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A CONTINUOUS TIME APPROACH

By

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Dynamic Revenue Maximization: A Continuous Time Approach*

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Abstract

We characterize the revenue-maximizing mechanism for time separable allocation problems in continuous time. The willingness-to-pay of each agent is private information and changes over time.

We derive the dynamic revenue-maximizing mechanism, analyze its qualitative structure and frequently derive its closed form solution. In the leading example of repeat sales of a good or service, we establish that commonly observed contract features such as flat rates, free consumption units and two-part tariffs emerge as part of the optimal contract. We investigate in detail the environments in which the type of each agent follows an arithmetic or geometric Brownian motion or a mean-reverting process. We analyze the allocative distortions and show that depending on the nature of the private information the distortion might increase or decrease over time.

Keywords: Dynamic Mechanism Design, Repeated Sales, Stochastic Flow, Flat Rates, Two-Part Tariffs, Leasing.

JEL Classification: D44, D82, D83.

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1 Introduction

1.1 Motivation

We analyze the nature of the revenue-maximizing contract in a dynamic environment with private information at the initial time of contracting as well as in all future periods. We consider a setting in continuous time and are mostly concerned with environments where the uncertainty, and in particular the private information of the agent is described by a Brownian motion. The present work makes progress by considering allocation problems that we refer to as *weakly time separable*. Namely, (i) the set of available allocations at time t is independent of the history of allocations and (ii) the flow utility functions of the agent and the principal at time t depend only the *initial* and the *current* private information of the agent (and hence the qualifier of weakly time separable).

With time separability, the allocation rule that maximizes the expected dynamic virtual surplus has the property that the allocation at time t is a function of the report of the agent at time 0 and time t only. As a result, at every time $t > 0$, each agent is only facing a static reporting problem since the current report is only used to determine the current allocation. A notable implication of this separability is that the incentive compatibility conditions can be decomposed completely into a time 0 problem and a sequence of static problem at all times $t > 0$. The restriction to time separable allocation problems is sufficiently mild to include many of the allocation problems explicitly analyzed in the literature so far, for the example the optimal quantity provision by the monopolist as in Battaglini (2005) or the auction environment of Eső and Szentes (2007).

The specific contribution of the continuous time setting to the analysis of the optimal mechanism arises *after* establishing the necessary conditions for optimality under time separability. And in fact, we obtain the first order conditions by using the envelope theorem using a small class of relevant deviations which is precisely the approach taken in discrete time, see for Eső and Szentes (2007) and Pavan, Segal, and Toikka (2014) for the seminal contributions. The resulting dynamic version of the virtual utility accounts for the influence that the present private information has on the future state of the world (and hence future private information of the agent) through a term that Pavan, Segal, and Toikka (2014) refer to as impulse response function. Now, in continuous time, the equivalent expression, which is commonly referred to as *stochastic flow*, is compact and summarizes the nature of the underlying stochastic process in an explicit formula. We then make use of the information conveyed by the stochastic flow in three distinct ways.

First, we explicitly derive the nature of the optimal allocation policy and the associated transfer rules. We consider in some detail a number of well-known stochastic processes, in particular the arithmetic and the geometric Brownian motion. The natural starting point here is to consider the case in which the private information of the agent is the current state of the process, in particular the initial state of the Brownian motion is private information, but we also analyze the problem when either the drift or volatility of the process are private information. In Section 5 we consider the nature of the optimal mechanism for repeated sales when the type of the agent follows a geometric Brownian motion. We establish that commonly observed contract features such as flat rates, free consumption units, two-part tariffs and leasing arrangements emerge as solutions to the optimal contract design.

Second, we derive sufficient conditions for the optimality of the dynamic mechanism in terms of the primitives of the stochastic process. This is demonstrated in detail in Section 6 where we, for example, derive sufficient conditions for optimality when the private information of agent cannot be ordered by first order stochastic dominance. In particular, we can allow the variance rather than the mean of the stochastic process to form the private information, and yet display transparent sufficient conditions for optimality. In much of the earlier literature, the types had to be assumed to be ordered according to first-order stochastic dominance in order to give rise to sufficient conditions for optimality.

Third, we systematically extend the analysis from Markovian settings where the initial private information (as well as any future private information) is the state of the stochastic process to settings in which the initial private information can present a parameter of the stochastic process, such the mean or variance of the Markov process. The subsequent private information continues to pertain to the state of the Markov process. This specification of the private information, the initial information about the parameter of the process and the ongoing information about the state of the process still conforms with our restriction to weakly time separable environments.

The initial private information may represent the drift or the volatility of the Brownian motion, or the long-run mean or the reversion rate of a mean-reverting Ornstein-Uhlenbeck process. The resulting informational term in the virtual utility, which is referred to as *generalized stochastic flow* in probability theory, still permits a compact representation that can be used for the determination of the optimal policy and/or for the sufficient conditions. With the notable exception of the recent papers by Boleslavsky and Said (2013) and Skrzypacz and Toikka (2015), and a discussion in the supplementary appendix of Pavan, Segal, and Toikka (2014), the earlier contributions with an infinite horizon did not allow for the possibility that the initial private information may pertain to a parameter of the stochastic process itself, such as the drift or the volatility. Interestingly, the continuous time version of the resulting generalized impulse response function is often a deterministic function of the initial state and time, whereas the corresponding discrete time process has a generalized impulse response function that depends on the realization of the entire sample path. This is shown for example in Section 5 where the initial private information is the mean of the geometric Brownian motion. The discrete time counterpart of this process, namely the multiplicative random walk, was analyzed earlier by Boleslavsky and Said (2013). Here the generalized impulse response term involves the number of *realized* upticks and downticks. In the continuous time equivalent, the generalized stochastic flow is simply the expected number of upticks or downticks which is a deterministic function of time and the initial state.

We should add that the current focus on time separable allocation problems is restrictive in that it excludes problems such as the optimal timing of a sale of a durable good, where the present decision, say a sale, naturally preempts certain future decision, say a sale, again. But our setting allows us to restrict attention to a small class of deviations, deviations that we call *consistent*. The consistent deviations, by themselves only necessary conditions, nonetheless completely describe the indirect utility of the agent in any incentive compatible mechanism. More precisely, at time zero the initial shock of the agent is drawn and the initial shock determines the probability measure of the entire future valuation process. If the agent deviates he changes the probability measure of the reported valuation process. To avoid working with the change in measures directly we restrict attention to consistent deviations. We call a deviation consistent if, after his initial misreport, say b instead of a, the agent reports his valuation as if it would follow the same Brownian motion as the one which drives his true valuation. As there is a true initial shock, namely b, which could have made these subsequent reports truthful, the principal cannot detect such a deviation and is forced to assign the allocation and transfer process of the imitated shock b. In particular, this allows us to evaluate the payoffs of the truthful and the consistently deviating agent with respect to the same expectation operator. Now, as we assume the initial shock to be one-dimensional and given that all deviations are parametrized over the time zero shock, standard mechanism design arguments deliver the smoothness of the value function of the agent.

Within the class of time separable allocation policies we can rewrite the sufficiency conditions exclusively in terms of the flow virtual utilities. By using the class of consistent deviations and allowing for time separable allocation policies, we can completely avoid the verification of the incentive compatibility conditions via backward induction methods which was the basic instrument to establish the sufficient conditions used in much of the preceding literature with dynamic adverse selection.

1.2 Related Literature

The analysis of the revenue-maximizing contract in an environment where the private information may change over time appears first in Baron and Besanko (1984). They considered a two period model of a regulator facing a monopolist with unknown, but in every period, constant marginal cost. Besanko (1985) offers an extension to a finite horizon environment with a general cost function, where the unknown parameter is either i.i.d. over time or follows a first-order autoregressive process. Since these early contributions, the literature has developed rapidly. Courty and Li (2000) consider the revenue-maximizing contract in a sequential screening problem where the preferences of the buyer change over time. Battaglini (2005) considered a quantity discriminating monopolist who provides a menu of choices to a consumer whose valuation can change over time according to a commonly known Markov process. In contrast to the earlier work, he explicitly considered an infinite time horizon and showed that the distortion due to the initial private information vanishes over time. Eső and Szentes (2007) rephrased the two period sequential screening problem by showing that the additional signal arriving in period two can always be represented by a signal that is orthogonal to the signal in period one. Eső and Szentes (2014) generalize this insight in an infinite horizon environment and show that the information rent of the agent is only due to his initial information.

Pavan, Segal, and Toikka (2014) consider a general environment in an infinite horizon setting and allowing for general allocation problems, encompassing the earlier literature (with continuous type spaces). They obtain general necessary conditions for incentive compatibility and present a variety of sufficient conditions for revenue-maximizing contracts for specific classes of environments. They also observed the beneficial implications of time separable environments for a tighter characterization of the optimal contract.

A feature common to almost all of the above contributions is that the private information of the agent is represented by the current state of a one-dimensional Markov process, and that the new information that the agent receives is controlled by the current state, and in turn, leads to a new state of the Markov process. Notably, Pavan, Segal, and Toikka (2014), Boleslavsky and Said (2013) and Skrzypacz and Toikka (2015) allowed for the possibility that the initial private information is about a parameter of the stochastic process itself.[1] For example, Boleslavsky and Said (2013) let the initial private information of the agent be the mean of a multiplicative random walk. Interestingly, this dramatically changes the impact that the initial private information has on the future allocations. In particular, the distortions in the future allocation may now increase over time rather than decline as in much of the earlier literature. The reason is that the influence of the parameter of the stochastic process, such as the drift or the variance, on the valuation may increase over time.[2] Finally, Kakade, Lobel, and Nazerzadeh (2013) consider a class of dynamic allocation problems, a suitable generalization of the single unit allocation problem and impose a separability condition (additive or multiplicative) on the interaction of the initial private information and all subsequent signals. The separability condition allows them to obtain an explicit characterization of the revenue-maximizing contract and derive transparent sufficient conditions for the optimal contract.

The remainder of the paper proceeds as follows. Section 2 presents the model. In Section 3 we derive the necessary and sufficient conditions for the revenue-maximizing contract. In Section 4 we analyze the implications of the revenue-maximizing contract for the structure of the intertemporal distortions. The nature of the optimal contract for repeat purchases of a product or service is analyzed in Section 5 in an environment where the type follows a geometric Brownian motion. Section 6 examines the optimal allocation among competing bidders when the private valuation is either driven by the arithmetic Brownian motion or the mean-reverting Ornstein-Uhlenbeck process.

[1] This is equivalent to assuming that the private information of the agent corresponds to the state of a two-dimensional Markov process, whose first component is constant after time zero, but influences the transitions of the second component.

[2] In a recent contribution, Garrett and Pavan (2012) also exhibit the possibility of increasing distortions over time, but the source there is a trade-off in the retention decision of a known agent versus a hiring decision of new, hence less well known agent.

Section 7 concludes. The Appendix contains some auxiliary proofs and additional results.

2 Model

There are n agents indexed by $i \in \{1, \ldots, n\} = N$. Time is continuous and indexed by $t \in [0, T]$, where the time horizon T can be finite or infinite. If the time horizon is infinite, then we assume a discount rate $r \in \mathbb{R}_+$ which is strictly positive, $r > 0$.

The flow preferences of agent i are represented by a quasilinear utility function:

$$v_t^i \cdot u^i(t, x_t^i) - p_t^i. \tag{1}$$

The function $u^i : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, \overline{u}]$ is continuous and strictly increasing in x, decreasing in t and satisfies $u^i(t, 0) = 0$ for all $t \in \mathbb{R}_+$. We refer to $u^i(t, x_t^i)$ as the *valuation* of $x_t^i \in [0, \overline{x}] \subset \mathbb{R}_+$ with $0 \leq \overline{x} < \infty$. The allocation x_t^i can be interpreted as either the quantity or quality of a good that is allocated to agent i at time t. The *type* of agent i in period t is given by $v_t^i \in \mathbb{R}$ and the flow utility in period t is given by the product of the type and the valuation. The payment in period t is denoted by $p_t^i \in \mathbb{R}$.

The type v_t^i of agent i at time t depends on his *initial shock* θ^i at time $t = 0$ and the contemporaneous shock W_t^i at time t:

$$v_t^i \triangleq \phi^i(t, \theta^i, W_t^i)\,. \tag{2}$$

The function $\phi^i : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ aggregates the initial shock θ^i and the contemporaneous W_t^i of agent i into his type v_t^i. The initial private information θ^i is not restricted to be the initial type v_0^i, but might be any other characteristic determining the probability measure over paths of the types $(v_t)_{t \in \mathbb{R}_+}$. In the case of the arithmetic or geometric Brownian motion, the initial shock θ^i could constitute the initial value v_0^i, but it could also be the drift μ^i or the variance $(\sigma^i)^2$ of the Brownian motion. Similarly, in the case of a mean reverting process, the initial shock θ^i could constitute the mean reversion speed or the long run-average of the stochastic process. In any event, at time zero each agent i privately learns his initial shock $\theta^i \in (\underline{\theta}, \bar{\theta}) = \Theta \subseteq \mathbb{R}$, which is drawn from a common prior distribution $F^i : \mathbb{R} \to [0, 1]$, independently across agents.

The distribution F^i has a strictly positive density $f^i > 0$ with decreasing inverse hazard rate $(1 - F^i)/f^i$. The contemporaneous shock is given by a random process $(W_t^i)_{t \in \mathbb{R}_+}$ of agent i that changes over time as a consequence of a sequence of incremental shocks and W_t^i is assumed to be independent of W_t^j for every $j \neq i$. In Sections 5 and 6, the valuation function $u^i(t, x_t^i)$ is simply a linear function $u^i(t, x_t^i) = x_t^i$ and the type v_t^i can then be directly interpreted as the *marginal willingness to pay* of agent i.

The function ϕ^i is twice differentiable in every direction and in the following we use a small annotation for partial derivatives:

$$\phi_\theta^i(t, \theta^i, w^i) \triangleq \frac{\partial \phi^i(t, \theta^i, w^i)}{\partial \theta}. \tag{3}$$

If θ^i is the initial value of the process of agent i, that is $v_0^i = \theta^i$, then the derivative ϕ_θ^i is commonly referred to as the *stochastic flow*; or *generalized stochastic flow* if θ^i determines the evolution of a diffusion by influencing the drift or variance term (see for example Kunita (1997)). The stochastic flow process $(\phi_\theta^i(t, \theta, W_t^i))_{t \in \mathbb{R}_+}$ is the analogue of the impulse response functions described in the discrete time dynamic mechanism design literature (see Pavan, Segal, and Toikka (2014), Definition 3). As we will see in the examples presented later the stochastic flow is of a very simple form for many classical continuous time diffusion processes, like the arithmetic and geometric Brownian motion.

We assume that for every agent i a higher initial shock θ^i leads to a higher type, $\phi_\theta^i(t, \theta^i, w^i) \geq 0$; and an agent i who observed a higher value of the process W_t^i has a higher type, $\phi_w^i(t, \theta^i, w^i) > 0$ for every $(t, \theta^i, w^i) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

Assumption 1 (Decreasing Influence of Initial Shock)**.**
The relative impact of the initial shock on the type:

$$\frac{\phi_\theta^i(t, \theta^i, w^i)}{\phi^i(t, \theta^i, w^i)} \tag{4}$$

is decreasing in w^i *for every* $(t, \theta^i, w^i) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

Assumption 2 (Decreasing Influence of Initial vs Contemporaneous Shock)**.**
The ratio of the marginal impact of initial and contemporaneous shocks:

$$\frac{\phi_\theta^i(t, \theta^i, w^i)}{\phi_{w^i}^i(t, \theta^i, w^i)} \tag{5}$$

is decreasing in θ^i *for every* $(t, \theta^i, w^i) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

The last assumption implies that the type with a large initial shock is influenced more by the contemporaneous shocks that arrive after time zero.

Assumption 3 (Finite Expected Impact of the Initial Shock)**.**
The expected influence of the initial shock on the type grows at most exponentially: there exists $C \in \mathbb{R}_+, q \in (0, r)$ *such that* $\mathbb{E}\left[\phi^i_\theta(t, \theta^i, W^i_t)\right] \leq Ce^{qt}$ *for all* $t \in \mathbb{R}_+$ *and* $\theta^i \in \Theta$.

Assumption 3 ensures that the effect of a marginal change in the agent's type on the sum of discounted expected future types is finite.

At every point in time t the principal chooses an allocation $x_t \in X$ from a compact, convex set $X \subset \mathbb{R}^n_+$, where x^i_t can be interpreted as the quantity or quality of a good that is allocated to agent i at time t. We assume that it is always possible to allocate zero to an agent:

$$x \in X \Rightarrow (x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \in X\,.$$

To ensure that the problem is well-posed we assume that every feasible allocation process $x^i = (x^i_t)$ gives finite expected utility to agent i,

$$\mathbb{E}\left[\int_0^T e^{-rt}\mathbf{1}_{\{v^i_t \geq 0\}} v^i_t u^i(t, x^i_t)\mathrm{d}t \,|\, \theta^i\right] < \infty,$$

for every θ^i in the support of F^i. The principal receives the sum of discounted flow payments $\sum_{i\in N} p^i_t$ minus the production costs $c(x_t)$:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} p^i_t - c\left(x_t\right)\right)\mathrm{d}t\right]. \tag{6}$$

The cost $c : X \to \mathbb{R}_+$ is continuous and increasing in every component with $c(0) = 0$. With minor abuse of language we shall refer throughout the paper to the net revenue (or profit) maximization problem given by (6) as simply the revenue maximization problem.

Definition 1 (Value Function)**.**
The indirect utility, or value function, $V^i(\theta^i)$ *of agent* i *given his initial shock* θ^i*, his consumption process* $(x^i_t)_{t\in\mathbb{R}_+}$ *and his payment process* $(p^i_t)_{t\in\mathbb{R}_+}$ *is*

$$V^i(\theta^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left(u^i(t, x^i_t)v^i_t - p^i_t\right)\mathrm{d}t \,|\, \theta^i\right]. \tag{7}$$

A contract specifies an allocation process $(x_t)_{t\in\mathbb{R}_+}$ and a payment process $(p_t)_{t\in\mathbb{R}_+}$. The allocation x_t and the payment p_t can depend on all types reported $(v_s^i)_{s\leq t, i\in N}$ by the agents prior to time t. We assume that the agent has an outside option of zero and thus require the following definition:

Definition 2 (Incentive and Participation Constraints)**.**
A contract $(x_t, p_t)_{t\in\mathbb{R}_+}$ is acceptable if for every agent i it is individually rational to accept the contract

$$V^i(\theta^i) \geq 0 \textit{ for all } \theta^i \in \Theta\,,$$

and it is optimal to report his shock θ^i and his type $(v_t^i)_{t\in\mathbb{R}_+}$ truthfully at every point in time $t \in \mathbb{R}_+$.

Given the transferable utility, we define the flow welfare function $s : \mathbb{R}_+ \times \mathbb{R}^n \times X \to \mathbb{R}$ that maps an allocation $x \in X$ and a vector of types $v \in \mathbb{R}^n$ into the associated flow of welfare

$$s(t, v, x) = \sum_{i\in N} v_t^i u^i(t, x^i) - c(x)\,. \tag{8}$$

The social value of the allocation process $(x_t)_{t\in[0,T]}$ aggregates the discounted flow of social welfare over time and is given by:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} v_t^i u^i(t, x_t^i) - c\left(x_t\right)\right)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt} s(t, v_t, x_t)\mathrm{d}t\right]. \tag{9}$$

As the allocation x_t at time t does not influence the future evolution of types or the set of possible future allocations the problem of finding a socially efficient allocation is time-separable. We define the optimal allocation function $x^\dagger : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ that maps a point in time t and a vector of types v into the set of optimal allocations

$$x^\dagger(t, v) = \arg\max_{x\in X} s(t, v, x)\,. \tag{10}$$

An allocation process $(x_t)_{t\in[0,T]}$ is welfare maximizing if and only if $x_t \in x^\dagger(t, v_t)$ almost surely for every $t \in [0, T]$.

Given the essentially static character of the social allocation problem, it follows immediately that the *welfare maximizing* allocation $x^\dagger$ can be implemented via a sequence of static Vickrey-Clarke-Groves mechanisms and associated payments:

$$p_t^{\dagger\, i} \triangleq p^{\dagger\, i}(t, v_t) = \max_{x\in X} \sum_{j\neq i}\left[u^j(t, x) - u^j(t, x^\dagger(t, v_t))\right] v_t^j - c(x) + c(x^\dagger(t, v_t))\,. \tag{11}$$

3 Revenue Maximization

In this section we derive a revenue-maximizing direct mechanism. Without loss of generality we restrict attention to direct mechanisms, where every agent i reports his initial shock θ^i and his type v_t^i truthfully. We first obtain a revenue equivalence result for incentive compatible mechanisms.

3.1 Necessity

We begin by establishing that the value function of the agent if he reports truthfully is Lipschitz continuous. As ϕ^i is strictly increasing in w^i we can implicitly define the function $\omega : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ by

$$v^i = \phi^i(t, \theta^i, \omega(t, \theta^i, v^i)) \text{ for all } (t, \theta^i) \in \mathbb{R}_+ \times \Theta\,. \tag{12}$$

Thus ω identifies the value that the contemporaneous shock W_t^i has to have at time t to generate a contemporaneous type v^i given the initial shock θ^i. We derive a necessary condition for incentive compatibility that is based only on the robustness of the mechanism to a small class of deviations, which we refer to as *consistent deviations.*

Definition 3 (Consistent Deviation)**.**
A deviation by agent i is referred to as a consistent deviation *if agent i with type $v_0^i = \phi^i(0, a, W_0^i)$ (and associated initial shock $a \in \Theta$) misreports $\widehat{v}_0^i = \phi^i(0, b, W_0^i)$ (and associated initial shock $b \in \Theta$) at $t = 0$ and continues to misreport:*

$$\widehat{v}_t^i = \phi^i(t, b, \omega(t, a, v_t^i)), \tag{13}$$

instead of his true type v_t^i at all future dates $t \in \mathbb{R}_+$.

Thus, an agent who misreports with a consistent deviation, continues to misreport his true type v_t^i in all future periods. More precisely, agent i's reported type $\hat{v}_t^i = \phi^i(t, b, W_t^i)$ equals the type he would have had if his initial shock would have been b instead of a. We note that the misreport generated by a consistent deviation has the property that the principal can infer from the misreport the true realized path of the contemporaneous shocks W_t^i. Now, since the allocation depends on the type v_t^i rather than the path of contemporaneous shocks W_t^i, the (inferred) truthfulness in the shocks is not of immediate use for the principal. We now show that this, one-dimensional, class of consistent deviations is sufficient to uniquely pin down the value function of the agent in any incentive compatible mechanism at time $t = 0$. The class of consistent deviations we consider here are not local deviations at one point in time, but rather represent a global deviation in the sense that the agent changes his reports at every point in time.

As $\phi^i(0, \theta^i, W_0)$ is strictly increasing in θ^i, it is convenient to describe the initial report directly in terms of the true initial shock a and the reported initial shock b. We thus define $V^i(a, b)$ to be the indirect utility of agent i with initial shock a but who reports shock b and misreports his type consistently as $\hat{v}_t^i = \phi^i(t, b, \omega(t, a, v_t^i))$. Note that by construction $W_t^i = \omega(t, a, v_t^i)$. Consequently the allocation agent i gets by consistently deviating and reporting b is the same allocation that he would get if his initial shock were b and he were to report it truthfully. Hence $V^i(a, b)$ is the indirect utility of an agent who has the initial shock a but reports initial shock b and misreports his type consistently and is given by:

$$V^i(a, b) = \mathbb{E}\left[\int_0^T e^{-rt}\left(u^i(t, x_t^i(b))\phi^i(t, a, W_t^i) - p_t^i(b)\right) \mathrm{d}t\right].$$

Note, that when restricted to consistent deviations the mechanism design problem turns into a standard one-dimensional problem, and the Envelope theorem yields the derivative of the indirect utility function of the agent:

Proposition 1 (Regularity of Value Function)**.**
The indirect utility function V^i of every agent $i \in N$ in any incentive compatible mechanism is Lipschitz continuous and has the weak derivative

$$V_\theta^i(\theta^i) = \mathbb{E}\left[\int_0^T e^{-rt} u^i(t, x_t^i(\theta))\phi_\theta^i(t, \theta^i, W_t^i)\mathrm{d}t\right] \; a.e.. \tag{14}$$

Proof. As the agent can always use consistent deviations, a necessary condition for incentive compatibility is $V(a, a) = \sup_b V(a, b)$. As ϕ^i is differentiable the derivative of V with respect to the first variable is given by

$$\begin{aligned} V_a(a, b) = & \frac{\partial}{\partial a}\mathbb{E}\left[\int_0^T e^{-rt}\left(u^i(t, x_t^i(b))\phi^i(t, a, W_t^i) - p_t^i(b)\right)\mathrm{d}t\right] \\ = & \;\mathbb{E}\left[\int_0^T e^{-rt}\left(u^i(t, x_t^i(b))\phi_\theta^i(t, a, W_t^i)\right)\mathrm{d}t\right] \leq \overline{u}\,\mathbb{E}\left[\int_0^T e^{-rt}\phi_\theta^i(t, a, W_t^i)\mathrm{d}t\right], \end{aligned}$$

which is bounded by a constant by Assumption 3. By the Envelope theorem (see Milgrom and Segal (2002), Theorem 1 and Theorem 2) we have that $V^i(\theta^i) = V^i(\theta^i, \theta^i)$ is absolutely continuous an the (weak) derivative is given by (14). As argued above (14) is bounded and thus V^i is Lipschitz continuous. □

We introduce a dynamic version of the virtual utility function $J^i : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ as:

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \phi_\theta^i(t, \theta^i, \omega(t, \theta^i, v_t^i)) \,. \tag{15}$$

We observe that the above virtual utility is modified relative to its static version only by the term of the stochastic flow ϕ_θ^i that multiplies the inverse hazard rate. Thus, the specific impact of the private information in the dynamic mechanism is going to arrive exclusively through the stochastic flow ϕ_θ^i (see (3)), the continuous time equivalent of the impulse response function. The properties of the virtual utility are summarized in the following proposition:

Proposition 2 (Monotonicity of Virtual Utility)**.**
If the virtual utility $J^i(t, \theta^i, v_t^i)$ is positive then it is non-decreasing in θ^i and v_t^i.

The proof of Proposition 2 given in the Appendix establishes the monotonicity of the virtual utility from Assumptions 1 and 2 using algebraic arguments. We observe that Proposition 2 establishes the monotonicity of the virtual utility only for the case that the virtual utility is positive. In fact, our assumptions are not strong enough to ensure the monotonicity of the virtual utility independent of its sign. The reason not to impose stronger monotonicity conditions is that for many important examples discussed later (for example the geometric Brownian motion with unknown initial value) the virtual utility is only monotone if positive.

We can now establish a revenue equivalence result that describes the revenue of the principal in any incentive compatible mechanism solely in terms of the allocation process $x = (x_t)_{t \in \mathbb{R}_+}$ and the expected time zero value the lowest type derives from the contract $V^i(\underline{\theta})$.

Theorem 1 (Revenue Equivalence)**.**
For any incentive compatible direct mechanism the expected payoff of the principal depends only on the allocation process $(x_t)_{t \in \mathbb{R}_+}$ and is given by the dynamic virtual surplus*:*

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i \in N} p_t^i - c(x_t)\right) \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i \in N} J^i(t, \theta_t^i, v_t^i) u^i(t, x_t^i) - c(x_t)\right) \mathrm{d}t\right] - \sum_{i \in N} V^i(\underline{\theta}). \tag{16}$$

Proof. Partial integration gives that in any incentive compatible mechanism (x,p) the expected transfer received by the principal from agent i equals the expected virtual utility of agent i :

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-rt}p_t^i \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T e^{-rt}u^i(t,x_t^i)v_t^i \mathrm{d}t\right] - \int_{\underline{\theta}}^{\overline{\theta}} f(\theta^i)V^i(\theta^i)\mathrm{d}\theta^{\mathrm{i}} \\ &= \mathbb{E}\left[\int_0^T e^{-rt}u^i(t,x_t^i)v_t^i \mathrm{d}t\right] - \int_{\underline{\theta}}^{\overline{\theta}} f^i(\theta^i)\frac{1-F^i(\theta^i)}{f^i(\theta^i)}V_\theta^i(\theta^i)\mathrm{d}\theta^i - V^i(\underline{\theta}) \\ &= \mathbb{E}\left[\int_0^T e^{-rt}u^i(t,x_t^i)\left(v_t^i - \frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi_\theta^i(t,\theta^i,W_t^i)\right)\mathrm{d}t\right] - V^i(\underline{\theta})\,.\end{aligned}$$

Summing up the transfers of all agents and subtracting the cost gives the result. □

As Theorem 1 provides a necessary condition for incentive compatibility it follows that if there exists an incentive compatible contract (x,p) such that the allocation process x maximizes the expected virtual surplus given by (16), then it maximizes the principal's surplus. Clearly, to maximize the virtual surplus it is optimal to set the transfer to the lowest initial shock equal to zero: $V^i(\underline{\theta}) = 0$ for all agents $i \in N$. We denote by $J(t,\theta,v_t) \in \mathbb{R}^n$ the vector of virtual utilities, $J(t,\theta,v_t)^i = J^i(t,\theta^i,v_t^i)$. The revenue of the principal defined by (16) equals the expected welfare when true utilities (types) v are replaced with virtual utilities J, hence referred to as the *dynamic virtual surplus*:

$$\mathbb{E}\left[\int_0^T e^{-rt}s(t,J(t,\theta_t,v_t),x_t)\mathrm{d}t\right] - \sum_{i\in N} V^i(\underline{\theta})\,, \tag{17}$$

where we defined the flow social value $s(\cdot)$ earlier in (8). In the next step we establish that there exists a direct mechanism that maximizes the dynamic virtual surplus defined in (17). To do so let us first state the following result which ensures that there exists a time separable allocation that maximizes the dynamic virtual surplus:

Proposition 3 (Virtual Surplus Maximizing Allocation)**.**
There exists an allocation function $x^\star : \mathbb{R}_+ \times \Theta \times \mathbb{R}^n \to X$ that maximizes the dynamic virtual surplus. Furthermore, the allocation $x^{\star i}(t,\theta,v_t)$ of agent i is non-decreasing in his type v_t^i and his initial shock θ^i.

Proof. For every t,θ,v_t there exists a non-empty set of allocations which maximize the flow of virtual surplus,

$$X^\star(t,\theta,v_t) = \arg\max_{x\in X}\{s(t,J(t,\theta,v_t),x)\} = \arg\max_{x\in X}\left\{\sum_{j\in N} J^j(t,\theta^j,v_t^j)u^j(t,x^j) - c(x)\right\}.$$

As u^i and c are increasing in x^i is is optimal to set the consumption of agent i to zero, $x^i = 0$, if his virtual utility $J^i(t, \theta^i, v_t^i)$ is negative. As u^i is increasing in x and J^i is increasing in θ^i and v^i by Proposition 2 it follows that the objective function of the principal $\sum_{i \in N} \max\{0, J^i(t, \theta^i, v_t^i)\} u^i(t, x^i) - c(x)$ is super-modular in (θ^i, x^i) and (v_t^i, x^i). By Topkis' theorem, there exists a quantity $x^\star(t, \theta, v_t) \in X^\star(t, \theta, v_t)$ that maximizes the flow virtual surplus such that the allocation $x^{\star i}(t, \theta, v_t)$ of agent i is non-decreasing in θ^i and v_t^i. As the virtual surplus of the principal at time t depends only on t, the initial reports θ, and the type v_t, this flow allocation that conditions only on (t, θ, v_t) is an optimal allocation process:

$$\sup_{(x_t)} \mathbb{E}\left[\int_0^T e^{-rt} s(t, J^i(t, \theta_t^i, v_t^i), x_t) \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt} \sup_{x \in X} s(t, J^i(t, \theta_t^i, v_t^i), x_t) \mathrm{d}t\right]. \qquad \square$$

3.2 Sufficiency

To prove incentive compatibility of the optimal allocation process let us first establish a version of a classic result in static mechanism design.

Proposition 4 (Static Implementation)**.**
Let $X \subset \mathbb{R}$ and let $V : X \times X \to \mathbb{R}$ be absolutely continuous in the first variable with weak derivative $V_1 : X \times X \to \mathbb{R}_+$ and let V_1 be increasing in the second variable. Then the payment

$$p(x) = V(x, x) - \int_0^x V_1(z, z) \mathrm{d}z\,.$$

ensures that truth-telling is optimal: $V(x, x) - p(x) \geq V(x, \widehat{x}) - p(\widehat{x})$ for all $x, \widehat{x} \in X$.

Proposition 4 is similar to Lemma 1 in Pavan, Segal, and Toikka (2014) and Proposition 2 in Rochet (1987) and differs only in the continuity requirements, namely absolute continuous here rather than Lipschitz continuous there.

In the first step we construct flow payments that make truthful reporting of types optimal (on and off the equilibrium path) if the virtual surplus maximizing allocation process $x^\star$ is implemented. Define the payment process $p_t \triangleq p(t, \theta, v_t)$ where the flow payment $p^i : t \times \Theta \times \mathbb{R}^n \to \mathbb{R}$ of agent i is given by:

$$p^i(t, \theta, v_t) \triangleq v_t^i u^i(t, x^{\star i}(t, \theta, v_t)) - \int_0^{v_t^i} u^i(t, x^{\star i}(t, \theta, (z, v_t^{-i}))) \mathrm{d}z\,. \tag{18}$$

Proposition 5 (Incentive Compatible Transfers).
In the contract $(x^\star, p)$ *it is optimal for every agent at every* $t > 0$ *to report his type* v_t^i *truthfully, irrespective of the reported shock* θ^i *and past reported types* $(v_s^i)_{s<t}$.

Proof. As the allocation $x^\star(t, \theta, v_t)$ and the payment $p(t, \theta, v_t)$ at time t are independent of all past reported types $(v_s)_{s<t}$ the reporting problem of each agent i is time-separable. As u^i is increasing in x, and $x^\star$ is increasing in v^i by Proposition 2, we can apply Proposition 4 to

$$(v^i, \hat{v}^i) \mapsto v^i u^i(t, x^\star(t, \theta, (\hat{v}^i, v^{-i}))) \, ,$$

and so guarantee that the payment scheme $p(t, \theta, v)$ makes truthful reporting of types optimal for all $t, \theta, v, \hat{v}^i$. □

It remains to augment the payments from Proposition 5 with additional payments that make it optimal for the agents to report their initial shocks θ truthfully. As the payments of Proposition 5, see (18) ensure truthful reporting of types even after initial misreports, we know how agents will behave even after an initial deviation. This insight transforms the time zero reporting problem into a static design problem in which the payments of Proposition 4 can be used to provide incentives at time $t = 0$.

We define the payment process for agent i as the sum of the flow incentive payment $p^i(t, \theta, v_t)$ and an *annuitized* payment $\pi^i(\theta)$ that depends only on the initial report θ :

$$P_t^{i\star} \triangleq p^i(t, \theta, v_t) + \pi^i(\theta) \tag{19}$$

where the annuitized payment $\pi^i : \Theta \to \mathbb{R}$ of agent i is given by:

$$\begin{aligned} \pi^i(\theta) = \mathbb{E}\Bigg[\int_0^T \frac{re^{-rt}}{1 - e^{-rT}} \Bigg[\int_0^{v_t^i} & u^i(t, x^{\star i}(t, \theta, (z, v_t^{-i})))\mathrm{d}z \\ & - \int_{\underline{\theta}}^{\theta^i} \phi_\theta^i(t, z, W_t^i) u^i(t, x^{\star i}(t, (z, \theta^{-i}), (\phi^i(t, z, W_t^i), v^{-i})))\mathrm{d}z \Bigg] \mathrm{d}t \Bigg] . \end{aligned} \tag{20}$$

Theorem 2 (Revenue Maximizing Contract).
The virtual surplus maximizing contract $(x^\star, P^\star)$ *maximizes the revenue of the principal. In the virtual surplus maximizing contract it is optimal for every agent* i *to report his shock* θ^i *and type* v_t^i *truthfully for all* $t \geq 0$, *irrespective of the reported shocks* θ^i *and past reported types* $(v_s^i)_{s<t}$.

Proof. We start with the flow payments p^i of Proposition 5 given by (19). By construction of the payments each agent reports his type truthfully independent of his initial report θ^i. Let $\hat{V}(\theta^i, \hat{\theta}^i)$ be the agent's value if his true initial shock is θ^i but he reports $\hat{\theta}^i$ and reports truthful after time zero

$$\hat{V}(\theta^i, \hat{\theta}^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left[v_t^i\, u^i(t, x^{\star i}(t, (\hat{\theta}^i, \theta^{-i}), v_t)) - p(t, (\hat{\theta}^i, \theta^{-i}), v_t)\right] \mathrm{d}t\right].$$

As it is optimal to report v_t^i truthfully we have that

$$\frac{\partial}{\partial v_t^i}\left(v_t^i\, u^i(t, x^{\star i}(t, (\hat{\theta}^i, \theta^{-i}), v_t)) - p(t, (\hat{\theta}^i, \theta^{-i}), v_t)\right) = u^i(t, x^{\star}(t, (\hat{\theta}^i, \theta^{-i}), v_t)).$$

Thus, the derivative of agent i's value with respect to his initial shock is given by

$$\hat{V}_\theta(\theta^i, \hat{\theta}^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left[\phi_\theta^i(t, \theta^i, W_t^i)\, u^i(t, x^{\star i}(t, (\hat{\theta}^i, \theta^{-i}), v_t))\right] \mathrm{d}t\right].$$

As ϕ_θ^i is positive, u^i is increasing in x, and $x^{\star i}$ is increasing in $\hat{\theta}^i$ by Proposition 2, Proposition 4 implies that truthful reporting of θ^i is optimal for agent i if he has to make a payment of $\pi^i(\theta)(1 - e^{-rT})/r$ at time zero. As the principal can commit to payments we can transform this payment into a constant flow payment with the same discounted present value by multiplying with $r/(1 - e^{-rT})$. Note, that as the payment $\pi(\theta)$ does not depend on the types it is optimal for the agent to report his types truthfully in the contract $(x^\star, P^\star)$ where $P_t^\star \triangleq p(t, \theta, v_t) + \pi(\theta)$. □

Theorem 2 describes a revenue-maximizing direct mechanism in which the agents report their types and the principal decides on an allocation and associated transfers at every point in time. The next result shows that in the case of a *single agent* there also exists a simple *indirect* mechanism in the form of a two-part tariff which maximizes the intertemporal revenues of the principal. In this mechanism the agent picks a specific contract at time zero and then chooses how much to consume at every point in time. The price paid by the agent at time t for his consumption x_t at time t depends only the initial contract choice through the fixed payment π and the level of consumption x_t at time t through the variable payment, and thus takes the form of a two-part tariff.

Proposition 6 (Two-Part Tariff)**.**
With a single agent there exists a revenue-maximizing two-part tariff: at time zero the agent chooses an fixed payment π and then at every point in time t chooses his consumption x_t and associated price $\tilde{p}(t, \pi, x_t)$.

Proof. Define the set of types such that a given allocation x is optimal at time t

$$V^{\star}(t,\theta,x)=\{v\in\mathbb{R}:x=x^{\star}(t,\theta,v)\}.$$

For every allocation x such that $V^{\star}(t,\theta,x)\neq\emptyset$ there exists at least one type v such that the agent would receive this allocation x if he reported v in the direct mechanism of Theorem 2. The payment of the mechanism described in Theorem 2 depends only on the allocation, but not on the type v. Thus, we have that the following payment implements the virtual surplus maximizing allocation in an indirect mechanism:

$$\tilde{p}(t,\theta,x)=\begin{cases}\inf\{p(t,\theta,v):v\in V^{\star}(t,\theta,x)\}, & \text{if } V^{\star}(t,\theta,x)\neq\emptyset;\\ \infty, & \text{otherwise}.\end{cases} \tag{21}$$

By convention, we assign an arbitrarily large payment, ∞, to the choice of an allocation that is never optimal and $V^{\star}=\emptyset$. By Theorem 2, there exists an incentive-compatible flow payment that is constant over time, $\pi(\theta)$ such that the agent reveals his true initial type, θ. Hence we can choose the allocation dependent payment $p(t,\theta,v)$ to depend on the corresponding fixed payment $\pi(\theta)$ and let $p(t,\pi(\theta),x)$ be the consumption dependent payment. If $\pi:\Theta\rightarrow\mathbb{R}$ fails to be invertible, then the agent can be offered a choice of menus across all $\tilde{p}(t,\theta',x)$ for all $\theta'\in\Theta$ such that the associated fixed payment $\pi(\theta)=\pi(\theta')$ as given by (20). □

The revenue-maximizing mechanism suggested by Proposition 6 is a menu over static contracts. This means that it is sufficient that the payments and allocations at time t depend only on the time t types and the time zero shocks θ.

3.3 The Relation between Discrete and Continuous Time Models

We should emphasize that the basic proof strategy to construct the optimal dynamic mechanism in continuous time mirrors the approach taken in discrete time, see Eső and Szentes (2007) and Pavan, Segal, and Toikka (2014). As in these earlier contributions, we obtain the first order conditions by using the envelope theorem using a small class of relevant deviations. Thus, the valuable insights from discrete time carry over to continuous time. Similarly, for the sufficient conditions, we use monotonicity conditions and time separability of the allocations to guarantee that it remains optimal for the agent to report truthfully after any misreport. Here, the continuous time version of the sufficiency arguments have the advantage that they can be expressed directly in terms of the primitives of the stochastic process which we will illustrate in Section 6.

A brief, but more detailed comparison with the discrete time arguments might be instructive at this point. Eső and Szentes (2007) and Pavan, Segal, and Toikka (2014) show that the additional signals arriving after the initial period can be represented as signals that are orthogonal to the past signals. In the present setting, the type v_t at every point in time is represented as a function ϕ^i of the initial shock θ^i, and an independent time t signal contribution (increment) dW_t, i.e. $v_t = \phi^i(t, \theta^i, W_t^i)$. Our use of consistent deviations is similar to the deviations used in Pavan, Segal, and Toikka (2014) and Eső and Szentes (2014) where each agent reports the shock W_t after time zero truthfully to establish revenue equivalence.

We can also relate the relevant conditions that guarantee the monotonicity of the type with respect to the initial shock. Indeed, our Assumptions 1 and 2 are closely related to the Assumptions 1 and 2 of Eső and Szentes. In particular, we show in the Appendix that our Assumption 1 is implied by Assumption 1 in Eső and Szentes and thus weaker. Furthermore, Assumption 2 of our setup is exactly equivalent to Assumption 2 in Eső and Szentes. Hence, the basic conditions on the payoffs and the shocks extend the conditions of Eső and Szentes directly to an environment with many periods and many (flow) allocation decisions.

Pavan, Segal, and Toikka (2014) observed in the context of a discrete time environment that time-separability of the allocation plus monotonicity of the virtual utility in θ^i and v_t^i is sufficient to ensure strong monotonicity of the virtual surplus maximizing allocation (monotonicity in θ^i and v_t^i after every history). Furthermore, they show that strong monotonicity is sufficient for the implementability of the virtual surplus maximizing allocation (Corollary 1). In Section 5 in the supplementary Appendix they use this insight to describe optimal mechanisms for discrete time situations where the private information of the agent is not the initial state of the process, but a parameter influencing the transitions.

As the allocation at time t does not change the set of possible allocations at later times our environment is time-separable. Our assumptions are similar to the assumptions made in the section discussing separable environments in Pavan, Segal, and Toikka (2014) in the sense that they ensure strong monotonicity which in turn implies implementability of the virtual surplus maximizing allocation.

4 Long-run Behavior of the Distortion

In this section we analyze how the allocative distortions behave in the long-run. We are interested in the expected social welfare generated by the revenue-maximizing allocation compared to the expected welfare generated by the socially optimal allocation. We begin with the following definition and recall that the flow social welfare $s(\cdot)$ is the sum of the flow utilities over all agents, see (8).

Definition 4 (Vanishing Distortion)**.**
The allocative distortion vanishes in the long-run if the social welfare generated by the revnue-maximizing allocation converges to the social welfare generated by the socially optimal allocation as $t \to \infty$*:*

$$\lim_{t\to\infty} \mathbb{E}\left[s\big(t, v_t, x(t, v_t)\big) - s\big(t, v_t, x(t, J(t, \theta, v_t))\big)\right] = 0\,.$$

The characterization of the long-run behavior comes in two parts. We first provide sufficient condition for the distortions to vanish in the long-run. Then we provide necessary conditions for persistence of allocative distortions in the long-run in the case of a single agent.

Proposition 7 (Long-run Behavior of the Distortion)**.**
The following two statements characterize the long-run behavior of the distortions:

(a) The distortion vanishes in the long run if the expected type of any initial shock converges to the expected type of the lowest shock, i.e.

$$\lim_{t\to\infty} \mathbb{E}\left[v_t \,|\, \theta^i = x\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] \to 0\,. \tag{22}$$

(b) If $n = 1$, $u(t, x) = x$, $c(x)$ *is twice continuously differentiable, strictly convex with* $0 < c''(x) \leq D$ *and the expected type for a (non-zero measure) set of shocks does not converge to the expected type of the lowest shock (i.e.* (22) *is not satisfied), then the allocative distortion does not vanish.*

Proof. First note that the difference in the expected type between a random and the lowest initial shock equals

$$\begin{aligned}
\mathbb{E}\left[v_t\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] &= \mathbb{E}\left[\phi^i(t, \theta^i, W_t^i) - \phi^i(t, \underline{\theta}, W_t^i)\right] \\
&= \mathbb{E}\left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{1 - F^i(z)}{f^i(z)} \phi_\theta^i(t, z, W_t^i) f(z) \mathrm{d}z\right] \\
&= \mathbb{E}\left[\frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \phi_\theta^i(t, \theta^i, W_t^i)\right]\,.
\end{aligned}$$

Part (a): We prove that the distortion vanishes if $\lim_{t\to\infty}\mathbb{E}\left[\frac{1-F^i(\theta^i)}{f(\theta^i)}\phi^i_\theta(t,\theta^i,W_t)\right]=0$. We first show that the welfare loss at a fixed point in time can be bounded by the difference between virtual utility $J\in\mathbb{R}^n$ and type $v\in\mathbb{R}^n$

$$\begin{aligned}
&s(t,v,x^\star(t,v)) - s(t,v,x^\star(t,J))\\
&\quad=\left(\sum_{i\in N} v^i u^i(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} v^i u^i(t,x^{\star i}(t,J)) - c(x^\star(t,J))\right)\\
&\quad=\left(\sum_{i\in N} v^i u^i(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} J^i u^i(t,x^{\star i}(t,J)) - c(x^\star(t,J))\right)\\
&\qquad\quad-\sum_{i\in N}(v_i - J^i)u^i(t,x^{\star i}(t,J))\\
&\quad\le\left(\sum_{i\in N} v^i u^i(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} J^i u^i(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right)\\
&\qquad\quad-\sum_{i\in N}(v_i - J^i)u^i(t,x^{\star i}(t,J))\\
&\quad=\sum_{i\in N}(v^i - J^i)(u^i(t,x^{\star i}(t,v)) - u^i(t,x^{\star i}(t,J)).
\end{aligned}$$

As the set of possible allocations X is compact and u^i is continuous there exists a constant $C>0$ such that

$$\sum_{i\in N}(v^i - J^i)(u^i(t,x^{\star i}(t,v)) - u^i(t,x^{\star i}(t,J)) \le C\sum_{i\in N}(v^i - J^i)\,.$$

Hence the welfare loss resulting from the revenue-maximizing allocation resulting from the revenue-maximizing allocation is linearly bounded by the difference between virtual utility and type. As the difference between v^i_t and J^i_t equals $\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi^i_\theta(t,\theta^i,W^i_t)$ it follows that

$$\begin{aligned}
\mathbb{E}\left[s(t,v_t,x^\star(t,v_t)) - s(t,v_t,x^\star(t,J_t))\right] &\le C\,\mathbb{E}\left[\sum_{i\in N}(v^i - J^i)\right]\\
&= C\,\mathbb{E}\left[\sum_{i\in N}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi^i_\theta(t,\theta^i,W^i_t)\right]\\
&= C\left(\mathbb{E}\left[v_t\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right]\right)\,.
\end{aligned}$$

Taking the limit $t\to\infty$ gives the result.

Part (b): We prove that the distortion does not vanish in the long run if the expected type of any initial shock does not converge to the expected type of the lowest initial shock. First, we prove that the distortion changes the allocation. As $u^i(t,x) = x$ is linear and c is convex this implies that the function $x \mapsto vx - c(x)$ is concave and has an interior maximizer for every (t,v). This implies that for every point in time t and every type v

$$0 = v - c'(x^\star(t,v))\,.$$

By the implicit function theorem

$$x^\star_v(t,v) = \frac{1}{c''(x^\star(t,v))} \geq \frac{1}{D}\,.$$

Intuitively this means that the allocation is responsive to the type v. We calculate the change in social welfare induced by the type v and the virtual valuation J

$$\begin{aligned} s(t,v,x^\star(t,v)) - s(t,v,x^\star(t,J)) &= [vx^\star(t,v) - c(x^\star(t,v))] - [vx^\star(t,J) - c(x^\star(t,J))] \\ &= \int_J^v x^\star(t,z)\mathrm{d}z - (v-J)x^\star(t,J) \\ &= \int_J^v x^\star(t,z) - x^\star(t,J)\mathrm{d}z \\ &\geq \frac{1}{D}\int_J^v (z-J)\mathrm{d}z = \frac{(v-J)^2}{2D}. \end{aligned}$$

As the difference between type and virtual utility is given by $\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi^i_\theta(t,\theta^i,W^i_t)$ taking expectations yields

$$\begin{aligned} \mathbb{E}\left[s(t,v,x(v)) - s(t,v,x(J))\right] &\geq \frac{1}{2D}\mathbb{E}\left[\left(\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi^i_\theta(t,\theta^i,W^i_t)\right)^2\right] \\ &\geq \frac{1}{2D}\mathbb{E}\left[\left(\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\phi^i_\theta(t,\theta^i,W^i_t)\right)\right]^2 = \frac{\left(\mathbb{E}\left[v_t\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right]\right)^2}{2D}\,, \end{aligned}$$

where the middle step follows from Jensen's inequality. As $\lim_{t\to\infty}\mathbb{E}\left[v_t \,|\theta = x\,\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] \neq 0$ for positive probability set of initial shock x it follows that $\lim_{t\to\infty}\mathbb{E}\left[v_t\,\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] \neq 0$. □

The sufficient condition for the allocative distortion to vanish requires that the conditional expectation of the type v_t at some distant horizon t converges for all initial realizations of the shock, θ, to the conditional expectation of the type v_t given the lowest initial shock $\underline{\theta}$. Clearly, in any model where the initial state θ is the current state of a recurrent Markov process, such as in Battaglini (2005), the sufficient condition will be satisfied as the influence of the initial state on the distribution of the future states of the Markov process is vanishing.

In turn, the failure of the sufficient condition is almost a necessary condition for the allocative distortion to persist. However, in addition we need to guarantee that the allocation problem is sufficiently responsive to the conditional expectation of the agent everywhere. This can be achieved by the linearity and convexity conditions in Proposition 4.2. We state the necessary conditions only for the problem with a single agent. With many agents, we would have to be concerned with the further complication that the distortion that each individual agent faces may be made obsolete by the distortion faced by the other agents, and thus a more stringent, and perhaps less transparent set of conditions would be required.

5 Repeated Sales

A common economic situation that gives rise to a dynamic mechanism design problem is the repeated sales problem where the buyer is unsure about his future valuation for the good. Examples of such situations are gym membership and phone contracts. At any given point in time the buyer knows how much he values making a call or going to the gym, but he might only have a probabilistic assessment on how much he values the service tomorrow or a year in the future. Usually, it is harder for the buyer to assess how much he values the good at times that are further in the future. Mathematically this uncertainty about future valuations can be captured by modelling the buyer's valuation as a stochastic process.

From the point of view of the seller the question arises whether the uncertainty of the buyer can be used to increase revenues by using a dynamic contract. A variety of dynamic contracts are used, for example for gym memberships and mobile phone contracts, as documented in DellaVigna and Malmendier (2006) or Grubb and Osborne (2015):

1. *Flat Rates* in which the buyer only pays a fixed fee regardless of his level consumption;

2. *Two-Part Tariffs* in which the buyer selects from a menu a fixed fee and a price of consumption. He pays the fixed fee independent of his level of consumption and a unit price for the realized consumption level. Tariffs with higher fixed fees feature lower unit prices of consumption

3. *Leasing Contracts* in which the buyer selects the length of the lease term and the price charged per unit of time.

While those dynamic contracts can be observed in a wide range of situations, their theoretical properties, surprisingly, have not been widely analyzed. Using a dynamic mechanism design perspective, we can explain why and under what circumstances these specific (and other) features of dynamic contracts and consumption plans might be offered. For the purpose of this section, we consider a single buyer, and hence omit the superscript i. We assume that $u(t, x_t) = x_t$ for all t, and the flow utility of the agent is described by

$$v_t \cdot x_t - p_t, \tag{23}$$

and hence v_t immediately represents the willingness-to-pay of the agent in period t.

In the following we describe the revenue-maximizing dynamic contract offered by a monopolistic seller. In general, dynamic contracts could have complicated features as the payments at time t could depend on all the past consumption decisions and messages sent by the agent. However we will show, using the results of the previous section, in particular Proposition 6 that offering a menu of simple static contracts is sufficient to maximize the expected intertemporal revenue.

5.1 Unknown Initial Value

We shall assume that the type $(v_t)_{t\in\mathbb{R}_+}$ of the buyer follows a geometric Brownian motion with zero drift, and possible shifted upwards by $\underline{v} \geq 0$:

$$\mathrm{d}v_t = (v_t - \underline{v})\sigma \mathrm{d}W_t\,, \tag{24}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and solution to the above differential equation is given by:

$$v_t = \phi(t, \theta, W_t) = v_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) + \underline{v}. \tag{25}$$

The choice of the *shifted geometric Brownian motion* as the type process ensures that the valuation v_t for the good will be greater than $\underline{v}$ at every point in time t. With zero drift, the valuation at time t is the agent's best estimate of his valuation at later times $s > t$:

$$v_t = \mathbb{E}[v_s \,|\, v_t]\,.$$

In this subsection, the initial shock θ^i is taken to be the initial valuation of the buyer $v_0 \in (\underline{v}, \infty)$. We assume that the distribution function F is such that

$$v_0 \mapsto \frac{1 - F(v_0)}{f(v_0)\, v_0} \tag{26}$$

is decreasing – a condition that is strictly weaker than the familiar increasing hazard rate condition – and that $f(\underline{v}) \geq 1/\underline{v}$.

At every point in time t the buyer chooses an amount of consumption $x_t \in X \subseteq \mathbb{R}_+$ and pays p_t such that his overall utility equals

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(v_t \cdot x_t - p_t)\mathrm{d}t\right].$$

To evaluate dynamic contracts from the sellers perspective, we assume that the seller faces continuous, non-decreasing production cost $c : X \to \mathbb{R}_+$, such that his overall payoff equals

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(p_t - c(x_t))\mathrm{d}t\right].$$

We can then specialize the form of the revenue-maximizing contract obtained earlier in Theorem 2 to the specific environment of the shifted geometric Brownian motion here. The *stochastic flow* of the shifted geometric Brownian motion is simply

$$\phi_\theta(t, \theta, W_t) = \frac{v_t - \underline{v}}{v_0},$$

and thus the virtual utility, derived earlier in its general form in (15), can now be written as:

$$J(t, v_0, v_t) = v_t\left(1 - \frac{1 - F(v_0)}{f(v_0)v_0}\right) + \frac{1 - F(v_0)}{f(v_0)v_0}\underline{v}\,. \tag{27}$$

By Theorem 2 it is then sufficient to verify that the above virtual surplus is increasing in v_0 and v_t to guarantee that a virtual surplus-maximizing contract exists.

Proposition 8 (Virtual Utility with Geometric Brownian Motion)**.**
The virtual utility $J(t, v_0, v_t)$ in the environment of the geometric Brownian motion defined by (25)-(26) is increasing in $v_0 \in \mathbb{R}_+$ and $v_t \in \mathbb{R}_+$ and a virtual surplus-maximizing contract exists.

As shown in Theorem 1 the seller aims to maximize

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(J_t x_t - c(x_t)\right)\right].$$

In the case of the geometric Brownian motion, the virtual utility, and hence the virtual surplus, are simply a linear function of the type v_t of the agent. The intercept and the slope of the function are determined by the value of the initial shock $\theta = v_0$ and the lower bound $\underline{v}$. We can therefore directly identify an indirect mechanism, a pricing mechanism, that aligns the preferences of the agents with those of the principal. Let us define

$$M(v_0) \triangleq \left(1 - \frac{1-F(v_0)}{f(v_0)v_0}\right)^{-1}, \tag{28}$$

and so we can express the virtual surplus of the principal as follows:

$$\begin{aligned} J(t, v_0, v_t)x_t - c(x) &= \left(v_t\left(1 - \frac{1-F(v_0)}{f(v_0)v_0}\right) + \frac{1-F(v_0)}{f(v_0)v_0}\underline{v}\right)x - c(x) \\ &= M(v_0)^{-1}\left(v_t x - M(v_0)c(x) + (M(v_0)-1)x\underline{v}\right). \end{aligned}$$

It follows that a consumption based payment $p_t\left(v_0, x_t\right)$ given by

$$p_t\left(v_0, x_t\right) \triangleq M(v_0)c(x_t) - (M(v_0)-1)x_t\underline{v},$$

perfectly aligns the interest of the buyer and the seller, the agent and the principal at every point in time $t > 0$. After all, it leads agent and principal to solve their respective optimality conditions at the same x_t. It remains to prove that it is incentive compatible for the buyer to report his time zero type truthfully. The following results describes optimal contracts (indirect mechanism) for the seller.

Proposition 9 (Revenue Maximizing Indirect Mechanism)**.**
A revenue-maximizing indirect mechanism *is given by a menu* $(\pi, p(\pi, x_t))$ *of membership fees* π *and consumption prices* $p(\pi, x_t)$ *of the form*

$$p(\pi, x_t) = M(\pi)c(x_t) - (M(\pi)-1)\underline{v}\,x_t. \tag{29}$$

Thus, the optimal contract is of the following form: At time zero the seller offers a *menu* of static contracts each consisting of a time independent and recurrent membership fee $\pi \geq 0$, and a consumption dependent payment:

$$p(\pi, x_t) = M(\pi)c(x_t) - \left[M(\pi)-1\right]\underline{v}\,x_t.$$

The consumption dependent payment p consists of a price of consumption of $M(\pi) \geq 1$, literally the *mark-up*, and a linear consumption discount $(M(\pi)-1)\underline{v}x_t$. If the buyer accepts a contract he has to pay a recurring membership fee $\pi \geq 0$ independent of his consumption. At the same time he has to pay $p(\pi, x_t)$ depending on his consumption x_t in period t such that his overall payment at time t equals

$$p_t = \pi + p(\pi, x_t) = \pi + M(\pi)c(x_t) - [(M(\pi) - 1]\, \underline{v} x_t\,. \tag{30}$$

The optimal fixed fee $\pi(v_0)$ that is chosen by the agent at the beginning of the contracts depends on the agent's initial valuation v_0. It will be such that

$$M(\pi(v_0)) = \left(1 - \frac{1 - F(v_0)}{f(v_0)v_0}\right)^{-1}.$$

With the general characterization of the optimal contract given by Proposition 9 we next establish under what conditions on the nature of the private information and the cost of delivering the service $c\,(x)$ the above mentioned contract features will arise as a part of an optimal contract.

5.1.1 Flat Rate Contracts

In a flat rate contract the payment $p_t = \pi$ is constant over time and independent of the level of consumption chosen by the buyer. Suppose that the set of possible allocation is given by $X = [0, 1]$, and that the minimal valuation $\underline{v}$ equals zero. Assume that the cost of production $c\,(x)$ is constant and normalized to zero, $c\,(x) = 0$. As the buyer's utility given by (23) increases linearly in the level of the consumption, he will always want to consume the good at the maximal possible level if he faces a flat rate. A direct consequence of the transfers described in (30) is the following result characterizing an optimal mechanism with zero (marginal) cost of production: The optimal mechanism is a flat rate where every agent who accepts the contract at time zero, makes a constant flow payment, independent of his consumption, and consumes the maximal possible amount: $x_t = 1$.

Now, if his current valuation v_t is below the flat rate $p_t = \pi$, not only is his current flow of utility negative, but so is his expected continuation utility of the contract:

$$\mathbb{E}\left[\int_t^{\infty} e^{-rs}(v_s - p_s)\mathrm{d}t \,|\, v_t\right] = \frac{v_t - \pi}{r}\,. \tag{31}$$

As a consequence of condition (31) only the agent with an initial valuation $v_0 \geq \pi$ will accept the contract. All agents with an initial valuation $v_0 < \pi$ reject the contract and never consume the good no matter how high the consumption utility is at times $t > 0$.

5.1.2 Two-Part Tariffs

With constant cost of production, the optimal contract leads to flat rate tariffs. We next consider the case of increasing and convex costs. We maintain the assumption that the minimal valuation $\underline{v}$ equals zero and assume that the cost function $c(x)$ is strictly increasing and convex for $x \in \mathbb{R}_+$.[3] In particular, we allow for a constant but positive marginal cost of production. By condition (30) a two-part tariff where the agent pays π independent of his consumption and $M(\pi)c(x)$ depending on his consumption x is a revenue-maximizing contract for the principal. It is worth emphasizing that a simple menu of static two-part tariffs can hence maximize the expected dynamic revenue of the principal.

We illustrate the structure of the two-part tariff with the following quadratic cost function $c(x) = x^2/2$ and assume that the initial valuation v_0 is exponentially distributed with mean μ:

$$F(v_0) = 1 - \exp(-\frac{v_0}{\mu}),$$

which will allow us to explicitly compute the terms of the contract. By Proposition 9, we know that if the agent decided on a contract $(\pi, M(\pi))$ then the optimal consumption of the agent at time t is given by

$$\{x_t\} = \arg\max_{x \geq 0} \left(x\, v_t - M(\pi)\frac{x^2}{2} \right) = \frac{v_t}{M(\pi)}\,. \tag{32}$$

Hence, the agent's expected time zero utility from the contract $(\pi, M(\pi))$ is

$$\begin{aligned} \max_{(x_t)_{t\in\mathbb{R}_+}} \mathbb{E}\left[\int_0^\infty e^{-rt}(v_t x_t - \pi - M(\pi)c(x_t))\right] &= \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\frac{v_t^2}{2M(\pi)} - \pi\right)\right] \\ &= \frac{v_0}{2M(\pi)(r-\sigma)} - \frac{\pi}{r}\,. \end{aligned} \tag{33}$$

Hence, the agent chooses his optimal contract at time 0 by maximizing his expected net utility (33) over π to select a contract $(\pi, M(\pi))$ based only on his time zero valuation v_0. Let us denote by $\pi(v_0)$ the fixed fee chosen by the agent of initial valuation v_0. By Proposition 8, in the optimal contract the mark-up $M(\pi(v_0))$ computed earlier in its general form as (28) is given as a function

[3] As we established the revenue equivalence theorem under the assumption that the quantity x is bounded we understand the model with unbounded quantities as the limit of optimal mechanisms when the bound on x converges to infinity.

of the initial valuation v_0 by:

$$M(\pi(v_0)) = \begin{cases} \frac{v_0}{v_0-\mu}, & \text{if } v_0 \geq \mu, \\ \infty, & \text{otherwise.} \end{cases} \tag{34}$$

Hence every buyer who has initially a valuation v_0 below the average time zero valuation μ will be excluded and never consume the good no matter how high his future valuation is. The mark-up decreases and converges to one, and hence the socially efficient allocation as $v_0 \to \infty$. Since the mark-up in the incentive compatible indirect mechanism has to satisfy (34) we can compute the membership fee π to be paid by the buyer with initial valuation v_0 from the incentive compatibility at time $t = 0$ based on (33) and get:

$$\pi\left(v_0\right) = \frac{r\mu}{2\left(r-\sigma\right)}\left(\ln\frac{v_0}{\mu}\right).$$

We illustrate the nature of the optimal tariff in Figure 1. The left panel expresses the membership fee π and the mark-up $M\left(\pi\right)$ in the two-part tariff as a function of the initial type (blue and red curve respectively). The right panel incentive compatible choice describes the equilibrium trade-off between the level of the membership π and the mark-up $M\left(\pi\right)$ which is given by:

$$M\left(\pi\right) = \left[1-\exp\left(\frac{-2\pi(r-\sigma)}{r\mu}\right)\right]^{-1}.$$

It follows that a lower mark-up $M\left(\pi\right)$ is purchased at the expense of higher membership fee π. In the optimal contract, an agent with a higher initial valuation is willing to purchase the rights to lower mark-up by means of higher membership fee. In consequence, an agent with a higher membership fee faces less distortion with respect to his flow consumption decision.

In Figure 2 we illustrate how the intertemporal distortions induced by the optimal contract influence the consumption choices over time. The solid lines are two path realizations of the geometric Brownian motion without drift. One path starts at an initial valuation of $v_0 = 2$ (blue) and one at $v_0 = 7$ (red). It so happens that both of these paths coincide after time $t = 1.75$. Now the dashed lines represent the consumption levels in the revenue-maximizing contract. Note, that even after the valuations coincide in the sample path, the consumption levels associated with different initial valuations differ. In particular, the optimal consumption level react with differential intensity to changes in the valuations. By contrast, the consumption of the agent in the social welfare maximizing allocation would exactly equal his valuation.

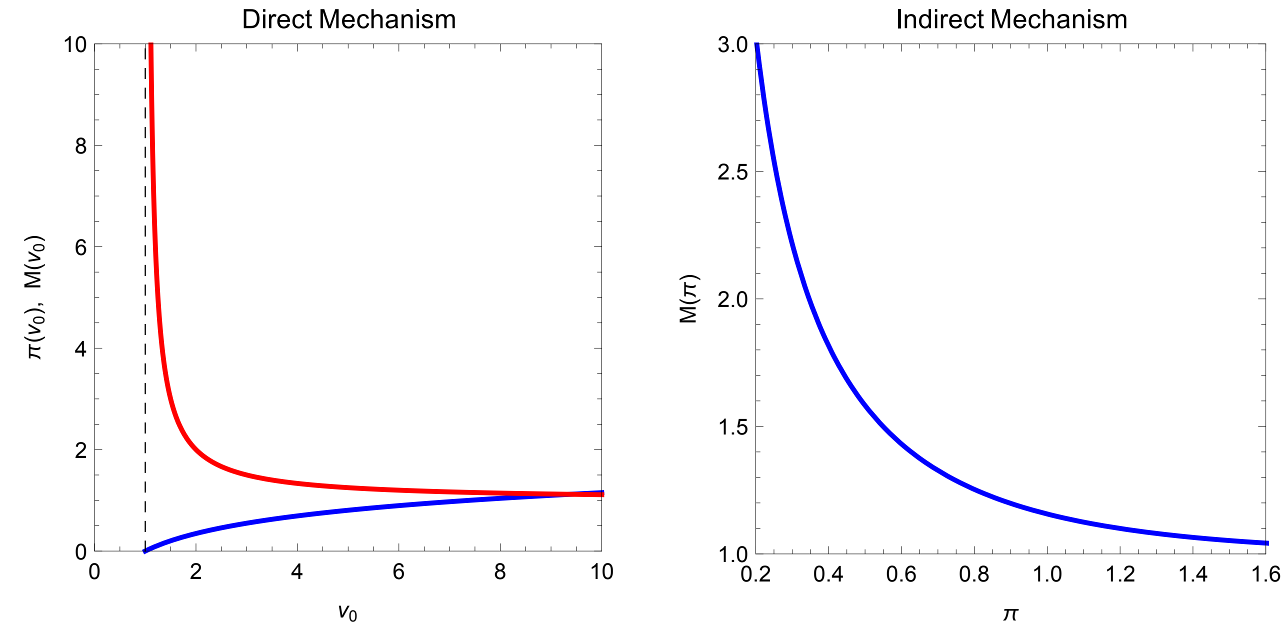

Figure 1: The two-part tariff in the direct mechanism (left) and in the indirect mechanism (right). The markup $M(v_0)$ is in red and the membership fee $\pi(v_0)$ in blue.

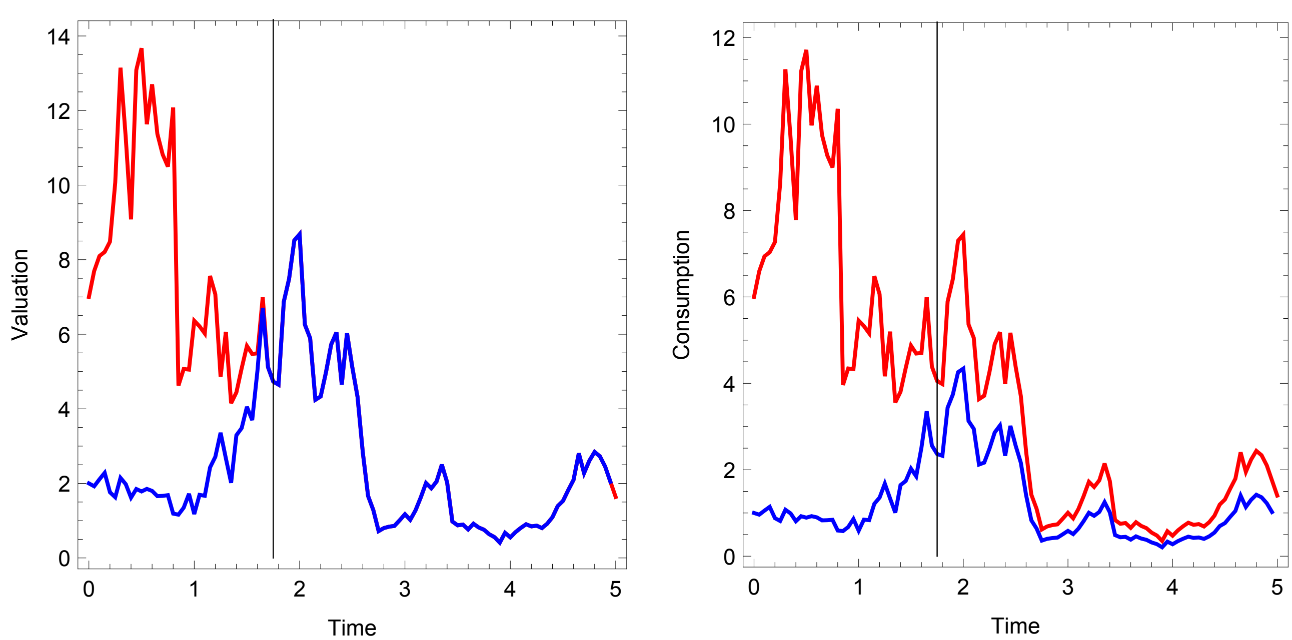

Figure 2: Consumption path and valuation if the initial valuation v_0 is exponentially distributed with mean one and the valuation evolves as a geometric Brownian motion without drift.

5.1.3 Free Minute Contract

We now consider the case in which the minimal type $\underline{v}$ of the agent is is strictly positive and that the density at the minimal valuation is bounded away from zero, or $f(\underline{v}) > 1/\underline{v}$. In addition we assume that the marginal cost of providing the good vanishes for small quantities, i.e. $c'(0) = 0$. When the agent decides how much to consume at time t he solves the maximization problem:

$$\max_x \{xv_t - (\pi + M(\pi)c(x) - (M(\pi) - 1)\underline{v}x)\} \; .$$

This leads to the first order condition:

$$0 = v_t - M(\pi)c'(x) + (M(\pi) - 1)\underline{v} \;\Leftrightarrow\; c'(x) = \underline{v} + \frac{(v_t - \underline{v})}{M(\pi)} \; .$$

As the marginal cost of providing the good vanishes if the quantity goes to zero it follows that the consumption of the agent is bounded from below at every point in time by $c'^{-1}(\underline{v})$. Hence we can interpret the amount $c'^{-1}(\underline{v})$ as a quantity provided to the agent for free. This is a feature that is common in mobile phone contracts. In such a contract the agent can consume a certain number of minutes for free and only has to pay for the consumption exceeding this amount.

5.2 Unknown Drift

We now return to the cost structure that lead to the flat rate contract earlier, namely, the cost of production is constant and normalized to zero and $x_t \in [0, 1]$. Different from the preceding analysis we now assume that the initial private information θ of the agent is about the drift of the geometric Brownian motion, or $\mu = \theta$, rather than the initial value of the geometric Brownian motion v_0 which we now assume to be public information. We set the lower bound on valuations $\underline{v}$ to zero. In consequence, the valuation v_t now evolves according to:

$$\mathrm{d}v_t = v_t\left(\theta \mathrm{d}t + \sigma \mathrm{d}W_t\right) \; ,$$

and the solution to the above differential equation is given by:

$$v_t = \phi(t, \theta, W_t) = v_0 \exp\left((\theta - \frac{\sigma^2}{2})t + \sigma W_t\right) . \tag{35}$$

The derivative of the type $v_t = \phi(t, \theta, W_t)$ with respect to θ, the *generalized stochastic flow,* now equals:

$$\phi_\theta = \phi t,$$

and the virtual utility is now given by:

$$J(t,\theta,v_t) = v_t\left(1 - \frac{1-F(\theta)}{f(\theta)}t\right). \tag{36}$$

Interestingly, the distortion is still formed on the basis of a multiplicative handicap, but now the handicap factor is increasing linearly in time as expressed by the second term of the virtual utility. It follows that in contrast to the above cases of unknown initial value, the distortion is now growing over time. As v_t is positive, it follows that the virtual valuation is strictly positive until a deterministic time T is reached which is precisely given by the hazard rate:

$$L(\theta) = \frac{f(\theta)}{1-F(\theta)},$$

and thereafter the virtual utility turns negative. Thus, the allocation of the object to agent i ends with probability one at time $L(\theta)$. The optimal contract can now be implemented by a constant leasing payment $p(\theta)$ the agent makes at every time $t \in [0, L(\theta)]$.

Proposition 10 (Leasing Contract)**.**
The revenue-maximizing mechanism allocates the object to the agent with initial shock θ if and only if $t \in [0, L(\theta)]$ and requires a payment of

$$p(t,\theta) = \begin{cases} \frac{r}{1-e^{-rL(\theta)}}\left(v_0\frac{e^{(\theta-r)L(\theta)}-1}{\theta-r} - \int_0^\theta \frac{e^{(z-r)L(z)}[L(z)(z-r)-1]}{(z-r)^2}\mathrm{d}z\right), & \textit{if } t \in [0, L(\theta)]; \\ 0, & \textit{otherwise.} \end{cases}$$

To establish the above formula for the payments we calculate the expected value that the agent with initial shock θ derives from getting the object until time $L(\theta)$. By the envelope theorem the payment equals this value minus the integral over the marginal value of those types with a lower initial shock.

In a recent paper, Boleslavsky and Said (2013) derive the revenue-maximizing contract in a discrete time setting where the private information of a single agent is the uptick probability of a multiplicative random walk. As it is well known, the geometric Brownian motion can be viewed as the continuous time limit of the discrete time multiplicative random walk stochastic process. Thus, it is interesting to compare their results to the implications following our analysis. In terms of the private information of the agent, the unknown drift in the geometric Brownian motion here represents the unknown uptick probability analyzed in Boleslavsky and Said (2013). As the general convergence result of the stochastic process itself would suggest, we can also establish, see the Appendix for the details, that the continuous time limit of the virtual valuation derived in Boleslavsky and Said (2013) is the virtual utility derived above by (36). However, in the continuous time limit the expression for the virtual utility, see (36), becomes notably easier to express and to interpret. The analysis in Boleslavsky and Said (2013) explicitly verifies the validity of the incentive constraints in the case of a single agent. With the general approach taken here, we can obtain sufficient conditions for the revenue optimal contract and associated allocations even in the presence of many agents. In fact, the next section considers such an allocation problem, namely the allocation of a single unit among competing bidders. This second class of allocation problems is notably more restrictive in terms of the cost of providing the service, namely constant for a single unit, but allow us to obtain some novel insight regarding the structure of the intertemporal distortion with many agents.

6 Sequential Auctions and Distortions

We illustrate the impact that the structure of the private information has on the intertemporal policies and the allocative distortion within the context of a sequential auction model. The allocation problem is as follows. At every point in time t, the owner of a single unit of a, possibly divisible, object wishes to allocate it among the competing bidders, $i = 1, ..., n$. The allocation space is given at every instant t by $x_t^i \in [0,1]$ and $\sum_{i=1}^{n} x_t^i \leq 1$. The marginal cost of providing the object is constant and normalized to zero. The flow utility of each agent i is given by $v_t^i \cdot x_t^i - p_t^i$.

We can interpret the allocation process as a process of intertemporal licensing where the current use of the object is determined on the basis of the past and current reports of the agents. In particular, the assignment of the object can move back and forth between the competing agents. Alternatively, the description of the valuation could be rephrased as a description of the marginal cost of producing a single good, and the associated allocation process is the solution to a long-term procurement contract with competing producers. As in the static theory of optimal procurement, the virtual utility would then be replaced by the virtual cost, but the structure of the allocation process would remain intact.

6.1 Arithmetic Brownian Motion

In the previous section we represented the valuation process by a geometric Brownian motion, now we consider the arithmetic Brownian motion, thus indicating the versatility of the current approach. The arithmetic Brownian motion v_t^i is completely described by its initial value v_0^i, the drift μ^i and the variance σ^i of the diffusion process W_t. The willingness to pay of agent i evolves according to:

$$\mathrm{d}v_t^i = \mu^i \mathrm{d}t + \sigma^i \mathrm{d}W_t^i,$$

and the willingness-to-pay of agent i, v_t^i, is:

$$v_t^i = v_0^i + \mu^i t + \sigma^i W_t^i. \tag{37}$$

We analyze the incentive problem when either one of the three determinants of the Brownian motion, the initial value, the drift or the variance is unknown, whereas the remaining two are commonly known. Surprisingly, we find that even though we consider the same stochastic process, the nature of the private information, i.e. about which aspect of the process the agent is privately informed, has a substantial impact on the optimal allocation. In particular, we find that the distortion is either constant, increasing or random (and increasing in expectation) depending on the precise nature of the private information.

Unknown Initial Value – Constant Distortion We begin with the case where the initial value of the Brownian motion, $v_0^i = \theta^i$, is private information to agent i, as are all future realizations v_t^i. In contrast, the drift μ^i and the variance σ^i of the Brownian motion are assumed to be commonly known. Given the representation of the Brownian motion (37), we have

$$v_t^i = \phi^i(t, \theta^i, W_t^i) = \theta^i + \mu^i t + \sigma^i W_t^i. \tag{38}$$

The partial derivative of ϕ^i with respect to θ^i is given by $\phi_\theta^i = 1$ and thus the virtual utility is given by:

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)}, \tag{39}$$

and the distortion imposed by the revenue-maximizing mechanism is *constant over time.* In every period, the object is allocated to the agent i_t^* with the highest virtual utility, provided that the valuation is positive. Thus, the allocation proceeds by finding the bidder with the highest valuation, after taking into account a handicap, that is determined once and for all through the report of the initial shock.

Earlier, we gave a general description of the payments decomposed into an annualized up-front payment π and a flow payment p_t. In the present auction environment, we can give an explicit description of the flow payments in terms of the virtual utility of the agents. The associated flow transfer of the bidders, p_t^i follows directly from the logic of the second price auction:

$$p_t^i = \begin{cases} \max_{j \neq i} \left\{ v_t^j - \frac{1-F^j(\theta^j)}{f^j(\theta^j)} \right\} + \frac{1-F^i(\theta^i)}{f^i(\theta^i)}, & \text{if} \quad i = i_t^*; \\ 0, & \text{if} \quad i \neq i_t^*. \end{cases} \tag{40}$$

Thus, it is only the winning bidder who incurs a flow payment. By rewriting (40), we find that the winning bidder has to pay his valuation, but receives a discount, namely his information rent, which is exactly equal to the difference in the virtual utility between the winning bidder and the next highest bidder, i.e.

$$p_t^{i^*} = v_t^{i^*} - \left(J^{i^*}(t, \theta^{i^*}, v_t^{i^*}) - \max_{j \neq i^*} \left\{ J^j(t, \theta^j, v_t^j) \right\} \right). \tag{41}$$

By construction of the transfer function, the flow net utility of the bidder is positive whenever he is assigned the object, as

$$v_t^{i^*} \geq v_t^j - \frac{1 - F^j(\theta^j)}{f^j(\theta^j)} + \frac{1 - F^i(\theta^{i^*})}{f^i(\theta^{i^*})}, \tag{42}$$

and thus, the flow allocation proceeds as a "handicap" second price auction, where the price of the winner is determined by the current value of the second highest bidder, as measured by the virtual utility. The "handicap" is computed as the difference between the constant handicap of the current winner and the current second highest bidder. The above version of the handicap auction appeared in Eső and Szentes (2007) in a two period model of a single unit auction. Similarly, Board (2007) develops a handicap auction in a discrete time, infinite horizon model, but one in which the object is allocated only once – at an optimal stopping time. There the handicap is represented – as it is here – by the constant terms, $\left(1 - F^j\left(\theta^j\right)\right) / f^j\left(\theta^j\right)$ and $\left(1 - F^i\left(\theta^i\right)\right) / f^i\left(\theta^i\right)$, but the second highest value is computed as the continuation value of the remaining bidders, as in Bergemann and Välimäki (2010).

Unknown Drift – Increasing Distortion We now consider the case where the initial private information is the drift of the Brownian motion. Let $v_t^i \in \mathbb{R}_+$ be an arithmetic Brownian motion with drift θ^i and known variance σ^i and known initial value, v_0^i:

$$v_t^i = \phi^i(t, \theta^i, W_t^i) = v_0^i + \theta^i t + \sigma^i W_t^i. \tag{43}$$

The derivative of the valuation ϕ^i with respect to the initial private information θ^i, which is now the drift of the Brownian motion, is given by $\phi_\theta^i = t$. Thus the virtual utility is now:

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} t. \tag{44}$$

The flow payment is of exactly the same form as (41), and the virtual utility function is given by (44). The distortion is still formed on the basis of the handicap, by the inverse hazard rate $(1 - F(\theta^i))/f(\theta^i)$, but now the handicap is *increasing linearly in time.* In contrast to the case of the unknown starting value, the distortion is *growing* deterministically over time, rather than vanishing over time. Since v_t^i might be growing as well, the deterministic increase in the distortion does not allow us to conclude that the assignment of the object is terminated with probability one at some finite time T, a conclusion that we arrived earlier in Section 5 where we considered the geometric Brownian motion.

Unknown Variance – Random Distortion We conclude the analysis with the case of unknown variance and the valuation v_t^i then evolves according to:

$$v_t^i = \phi^i(t, \theta^i, W_t^i) = v_0^i + \mu^i t + \theta^i W_t^i. \tag{45}$$

Now, the initial private information θ^i represents the volatility of the Brownian motion. The derivative of the valuation ϕ^i with respect to the initial private information θ^i now takes the form:

$$\phi_\theta^i = \frac{\phi^i - v_0^i - \mu^i t}{\theta^i} = W_t^i$$

In consequence the virtual utility of agent i can be expressed as:

$$\begin{aligned} J^i(t, \theta^i, v_t^i) &= v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} W_t^i \\ &= v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \frac{v_t^i - v_0^i - \mu^i t}{\theta^i} \\ &= v_t^i \left(1 - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)\theta^i}\right) + \frac{1 - F^i(\theta^i)}{f^i(\theta^i)\theta^i}(v_0^i + \mu^i t). \end{aligned} \tag{46}$$

We observe from the first line of the above virtual utility that the expected virtual utility equals the initial value v_0^i for any time zero shock θ^i. The variance of the Brownian motion does not lend itself to an ordering along the first order stochastic dominance criterion, rather it is ordered by second order stochastic dominance. Formally, in the case of unknown variance ϕ^i is not increasing in θ_i and fails Assumption 2. But as those assumptions were only used to establish that the virtual utility is increasing in θ^i, v_t^i if it takes positive values, we can dispense with them here as we can ensure monotonicity here by requiring that $\mu^i, v_0^i \leq 0$.

The basic proof idea is to use the convexity of the objective function to guarantee that an increase in variance leads to an increase in the expected (virtual) valuation. After all, if the virtual utility turns negative, the seller does not want to assign the object to the buyer, thus the revenue is flat and equal to zero. It therefore follows that the revenue of the seller has a convex like property. But in contrast to the utility of the buyer, which is linear in v_t^i, and hence strictly convex if truncated below by zero, the virtual surplus of the seller has additional terms, as displayed by (46) which need to be controlled to guarantee the monotonicity of the virtual utility. From the expression of the virtual utility function we can immediately derive sufficient conditions for the monotonicity. Thus if we assume that the initial value v_0^i is negative, $v_0^i \leq 0$, and the arithmetic Brownian motion has a negative drift $\mu^i \leq 0$, then we are guaranteed that the convexity argument is sufficiently strong.

Formally, let $\hat{\theta}^i$ be the solution to $\hat{\theta}^i - \frac{1-F^i(\hat{\theta}^i)}{f^i(\hat{\theta}^i)} = 0$. As

$$J^i(t, \theta^i, v_t^i) \leq v_t^i \left(1 - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)\theta^i}\right)$$

the virtual utility $J^i(t, \theta^i, v_t^i)$ is only positive if the valuation v_t^i is negative, for all $\theta^i < \hat{\theta}^i$. But this implies that the gross expected utility of all agents with initial shock $\theta^i < \hat{\theta}^i$ is negative, and hence they cannot generate a nonnegative revenue due to the ex ante participation constraint, and hence, it can never be optimal to allocate to an agent with variance $\theta^i < \hat{\theta}^i$. Thus, we ignore agents with low variance $\theta^i < \hat{\theta}^i$ and never allocate the object to them. As $\frac{1-F^i(\theta^i)}{f^i(\theta^i)}$ is decreasing we have that $1 - \frac{1-F^i(\theta^i)}{f^i(\theta^i)\theta^i} > 0$ for all $\theta^i > \hat{\theta}$ and hence $J^i(t, \theta^i, v_t^i)$ is increasing in v_t^i and θ^i for all $v_t^i > 0\,, \theta^i > \hat{\theta}^i$. Hence, by the argument of Proposition 5, there exists a payment such that truthful reporting of valuations becomes optimal irrespective of the reported types. As the virtual utility

$$J^i(t, \theta^i, \phi^i(t, \theta^i, W_t^i)) = W_t^i(\theta^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)}) + \mu^i t + v_0^i$$

is increasing in θ^i whenever $W_t^i > 0$ and decreasing whenever $W_t^i < 0$ it follows that the product

$$W_t^i u^i(t, x^{\star i}(t, (\hat{\theta}^i, \theta^{-i}), v_t^i))$$

is increasing in the reported shock $\hat{\theta}^i$. The derivative of the agents utility with respect to his initial shock simplifies to

$$\mathbb{E}\left[\int_0^T e^{-rt}\left[W_t^i\, u^i(t, x^{\star i}(t, (\hat{\theta}^i, \theta^{-i}), v_t^i))\right] \mathrm{d}t\right]$$

and thus, by the argument of Theorem 2, the virtual surplus maximizing allocation for the shocks $\theta^i > \hat{\theta}^i$ is incentive compatible.

The last two examples emphasize that our approach can accommodate not only private information about the initial state of a random process, but also private information about a parameter of the stochastic process, such as the mean or the variance of the process.

6.2 Ornstein-Uhlenbeck process

Finally we describe the implications for the revenue-maximizing allocation if the stochastic process is given by the Ornstein-Uhlenbeck process, which is the continuous-time analogue of the discrete-time AR(1) process. This example is closely connected to the discrete time literature. Besanko (1985) showed that the distortions induced by the discrete-time AR(1) process vanish for privately known initial values of the process if and only if the process is mean-reverting. Furthermore, the AR(1) process, was the leading example in the analysis of the impulse response function in Pavan, Segal, and Toikka (2014).

The Ornstein-Uhlenbeck process v_t^i is completely described by its initial value v_0^i, the mean reversion level μ, the mean reversion speed $M \geq 0$ and the variance $\sigma \geq 0$ of the diffusion process B_t. The willingness to pay of agent i evolves according to the stochastic differential equation:

$$\mathrm{d}v_t^i = m(\mu - v_t)\mathrm{d}t + \sigma \mathrm{d}B_t^i\,,$$

where B_t is a standard Brownian motion. The Ornstein-Uhlenbeck process can be represented using a distinct Brownian motion $\tilde{B}$ as:

$$v_t = v_0 e^{-mt} + \mu(1 - e^{-mt}) + \frac{\sigma e^{-mt}}{\sqrt{2m}} \tilde{B}_{2^{mt}-1}\,. \tag{47}$$

Hence we can define the process W as a time-changed Brownian motion by

$$W_t^m = \frac{e^{-mt}}{\sqrt{2m}} \tilde{B}_{2^{mt}-1} \, .$$

Using W we can represent the valuation of the agent as

$$v_t = v_0 e^{-mt} + \mu(1 - e^{-mt}) + \sigma W_t^m \, .$$

Unknown Initial Value Consider the case where the valuation process is an Ornstein-Uhlenbeck process and the initial valuation is private information, i.e. $v_0^i = \theta^i$. Given the representation (47) it follows that

$$\frac{\partial \phi^i(t, \theta^i, W_T^i)}{\partial \theta^i} = e^{-mt}.$$

Thus, Assumption 1 and 2 are satisfied. The virtual utility J^i equals

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} e^{-mt} \, .$$

Hence the optimal mechanism is a handicap mechanism with a deterministic handicap that is exponentially decreasing over time. As the Ornstein-Uhlenbeck process converges to a stationary distribution which is independent of the starting value θ^i, Proposition 7 applies and the distortion vanishes in the long run. Intuitively the initial valuation does not change the expected valuation in the long run.

Unknown Long Run Average We can also take a parameter of the stochastic process to be the private information of the agent, that is we can take the expected long run average of the process to be the private information of agent i, i.e. $\mu = \theta^i$. Given the representation (47) it follows that

$$\frac{\partial \phi_t^i}{\partial \theta^i} = 1 - e^{-mt}.$$

Thus, Assumption 1 and 2 are satisfied. The virtual utility J^i equals

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} (1 - e^{-mt}) \, .$$

Hence the optimal mechanism is a handicap mechanism with a deterministic handicap that is increasing over time. As the Ornstein-Uhlenbeck process converges in the long run to a stationary distribution which depends on the long run average θ^i the distortion increases in the long run.

Intuitively, the expected valuation converges to the long run average θ^i, and so does the virtual utility, it converges to the long rune average of the virtual utility as well. In a notable recent contribution, Skrzypacz and Toikka (2015) consider dynamic mechanisms for repeated trade under private information. In particular they analyze the discrete time version of the mean-reverting process in which the persistence of the stochastic process is private information, the equivalent of the mean-reversion speed m here. They establish that the allocative distortion is increasing over time rather than decreasing as it is when the initial state of the stochastic process constitutes the private information.

7 Conclusion

We analyzed a class of dynamic allocation problems with private information in continuous time. In contrast to much of the received literature in dynamic mechanism design, the private information of each agent was not restricted to the current state of the Markov process. In particular, the initial private information was allowed to pertain to a one-dimensional parameter of the stochastic process such as the drift of the arithmetic or geometric Brownian motion, or the speed of the mean-reverting process. By allowing for a richer class of private information structures, we gained a better understanding about the nature of the distortion due the private information. In contrast to the settings where the private information always pertains to the state of the Markov process and where the distortions induced by the revenue-maximizing allocation are typically vanishing over time, we have shown that the distortion can be constant, increasing or decreasing over time. The analysis of the private information in terms of the stochastic flow, the equivalent of the impulse response functions in continuous time, allowed us directly link the nature of the private information to the nature of the intertemporal distortion.

A distinct advantage of the continuous and time-separable approach taken here is that we could offer explicit solutions, in terms of the optimal allocation, the level of distortion and the transfer payments. We highlighted this advantage in the analysis of the repeated sales environment in which we gave complete, explicit and surprisingly simple solutions to a class of sales or licensing problems. In particular, we showed that we can implement the dynamic optimal contract by means of an essentially static contract, a membership contract, that displayed such common empirical features as flat rates, free consumption units and two-part tariffs.

Appendix

Proof of Proposition 2. As there is no risk of confusing agents we drop the upper indices in the proof and denote by (θ, v) the type and the type of agent i. Assume that the virtual utility is positive $J(t,\theta,v) > 0$. We first prove the monotonicity in v and than in θ.

Part 1: $J(t,\theta,v) > 0 \Rightarrow J_v(t,\theta,v) \geq 0$:

Note that

$$J(t,\theta,v) = v - \frac{1-F(\theta)}{f(\theta)}\phi^i_\theta(t,\theta,w(t,\theta,v)) = v\left(1 - \frac{1-F(\theta)}{f(\theta)}\frac{\phi^i_\theta(t,\theta,w(t,\theta,v))}{\phi^i(t,\theta,w(t,\theta,v))}\right).$$

As $\phi^i_\theta > 0$ it follows that $J(t,\theta,v) \leq v$ and hence $v \geq 0$. Consequently the second term needs to be positive as well. Clearly, $v \mapsto v$ is non-decreasing. As ϕ^i_θ/ϕ^i is decreasing in w by (4) and $w(t,\theta,v)$ is increasing in v, so the second term is increasing in v.

Part 2: $J(t,\theta,v) > 0 \Rightarrow J_\theta(t,\theta,v) \geq 0$:

It remains to prove that the virtual utility $J(t,\theta,v) = v - \frac{1-F(\theta)}{f(\theta)}\phi^i_\theta(t,\theta,w(t,\theta,v))$ is non-decreasing in θ. First, note that $\frac{1-F(\theta)}{f(\theta)}$ is decreasing in θ by assumption. Second, note that $0 = \phi^i_\theta + \phi^i_w w_\theta$ and hence

$$\begin{aligned}\frac{\partial}{\partial\theta}\phi^i_\theta(t,\theta,w(t,\theta,v)) &= \phi^i_{\theta\theta}(t,\theta,w(t,\theta,v)) + \phi^i_{\theta w}(t,\theta,w(t,\theta,v))w_\theta(t,\theta,v) \\ &= \phi^i_{\theta\theta}(t,\theta,w(t,\theta,v)) - \phi^i_{\theta w}(t,\theta,w(t,\theta,v))\frac{\phi^i_\theta(t,\theta,w(t,\theta,v))}{\phi^i_w(t,\theta,w(t,\theta,v))}.\end{aligned}$$

Now we replace $w(t,\theta,v)$ by w and prove that the derivative is negative for any $w \in \mathbb{R}$:

$$\begin{aligned}&= \phi^i_\theta(t,\theta,w)\left(\frac{\phi^i_{\theta\theta}(t,\theta,w)}{\phi^i_\theta(t,\theta,w)} - \frac{\phi^i_{\theta w}(t,\theta,w)}{\phi^i_w(t,\theta,w)}\right) \\ &= \phi^i_\theta(t,\theta,w)\left(\frac{\partial}{\partial\theta}\log(\phi^i_\theta(t,\theta,w)) - \frac{\partial}{\partial\theta}\log(\phi^i_w(t,\theta,w)\right) \\ &= \phi^i_\theta(t,\theta,w)\frac{\partial}{\partial\theta}\log\left(\frac{\phi^i_\theta(t,\theta,w)}{\phi^i_w(t,\theta,w)}\right) \\ &\leq 0\,.\end{aligned}$$

The last step follows as $\frac{\phi^i_\theta(t,\theta,w)}{\phi^i_w(t,\theta,w)}$ is decreasing in θ by (5), and so the logarithm is decreasing as well. □

Proof of Proposition 4. We have that

$$V(x,\widehat{x}) - p(\widehat{x}) = V(x,\widehat{x}) - V(\widehat{x},\widehat{x}) + \int_0^{\widehat{x}} V_1(z,z)\mathrm{d}z = \int_{\widehat{x}}^{x} V_1(z,\widehat{x})\mathrm{d}z + \int_0^{\widehat{x}} V_1(z,z)\mathrm{d}z$$
$$= \int_{\widehat{x}}^{x} \left(V_1(z,\widehat{x}) - V_1(z,z)\right)\mathrm{d}z + \int_0^{x} V_1(z,z)\mathrm{d}z \le \int_0^{x} V_1(z,z)\mathrm{d}z = V(x,x) - p(x)\,. \square$$

Proof of Proposition 8. We begin with the case where $\underline{v} = 0$. Note that in this case Assumption 1 and 2 are satisfied and thus Proposition 2 yields the monotonicity of the virtual valuation $J(t, v_0, v_t)$ in v_0 and v_t conditional on $J_t \ge 0$.

If $\underline{v}$ is greater than zero it follows from $f(\underline{v}) > 1/\underline{v}$ and the monotonicity of $\frac{1-F(v_0)}{f(v_0)v_0}$ that for all $v_0 \ge \underline{v}$

$$1 - \frac{1-F(v_0)}{f(v_0)v_0} > 0\,.$$

Hence, the virtual utility defined in (27) is increasing in v_t and v_0. The proof of Theorem 2 show that this is sufficient for for the existence of a payment that makes it incentive compatible to report the time zero valuation truthfully. $\square$

Proof of Proposition 10. We can explicitly calculate the time zero expected utility the agent derives from consuming the good when she reported a shock $\hat{\theta}$ if her true shock equals θ

$$\hat{V}(\theta,\hat{\theta}) = \mathbb{E}\left[\int_0^{L(\hat{\theta})} e^{-r\,t} v_t \mathrm{d}t\right] = \int_0^{L(\hat{\theta})} e^{-r\,t}\,\mathbb{E}\left[v_t\right]\mathrm{d}t = \int_0^{L(\hat{\theta})} e^{-r\,t} e^{\theta\,t} v_0 \mathrm{d}t$$
$$= v_0 \left[\frac{e^{(\theta-r)t}}{\theta - r}\right]_{t=0}^{t=L(\hat{\theta})} = v_0 \frac{e^{(\theta-r)L(\hat{\theta})} - 1}{\theta - r}\,.$$

Thus, time zero transfers that make this allocation incentive compatible are given by

$$\hat{V}(\theta,\theta) - \int_0^{\theta} \frac{\partial V}{\partial \hat{\theta}}(z,z)\mathrm{d}z = v_0 \frac{e^{(\theta-r)L(\theta)} - 1}{\theta - r} - \int_0^{\theta} \frac{e^{(z-r)L(z)}\left[L(z)(z-r) - 1\right]}{(z-r)^2}\mathrm{d}z\,.$$

If payment is made as a flow transfer on the time interval $[0, L(\theta)]$ we need to adjust it by multiplying with $r(1 - e^{-rL(\theta)})^{-1}$. $\square$

Relationship to Eső and Szentes (2007)

In Lemma 2 Eső and Szentes show that their Assumption 1 is equivalent to (in our notation)

$$\phi^i_{\theta w}(t, \theta, w) \le 0, \tag{A}$$

and their Assumption 2 is equivalent to (in our notation)

$$\frac{\phi^i_{\theta\theta}(t,\theta,w)}{\phi^i_{\theta}(t,\theta,w)} \le \frac{\phi^i_{\theta w}(t,\theta,w)}{\phi^i_{w}(t,\theta,w)}. \tag{B}$$

As

$$\frac{\partial}{\partial w}\frac{\phi^i_\theta}{\phi^i} = \frac{\phi^i_{\theta w}\phi^i - \phi^i_\theta \phi^i_w}{\phi^{i2}},$$

Assumption 1 of Eső and Szentes implies our Assumption 1 and is thus stronger. As

$$\frac{\partial}{\partial \theta}\frac{\phi^i_\theta}{\phi^i_w} = \frac{\phi^i_{\theta\theta}\phi^i_w - \phi^i_\theta \phi^i_{\theta w}}{\phi^{i2}_w} = \frac{\phi^i_\theta}{\phi^i_w}\left(\frac{\phi^i_{\theta\theta}}{\phi^i_\theta} - \frac{\phi^i_{\theta w}}{\phi^i_w}\right).$$

Hence, Assumption 2 of our setup is exactly equivalent to Assumption 2 in Eső and Szentes.

Relationship to Boleslavsky and Said (2013)

We briefly establish the relationship between the multiplicative random walk in the discrete time environment of Boleslavsky and Said (2013) and the geometric Brownian motion analyzed here. Let $(X_k)_{k\in\mathbb{N}}$ be a multiplicative random walk, i.e.

$$X_{k+1} = \begin{cases} u\,X_k, & \text{with probability } \theta, \\ d\,X_k, & \text{with probability } 1-\theta\,; \end{cases}$$

for some $d < 1 < u$ and let the uptick probability $\theta \in (0,1)$ be the private information. Boleslavsky and Said (2013) show, see page 11, equation (7), that the virtual utility in period k equals[4]

$$v^i_k\left(1 - \sum_{s\le k}\mathbf{1}_{\{X_s = dX_{s-1}\}}\frac{u-d}{d(1-\theta)}\frac{1-F^i(\theta)}{f^i(\theta)}\right).$$

In the next step we let the period length Δ go to zero. To do so let $d \equiv d^\Delta, u \equiv u^\Delta$ and $t \equiv \Delta k \in \mathbb{N}$. The virtual utility at the physical time t thus equals

$$v^i_t\left(1 - \sum_{s\le \frac{t}{\Delta}}\mathbf{1}_{\{X_s = dX_{s-1}\}}\left((\frac{u}{d})^\Delta - 1\right)\frac{1-F^i(\theta)}{f^i(\theta)(1-\theta)}\right).$$

[4]For convenience we translated their result into our notation. We use k for the period to clearly differentiate between periods and physical time.

Note that $\sum_{s\leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s=dX_{s-1}\}}$ is Binomial distributed and converges to its expectation for $\Delta \to 0$, i.e.

$$\lim_{\Delta\to 0} \sum_{s\leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s=dX_{s-1}\}} = \mathbb{E}\left[\sum_{s\leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s=dX_{s-1}\}}\right] = (1-\theta)\frac{t}{\Delta}.$$

As $\lim_{\Delta\to 0} \frac{1}{\Delta}\left((\frac{u}{d})^{\Delta} - 1\right) = 1$ we have that the virtual utility goes to:

$$\begin{aligned} v_t^i \left(1 - (1-\theta)\frac{t}{\Delta}\left((\frac{u}{d})^{\Delta} - 1\right)\frac{1-F^i(\theta)}{f^i(\theta)(1-\theta)}\right) &= v_t^i\left(1 - t\frac{1}{\Delta}\left((\frac{u}{d})^{\Delta} - 1\right)\frac{1-F^i(\theta)}{f^i(\theta)}\right) \\ &= v_t^i\left(1 - \frac{1-F^i(\theta)}{f^i(\theta)}t\right), \end{aligned}$$

which establishes the convergence to the virtual utility derived earlier in (36).

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