z^S to $qk\pi$ and setting $\mathbf{1}_S = 1$. Hence, we can assume that $\mathbf{1}_S = 1$. There is then no loss of generality in assuming that z_S is set so that the incentive constraint binds. This gives the problem

$$\max_{z^S, W^S, V^S, W^F, V^F} W + V$$

$$\begin{aligned} s.t. \qquad W &= (1-\delta)pz^S + \delta(pW^S + (1-p)W^F) \\ V &= (1-\delta)q(p\pi - c) + \delta(pW^S + (1-p)W^F) \\ (1-\delta)z^S + \delta W^S &= (1-\delta)qk\pi + \delta W^F, \end{aligned}$$

subject to the constraint that W^S, V^S, W^F, and V^F are equilibrium payoffs and that W and V are nonnegative. Solving the incentive constraint gives $(1-\delta)pz^S = (1-\delta)qpk\pi + \delta p(W^F - W^S)$, and inserting into the expressions for W and V, we have

$$\max_{W^S, V^S, W^F, V^F} W + V$$

$$\begin{aligned} s.t. \qquad W &= (1-\delta)qpk\pi + \delta(pW^S + (1-p)W^F) + \delta p(W^F - W^S) \\ &= (1-\delta)qpk\pi + \delta W^F \\ V &= (1-\delta)q(p(1-k)\pi - c) + \delta(pV^S + (1-p)V^F) - \delta p(W^F - W^S) \\ &= (1-\delta)q(p(1-k)\pi - c) + \delta(p(W^S + V^S) + (1-p)(W^F + V^F) - W^F) \end{aligned}$$

subject to the familiar constraints. Recalling that $\hat{S}$ is the solution to the surplus maximization problem, we have

$$(p(W^S + V^S) + (1-p)(W^F + V^F) - W^F) \leq \hat{S} - W^F \leq \hat{S} - \underline{W},$$

where $\underline{W}$ is the smallest equilibrium payoff for the agent. An upper bound on the principal's equilibrium payoff is then given by $\overline{V}$, where

$$\overline{V} \leq (1-\delta)q[p(1-k)\pi - c] + \delta\overline{V}$$

for some $q \in [0,1]$. Since $p(1-k)\pi - c < 0$, this gives $\overline{V} = 0$, completing the proof. ■

5.2 Proof of Lemma 5

To simplify the notation, let $w := W/W^*$ and $s(w) := S(W)/(p\pi - c)$. Hence, we are interested in the behavior of the function $s(w)$ on $w \in [0,1]$ (corresponding to the behavior of $S(W)$ on $[0, W^*]$). The function s is the solution to the problem $\mathcal{P}$: solve for $s : \mathbf{R}_+ \to \mathbf{R}_+$, given by

$$s(w) = \max_{q \in [0, \min\{w/(1-\delta),1\}]} \left\{ (1-\delta)q + \delta p s\left(\frac{w + (1-\delta)q(1-p)/p}{\delta}\right) + \delta(1-p)s\left(\frac{w-(1-\delta)q}{\delta}\right)\right\},$$

with boundary conditions $s(0) = 0$, $s(w) = 1$ for $w \geq 1$.[10]

[5.1] First, we show that s has an infinite right derivative at $w = 0$. To see that $\lim_{w \downarrow 0} \frac{s(w)}{w} = +\infty$, consider the strategy that sets q so that $w^F = w$ as long as $w \in (0,1)$, that is, the strategy that sets $q = w$ (until $w \geq 1$, if ever, in which case the optimal strategy is followed, for a payoff of 1). This gives us a lower bound on the value, call it $\tilde{s}$. This lower bound satisfies the recursion

$$\tilde{s}(w) = \frac{1-\delta}{1-\delta+\delta p} w + \frac{\delta p}{1-\delta+\delta p} \tilde{s}\left(\frac{1+(1-\delta)\frac{1-p}{p}}{\delta} w\right).$$

Fixing w, we can compute the number n of successive successes required to push the resulting continuation payoff for the agent above 1, namely, the lowest integer such that

$$\left(\frac{1+(1-\delta)\frac{1-p}{p}}{\delta}\right)^n w \geq 1,$$

or

$$n = \lfloor -\frac{\ln w}{\ln\left(1+\frac{1-\delta}{\delta p}\right)} \rfloor.$$

Plugging back into the recursion for $\tilde{s}$, we obtain

$$\frac{\tilde{s}(w)}{w} = \frac{(1-\delta)n + \delta p}{1-\delta+\delta p},$$

and it is readily verified that the right-hand tends to infinity as $w \to 0$, yielding the result.

Next, we show that $s(w)$ has a zero left derivative at $w = 1$. Since the function s is constant to the right of $w = 1$, this ensures that s is differentiable at 1. To show that

[10]Note that we have already taken into account in the range of possible values of q (by requiring $q \leq w/(1-\delta)$) the constraint that $w^F \geq 0$.

$\lim_{w\uparrow 1}\frac{s(1)-s(w)}{1-w}=0$, we proceed as for the previous claim, but use another strategy for this end. Consider the strategy which, starting from $w\in\left(\delta-(1-\delta)\frac{1-p}{p},1\right)$, sets $q=1$ as long as the continuation payoff lies in this interval, and reverts to the optimal strategy as soon as this interval is exited. Note that, by choice of this interval, a single success yields $w^S\geq 1$ independently of w within this interval. We derive a lower bound $\tilde{s}$ for the value by considering that the payoff is 0 if this interval is ever exited from below. Starting from a given w within this interval, it takes n successive failures for this happen, where n is given by

$$n=\lfloor\frac{\ln\frac{p(1-w)}{1-\delta}}{\ln\delta}\rfloor.$$

We can solve the recursion that characterizes the lower bound and check that the loss $(s(1)-\tilde{s}(w))/(1-w)$ is given by

$$\frac{p(1-p)^n\delta^n}{1-\delta}\to_{w\to 1}0,$$

as was to be shown.

We now argue that s is differentiable on $(0,1)$ (and hence also at $w=1$). Fix $w\in(0,1)$, and correspondingly w^S,w^F, and the equilibrium q at this value of w. Consider the following strategy available to the principal. At the beginning of the period, the principal uses the public randomization device to flip a coin. With probability ε^2, where ε is a small number, not necessarily positive, no funds are advanced ($q=0$), no payment is made, and we move to the next period with continuation promise w^S. With probability

$$\varepsilon^2\frac{\delta w^S-w}{w-\delta w^F}+\varepsilon^3,$$

no funds are advanced, no payment is made, and we move to the next period with the agent's payoff unchanged at w. We note that, because $q>0$ and so $\delta w^S-w>0$, we have $w-\delta w^F>0$. Finally, with the remaining probability, play proceeds as in the equilibrium given w. First, note that because the coin flip is publicly observable, the incentive compatibility condition does not change: it only applies in the last of the three events, and conditional on this event, everything remains unchanged. Given this policy, the utility of the agent, denoted by w_ε, satisfies

$$w_\varepsilon=\delta\varepsilon^2w^S+\delta\left(\varepsilon^2\frac{\delta w^S-w}{w-\delta w^F}+\varepsilon^3\right)w+\delta\left(1-\varepsilon^2-\varepsilon^2\frac{\delta w^S-w}{w-\delta w^F}-\varepsilon^3\right)(pw^S+(1-p)w^F),$$

and it is easily verified that $w_\varepsilon\lesseqgtr w$ iff $\varepsilon\lesseqgtr 0$. Consider applying this strategy (parametrized by ε) at w_ε in some neighborhood $(w-\bar{\varepsilon},w+\bar{\varepsilon})$. We obtain a surplus $\tilde{s}$ that is (i) below s (since this cannot outperform the best strategy), (ii) continuously differentiable in ε, (iii) coincides with s at w. Because s is weakly concave, it follows as in Benveniste and Scheinkman [2] that s is continuously differentiable.

[5.2] We show that for any $w \in (0,1)$, we have $s'(w^F) \neq s'(w^S)$. There are two cases. First, suppose that $q < 1$. Then the first-order condition must hold (recall that $q > 0$), namely

$$1 + (1-p)(s'(w^S) - s'(w^F)) = 0,$$

from which it immediately follows that $s'(w^F) \neq s'(w^S)$. Suppose instead that $q = 1$, and $s'(w^F) = s'(w^S) =: \alpha$. We note that, because $q = 1$ and $w < 1$, we must have $w^F < w < w^S$. We also note that, given that $q = 1$, for any $\hat{w} \in [w^F, w]$, it holds that $\hat{w}^S \leq w^S$, and so since $\hat{w}^S \geq \hat{w}$, we have also $s'(\hat{w}) = s'(\hat{w}^S) = \alpha$. From the envelope theorem, at $\hat{w}$, we have $s'(\hat{w}) = ps'(\hat{w}^S) + (1-p)s'(\hat{w}^F)$, and so it follows that also $s'(\hat{w}^F) = \alpha$. Hence, from the first-order condition, $\hat{q}$, the scale chosen at $\hat{w}$ must also be equal to 1. Hence $\hat{w}^F < w^F$. In fact, because we can take $\hat{w} = w^F$, it follows that $s'((w^F)^F) = \alpha$, with $q(w) = 1$, $q(w^F) = 1$, and this argument can be repeated. Given that $q(w) = q(w^F) = q((w^F)^F) = \cdots = 1$, the sequence $(w^F, (w^F)^F, \ldots)$ eventually hits 0, and so the slope must be α at arbitrarily small values of w, a contradiction to our finding that $s'(0) = \infty$.

[5.3] This in turn allows us to argue that $q > w$. Suppose that $q \leq w$. Then $w^F \geq w$. But then, by familiar reasoning, $s'(w^F) = s'(w^S) = s'(w)$, contradicting our finding that these derivatives must be different.

[5.4] The function s is strictly concave on $(0,1)$. Suppose not, and consider an interval $[w_1, w_2]$ of maximal length $w_2 - w_1 =: \lambda > 0$ over which it is affine. Because of $s'(w^F) \neq s'(w^S)$, we must have $w_1^F \leq w_1$, $w_2^F \leq w_1$, and similarly $w_2^S \geq w_2$, $w_1^S \geq w_2$. It is readily verified from the formulas for w_k^S, w_k^F that

$$\max\{|w_1^F - w_2^F|, |w_1^S - w_2^S|\} > \lambda.$$

Hence, by definition, s is not affine over one of the two intervals $[\min\{w_1^F, w_2^F\}, \max\{w_1^F, w_2^F\}]$, $[\min\{w_1^S, w_2^S\}, \max\{w_1^S, w_2^S\}]$. Consider now the strategy that at $w = (w_1 + w_2)/2$, picks $q = (q_1 + q_2)/2$ (where q_k is the scale chosen at w_k), and continuation payoffs $w^F = (w_1^F + w_2^F)/2$, $w^S = (w_1^S + w_2^S)/2$, and reverts back to optimal play thereafter. Because the payoff in the first period is an average of the payoffs starting at w_1 and w_2, and because the continuation payoff is at least as much as the corresponding average (with one strict inequality, corresponding to the outcome that takes the continuation payoff into the interval over which s is not linear), this gives a payoff that is strictly higher than the average over the payoffs starting at w_1 and w_2.

[5.5] Finally, because s is strictly concave and differentiable on $(0,1)$, with $s'(w)$ being an average of $s'(w^F)$ and $s'(w^S)$, and $w^F \neq w^S$, it follows that $w^F < w < w^S$ for every $w \in (0,1)$. ■

5.3 Proof of Proposition 3

We maintain the normalizations introduced in Section 5.2.

Because s' is strictly decreasing, it is differentiable almost everywhere. In addition, s' is continuous and $\{w : s'(w) = \infty\}$ is a zero-measure set (namely $\{0\}$), and hence s' is absolutely continuous.[11] If $q \in (0,1)$ is optimal at w, it must hold that

$$s'(w^S(q)) - s'(w^F(q)) = \int_{w^F(q)}^{w^S(q)} s''(w)\mathrm{d}w = -\frac{1-\delta}{1-p},$$

(with the obvious notation $w^S(q), w^F(q)$), and we note that there can be at most one solution to such an equation, since w^F is decreasing and w^S increasing in q. By the same reasoning, if a solution $q \in (0,1)$ exists, it cannot be that $q = 1$ is also optimal, as this would require $\int_{w^F(q)}^{w^S(q)} s''(w)\mathrm{d}w \geq -\frac{1-\delta}{1-p} = \int_{w^F(1)}^{w^S(1)} s''(w)\mathrm{d}w$. Hence, the (upper hemicontinuous) correspondence $q : [0,1] \rightrightarrows [0,1]$ mapping w into the set of maximizers of s is a continuous function. Then we can decompose the interval $[0,1]$ into a partition of intervals $\{I_k\}$, $\{J_k\}$, with $q < 1$ on each I_k, and $q = 1$ on each J_k, and it follows from Santos [14], for instance, that q is differentiable on the interior of each I_k.

We already know that there exists $a > 1-\delta$ such that $q \in (0,1)$ for all $w \in (0,a)$ (this is because we must have $w^F > 0$). Similarly, there exists $b > 0$ such that $q = 1$ for all $w \in [1-b, 1]$. Indeed, note that, with $q = 1$, the first-order condition for any w such that $w^S \geq 1$ becomes $1 - \delta - (1-p)s'(w^F) \geq 0$, which is true for w close enough to 1, because $\lim_{w\to 1} w^F = 1$ and Lemma 5.1 (establishing the zero derivative at $s = 1$).

It remains to show that the partition I_k, J_k has a simple structure: $q < 1$ if and only if $w \leq \hat{w}$, for some $\hat{w} \in (0,1)$. This will be established by value iteration, making the following induction hypothesis. Let there be given a function s_n on $(0,1)$, with the following properties: it is continuously differentiable, with $s_n'(w) > 0$ for all $w \in [0,1]$, with s_n' being almost everywhere twice differentiable, with $s_n''(w) < 0$, and $s_n'''(w) > -s_n''(w)$, $s_n(0) = 0$, $s_n(1) = 1$, $\lim_{w\to 0} s_n'(w) = +\infty$, $\lim_{w\to 1} s_n'(w) = 0$ (and we extend it to $s_n(w) = 1$ for $w \geq 1$). We establish that these properties also hold for s_{n+1}, the function resulting from maximizing

[11] See, *e.g.*, Leoni [8, Exercise 3.20].

over q the payoff from $\mathcal{P}$, taking s_n as the continuation payoff. Let q_{n+1} be the maximizing correspondence (a function, by the same arguments as before).

We divide the analysis according to whether $q = q_{n+1}(w) = 1$ or $q < 1$—that is, we consider in turn open intervals of values of w for which $q = 1$ and then $q < 1$. That there is such partition follows from arguments entirely analogous to the previous ones, and omitted (as is the proof that s'_{n+1} is continuously differentiable, and that $\lim_{w\to 0} s'_{n+1}(w) = +\infty$, $\lim_{w\to 1} s'_{n+1}(w) = 0$).

First, suppose that $q = 1$ (which enters the definition of w^S and w^F). Then we have from the envelope theorem,

$$s'_{n+1}(w) = (1-p)s'_n(w^F) + ps'_n(w^S),$$

and, obviously, since $q = 1$ identically on such an (open) interval, we have more generally, for $k = 1, 2, 3$,

$$\delta^{k-1} s^{(k)}_{n+1}(w) = (1-p)s^{(k)}_n(w^F) + ps^{(k)}_n(w^S),$$

where $f^{(k)}$ refers to the k-th derivative of f (and differentiability follows from the differentiability of $s^{(k)}_n$). So, in particular,

$$\begin{aligned} s^{(3)}_{n+1}(w) &= \frac{1}{\delta^2}((1-p)s^{(3)}_n(w^F) + ps^{(3)}_n(w^S)) \\ &\geq -\frac{1}{\delta^2}(-(1-p)s^{(2)}_n(w^F) + ps^{(2)}_n(w^S)) \\ &\geq -\frac{1}{\delta}(-(1-p)s^{(2)}_n(w^F) + ps^{(2)}_n(w^S)) = -s''_{n+1}(w). \end{aligned}$$

Consider now an interval over which $q < 1$ (obviously, q need not be constant). The necessary first-order condition with respect to q gives

$$1 + (1-p)(s'_n(w^S) - s'_n(w^F)) = 0,$$

and differentiating with respect to w (differentiability of q'_{n+1} follows from the implicit function theorem), we get

$$q'_{n+1}(w) = \frac{p}{1-\delta} \frac{s''_n(w^F) - s''_n(w^S)}{ps''_n(w^F) + (1-p)s''_n(w^S)}.$$

The envelope theorem gives

$$s'_{n+1}(w) = (1-p)s'_n(w^F) + ps'_n(w^S),$$

which implies $s'_{n+1} > 0$. This equation holding identically, we differentiate again with respect to w (not forgetting to insert $q'_{n+1}(w)$) to get

$$s''_{n+1}(w) = \frac{1}{\delta} \frac{s''_n(w^F)s''_n(w^S)}{ps''_n(w^F) + (1-p)s''_n(w^S)},$$

yielding $s''_{n+1} < 0$. We repeat this exercise once more, to get

$$s'''_{n+1}(w) = \frac{1}{\delta^2} \frac{(1-p)s''_n(w^S)^3 s'''_n(w^F) + p s''_n(w^F)^3 s'''_n(w^S)}{(p s''_n(w^F) + (1-p)s''_n(w^S))^3}.$$

It remains to show that

$$s'''_{n+1}(w) \geq -s''_{n+1}(w),$$

which will follow if we show

$$\frac{(1-p)s''_n(w^S)^3 s'''_n(w^F) + p s''_n(w_F)^3 s'''_n(w^S)}{(p s''_n(w^F) + (1-p)s''_n(w^S))^3} \geq -\frac{s''_n(w^F) s''_n(w^S)}{p s''_n(w^F) + (1-p)s''_n(w^S)}.$$

By the induction hypothesis,

$$\frac{(1-p)s''_n(w^S)^3 s'''_n(w^F) + p s''_n(w_F)^3 s'''_n(w^S)}{(p s''_n(w^F) + (1-p)s''_n(w^S))^3} \geq -s''_n(w^F) s''_n(w^S) \frac{(1-p)s''_n(w^S)^2 + p s''_n(w^F)^2}{(p s''_n(w^F) + (1-p)s''_n(w^S))^3},$$

hence it suffices to show that

$$-\frac{(1-p)s''_n(w^S)^2 + p s''_n(w^F)^2}{(p s''_n(w^F) + (1-p)s''_n(w^S))^3} \geq -\frac{1}{p s''_n(w^F) + (1-p)s''_n(w^S)},$$

or

$$(1-p)s''_n(w^S)^2 + p s''_n(w^F)^2 \geq (p s''_n(w^F) + (1-p)s''_n(w^S))^2,$$

which is equivalent to

$$p(1-p)(s''_n(w^S)^2 + p s''_n(w^F)^2) \geq 2p(1-p)s''_n(w^F) s''_n(w^S),$$

or

$$p(1-p)(s''_n(w^S) - s''_n(w^F))^2 \geq 0,$$

which holds trivially. Hence,

$$\begin{aligned} q'_{n+1}(w) &= -\frac{p}{1-\delta} \frac{\int_{w^F}^{w^S} s'''_n(w)dw}{p s''_n(w^F) + (1-p)s''_n(w^S)} \\ &\geq \frac{p}{1-\delta} \frac{\int_{w^F}^{w^S} s''_n(w)dw}{p s''_n(w^F) + (1-p)s''_n(w^S)} = \frac{s'_n(w^S) - s'_n(w^F)}{p s''_n(w^F) + (1-p)s''_n(w^S)} \geq 0, \end{aligned}$$

where we use concavity and the continuous differentiability of s'.[12] It follows that q_{n+1} is continuously increasing on some interval $[0, \hat{w}]$, and equal to 1 on $[\hat{w}, 1]$.

[12] Unlike s'_{n+1}, s''_{n+1} is *not* continuously differentiable. Hence, $\int_{w'}^{w''} s'''_{n+1}(w)dw \neq s''_{n+1}(w'') - s''_{n+1}(w')$ for some values of w', w'', so that assuming $s'''_n \geq 0$ as induction hypothesis is not enough for the proof.

By value iteration then, we have that $s^n \to s$, and q^n (which converges along some subsequence, by Helly's selection theorem, given that q^n is monotone) converges to a policy q that is optimal with an infinite horizon (see, e.g., Schäl [15]). In particular, it follows that q is non-decreasing on some $[0, \hat{w}]$, and equal to 1 on $[\hat{w}, 1]$. By our prior results, we know that it must be continuous.

5.4 Calculations, Remark 2

Consider a continuation history beginning with value W, and with the next two periods referred to as periods 1 and 2. Recalling that

$$\begin{aligned} W^F &= \frac{W}{\delta} - p\frac{1-\delta}{\delta}qk\pi \\ W^S &= \frac{W}{\delta} + (1-p)\frac{1-\delta}{\delta}qk\pi, \end{aligned}$$

we can write

$$\begin{aligned} W^{SS} &= \frac{W}{\delta^2} + (1-p)\frac{1-\delta}{\delta^2}q_1k\pi + (1-p)\frac{1-\delta}{\delta}\overline{q}_2k\pi \\ W^{SF} &= \frac{W}{\delta^2} + (1-p)\frac{1-\delta}{\delta^2}q_1k\pi - p\frac{1-\delta}{\delta}\overline{q}_2k\pi \\ W^{FS} &= \frac{W}{\delta^2} - p\frac{1-\delta}{\delta^2}q_1k\pi + (1-p)\frac{1-\delta}{\delta}\underline{q}_2k\pi \\ W^{FF} &= \frac{W}{\delta^2} - p\frac{1-\delta}{\delta^2}q_1k\pi - p\frac{1-\delta}{\delta}\underline{q}_2k\pi, \end{aligned}$$

where $\overline{q}_2$ is the second period value of q following a success in period 1, and $\underline{q}_2$ the value following a failure, and where W^{SF} (for example) is the continuation value following first a success and then a failure. Now suppose we contemplate adjustments in the values of q_1 and q_2 satisfying

$$\frac{d\underline{q}_2}{dq_1} = \frac{d\overline{q}_2}{dq_1} = -\frac{1}{\delta}.$$

Then this adjustment preserves the principal's expected payoff in periods 1 and 2, and also preserves the values of W^{SS} and W^{FF}. The effect of this adjustment on $S(W)$ then arises entirely out of its effect on W^{SF} and W^{FS}. Here, we have

$$\begin{aligned} \frac{dW^{SF}}{dq_1} &= \frac{1-\delta}{\delta}k\pi\left(\frac{1-p}{\delta} + \frac{p}{\delta}\right) = \frac{1-\delta}{\delta^2}k\pi \\ \frac{dW^{FS}}{dq_1} &= \frac{1-\delta}{\delta}k\pi\left(\frac{-p}{\delta} - \frac{(1-p)}{\delta}\right) = -\frac{1-\delta}{\delta^2}k\pi. \end{aligned}$$

Hence, increasing q_1 gives us an increase in W^{SF} and a like-sized decrease in W^{FS}, with a decrease in q_1 having the reverse effect. Because $S(W)$ is concave, and since a success followed by failure and a failure followed by success are equiprobable, it is always advantageous to adjust q_1 (with the corresponding adjustments in $\underline{q}_2$ and $\overline{q}_2$) so as to push W^{SF} and W^{FS} closer together. Now notice the following string of equivalent statements:

$$\begin{aligned} W^{SF} &> W^{FS} \\ (1-p)\frac{q_1}{\delta} - p\overline{q}_2 &> -p\frac{q_1}{\delta} + (1-p)\underline{q}_2 \\ p(q_1 - \delta\overline{q}_2) &> (1-p)(\delta\underline{q}_2 - q_1) \\ p(\delta\overline{q}_2 - q_1) &< (1-p)(q_1 - \delta\underline{q}_2). \end{aligned}$$

Hence, in the absence of any constraints, we must have

$$p(\delta\overline{q}_2 - q_1) = (1-p)(q_1 - \delta\underline{q}_2),$$

with

$$p(\delta\overline{q}_2 - q_1) > (1-p)(q_1 - \delta\underline{q}_2)$$

corresponding to pressure to increase q_1 and

$$p(\delta\overline{q}_2 - q_1) < (1-p)(q_1 - \delta\underline{q}_2)$$

corresponding to pressure to decrease q_1.

Now suppose $q_1 = 1$. Then surely we must have

$$p(\delta\overline{q}_2 - q_1) < (1-p)(q_1 - \delta\underline{q}_2),$$

since the left side is necessarily negative and the right side positive. Then optimality calls for a value of $q_1 < 1$, a contradiction, in the absence of some constraint.

Hence, if q_1, $\underline{q}_2$ and $\overline{q}_2$ are all interior, then

$$\delta\overline{q}_2 - q_1 = q_1 - \delta\underline{q}_2,$$

which gives $W^{SF} = W^{FS}$, which suffices for the result.

5.5 Proof of Proposition 4

We maintain the normalizations introduced in Section 5.2.

First, we claim that $\delta' > \delta$ implies that $s_{\delta'} \geq s_\delta$. Let $q = q_\delta(w)$ be the maximizer given δ. Note that, given δ', we can set $q' = \frac{1-\delta}{1-\delta'}q$ (a feasible policy, as we have $w^F_{\delta'}(q') > 0$ if and only if $w^F_\delta(q) > 0$). Because the Bellman operator is monotone, consider applying q' with a continuation payoff $s = s_\delta$: we claim that the resulting payoff exceeds $s(w)$, so that the fixed point $s_{\delta'}$ lies above s_δ, strictly for $w \in (0,1)$. Indeed, because $s(0) = 0$, and s is strictly concave, we have that, for $\alpha > 1$, $w \in (0,1)$,

$$\alpha s\left(\frac{w}{\alpha}\right) > s(w),$$

hence, for $w' = w + q(1-\delta)(1-p)/p = w + q'(1-\delta')(1-p)/p$, as well as $w' = w - q(1-\delta) = w - q'(1-\delta')$,

$$\delta' s\left(\frac{w'}{\delta'}\right) = \delta\frac{\delta'}{\delta}s\left(\frac{w'}{\delta\frac{\delta'}{\delta}}\right) > \delta s\left(\frac{\delta'}{\delta}\frac{w'}{\delta\frac{\delta'}{\delta}}\right) = \delta s\left(\frac{w'}{\delta}\right),$$

and the result follows from

$$\begin{aligned} & (1-\delta)q' + \delta' p s\left(\frac{w + q'(1-\delta')(1-p)/p}{\delta'}\right) + \delta'(1-p)s\left(\frac{w - q'(1-\delta')}{\delta'}\right) \\ \geq \; & (1-\delta)q + \delta p s\left(\frac{w + q(1-\delta)(1-p)/p}{\delta}\right) + \delta(1-p)s\left(\frac{w - q(1-\delta)}{\delta}\right). \end{aligned}$$

Next, we show that $\lim_{\delta\to 1} s_\delta(w) = 1$ for all $w > 0$. Consider the strategy that sets $q(w) = 1$ for all $w > 1-\delta$, and $q(w) = w/(1-\delta)$ otherwise. Note that, because the payoff from this strategy is non-decreasing in w, we have that, for $w > 1-\delta$,

$$\begin{aligned} s(w) &= (1-\delta) + \delta p s\left(\frac{w + (1-\delta)(1-p)/p}{\delta}\right) + \delta(1-p)s\left(\frac{w - (1-\delta)}{\delta}\right) \\ &\geq (1-\delta) + \delta p s\left(w + \frac{(1-\delta)(1-p)/p}{\delta}\right) + \delta(1-p)s\left(w - \frac{(1-\delta)}{\delta}\right), \end{aligned}$$

and we note that the process that takes value $w + \frac{(1-\delta)(1-p)/p}{\delta}$ with probability p and takes value $w - \frac{(1-\delta)}{\delta}$ with probability $1-p$ is a martingale (for $w > 1-\delta$). Hence, by Azuma-Hoeffding's inequality, starting from w, the probability that the process hits $\{w : w \leq 1-\delta\}$ in exactly N steps is majorized by

$$\exp\left(\frac{-w^2}{2N\max\{1,(1-p)/p\}\frac{(1-\delta)^2}{\delta^2}}\right),$$

so that the loss from this policy (relative to 1) is majorized by

$$\sum_{N=1}^{\infty}\delta^N \exp\left(\frac{-w^2}{2N\max\{1,(1-p)/p\}\frac{(1-\delta)^2}{\delta^2}}\right),$$

which tends to 0 as $\delta \to 1$.

Finally, we note that this also implies that the optimal policy converges pointwise to 1. Given concavity and convergence of w to 1, it follows from the first-order condition

$$1 + (1-p)(s'(w^S) - s'(w^F)) > 0 \Rightarrow q(w) = 1,$$

that, fixing w', and picking δ high enough so that necessarily $s'(w^S) - s'(w^F) > -1/(1-p)$ for all $w > w'$, $q(w) = 1$. ■

5.6 Calculations, Section 4.2

We first argue that the lowest equilibrium payoffs $(\underline{W}, \underline{V})$ can be jointly obtained and lie on the function h. Suppose they are not jointly obtained. Then we have an equilibrium with payoffs $(\underline{W}, \tilde{V})$ that achieves the lowest equilibrium payoff for the agent and for which $\tilde{V}$ is the minimum payoff for the principal consistent with the agent receiving $\underline{W}$, but which features $\tilde{V} > \underline{V}$. Hence, there exist feasible continuation payoffs (W^F, V^F) and (W^S, V^S) such that the values $(\underline{W}, \tilde{V})$ satisfy

$$\begin{array}{rcl}
\underline{W} & = & (1-\delta)[p\mathbf{1}_S z^S + p(1-\mathbf{1}_S)qk\pi + (1-p\mathbf{1}_S)z^F] + \delta[pW^S + (1-p)W^F] \\
\tilde{V} & = & (1-\delta)[p\mathbf{1}_S(q\pi - z^S) - (1-p\mathbf{1}_S)z^F - qc] + \delta[pV^S + (1-p)V^F] \\
\mathbf{1}_S = 1 & \Longleftarrow & (1-\delta)z^S + \delta W^S > (1-\delta)(z^F + qk\pi) + \delta W^F \\
\mathbf{1}_S = 0 & \Longleftarrow & (1-\delta)z^S + \delta W^S < (1-\delta)(z^F + qk\pi) + \delta W^F \\
\tilde{V} & > & h(\underline{W}).
\end{array}$$

Suppose $\mathbf{1}_S = 1$. Then we can rewrite this as

$$\begin{array}{rcl}
\underline{W} & = & (1-\delta)[pz^S + (1-p)z^F] + \delta[pW^S + (1-p)W^F] \\
\tilde{V} & = & (1-\delta)[p(q\pi - z^S) - (1-p)z^F - qc] + \delta[pV^S + (1-p)V^F] \\
 & & (1-\delta)z^S + \delta W^S \geq (1-\delta)(z^F + qk\pi) + \delta W^F \\
\tilde{V} & > & h(\underline{W}).
\end{array}$$

We now make use of the fact that (by assumption) the final constraint does not bind. First, we must have $z^F = 0$, since reducing z^F reduces $\underline{W}$ while preserving all of the constraints, contradicting that $\underline{W}$ is the agent's lowest equilibrium payoff. Next, $W^F = \underline{W}$. Otherwise, we could replace (W^F, V^F) with $(\lambda\underline{W} + (1-\lambda)W^F, \lambda\tilde{V} + (1-\lambda)V^F)$, which preserves the constraints and the feasibility of continuation payoffs, increasing λ and hence decreasing W^F

until either the final constraint binds or $W^F = \underline{W}$. Next, the agent's incentive constraint must bind, since otherwise we can reduce z^S, possibly until it hits zero, and then replace (W^S, V^S) by $(\lambda\underline{W}+(1-\lambda)W^S, \lambda\tilde{V}+(1-\lambda)V^S)$, again in the process preserving feasibility and reducing W^S and continuing until either the final constraint binds or the agent's incentive constraint binds. Hence, we have

$$\begin{aligned}
\underline{W} &= (1-\delta)pz^S + \delta[pW^S + (1-p)\underline{W}] \\
\tilde{V} &= (1-\delta)[p(q\pi - z^S) - qc] + \delta[pV^S + (1-p)V^F] \\
&\quad (1-\delta)z^S + \delta W^S = (1-\delta)(qk\pi) + \delta\underline{W} \\
\tilde{V} &> h(\underline{W}).
\end{aligned}$$

Now suppose $W^S > \underline{W}$. Then we can decrease W^S, preserving feasibility by replacing (W^S, V^S) with $(\lambda\underline{W}+(1-\lambda)W^S, \lambda\tilde{V}+(1-\lambda)V^S)$, and also increasing z^S so as to preserve both the incentive constraint and the agent's payoff, continuing until either the final constraint binds or $W^S = \underline{W}$. Our problem is then

$$\begin{aligned}
\underline{W} &= (1-\delta)pz^S + \delta\underline{W} \\
\tilde{V} &= (1-\delta)[p(q\pi - z^S) - qc] + \delta[pV^S + (1-p)V^F] \\
&\quad z^S = qk\pi \\
\tilde{V} &> h(\underline{W}).
\end{aligned}$$

Now a reduction in q and a proportional reduction in z^S gives us an equilibrium with a smaller value for the agent, a contradiction to $\underline{W}$ being the lowest equilibrium payoff for the agent, unless $q = 0$. But then we have $\underline{W} = 0$, also a contradiction.

Suppose instead $\mathbf{1}_S = 0$. Then we can rewrite this as

$$\begin{aligned}
\underline{W} &= (1-\delta)z^F + \delta W^F \\
\tilde{V} &= (1-\delta)[-z^F - qc] + \delta V^F \\
&\quad (1-\delta)z^S + \delta W^S \leq (1-\delta)(z^F + qk\pi) + \delta W^F \\
\tilde{V} &> h(\underline{W}).
\end{aligned}$$

We can take $z^S = q = 0$ and can let the continuation payoffs following a success equal those following a failure, thereby setting W^S equal to W^F and hence eliminating the agent's incentive constraint, making this

$$\begin{aligned}
\underline{W} &= \delta W^F \\
\tilde{V} &= (1-\delta)[-z^F] + \delta V^F \\
\tilde{V} &> h(\underline{W}).
\end{aligned}$$

But then either the final constraint binds or, by replacing (W^F, V^F) with $(\lambda \underline{W} + (1-\lambda)W^F, \lambda\tilde{V} + (1-\lambda)V^F)$, we can take $W^F = \underline{W}$, giving $\underline{W} = 0$, a contradiction.

We thus know that there exists a payoff $(\underline{W}, \underline{V})$, drawn from the function $V = h(W)$, that is the lowest equilibrium payoff for each player. From our previous reasoning, we have that we can take $z^F = 0$, that the function $\underline{V} = h(\underline{W})$ binds, and that the agent's incentive constraint binds. The outcome of an equilibrium with $\mathbf{1}_S = 0$ can be duplicated by an equilibrium with $\mathbf{1}_S = 1$ and $q = 0$, and so $(\underline{W}, \underline{V})$ solves

$$\begin{aligned}
\underline{W} &= (1-\delta)pz^S + \delta[pW^S + (1-p)W^F] \\
\underline{V} &= (1-\delta)[p(q\pi - z^S) - qc] + \delta[pV^S + (1-p)V^F] \\
& \quad (1-\delta)z^S + \delta W^S > (1-\delta)qk\pi + \delta W^F \\
\underline{V} &= h(\underline{W}).
\end{aligned}$$

The current-period expected payoff is given by $(pz^S, p(qk\pi - z^S)) =: \Theta$, and we have the convex combination

$$(\underline{W}, \underline{V}) = (1-\delta)\Theta + \delta(1-p)(\underline{W}, \underline{V}) + \delta p(W^S, W^F). \tag{12}$$

We now note that[13]

- If $pqk\pi > h(0)$, then we can adjust z^S, with a corresponding adjustment in (W^S, V^W) to preserve (12), noting that this also necessarily preserves the incentive constraint, until we have $z^S = h(pqk\pi)$, *i.e.*, until Θ lies on the function h.
- $pqk\pi \leq h(0)$, then we can adjust z^S, with a corresponding adjustment in (W^S, V^W) to preserve (12), noting that this also necessarily preserves the incentive constraint, until we have $z^S = 0$.

Hence, we can assume that Θ is drawn either from the function h or from the line segment connecting the original to the vertical intercept of h. We can then note that if Θ is drawn from the graph of the function h, it must be that $z^S = 0$. If not, we can preserve the continuation payoffs (W^S, W^F) and replace Θ with the vertical intercept of h, with the resulting solution to (12) allowing us to reduce the putative lowest equilibrium payoffs, a contradiction. Hence, we can assume that Θ is drawn from the line segment connecting the origin with the vertical intercept of h.

[13] One might in each of the following cases wonder whether these adjustments preserve the feasibility of (W^S, V^S). In each case, the resulting values remain within the convex hull of four payoff vectors, namely $(\underline{W} < \underline{V})$, $(\underline{V} - \underline{W}, \underline{W})$, (W^*, V^*), and the original values of (W^S, V^S).

For large values of δ, we can now derive an upper bound on the lowest equilibrium payoffs. Suppose we set Θ equal to the vertical intercept of h. Then from (12), the agent's incentive constraint, and the equation for h, we have, respectively

$$\begin{aligned} W^S &= \frac{\delta p}{1-\delta(1-p)}\underline{W} \\ W^S &= \frac{qk\pi}{\delta} + \underline{W} \\ pqk\pi &= \frac{p(1-k)\pi - c - \frac{p\delta}{1-\delta}(p\pi - c)}{1 - \frac{p\delta}{1-\delta}}. \end{aligned}$$

The third equation fixes the value of q. Given this, the first two equations joint determine $\overline{W}$ and W^S. The resulting solution for the value of $\overline{W}$, with the accompanying value is an upper bound on the value of the lowest equilibrium payoffs, since we have constructed this value subject to the restriction that π be given by the vertical intercept of the function h.

As δ declines to zero, value of W^S associated with this upper bound explodes. Hence, there exists value $\underline{\delta}$ at which the attendant value of W^S hits $pk\pi$. Above this value of δ our current construction will yield an equilibrium, and hence the upper bound no longer applies.

We can show that if δ is sufficiently small but nonzero, then the lowest equilibrium payoffs are (W^*, V^*). Substituting the agent's incentive constraint into the expression for $\underline{W}$, we have $\underline{W} \geq (1-\delta)qpk\pi + \delta W^F$, which implies that $qpk\pi \leq \underline{W}$, and hence

$$q < \frac{\underline{W}}{pk\pi}.$$

Feasibility requires

$$\begin{aligned} \underline{W} + h(\underline{W}) &\leq (1-\delta)q(p\pi - c) + \delta(p\pi - c) \\ &\leq (1-\delta)\underline{W}\frac{p\pi - c}{pk\pi} + \delta(p\pi - c). \end{aligned}$$

Both sides of this weak inequality are linear and increasing in $\underline{W}$. The relationship holds with equality if $\underline{W} = pk\pi$. We are interested in its smallest solution. This smallest solution will be $pk\pi$ if the slope of the left side falls short of that of the right, *i.e.*, if

$$1 + \frac{\frac{p\delta}{1-\delta}}{1 - \frac{p\delta}{1-\delta}} < (1-\delta)\frac{p\pi - c}{pk\pi},$$

which gives

$$\delta(1+p)(p\pi - c) < p(1-k)\pi - c.$$

This will clearly be satisfied, and hence $(\underline{W}, \underline{V}) = (W^*, V^*)$, for sufficiently small but positive δ.

We thus have that for all

$$\delta < \frac{p(1-k)\pi - c}{(1+p)(p\pi - c)},$$

the lowest equilibrium payoffs are (W^*, V^*). This is a sufficient but not necessary condition for lowest equilibrium payoffs to be (W^*, V^*), since our feasibility calculation assumed that the continuation payoffs could achieve the efficient surplus, which is an overestimate.

Finally, when it is nontrivial, what determines the location of the $(\underline{W}, \underline{V})$ on the function h? It will be the solution to the program

$$\begin{aligned}
&\min_{q\in[0,1]} \underline{W} \\
&s.t. \qquad W^S = \frac{\delta p}{1-\delta(1-p)}\underline{W} \\
&\qquad\qquad W^S = \frac{qk\pi}{\delta} + \underline{W} \\
&\qquad\qquad (W^S, V^S) \text{ feasible.}
\end{aligned}$$

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