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Approximate Solutions of the Walrasian Equilibrium Inequalities with Bounded Marginal Utilities of Income

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Abstract

Recently Cherchye et al. (2011) reformulated the Walrasian equilibrium inequalities, introduced by Brown and Matzkin (1996), as an integer programming problem and proved that solving the Walrasian equilibrium inequalities is NP-hard. Brown and Shannon (2002) derived an equivalent system of equilibrium inequalities ,i.e., the dual Walrasian equilibrium inequalities. That is, the Walrasian equilibrium inequalities are solvable iff the dual Walrasian equilibrium inequalities are solvable.

We show that solving the dual Walrsian equilibrium inequalities is equivalent to solving a NP-hard minimization problem. Approximation theorems are polynomial time algorithms for computing approximate solutions of NP-hard minimization problems. The primary contribution of this paper is an approximation theorem for the equivalent NP-hard minimization problem. In this theorem, we derive explicit bounds, where the degree of approximation is determined by observable market data.

Keywords: Rationalizable Walrasian markets, NP-hard minimization problems, Approximation theorems

JEL Classification: B41, C68, D46

1 Introduction

The Brown-Matzkin (1996) theory of rationalizing market data with Walrasian markets, where consumers are price-taking, utility maximizers subject to budget constraints, consists of market data sets and the Walrasian equilibrium inequalities. A market data set is a finite number of observations on market prices, income distributions and social endowments. The Walrasian equilibrium inequalities are the Afriat inequalities for each consumer–see Afriat (1967) and Varian (1982); the budget constraints for each consumer and the market clearing conditions in each observation. The unknowns in the Walrasian equilibrium inequalities are the utility levels, the marginal utilities of income and the Marshallian demands of individual consumers in each observation. The parameters are the observable market data: market prices, income distributions and social endowments in each observation. The Walrasian equilibrium inequalities are said to rationalize the observable market data if the Walrasian equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of individual consumers, where aggregate Marshallian demands are equal to the social endowments in every observation. Brown and Matzkin show that the observed market data is consistent with the Walrasian paradigm, as articulated by Arrow and Debreu (1954), iff the Walrasian equilibrium inequalities rationalize the observed market data. As such, the Brown–Matzkin theory of rationalizing market data with Walrasian markets requires an efficient algorithm for solving the Walrasian equilibrium inequalities.

The Walrasian equilibrium inequalities are multivariate polynomial inequalities. The Tarski–Seidenberg theorem, Tarski (1951), provides an algorithm, "quantifier elimination," that derives a finite family of multivariate polynomial inequalities, i.e., the "revealed Walrasian equilibrium inequalities"from the Walrasian equilibrium inequalities, where the unknowns are the observable market data: market prices, income distributions and the social endowments in each observation. It follows from the Tarski–Seidenberg theorem that the revealed Walrasian equilibrium inequalities are solvable for the observed market data iff the Walrasian equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of consumers.

An important example is the special case of the Walrasian equilibrium inequalities, recently introduced by Brown and Calsamiglia (2014). They propose necessary and sufficient conditions on observable market data to rationalize the market data with consumers endowed with utility functions, where the marginal utilities of income are constant: the so-called "strong law of demand". The strong law of demand is a finite family of linear inequalities on the observed market data, hence solvable in polynomial time. See their paper for details.

Unfortunately, in general, the computational complexity of the Tarski–Seidenberg algorithm, is known to be doubly exponential in the worse case. Hence we are forced to consider approximate solutions of the Walrasian equilibrium inequalities. See Basu (2011) for a discussion of the Tarski–Seidenberg theorem and the computational complexity of quantifier elimination.

A decision problem in computer science is a problem where the answer is "yes" or "no." In this paper, the decision problem is: Can the observed market data be rationalized with Walrasian equilibrium inequalities? That is, are the Walrasian equilibrium inequalities solvable if the values of the parameters are derived from the observed market data? A decision problem is said to have polynomial complexity, i.e., the problem is in class P, if there exists an algorithm that solves each instance of the problem in time that is polynomial in some measure of the size of the problem instance. In the literature on computational complexity, polynomial time algorithms are referred to as "efficient" algorithms. A decision problem is said to be in NP, if there exists an algorithm that verifies, in polynomial time, if a proposal is a solution of the problem instance Clearly, $P \subset NP$ but it is widely conjectured by computer scientists that $P \neq NP$. The decision problem A is said to be NP-*hard*, if every problem in NP can be reduced in polynomial time to A. That is, if we can decide the NP-*hard* problem A in polynomial time then we can decide every NP problem in polynomial time. In this case, contrary to the current beliefs of computer scientists, $P = NP$.

What is the computational complexity of solving the Walrasian equilibrium inequalities? This important question was first addressed by Cherchye et al. (2011). They proved that solving the Walrasian equilibrium inequalities, reformulated as an integer programming problem, is NP-*hard*. We show that approximate solutions of the Walrasian equilibrium inequalities, reformulated as the dual Walrasian equilibrium inequalities introduced by Shannon and Brown (2000), can be computed in polynomial time. In the Brown-Shannon theory of rationalizing market data with Walrasian markets, the Afriat inequalities are replaced by the dual Afriat inequalities for minimizing the consumer's monotone, strictly convex, indirect utility function over prices subject to her budget constraint, defined by her Marshallian demand at the equilibrium market prices. The dual Walrasian equilibrium inequalities are said to rationalize the observed market data if the inequalities are solvable for some family of indirect utility levels, marginal indirect utilities and Marshallian demands of individual consumers, derived from Roy's identity, where the aggregate Marshallian demands are equal to the social endowment in every observation.

Brown and Shannon proved that the Walrasian equilibrium inequalities are solvable iff the dual Walrasian equilibrium inequalities are solvable. We show that solving the dual Walrasian equilibrium inequalities is equivalent to solving a $NP-hard$ minimization problem. Approximation theorems are polynomial time algorithms for computing approximate solutions of a $NP-hard$ minimization problem, where there are explicit a priori bounds on the degree of approximation. The primary contribution of this paper is an approximation theorem for a NP-*hard* minimization problem equivalent to solving Walrasian equilibrium inequalities with uniformly bounded marginal utilities of income.

2 The Dual Walrasian Equilibrium Inequalities

In this section, **following the suggestions of the referees on notation and exposition**, we review and summarize the dual Walrasian equilibrium inequalities proposed by Brown and Shannon. We consider an exchange economy, with $i \in \{1, 2, ..., M\}$ consumers. For each observation $j \in \{1, 2, ..., N\}$, p_j is a vector of prices in R^L_{++} , η_j is a vector of aggregate endowments of commodities in R^L_{++} and $\{I_{1,j}, I_{2,j}, ..., I_{M,j}\}$ is the distribution of positive incomes of consumers in observation j, where$\sum_{i=1}^{i=M} I_{i,j} = p_j \cdot \eta_j$ for $j = 1, 2, .., N$. Brown and Shannon show that there exist strictly convex indirect utility functions $V_i(\frac{p}{I})$ for the i^{th} consumer and Marshallian demand vectors $x_{ij} \in R^L_{++}$ for the i^{th} consumer in the j^{th} observation that constitute a competitive equlibrium in the j^{th} observation with respect to the observed data iff there exists numbers $V_{i,j}$ and $\lambda_{i,j} > 0$ and vectors $q_{i,j} << 0$ such that Eqs. (**1**) and **(2)** hold:

$$\textbf{(1)}\ V_{i,k} > V_{i,j} + q_{i,j} \cdot (\frac{1}{I_{i,k}} p_k - \frac{1}{I_{i,j}} p_j) \textbf{ Dual Afriat Inequalities}$$

$$\textbf{(2)}\ \sum_{i=1}^{i=M} \frac{-1}{\lambda_{i,j} I_{i,j}} q_{i,j} \ \leq \eta_j \textbf{ Market Clearing}$$

$$\textbf{(3)}\ \frac{p_j \cdot -q_{i,j}}{I^2_{i,j}} = \lambda_{i,j} \textbf{ FOC}$$

for all $i \in \{1, 2, ..., M\}$ and for all $j, k \in \{1, 2, ..., N\}$, $j \neq k$.

To derive the explicit expression of the marginal utilities of income in Eq. **3**, we consider the dual optimization problem of a consumer in an exchange economy, where she minimizes a monotone, smooth, strictly convex indirect utility function subject to her budget constraint. The budget constraint is defined by her income and her Marshallian demand, given by her income and market prices. That is, for fixed $(p_j, I_{i,j})$, the i^{th} consumer in observation j, solves the primal concave maximization problem $(P_{i,j})$:

$$\max_{\{x \in R^L_{++}:\ p \cdot x\ \leq\ I_{i,j}\}} U_i(x) = U_i(x_{i,j})$$

, where $x_{i,j}$ is the Marshallian demand at $(p_j, I_{i,j})$. In the dual problem for fixed $(x_{i,j}, I_{i,j})$ the consumer solves the dual convex minimization problem $(D_{i,j})$:

$$\min_{\{p \in R^L_{++}:\ p \cdot x_{i,j}\ \leq\ I_{i,j}\}} V_i(\frac{p}{I_{i,j}}) = V_i(\frac{p_j}{I_{i,j}})$$

where by Roy's identity

$$x_{i,j} = -\frac{\nabla_p V_i(\frac{p_j}{I_{i,j}})}{\nabla_I V_i(\frac{p_j}{I_{i,j}})} = -\frac{\frac{q_{i,j}}{I_{i,j}}}{-\frac{p_j \cdot q_{i,j}}{I^2_{i,j}}} = \frac{I_{i,j} q_{i,j}}{p_j \cdot q_{i,j}}$$

It follows from Slater's constraint qualification that there exist $\mu_{i,j} \geq 0$ such that the Lagrangian for $D_{i,j}$,

$$L(p; \mu_{i,j}) = V_i(\frac{p}{I_{i,j}}) + \mu_{i,j}(\frac{p}{I_{i,j}} \cdot x_{i,j} - 1)$$

Hence the necessary and sufficient first order conditions for minimizing $L(p;\mu_{i,j})$ are:

$$\frac{q_{i,j}}{I_{i,j}} = \nabla_p V_i(\frac{p_j}{I_{i,j}}) = -\mu_{i,j}x_{i,j} = -\mu_{i,j}\frac{I_{i,j}q_{i,j}}{p_j \cdot q_{i,j}} = \mu_{i,j}\frac{\frac{q_{i,j}}{I_{i,j}}}{\lambda_{i,j}} = \text{ iff } \mu_{i,j} = \lambda_{i,j}$$

,where

$$\lambda_{i,j} = \frac{p_j \cdot -q_{i,j}}{I_{i,j}^2}$$

As noted by the referees, the intuition of this specification is immediate: $V_{i,j}$ is the i^{th} consumer's utility of $x_{i,j}$ in observation j; $\lambda_{i,j}$ is her marginal utility of income in observation j; $q_{i,j}$ is the gradient of her indirect utility function with respect to $\frac{1}{I_{i,j}}p_j$ in observation j; Eq. **(1)** is the dual Afriat inequalities for minimizing her strictly convex, indirect utility function subject to her budget constraint in each observation; Eq. **(3)** is the first order conditions of consumer i in observation j for minimizing her strictly convex, indirect utility function subject to her budget constraint and Eq.**(2)** are the market clearing conditions in observation j.

The system of inequalities defined by Eqs. **(1)** and **(3)** are linear in the unknown utility levels $V_{i,j}$, marginal utilities of income $\lambda_{i,j}$ and marginal indirect utilities $q_{i,j}$ .Unfortunately, Eq. **(2)** is nonlinear in $\lambda_{i,j}$ and $q_{i,j}$ .In fact, this nonlinearity is the cause of the $NP-hard$ computational complexity first observed by Cherchye et al.

3 Bounds on the Marginal Utilities of Income

There is a special case of the dual Walrasian equilibrium inequalities where the computational complexity is polynomial. If we restrict attention to exchange economies where $\lambda_{i,j}$ is 1 for all i and j, as in Brown and Calsamiglia, then Eqs. **(2)** and **(3)** can be rewritten, respectively, as $\sum_{i=1}^{i=M}\frac{-1}{I_{i,j}}q_{i,j} \leq \eta_j$ and$\lambda_{i,j} = \frac{p_j\cdot -q_{i,j}}{I_{i,j}^2} = 1$ In this case, the dual Walrasian equilibrium inequalities : Eqs. (**1**), (**2**) and (**3**), are linear inequalities in the unknowns $\Omega \equiv \{(V_{i,j}, q_{i,j})$: Eq. **(1)** holds and $i = 1, 2, ..., M$; $j = 1, 2, ..., N\}$. Hence solvable in polynomial time.

Theorem 1 *To derive an upper bound for $\lambda_{i,j}$, we normalize $V_i(\frac{p_j}{I_{i,j}})$ by multiplying $V_i(\frac{p_j}{I_{i,j}})$ by*

$$[\max_{r,s}\left\|\frac{-q_{r,s}}{I_{r,s}}\right\|_2]^{-1}$$

$\overline{q_{i,j}}$ are defined as the gradients of the normalized indirect utility functions, where

$$\overline{q_{i,j}} = [\max_{r,s}\left\|\frac{-q_{r,s}}{I_{r,s}}\right\|_2]^{-1}\frac{q_{i,j}}{I_{i,j}}$$

If

$$\Theta_W \equiv \max\{1, \max_{r,s}\left\|\frac{p_s}{I_{r,s}^2}\right\|_1\}$$

then

$$\Theta_W \geq \max\{1; \lambda_{ij}(\omega) : 1 \leq i \leq M; 1 \leq j \leq N; \omega \in \Omega\}$$

Proof.

$$\textbf{(4)}\ \lambda_{i,j} = -\overline{q_{i,j}} \cdot \frac{p_j}{I_{i.j}^2} = [\max_{r,s} \left\| \frac{-q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{p_j}{I_{i.j}^2} \cdot \frac{-q_{i,j}}{I_{i,j}} \leq$$

$$[\max_{r,s} \left\| \frac{-q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \left\| \frac{-q_{i,j}}{I_{i,j}} \right\|_2 \left\| \frac{p_j}{I_{i.j}^2} \right\|_2 \leq \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}^2} \right\|_1 \} \equiv \Theta_W \ \textbf{Upper Bound}$$

■

We now have a uniform upper bound on the marginal utilities of income, where $\lambda_{i,j} = 1$ is feasible. That is, $1 \leq \Theta_W$ and for $1 \leq i \leq N$ and $1 \leq j \leq M$: $\lambda_{i,j} \leq \Theta_W$.

4 Approximation Theorem

Definition 2 *An approximation theorem for a $NP-hard$ minimization problem, with optimal value $OPT(\beta)$ for each input β, is a polynomial time algorithm for computing $\widehat{OPT(\beta)}$, the optimal value of the approximating minimization problem for the input β, and the approximation ratio $\alpha(\beta) \geq 1$, where*

$$OPT(\beta) \leq \widehat{OPT(\beta)} \leq \alpha(\beta) OPT(\beta)$$

That is,

$$\frac{\widehat{OPT(\beta)}}{\alpha(\beta)} \leq OPT(\beta) \leq \widehat{OPT(\beta)}$$

This definition was taken from the survey paper by Arora (1998) on the theory and application of approximation theorems in combinatorial optimization.

Theorem 3 *If $\Theta \geq \Theta_W$ and Δ_W is the optimal value of the nonconvex program S_W, where*

$$\textit{(5)}\ \Delta_W \equiv \min_{\omega \in \Omega, s_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} s_j : \textit{Eqs. (1) to (4) hold and} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}\lambda_{i,j}(\omega)} q_{i,j} \ \leq s_j \eta_j \ \textit{for } 1 \leq j \leq N \} : S_W$$

Γ_W is the optimal value of the approximating linear program R,where

$$\textit{(6)}\ \Gamma_W \equiv \min_{\omega \in \Omega, r_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} r_j : \textit{Eqs.(1) to (4) hold and} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} q_{i,j} \ \leq r_j \eta_j \ \textit{for } 1 \leq j \leq N \} : R_W$$

Ψ_W is the optimal value of the nonconvex program T, where

$$\textit{(7)}\ \Psi_W \equiv \min_{\omega \in \Omega, t_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} t_j : \textit{Eqs. (1) to (4) hold and } \Theta \sum_{i=1}^{i=M} \frac{-1}{I_{ij}\lambda_{ij}(\omega)} q_{ij} \leq t_j \eta_j \ \textit{for } 1 \leq j \leq N \} : T_W$$

then

$$(\textit{8})\ \Psi_W \geq \Gamma_W \geq \Delta_W$$

and

$$(\textit{9})\ \Psi_W = \Theta\Delta_W$$

Hence

$$\textit{(10)}\ \Theta\Delta_W \geq \Gamma_W \geq \Delta_W \Longleftrightarrow \Gamma_W \geq \Delta_W \geq \frac{\Gamma_W}{\Theta}$$

, where Θ is the the approximation rato.

Proof. (i) If r_j is feasible in R_W, then r_j is feasible in S_W and if t_j is feasible in T then t_j is feasible in R_W.To prove (ii) note the $1-1$ correspondence between s_j and t_j, where

$$t_j \rightarrow \frac{t_j}{\Theta} \equiv s_j \text{ and } s_j \rightarrow s_j\Theta \equiv t_j$$

That is,

$$\frac{\Psi_W}{\Theta} \equiv \min_{\omega\in\Omega, t_j\geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} t_j : \text{Eqs. (1) to (5) hold and } \sum_{i=1}^{i=M} \frac{-1}{I_{ij}\lambda_{ij}(\omega)} q_{ij} \leq \frac{t_j}{\Theta}\ \eta_j \text{ for } 1 \leq j \leq N\} =$$

$$\Delta_W \equiv \min_{\omega\in\Omega, s_j\geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} s_j : \text{Eqs. (1) to (5) hold and } \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}\lambda_{i,j}(\omega)} q_{i,j}\ \leq s_j\eta_j \text{ for } 1 \leq j \leq N\}$$

Hence

$$\Psi_W = \Theta\Delta_W$$

■

Corollary 4

$$\Theta_W\Delta_W \geq \Gamma_W \geq \Delta_W \Longleftrightarrow \Gamma_W \geq \Delta_W \geq \frac{\Gamma_W}{\Theta_W}$$

Corollary 5 *(a) $\Psi_W = 1$, iff the Walrasian equilibrium inequalities with constant marginal utilities of income rationalize the observed market data. (b) It follows from the Brown and Calsamiglia paper that $\Psi_W = 1$ iff the observed market data satisfies the strong law of demand. (c) If $\Psi_W = 1$ then $\Delta_W = 1$. (d) $\Delta_W = 1$, iff the Walrasian equilibrium inequalities with uniformly bounded marginal utilities of income rationalize the observed market data*

5 The Gorman Polar Form Equilibrium Inequalities

In Gorman's seminal (1961) paper, representative agent eonomies are characterized as exchange economies, where each consumer's indirect utility function $V_i(p, I)$ can be expressed as $V_i(p, I) = \frac{I-a_i(p)}{b(p)}$ where $a_i(p)$ and $b(p)$ are concave and homogeneous of degree 1. Hence all consumers have the same marginal utility of income, $\frac{1}{b(p)}$.

The $V_i(p, I)$ are quasiconvex in p, but we make the stronger assumption that they are convex in p. As suggested by Varian(1992) in section 9.4, where he implicitly uses well known results in fractional programming on ratios of convex and concave functions, we express utility functions in Gorman polar form as follows:

$$\textbf{(G)} \text{ If } V_i(p, I) = \frac{I - a_i(p)}{b(p)}, \text{ then } V_i(p, I) = Ie(p) + f_i(p),$$

$$\text{where } e(p) \equiv \frac{1}{b(p)} \text{and } f_i(p) \equiv \frac{-a(p)}{b(p)} \text{are convex in } p$$

Theorem 6 *To derive an upper bound for $\lambda_{,j}$, we normalize $V_i(\frac{p_j}{I_{,j}})$ by multiplying $V_i(\frac{p_j}{I_{,j}})$ by*

$$[\max_{r,s} \left\| \frac{-d_{r,s}}{I_{r,s}} \right\|_2]^{-1}$$

$\overline{q_{i,j}}$ are defined as the gradients of the normalized indirect utility functions, where

$$\overline{d_{i,j}} = [\max_{r,s} \left\| \frac{-d_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{d_{i,j}}{I_{i,j}}$$

If

$$\Theta_G \equiv \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}^2} \right\|_1 \}$$

then

$$\Theta_G \geq \max\{1; \lambda_j(\omega) : 1 \leq i \leq M; 1 \leq j \leq N; \omega \in \Omega\}$$

Proof.

$$\textbf{(14)}\ \lambda_{,j} = -\overline{d_{i,j}} \cdot \frac{p_j}{I_{i.j}^2} = [\max_{r,s} \left\| \frac{-q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{p_j}{I_{i.j}^2} \cdot \frac{-d_{i,j}}{I_{i,j}} \leq$$

$$[\max_{r,s} \left\| \frac{-d_{r,s}}{I_{r,s}} \right\|_2]^{-1} \left\| \frac{-d_{i,j}}{I_{i,j}} \right\|_2 \left\| \frac{p_j}{I_{i.j}^2} \right\|_2 \leq \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}^2} \right\|_1 \} \equiv \Theta_G \textbf{ Upper Bound}$$

■

We now have a uniform upper bound on the marginal utilities of income, where $\lambda_{,j} = 1$ is feasible. That is, $1 \leq \Theta_G$ and for $1 \leq i \leq N$ and $1 \leq j \leq M$: $\lambda_{,j} \leq \Theta_G$.

Theorem 7 *The Gorman Polar Form Equilibrium Inequalities:Eqs.* ***(11)***, ***(12)***, ***(13)***, ***(14)*** *and* ***(G1)***, ***(G2)***, ***(G3)***, ***(G4)*** ***(G5)*** *are necessary and sufficient conditions for rationalizing the observed market data with an exchange economy where consumers are endowed with Gorman polar form utility functions.*

$$\textit{\textbf{(11)}}\ V_{i,k} > V_{i,j} + d_{i,j} \cdot (p_k - p_j)\ + \lambda_j(I_{i,k} - I_{i,j})\ \textit{\textbf{Afriat Inequalities}}$$

$$\textit{\textbf{(12)}}\ \sum_{i=1}^{i=M} \frac{-1}{\lambda_j} d_{i,j}\ \leq \eta_j\ \textit{\textbf{Market Clearing}}$$

$$\textbf{\textit{(13)}}\ \frac{p_j \cdot -d_{i,j}}{I_{i,j}} = \lambda_j\ \textbf{\textit{FOC}}$$

$$\textbf{\textit{(14)}}\ \lambda_j = -\overline{d_{i,j}} \cdot \frac{p_j}{I_{i.j}} = [\max_{r,s} \|-d_{r,s}\|_2]^{-1} \frac{p_j}{I_{i.j}} \cdot -d_{i,j} \leq$$

$$[\max_{r,s} \|-d_{r,s}\|_2]^{-1} \|-d_{i,j}\|_2 \left\| \frac{p_j}{I_{i.j}} \right\|_2 \leq \max\{1, \max_{r,s} \left\| \frac{p_j}{I_{i.j}} \right\|_1\} \equiv \Theta_G\ \textbf{\textit{Upper Bound}}$$

If $e_i(p)$ and $f_i(p)$ are smooth and strictly convex, then for $k \neq j$ and for $1 \leq i \leq N$ and $1 \leq j \leq M$:

$$\textbf{\textit{(G1)}}\ e(p_k) > e(p_j) + \nabla_p e(p_j) \cdot (p_k - p_j)$$

$$\textbf{\textit{(G2)}}\ f_i(p_k) > f_i(p_j) + \nabla_p f_i(p_j) \cdot (p_k - p_j)$$

$$\textbf{\textit{(G3)}}\ V_{i,j} = I_{i,j} e(p_j) + f_i(p_j)$$

$$\textbf{\textit{(G4)}}\ \overline{d_{i,j}} = I_{i,j} \nabla_p e(p_j) + \nabla_p f_i(p_j)$$

$$\textbf{\textit{(G5)}}\ \lambda_j = e(p_j)$$

Proof. Necessity is obvious. To prove sufficiency, we use Afriat's construction to derive piecewise linear convex indirect utility functions $V_i(p, I)$ satisfying Eqs.**11, 13, 14** and piecewise linear convex functions: $e(p)$ and $f_i(p)$ satisfying Eqs.**G1 to G5**. That is, if $z_i - z_j \geq \nabla_a z_j \cdot (a_i - a_j)$ for $1 \leq i, j \leq L$ is solvable for z_l and $\nabla z_l \in R^S_{++}$, where $a_l \in R^S_{++}$, then $z(a) \equiv \max_{j=1}^{j=L}\{z_j + z_j \cdot (a - a_j)\}$ is a piecewise linear convex function. See Theorem 2.49 in Rockfellar and Wets (1998). $\partial z(a) =$ convex hull$\{\nabla_a[z_k + z_k \cdot (a - a_k) : z(a) = [z_k + z_k \cdot (a - a_k)\}$ is the subdifferential at a, i.e., the set of subgradients of z at point a. See Exercise 8.31 in Rockafellar and Wets. If $W_i(p, I) \equiv Ie(p) + f_i(p)$, then $V_i(p, I)$ and $W_i(p, I)$ have the same subdifferential . As is well known, convex functions with the same subdifferential differ by at most a constant. See Theorem 24.9 in Rockfellar (1970). Hence, $V_i(p, I) = W_i(p, I) + K_i$. That is, $V_i(p, I)$ and $W_i(p, I)$ define the same family of indifference curves. ■

We now prove an approximation theorem for the Gorman Polar Form Equilibrium Inequalities.

Theorem 8 *If $\Theta \geq \Theta_G$ and Δ_G is the optimal value of the nonconvex program S_G, where*

$$\textbf{\textit{(15)}}\ \Delta_G \equiv \min_{\omega \in \Omega, s_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} s_j : \textit{Eqs. } (11) \textit{ to } (G5) \textit{ hold and } \frac{1}{\lambda_j(\omega)} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} d_{i,j} \leq s_j \eta_j \textit{ for } 1 \leq j \leq N\} : S_G$$

Γ_G is the optimal value of the approximating linear program R_G,where

$$\textbf{\textit{(16)}}\ \Gamma_G \equiv \min_{\omega \in \Omega, r_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} r_j : \textit{Eqs.}(11) \textit{ to } (G5) \textit{ hold and} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} d_{i,j} \leq r_j \eta_j \textit{ for } 1 \leq j \leq N\} : R_G$$

Ψ_G is the optimal value of the nonconvex program T_G, where

$$\textbf{\textit{(17)}}\ \Psi_G \equiv \min_{\omega \in \Omega, t_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} t_j : \textit{Eqs. } (11) \textit{ to } (G5) \textit{ hold and } \frac{\Theta}{\lambda_j(\omega)} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} d_{i,j} \leq t_j \eta_j \textit{ for } 1 \leq j \leq N\} : T_G$$

then

$$(\textbf{\textit{18}})\ \Psi_G \geq \Gamma_G \geq \Delta_G$$

and

$$(\textbf{\textit{19}})\ \Psi_G = \Theta\Delta_G$$

Hence

$$\textbf{\textit{(20)}}\ \Theta\Delta_G \geq \Gamma_G \geq \Delta_G \Longleftrightarrow \Gamma_G \geq \Delta_G \geq \frac{\Gamma_G}{\Theta}$$

Proof. See the proof of Theorem 3. ∎

Corollary 9

$$\Theta_G\Delta_G \geq \Gamma_G \geq \Delta_G \Longleftrightarrow \Gamma_G \geq \Delta_G \geq \frac{\Gamma_G}{\Theta_G}$$

Corollary 10 *(a)* $\Psi_G = 1$, *iff the Gorman Polar Form equilibrium inequalities with constant marginal utilities of income rationalize the observed market data. (b) It follows from the Brown and Calsamiglia paper that* $\Psi_G = 1$ *iff the observed market data satisfies the strong law of demand. (c) If* $\Psi_G = 1$ *then* $\Delta_G = 1$. *(d)* $\Delta_G = 1$, *iff the Gorman Polar Form equilibrium inequalities with uniformly bounded marginal utilities of income rationalize the observed market data.*

6 Discussion

In this final section of the paper, we wish to describe our contribution to the growing literature on Algorithmic Game Theory (AGT) or more precisely to the literature on Algorithmic General Equilibrium (AGE) that predates AGT. AGE begins with Scarf's seminal (1967) article on computing approximate fixed points, followed by his classic (1973) monograph: The Computation of Economic Equilibria.

Codenotti and Varadarajan (2007) review the literature on polynomial time algorithms for computing competitive equilibria of restricted classes of exchange economies, where the set of competitive equilibria is a convex set. It is the convexity of the equilibrium set that allows the use of algorithms devised for solving convex programs. These specifications are inspired by models of exchange proposed by Fisher (1891) – see Brainard and Scarf (2000) – and Eisenberg (1961). In general, i.e., exchange economies with nonconvex sets of competitive equilibria, the authors conclude that the computational complexity of these models is unlikely to be polynomial, even in the special case where consumers are endowed with Leontief utility functions.

In their recent (2009) survey of the computational complexity of fixed point methods and their application to general equilibrium price adjustment mechanisms, Papadimitriou and Yannakakis show: Price adjusment mechanisms that find prices which approximately clear the market, cannot converge in time less than exponential in the number of goods. It is important to note that Scarf's algorithm is not a price adjustment mechanism, as defined in their paper. For economists, the analysis in this paper is an exploration of the complexity of Smale's (1976) price adjustment mechanism.

Scarf never explicitly addressed the issue of the computational complexity of, what is now called, the Scarf algorithm, nor did his graduate students. His primary research agenda was the computation of economic equilibria in real world economies. This project is best illustrated in the (1992) monograph of Shoven and Whalley, two of Scarf's graduate students, who coined the phrase: Applied General Equilibrium, but following the recent literature, we will call this class of general equilibrium models:Computable general equilibrium models (CGE) models There are now several dozen CGE models used by policy makers around the world for estimating the economic impact of proposed taxes, quotas, tariffs, price controls, changes in social insurance, global warming, agricultural subsidies, ... These models are now the primary tools for counterfactual economic policy analysis. See the (2012) Handbook of Computable General Equilibrium Modeling edited by Dixon and Jorgenson.

CGE models use parametric specifications of utility functions and cost functions.The parameters are often estimated using a method called "calibration". That is, choosing parameter values such that the CGE model replicates the observed equilibrium prices and observed market demands in a single bench mark data set. As you might expect there is some debate about the efficacy of this methodology among academic economists. In response to the obvious limitations of calibration and parametric specification, the Walrasian equilibrium inequalities were proposed by Brown and Matzkin as a methodology for nonparametric estimation of CGE models using several bench mark data sets. They proposed to extend the revealed preference analysis of individual consumer demand introduced by Samuleson (1937) i.e., the Weak Axiom of Revealed Preference, to rationalize the aggregate market data in each of the several bench mark data sets: market prices, income distributions and social endowments., by proposing necessary and sufficient conditions for the existence of consumers who maximize utility subject to budget constraints defined by the market prices and the income distributions in each benchmark data set such that aggregate consumer demand is equal to the social endowment. That is, revealed general equilibrium (RGE).

Brown and Kannan (2008) initiated the complexity analysis of searching for solutions of the Walrasian equilibrium inequalities. Subsequently, Cherchye et al (2011) showed that the search problem for solving the Walrasian equilibrium inequalities, formulated as an integer programming problem, is NP-complete. Unlike the computation of approximate fixed points or the equivalent problem of computing approximate economic equilibria, the search problem for the Walrasian equilibrium inequalities has a natural formulation within the literature on computational complexity, as demonstrated by Cherchye et al.

Hence, AGE consists of two computable classes of general equilibrium models : The parametric CGE models of Scarf-Shoven-Whalley and the nonparametric RGE models of Brown-Matzkin-Shannon. Both classes of models admit counterfactual policy analysis. The Lee-Brown (2007) model of monopoly pricing is an example of RGE counterfactual analysis. In general, both classes of models lack the polynomial time algorithms necessary for efficient solution, hence they require approximation theorems to carry out effective counterfactual policy analysis. Both classes of models contain special cases solvable in polynomial time. In the CGE models, homothetic exchange economies satisfying the assumptions of Eisenberg's (1961) aggregation theorem are solvable in polynomial time, using algorithmns from convex programming. In the RGE models, the equilibrium inequalities for quasilinear exchange economies are linear, hence solvable in polynomial time, using interior point methods from linear programming. See Brown and Calsamiglia (2014).

We have presented two families of equilibrium inequalities as possible rationalizations of the observed market data. In each instance, we derived an approximation theorem, where consumers in the approximating exchange economies are endowed with quasilinear utilities. To test the null hypotheses that the observed market data is rationalized by the Walrasian equilibrium inequalities or rationalized by the Gorman Polar Form equilibrium inequalities, we compute the "confidence intervals" $[\Gamma_G - \frac{\Gamma_G}{\Theta_G}]$ and $[\Gamma_w - \frac{\Gamma_W}{\Theta_W}]$. It follows from the approximation theorems that $\Delta_W \in [\Gamma_w - \frac{\Gamma_W}{\Theta_W}]$ and $\Delta_G \in [\Gamma_G - \frac{\Gamma_G}{\Theta_G}]$.

If the null hypothesis, $H_{W,0}$: the Walrasian equilibrium inequalities with marginal utilities of income uniformly bounded by Θ_W rationalizes the observed data set, then the null hypothesis is rejected if $1 \notin [\Gamma_w - \frac{\Gamma_W}{\Theta_W}]$and we accept the alternative hypothesis, $H_{W,A}$: the Walrasian equilibrium inequalities with marginal utilities of income uniformly bounded by Θ_W is refuted for the observed data set. Similarly, if the null hypothesis, $H_{G,0}$: the Gorman Polar Form equilibrium inequalities with marginal utilities of income uniformly bounded by Θ_G rationalizes the observed data set, then the null hypothesis is rejected if $1 \notin [\Gamma_G - \frac{\Gamma_G}{\Theta_G}]$and we accept the alternative hypothesis, $H_{G,A}$: the Gorman Polar Form equilibrium inequalities with marginal utilities of income uniformly bounded by Θ_W is refuted for the observed data set.

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