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Georg Nöldeke

Larry Samuelson

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By

Georg Nöldeke and Larry Samuelson

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Box 208281
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Investment and Competitive Matching*

Georg Nöldeke
Faculty of Business and Economics
University of Basel
4002 Basel, Switzerland
georg.noeldeke@unibas.ch

Larry Samuelson
Department of Economics
Yale University
New Haven, CT 06520 USA
Larry.Samuelson@yale.edu

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Abstract: We study markets in which agents first make investments and are then matched into potentially productive partnerships. Equilibrium investments and the equilibrium matching will be efficient if agents can simultaneously negotiate investments and matches, but we focus on markets in which agents must first sink their investments before matching. Additional equilibria may arise in this sunk-investment setting, even though our matching market is competitive. These equilibria exhibit inefficiencies that we can interpret as coordination failures. All allocations satisfying a constrained efficiency property are equilibria, and the converse holds if preferences satisfy a separability condition. We identify sufficient conditions (most notably, quasiconcave utilities) for the investments of matched agents to satisfy an exchange efficiency property as well as sufficient conditions (most notably, a single crossing property) for agents to be matched positive assortatively, with these conditions then forming the core of sufficient conditions for the efficiency of equilibrium allocations.

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Investment and Competitive Matching

1 Introduction

There are many markets whose participants make investments before entering. Employers create firms before hiring employees, scientists develop inventions before taking them to market, developers construct commercial buildings and homes before finding buyers, people acquire human capital before embarking on careers, and so on. The agents in these markets are typically heterogeneous, both in their underlying characteristics and their investments, and hence the market must solve a matching problem rather than simply setting a market-clearing price. Perhaps the most obvious example is the market for skilled labor, requiring years of investment on the part of workers and the marshalling of significant physical and institutional capital on the part of firms, all before it is known who will match with whom.

The outcomes agents receive in the matching market will depend on their investments and hence will affect their investment incentives. A large literature has considered the question of how imperfections in the matching market will interact with the noncooperative nature of investment choices to yield inefficient investments. For example, Acemoglu and Shimer (1999), Cole, Mailath, and Postlewaite (2001a), de Meza and Lockwood (2010), and Felli and Roberts (2012), study the hold-up problems (Grossman and Hart (1986) and Williamson (1985)) that can arise as a consequence of bargaining power at the matching stage. Bidner (2010), Cole, Mailath, and Postlewaite (1995), Hopkins (2012), Hoppe, Moldovanu, and Sela (2009) and Rege (2008) study the consequences of imperfect information at the matching stage. Burdett and Coles (2001) and Mailath, Samuelson, and Shaked (2000) study models in which it is costly to search for a partner after one has invested.

While it is obviously important to understand how imperfections in the matching market affect incentives for investment, we examine a more basic question—even in the absence of such imperfections, can we expect investments to be efficient? We accordingly work throughout this paper with an economy whose matching market is competitive, in the sense that agents treat as fixed the utilities that must be provided to potential matching partners. In particular, we study equilibria in economies in which agents first make investments and then enter the matching market, where they form pairs whose productivity depends on their underlying characteristics as well as the investments they bring to the market. The structure of the underlying production process for a matched pair may give rise to imperfectly transferable utilities (as argued by Legros and Newman (2007b) and as in Iyigun and Walsh (2007)), and so we allow utility to be imperfectly transferable within a pair. Perfectly transferable transferable utility is a special case. We identify when such economies will yield efficient outcomes and characterize the nature and causes of inefficiencies.

We first formulate a benchmark "ex ante" equilibrium concept in which agents can simultaneously choose investments and matching partners. Markets are complete in this economy, and forces analogous to those lying behind the familiar welfare theorems lead to the expected result that an allocation is an ex ante equilibrium if and only if it satisfies an appropriate (pairwise) efficiency condition. We then formulate an "ex post" equilibrium concept to capture the case in which investments must be sunk before matches are formed. We show that ex ante equilibria are also ex post equilibria, implying that efficient ex post equilibria exist whenever efficient allocations exist. The reasoning here is straightforward—the ex post setting affords agents fewer opportunities to deviate from a putative equilibrium allocation. Agents have no profitable deviations from an ex ante equilibrium allocation, and so must continue to have no profitable deviations from such an allocation in the ex post setting. Hence, sunk investments per se do not *preclude* efficiency in competitive markets.

Alas, not all ex post equilibria are efficient. The difficulty is that markets are incomplete—agents cannot simultaneously determine both investments and matches. There is no necessary link between competition and efficiency in the absence of complete markets. Which markets are available at the matching stage is determined endogenously by the agents' investment decisions. This gives rise to a coordination problem, with coordination failures leading to inefficient ex post equilibria. We formulate a "constrained efficiency" notion reflecting the more limited opportunities available to agents in the ex post setting, and show that all constrained efficient allocations are ex post equilibria, and that if the agents' preferences satisfy a separability condition, then all ex post equilibria are constrained efficient.

Ex post equilibria can be inefficient for any of three reasons. Matched agents may fail to coordinate on efficient investments, agents may have inadequate incentives to participate in the market, and agents may match with the "wrong" partners. We identify (independent) assumptions on the economy that suffice to eliminate each of these problems. First, we show that a quasiconcavity assumption on agents' utility functions suffices to ensure that the investments of matched agents are efficient. Second, an assumption that the optimal investments of unmatched agents allow them to be matched productively suffices to rule out inefficiencies stemming from too little participation in the market. Third, we examine mismatch in a model featuring unidimensional types and investments and satisfying our separability assumption. The first step is to show that if the utility frontiers satisfy a single crossing condition, then agents in an ex ante equilibrium must be positive assortatively matched. We then use separability and the constrained efficiency of ex post equilibria to show that the latter can be viewed as ex ante equilibria in an economy with restricted sets of possible investments, and hence must also be positive assortatively matched. We thus have conditions under which there can be no mismatch: every ex post equilibrium matches the agents just as does a pairwise efficient allocation. Finally, combining the assumptions that rule out each of the three sources of inefficiency gives us sufficient conditions for the Pareto efficiency of ex post equilibria.

The existing literature has considered the issues analyzed in this paper in a number of specific contexts. Cole, Mailath, and Postlewaite (2001b) were among the first to study the investment incentives generated by a competitive matching market, using an equilibrium concept akin to our ex post equilibrium. They consider a model with perfectly transferable utility, satisfying our single crossing crossing and separability conditions, obtaining a counterpart of our constrained efficiency result. They identify cases in which constrained efficiency in itself eliminates the possibility that agents coordinate on inefficient investments (which is the only source of inefficiency that may arise in their setting). Dizdar (2012) notes that the efficiency result of Cole, Mailath, and Postlewaite (2001b) can fail in the absence of a counterpart to our quasiconcavity condition and also presents examples showing that mismatch may arise in the absence of a single crossing property. Iyigun and Walsh (2007) consider a model in which consumption sharing within a match may give rise to imperfectly transferable utility, and argue that (their counterpart to) ex post equilibria are efficient. We explain in Section 2.1.4 how these and other examples fit into our framework. Our analysis unifies and extends these existing studies of investment in competitive matching markets, characterizing the nature and causes of inefficiency and identifying conditions under which equilibrium outcomes will be efficient.

Peters and Siow (2002) assume that it is *impossible* to transfer utility ex post. We find it convenient to initially exclude such nontransferable utility models. Section 5 explains how our analysis can be extended to the nontransferable case. Most of our results carry over, with one notable exception. In the absence of transfers, there is no counterpart to our result that agent with quasiconcave utility functions will necessarily coordinate on efficient investments. Perfect transferablility is thus not critical to the primary results in the literature, but it is important that the agents have at least *some* ability to make ex post utility transfers,

2 The Model

2.1 The Economy

2.1.1 The Technology

There is a set of buyers (he), with names (indexed by i) distributed on a compact subset N of a Euclidean space, and a set of sellers (she), with names (indexed by j) identically distributed.[1] In many of our examples we take N to be an interval in the real numbers. When N is infinite, we assume names are distributed according to Lebesgue measure. When N is finite, we have a model with finite, identical numbers of buyers and seller. We refer to this as the *finite case*.

There are two functions $\boldsymbol{\beta} : N \to \mathfrak{B}$ and $\boldsymbol{\sigma} : N \to \mathfrak{S}$ which map each buyer i into his type $\boldsymbol{\beta}(i) \in \mathfrak{B}$ and each seller j into her type $\boldsymbol{\sigma}(j) \in \mathfrak{S}$, where $\mathfrak{B}$ and $\mathfrak{S}$ are compact subsets of a (possibly multidimensional) Euclidean space. Each buyer chooses an investment $b \in B$ and each seller chooses an investment $s \in S$, with B and S again being compact subsets of a (possibly multidimensional) Euclidean space. Then they match, with each match pairing a single buyer with a single seller. A matched buyer and seller can make a transfer $t \in \mathbb{R}$.

We view our model as applying to a wide variety of matches or partnerships. Depending on the circumstances, the matches in the model may be interpreted as involving buyers and sellers, firms and workers, men and women, and so on. We find it convenient to work with a consistent set of terms throughout, and refer to the agents as buyers and sellers.

When a buyer of type β who chooses investment b matches with a seller of type σ who chooses investment s and the two agents agree on a transfer t, the resulting utility for the buyer is denoted by $U(b, s, \beta, \sigma, t)$ and the resulting utility for the seller by $V(s, b, \sigma, \beta, t)$.[2] It is natural to interpret the transfer t as a monetary payment from the buyer to the seller, but we might also think of t as describing the allocation of effort in a joint production process, the allocation of consumption in a marriage, or the division of joint output. While the transfer can be used to shift utility between the buyer and seller, unless explicitly mentioned, we do not assume that utility is perfectly transferable, i.e., we do not assume that U and V are linear in t.

An agent may remain unmatched. A buyer of type β who chooses investment $b \in B$ and remains unmatched receives utility $\underline{U}(b, \beta)$. A seller of type σ who chooses investment $s \in S$ and remains unmatched receives utility $\underline{V}(s, \sigma)$. We thus model unmatched agents

[1] The assumption that the sets of names for buyers and sellers are identical and have the same measure is a convenient simplification, maintained in most of the related literature. Remark 7 in Section 2.2.3 comments on the most important implication of allowing unequal measures of buyers and sellers.

[2] These utility functions incorporate a practice that we follow whenever possible, of reversing the order of arguments in pairs of functions that have comparable roles, one from the perspective of the buyer and one from the perspective of the seller.

as choosing investments from the same sets B and S as do matched agents. As a result, a matched buyer (for example) can consider a potentially productive match with an unmatched seller.

2.1.2 Assumptions

Throughout, we take "increasing" to mean "weakly increasing."

Assumption 1.

[1.1] The functions $U : B \times S \times \mathfrak{B} \times \mathfrak{S} \times \mathbb{R} \to \mathbb{R}$, $V : S \times B \times \mathfrak{S} \times \mathfrak{B} \times \mathbb{R} \to \mathbb{R}$, $\underline{U} : B \times \mathfrak{B} \to \mathbb{R}$, *and* $\underline{V} : S \times \mathfrak{S} \to \mathbb{R}$ *are continuous.*

[1.2] The function U *is strictly decreasing in* t *and for each* (b, s, β, σ) *has* $\mathbb{R}$ *as its image.*

[1.3] The function V *is strictly increasing in* t *and for each* (s, b, σ, β) *has* $\mathbb{R}$ *as its image.*

In conjunction with our compactness assumptions, Assumption 1.1 ensures that solutions exist to the maximization problems (appearing in Sections 2.2.1 and 2.2.2) defining the utility possibilities available to pairs of matched agents.

The requirement that U and V have $\mathbb{R}$ as their range in Assumptions 1.2–1.3 eliminates some special cases that we would otherwise have to explicitly address.

We say that the utility functions U and V have the *strict Pareto property* if for any (i) pair of types (β, σ), (ii) investments and transfer (b, s, t), and (ii) utility levels (u, v) satisfying

$$U(b, s, \beta, \sigma, t) \geq u, \qquad V(s, b, \sigma, \beta, t) \geq v, \tag{1}$$

with at least one strict inequality, there exists t' such that

$$U(b, s, \beta, \sigma, t') > u, \qquad V(s, b, \sigma, \beta, t') > v. \tag{2}$$

Assumption 1, in particular the requirement that U and V are continuous and strictly monotonic in t, implies the strict Pareto property.

The key role of transfers in our arguments is to ensure the strict Pareto property. As long as this property holds, we could just as well have allowed transfers to be multidimensional, in the process perhaps better accommodating interpretations that involve the allocation of effort or consumption.

2.1.3 Allocations

An allocation specifies for each buyer i a triple $(J(i), \boldsymbol{b}(i), \boldsymbol{u}(i))$ identifying the seller $J(i)$ (if any) with whom buyer i is matched and otherwise specifying that buyer i is unmatched $(J(i) = \emptyset)$, the investment $\boldsymbol{b}(i)$ chosen by buyer i, and the level of utility $\boldsymbol{u}(i)$ received by buyer i. An allocation also specifies an analogous triple $(I(j), \boldsymbol{s}(j), \boldsymbol{v}(j))$ for each seller j.

Definition 1. *An allocation is a sextuple* $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *of functions*

$$\begin{aligned}
J &: N \to N \cup \emptyset \\
I &: N \to N \cup \emptyset \\
\boldsymbol{b} &: N \to B \\
\boldsymbol{s} &: N \to S \\
\boldsymbol{u} &: N \to \mathbb{R} \\
\boldsymbol{v} &: N \to \mathbb{R}\ .
\end{aligned}$$

An allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is feasible if*

$$I(J(i)) = i \quad \forall i \in N \text{ s.t. } J(i) \neq \emptyset, \qquad J(I(j)) = j \quad \forall j \in N \text{ s.t. } I(j) \neq \emptyset, \tag{3}$$

$$I \text{ and } J \text{ are measure–preserving on } \{i \in N : J(i) \neq \emptyset\} \text{ and } \{j \in N : I(j) \neq \emptyset\}, \tag{4}$$

and, for all (i, j) *with* $J(i) = j \in N$ *(or, equivalently given* (3), $I(j) = i \in N$*), there exists* $t \in \mathbb{R}$ *such that*

$$\boldsymbol{u}(i) = U(\boldsymbol{b}(i), \boldsymbol{s}(j), \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) \tag{5}$$

$$\boldsymbol{v}(j) = V(\boldsymbol{s}(j), \boldsymbol{b}(i), \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t), \tag{6}$$

and, for all $i \in N$ *with* $J(i) = \emptyset$ *and for all* $j \in N$ *with* $I(j) = \emptyset$,

$$\boldsymbol{u}(i) = \underline{U}(\boldsymbol{b}(i), \boldsymbol{\beta}(i)) \tag{7}$$

$$\boldsymbol{v}(j) = \underline{V}(\boldsymbol{s}(j), \boldsymbol{\sigma}(j)), \tag{8}$$

and there exist measure-preserving bijections $\hat{J} : N \to N$ *and* $\hat{I} : N \to N$ *that are inverses and for which* $\hat{J}(i) = J(i)$ *whenever* $J(i) \neq \emptyset$ *and* $\hat{I}(j) = I(j)$ *whenever* $I(j) \neq \emptyset$.

Conditions (3)–(4) are the market balance conditions that matches are reciprocal and that any measurable set of buyers is matched with an equal-measure set of sellers. Conditions (5)–(6) ensure that the utility levels of matched agents are feasible given the investments and utility functions. Conditions (7)–(8) ensure that the utilities of unmatched agents are feasible. The final requirement, that there exist measure-preserving bijections $\hat{J}$ and $\hat{I}$ coinciding with J and I for matched agents, is a technical condition excluding counterintuitive constructions that arise out of the quirks of the continuum.[3] Intuitively, this final condition requires that a buyer can be unmatched only if some seller is also unmatched. We could interpret this by thinking of a first stage described by $\hat{J}$ and $\hat{I}$ in which the buyers and sellers are completely sorted into potential pairs, with some such pairs then electing to remain unmatched while the remaining matches are described by J and I. In the finite case, this requirement and condition (4) are satisfied by any pair of functions J and I satisfying (3). We return to the role played by $\hat{J}$ and $\hat{I}$ in Remark 3 of Section 2.1.5.

Given a feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ we let $M \subset N \times N$ identify the collection of matched pairs, so that $(i, j) \in M$ whenever $j = J(i)$ or (equivalently) $i = I(j)$ holds. For every matched pair (i, j), we can identify from (5)–(6) the transfer t made by this pair. We refer to the corresponding (b, s, t) as the *exchange* made by the pair (i, j). An alternative formulation would be to express an allocation in terms of the matching and exchanges, which would in turn imply utilities.

Let $\underline{u}(i) = \max_{b \in B} \underline{U}(b, \boldsymbol{\beta}(i))$ and $\underline{v}(j) = \max_{s \in S} \underline{V}(s, \boldsymbol{\sigma}(j))$ denote the *outside options* of buyers i and sellers j. Assumption 1.1 ensures that outside options are well defined. We refer to any $b \in B$ satisfying $\underline{u}(i) = \underline{U}(b, \boldsymbol{\beta}(i))$ as an *autarchy investment* of buyer i and to any $s \in S$ satisfying $\underline{v}(j) = \underline{V}(s, \boldsymbol{\sigma}(j))$ as an autarchy investment of seller j.

[3] For example, the requirement that $\hat{J}$ and $\hat{I}$ are bijections excludes the possibility that $N = [0, 1]$ and the matching is described by the identity function except for agents $\{1, 1/2, 1/3, 1/4, \ldots\}$, with buyer 1 being matched with seller 1/2, buyer 1/2 with seller 1/3, and so on. This arrangement leaves every agent matched except seller 1.

A feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is *individually rational* if it satisfies the *individual rationality* conditions

$$\boldsymbol{u}(i) \geq \underline{u}(i) \text{ for all } i \in N \text{ and } \boldsymbol{v}(j) \geq \underline{v}(j) \text{ for all } j \in N. \tag{9}$$

Throughout the following we will focus on individually rational allocations, reflecting the idea that all agents are free to remain unmatched and choose an autarchy investment. The feasible allocation that results if all agents choose to exercise this option and then receive their outside options is the *autarchy allocation.*

A feasible allocation is *fully matched* if J (and hence also I) maps onto N. In this case, $\hat{J}$ and J coincide, as do $\hat{I}$ and I. Feasible allocations $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ and $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ are *payoff equivalent* if $\boldsymbol{u}'(i) = \boldsymbol{u}(i)$ and $\boldsymbol{v}'(j) = \boldsymbol{v}(j)$ hold for all $i, j \in N$.

2.1.4 Special Cases

A special case of our model that has played a prominent role in the literature is that in which utilities depend linearly on transfers, that is, there exist functions $\tilde{U} : B \times S \times \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}$ and $\tilde{V} : S \times B \times \mathfrak{S} \times \mathfrak{B} \to \mathbb{R}$ such that for all (b, s, β, σ, t), we have

$$\begin{aligned} U(b, s, \beta, \sigma, t) &= \tilde{U}(b, s, \beta, \sigma) - t \\ V(s, b, \sigma, \beta, t) &= \tilde{V}(s, b, \sigma, \beta) + t. \end{aligned}$$

We say that utility is *perfectly transferable* in this case. The requirement in Assumptions 1.2-1.3 that U and V have all of $\mathbb{R}$ as their range is automatic when utility is perfectly transferable.

When utility is perfectly transferable, we can represent the sum of the agents' utilities by a *value function* $Z : B \times S \times \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}$ with

$$\tilde{U}(b, s, \beta, \sigma) + \tilde{V}(s, b, \sigma, \beta) = Z(b, s, \beta, \sigma).$$

If buyer i and seller j match and choose investments (b, s), then there exists a transfer such that (5)–(6) hold if and only if $\boldsymbol{u}(i) + \boldsymbol{v}(j) = Z(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j))$. The important information about preferences is contained in the value function $Z(b, s, \beta, \sigma)$, in the following sense. Fix a quadruple of utility functions $(\tilde{U}, \tilde{V}, \underline{U}, \underline{V})$ and hence the attendant value function Z. Any other quadruple $(\tilde{U}', \tilde{V}', \underline{U}, \underline{V})$ of utility functions giving rise to the same function Z also gives rise to the same sets of (ex ante and ex post, defined below) equilibria.

At the other extreme from perfectly transferable utility is a model in which t does not enter the utility functions U and V as an argument, so that there are effectively no transfers, as in Peters and Siow (2002). The utilities in a match between types (β, σ) are then completely determined by their investments (b, s). Assumptions 1.2–1.3 exclude such this case. It keeps the exposition uncluttered to postpone the consideration of such *nontransferable* utility models to Section 5. We refer to the case in which Assumption 1 holds, but utility is not perfectly transferable, as the case of *imperfectly transferable utility.*

The literature has overwhelmingly focussed on models with *separable* preferences. Intuitively, preferences are separable if agents' payoffs depend on the investments their partners have chosen, but not on the types of the partners choosing those investments. A marriage market may be separable because a man (for example) may care about the wealth with which his spouse has been endowed by her parents, but not the cost at which her parents amassed such wealth. A labor market may be nonseparable because firms are willing to hire software engineers who have invested relatively little in learning the relevant programming languages, but who nonetheless have great natural talent for programming.

We offer the general definition of separability in Section 3.2.2, noting here that a sufficient condition for separability is that preferences are *additively separable*, meaning that there are functions $\hat{f}$, $\hat{g}$, $\underline{f}$, $\underline{g}$, $\mathfrak{f}$ and $\mathfrak{g}$ such that

$$\begin{aligned} U(b,s,\beta,\sigma,t) &= \hat{f}(b,s,t)-\mathfrak{f}(b,\beta) && (10)\\ V(s,b,\sigma,\beta,t) &= \hat{g}(s,b,t)-\mathfrak{g}(s,\sigma) && (11)\\ \underline{U}(b,s) &= \underline{f}(b)-\mathfrak{f}(b,\beta) && (12)\\ \underline{V}(s,\sigma) &= \underline{g}(s)-\mathfrak{g}(s,\sigma). && (13)\end{aligned}$$

We can interpret $\hat{f}$, $\hat{g}$, $\underline{f}$, and $\underline{g}$ as return functions and $\mathfrak{f}$ and $\mathfrak{g}$ as cost-of-investment functions, so that the payoffs of agents are additively separable in returns and costs. In our examples considering additively separable preferences we typically assume that the unmatched return functions $\underline{f}(b)$ and $\underline{g}(s)$ appearing in (12)–(13) are identically equal to zero.[4]

If utility is perfectly transferable, then there exist functions $\tilde{f}$ and $\tilde{g}$ such that (10)–(11) can be written as

$$\begin{aligned} U(b,s,\beta,\sigma,t) &= \tilde{f}(b,s)-\mathfrak{f}(b,\beta)-t && (14)\\ V(s,b,\sigma,\beta,t) &= \tilde{g}(s,b)-\mathfrak{g}(s,\sigma)+t. && (15)\end{aligned}$$

Indeed, Section 3.2.2 confirms that in the case of transferable utility a representation satisfying (14)–(15) and (12)–(13) is necessary and sufficient for *separable* (rather than additively separable) preferences. When utility is perfectly transferable and preferences are separable, we can define the *surplus function* $z(b,s)=\tilde{f}(b,s)+\tilde{g}(b,s)$ and write the value function as

$$Z(b,s,\beta,\sigma)=z(b,s)-\mathfrak{f}(b,\beta)-\mathfrak{g}(s,\sigma). \tag{16}$$

Referring to z as the surplus function is particularly apt when we invoke the normalization $\underline{f}(b)=\underline{g}(s)=0$ for all b and s. Then $z(b,s)$ identifies the surplus created by entering a match with investments (b,s), relative to choosing the same investments but remaining unmatched.

Three further ways that we might specialize the model have played prominent roles in the literature and reappear in parts of our analysis.

First, it is common to restrict names, types and investments to be unidimensional, i.e., to restrict the sets N of names, $\mathfrak{B}$ and $\mathfrak{S}$ of types, and B and S of investments to be subsets of $\mathbb{R}$.

Second, much of the matching literature has followed the lead of Becker (1973) in focusing on conditions under which equilibrium matchings will be positive assortative, i.e., higher buyers are matched with higher sellers. When names are unidimensional, we say that an allocation is *positive assortative* (or satisfies *positive assortative matching*) if J (or, equivalently I) is the identity map. Building on insights from Legros and Newman (2007b), Section 4.3 identifies an appropriate single crossing property that ensures positive assortative matching in ex post equilibria. When utility is perfectly transferable, the assumption that the function $Z(b,s,\beta,\sigma)$ is supermodular plays an important role in ensuring this single crossing

[4] Doing so is without loss of generality as we may redefine the cost functions $\mathfrak{f}(b,\beta)$ and $\mathfrak{g}(s,\sigma)$ to coincide with $\underline{U}(b,s)$ and $\underline{V}(s,\sigma)$ and then redefine the return functions for matched agents by deducting $\underline{f}(b)$ from $\hat{f}(b,s,t)$ and $\underline{g}(s)$ from $\hat{g}(b,s,t)$ to obtain an equivalent model.

property. If we also have separability, then (because the sum of supermodular functions is supermodular) the value function Z will be supermodular if the functions z, $-\mathfrak{f}$, and $-\mathfrak{g}$ appearing in (16) are supermodular.

Third, we have taken the sets of buyers and sellers to be identical and to have equal measure, ensuring the existence of fully matched allocations. Much of the attention in the literature has focussed on fully matched allocations, and we will also do so when convenient. We note that, by the definition given above, positive assortative allocations are fully matched. Remark 7 in Section 2.2.3 comments on the most important implication of allowing unequal measures of buyers and sellers. Section 4.2 gives conditions under which all equilibria are (payoff equivalent to) fully matched allocations.

We can now indicate how several existing models fit into our framework:

1. Cole, Mailath, and Postlewaite (2001b) work with unidimensional types and investments, perfectly transferable utility, additively separable preferences, a supermodular value function $Z(b, s, \beta, \sigma)$ whose form is given by (16), and fully matched equilibria. Very similar assumptions on preferences are maintained in Cole, Mailath, and Postlewaite (2001a) and Felli and Roberts (2012), who study the finite case (without assuming matching to be competitive).

   Cole, Mailath, and Postlewaite (2001b) introduce the concepts of ex ante contracting and ex post contracting equilibria, differing in technical details but analogous to our ex ante and ex post equilibria (Sections 2.2.1 and 2.2.2 below). They show that in their setting, ex ante contracting equilibria exist, that ex ante contracting equilibria are efficient and are also ex post contracting equilibria, and that inefficient "coordination failure" ex post contracting equilibria also exist. They obtain a counterpart of to our constrained efficiency result and identify cases in which constrained efficiency in itself eliminates the possibility that agents coordinate on inefficient investments

2. Peters and Siow (2002) work with unidimensional types and investments, nontransferable utility, preferences that are separable and satisfy a single crossing condition, and fully matched equilibria. They introduce the notion of a rational expectations equilibrium and show that such equilibria are efficient. We discuss this result in Section 5.

3. Dizdar (2012) works with multidimensional types and investments, while otherwise maintaining the framework from Cole, Mailath, and Postlewaite (2001b). Dizdar (2012) notes that the efficiency result of Cole, Mailath, and Postlewaite (2001b) can fail in the absence of a counterpart to our quasiconcavity condition and offers a sufficient condition for matched agents to avoid coordination failures in investments, which we discuss in Section 4.1. He shows that when investments and types are multidimensional, there exist ex post equilibria featuring a different matching than the one obtained in ex ante equilibrium, an impossibility in Cole, Mailath, and Postlewaite (2001b). We discuss this result in Appendix G.3.

4. Acemoglu (1996) works with unidimensional types and investments, perfectly transferable utility, additively separable preferences and a supermodular value function. Buyers, corresponding to firms in his model, are ex ante identical, which in our setting corresponds to the assumption that the function $\boldsymbol{\beta}$ is constant. Acemoglu (1996) defines the concept of a Walrasian equilibrium and shows that there is a unique Walrasian

equilibrium in his model. His Walrasian equilibrium is the counterpart of a collection of prices supporting a fully matched ex ante equilibrium (cf. Section 3.1.5 below), and is efficient. Acemoglu (1996, footnote 7) mentions the issue which is at the center of our paper, namely the incompleteness of markets when investments are chosen before markets operate, but his analysis concentrates on the implications of search and bargaining frictions that do not arise in our analysis (or the other papers cited here).

5. Iyigun and Walsh (2007) work with unidimensional types and investments, and fully matched equilibria. Utility functions in their model can be written as

$$\begin{aligned} U(b,s,\beta,\sigma,t) &= u_1(\beta - b) + u_2\left(f(b,s) + k - t\right) \\ V(s,b,\sigma,\beta,t) &= v_1(\sigma - s) + v_2\left(g(s,b) + k + t\right) \\ \underline{U}(b,\beta) &= u_1(\beta - b) + u_2\left(f(b,0)\right) \\ \underline{V}(s,\sigma) &= v_2(\sigma - s) + v_2\left(g(s,0)\right), \end{aligned}$$

where $k \geq 0$ is a constant. The interpretation is that agents' types correspond to their initial wealth, which they split between consumption in the first period and an investment into a technology. This technology produces second period consumption $f(b,0)$ for an unmatched buyer and $g(s,0)$ for an unmatched seller (with our buyers and sellers corresponding to men and women in Iyigun and Walsh (2007)). If a buyer and a seller match, the technology yields the amount $f(b,s) + g(s,b) + 2k$ of the second period consumption good, which the matched agents can share in any way they want.[5]

Preferences in this model are additively separable with $\mathfrak{f}(b,\beta) = -u_1(\beta - b)$, $\hat{f}(b,s,t) = u_2\left(f(b,s) + k - t\right)$, and $\underline{f}(b) = u_2\left(f(b,0)\right)$ for the buyers and an analogous specification for the sellers. If the functions u_2 and v_2 were linear, this would be a model with perfectly transferable utility, but instead the functions u_1, u_2, v_1, v_2 are all assumed to be strictly concave, resulting in a model with imperfectly transferable utility. Section 4 discusses conditions under which equilibria in such a model feature positive assortative matching and are efficient, extending corresponding results in Iyigun and Walsh (2007).

6. Han (2002) works with unidimensional types and investments and perfectly transferable utility, but with preferences that are not separable, developing conditions under which matching must be positive assortative. We return to this model after giving a precise definition of separability in Section 3.2.2.

2.1.5 Modeling Competitive Matching

Here we offer some remarks on the modeling choices we have made and relate them to alternatives that have been pursued in the literature.

Remark 1. We have followed the practice, common since Aumann (1964) when studying continuum economies, of starting with a measure space of agents whose names have no direct economic implications, and then relying on functions such as $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ to assign the relevant economic characteristics to agents. Doing so simplifies many of our definitions and allows us

[5] The model in Iyigun and Walsh (2007) does not satisfy our Assumption 1 because they assume that first and second period consumption must be positive for all agents, implying that investments are restricted to $b \in [0,\beta]$ and $s \in [0,\sigma]$ and, similarly, that transfers are restricted to the interval $[-g(s,b), f(b,s)]$. We could write a more general (though more tedious) version of Assumption 1 that would accommodate this setting.

to avoid making assumptions on the functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$, assigning types to agents, throughout much of our analysis. ■

Remark 2. Cole, Mailath, and Postlewaite (2001b), Mailath, Postlewaite, and Samuelson (2013a,b), and Peters and Siow (2002) describe their market as matching buyer investments with seller investments, rather than matching buyers with sellers. Our counterpart of this, given that we do not assume preferences are separable, would be to describe the market as matching pairs (b, β) with pairs (s, σ).

The difficulty in working with such a formulation is that with an infinite number of agents we cannot be sure that the sets of investments chosen by buyers and sellers are well behaved, raising questions as to how one should define an equilibrium. Cole, Mailath, and Postlewaite (2001b) solve this problem by focusing attention on strictly increasing equilibrium investment functions $\boldsymbol{b}$ and $\boldsymbol{s}$ and defining the payoffs from agents' matches in terms of the limits of payoffs obtained from converging sequences of nearby matches. Mailath, Postlewaite, and Samuelson (2013a,b) again place conditions on equilibrium functions $\boldsymbol{b}$ and $\boldsymbol{s}$, and define the matching function on the closure of the set of investments chosen by the agents. Peters and Siow (2002) require their matching function to hold only almost everywhere. ■

Remark 3. We have formulated our feasibility condition as a collection of pointwise requirements—feasibility places restrictions on the match of *every* agent and on the utility of *every* agent—rather than defining feasibility in terms of conditions that are required to hold only for almost all agents. These two approaches to feasibility coincide when the number of agents is finite, and we view it appropriate to use whichever most conveniently addresses the questions of interest. Our formulation allows us to avoid a host of measure-theoretic technicalities. In particular, having excluded some perverse cases (cf. footnote 3) by building $\hat{J}$ and $\hat{I}$ into the definition of a feasible allocation, we find that our formulation significantly simplifies many of the arguments. In particular, we use the existence of $\hat{J}$ and $\hat{I}$ in establishing sufficient for the Pareto efficiency of ex post equilibria in Proposition 8. ■

Remark 4. An alternative approach to defining a feasible allocation is to dispense with the functions I and J, specifying who is matched with whom, and instead to characterize the matching in terms of a measure on the product space of buyers and sellers. Dizdar (2012) follows this approach and Cole, Mailath, and Postlewaite (2001b, Appendix B) also consider this possibility. When utility is perfectly transferable this approach makes powerful techniques from the optimal transport literature available, but it is less obviously useful in when utility is imperfectly transferable. Section 6.1 provides further discussion. ■

2.2 Equilibrium

In this section we define two equilibrium notions, ex ante equilibrium and ex post equilibrium. The technology is the same in either case, requiring that investments be chosen before matches become productive. The ex ante equilibrium concept is appropriate for situations in which bilateral contracts, specifying matching partners and utilities, can be determined before investment decisions are made, while the ex post equilibrium concept is appropriate when investments must be made before matches are determined. We are primarily interested in the latter, with the simpler concept of an ex ante equilibrium serving as a useful benchmark.

We view both equilibria as being the functional equivalent of a competitive equilibrium, with complete markets in the case of ex ante equilibria and incomplete markets in the case of ex post equilibria. The standard notion of a competitive equilibrium combines three features:

(i) prices, identifying the terms under which agents can trade the goods in the economy, with each agent viewing these prices as exogenously fixed, (ii) optimization, in the form of a requirement that each agent maximize utility, given the constraints imposed by the prices and (iii) market clearing, requiring that the excess demands emerging from the various agents' optimization problems balance, thus ensuring feasibility.

The counterpart of prices in the equilibrium definitions we give below is a pair of utility schedules $\boldsymbol{u}$ and $\boldsymbol{v}$, with $\boldsymbol{u}(i)$ identifying the "utility price" at which a seller can match with buyer type i and $\boldsymbol{v}(j)$ identifying the utility price at which a buyer can match with seller type j. The optimization requirement is that each buyer chooses a utility maximizing exchange and partner, given the utility possibilities presented by the schedule $\boldsymbol{v}$ (with sellers behaving similarly). Market-clearing is captured by the requirement that agents' choices yield a feasible allocation.

The key difference between ex ante equilibrium and ex post equilibrium is that in ex post equilibrium agents not only take the utility, but also the investments, of their potential partners as given when choosing a utility maximizing exchange and partner. The latter constraint is not present in ex ante equilibrium.

As we explain in Section 3.1.5, we could also think of agents facing a price schedule (rather than utility schedules), specifying what transfer t a particular buyer, who chooses investment b, must make to obtain a match with a particular seller making investment s (and vice versa). However, it clarifies the arguments, including the extension to the nontransferable-utility case, to express the equilibrium conditions in terms of utilities.

2.2.1 Ex Ante Equilibrium

To define ex ante equilibrium, we find it convenient to formalize the maximization problem faced by the agents in two stages. We describe agents as first determining their optimal exchange conditional on matching with a particular partner and providing that partner with a particular utility level, and then, given the schedule of induced utilities from matching with various partners, deciding on the optimal partner (or choosing to stay unmatched).

To make this precise, let

$$\phi(i,j,v) \quad = \max_{(b,s,t)\in B\times S\times\mathbb{R}} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t) \quad s.t. \quad V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t) \geq v \qquad (17)$$

$$\psi(j,i,u) \quad = \max_{(s,b,t)\in S\times B\times\mathbb{R}} V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t) \quad s.t. \quad U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t) \geq u. \qquad (18)$$

Hence, $\phi : N \times N \times \mathbb{R} \to \mathbb{R}$ identifies the maximum utility a buyer of type i can achieve when matched with a seller of type j to whom he must provide utility v. The function $\psi : N \times N \times \mathbb{R} \to \mathbb{R}$ has an analogous interpretation. Assumption 1 ensures that these functions are well-defined and have the properties asserted in the following lemma. The straightforward proof is in Appendix A.

Lemma 1. *Let Assumption 1 hold. Then for every* $(i,j) \in N^2$*,*

[1.1] ϕ *is strictly decreasing in* v *and* ψ *is strictly decreasing in* u*,*

[1.2] ϕ *and* ψ *are inverse:* $u = \phi(i,j,\psi(j,i,u))$ *for all* $u \in \mathbb{R}$ *and* $v = \psi(j,i,\phi(i,j,v))$ *for all* $v \in \mathbb{R}$*, and*

[1.3] ϕ *is continuous in* v *and* ψ *is continuous in* u*.*

The interpretation of Lemma 1.2 is that for a given pair of types (i,j) the functions ϕ and ψ provide two equivalent ways of describing the Pareto frontier of the set of utilities available to this pair when forming a match. The Pareto frontier is strictly decreasing (Lemma 1.1) and continuous (Lemma 1.3).

To conserve on notation, if the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is such that $\sup_{j \in N} \phi(i, j, \boldsymbol{v}(j)) \leq \underline{u}(i)$, we say that $\emptyset$ maximizes $\phi(i, j, \boldsymbol{v}(j))$ over the set $N \cup \{\emptyset\}$ and that $\max_{j \in N \cup \{\emptyset\}} \phi(i, j, \boldsymbol{v}(i)) = \underline{u}(i)$. We adopt a similar convention for ψ. This gives us a convenient way of describing the maximization problem in which buyer i maximizes his utility by either choosing a seller with whom to match or choosing to remain unmatched.

Definition 2. *An ex ante equilibrium is a feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ satisfying, for all $i \in N$ and $j \in N$,*

$$J(i) \in \operatorname*{argmax}_{j \in N \cup \{\emptyset\}} \phi(i, j, \boldsymbol{v}(j)) \text{ and } \boldsymbol{u}(i) = \max_{j \in N \cup \{\emptyset\}} \phi(i, j, \boldsymbol{v}(j)) \tag{19}$$

$$I(j) \in \operatorname*{argmax}_{i \in N \cup \{\emptyset\}} \psi(j, i, \boldsymbol{u}(i)) \text{ and } \boldsymbol{v}(j) = \max_{i \in N \cup \{\emptyset\}} \psi(j, i, \boldsymbol{u}(i)). \tag{20}$$

Notice that one of the requirements for equilibrium is that the maxima in (19)–(20) exist.

The (utility)-price-taking feature of competitive equilibrium appears in the incentive constraints (19)–(20), where each buyer i (for example) views the function $\boldsymbol{v}$ as a constraint requiring that the match (i, j) can form only if seller j receives at least utility $\boldsymbol{v}(j)$ from the match. Cole, Mailath, and Postlewaite (2001b), Mailath, Postlewaite, and Samuelson (2013a,b), and Peters and Siow (2002) build analogous competition assumptions into their models.

The incentive conditions (19)–(20) incorporate the individual rationality conditions (9). A fully matched allocation is an ex ante equilibrium if and only if the individual rationality conditions hold and, for all $i \in N$ and $j \in N$,

$$J(i) \in \operatorname*{argmax}_{j \in N} \phi(i, j, \boldsymbol{v}(j)) \quad \text{and} \quad \boldsymbol{u}(i) = \max_{j \in N} \phi(i, j, \boldsymbol{v}(j)) \tag{21}$$

$$I(j) \in \operatorname*{argmax}_{i \in N} \psi(j, i, \boldsymbol{u}(i)) \quad \text{and} \quad \boldsymbol{v}(j) = \max_{i \in N} \psi(j, i, \boldsymbol{u}(i)). \tag{22}$$

The investment functions $\boldsymbol{b}$ and $\boldsymbol{s}$ (and the associated transfers) enter the equilibrium conditions through (19)–(20) and the requirement that the allocation be feasible. In particular, (19)–(20) imply that an ex ante equilibrium $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ satisfies

$$\boldsymbol{u}(i) = \phi(i, j, \boldsymbol{v}(j)) \qquad \text{and} \qquad \boldsymbol{v}(j) = \psi(j, i, \boldsymbol{u}(i)) \quad \forall (i, j) \in M, \tag{23}$$

so that for every matched pair (i, j), there exists a transfer t such that the equilibrium utilities $\boldsymbol{u}(i) = U(\boldsymbol{b}(i), \boldsymbol{s}(j), \boldsymbol{\beta}(i), \sigma(j), t)$ and $\boldsymbol{v}(j) = V(\boldsymbol{s}(j), \boldsymbol{b}(i), \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t)$ lie on the utility frontier defined in (17)–(18). We say that an allocation satisfying (23) is *exchange efficient* and note that exchange efficiency is a necessary condition for a feasible allocation to be an ex ante equilibrium.

More generally, we find it useful to say that (b, s, t) is *exchange efficient for the pair* (i, j) if the exchange (b, s, t) solves both of the maximization problems appearing in (17)–(18) given the utility levels induced by (b, s, t), that is,

$$U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) = \phi(i, j, V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t)) \tag{24}$$

$$V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) = \psi(j, i, U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t)). \tag{25}$$

By Lemma 1.2 the functions ϕ and ψ are inverses, so that an exchange (b, s, t) is efficient for the pair (i, j) if and only if *one* of the two conditions appearing in (24)–(25) holds. Consequently, one of these two conditions is redundant. A corresponding observation applies to the two conditions for the exchange efficiency of an allocation given in (23), and also applies to the incentive constraints (21)–(22) for a fully matched equilibrium. We will exploit these equivalences in our proofs. Such equivalence relations do not hold in the nontransferable utility case (Section 5) and for the corresponding conditions for ex post equilibria (Section 2.2.2), making it helpful later on to have stated the above pairs of conditions (rather than only one of each pair) here.

When utility is perfectly transferable, the two conditions (24)–(25) for an exchange (b, s, t) to be exchange efficient for a pair (i, j) reduces to the requirement that the pair of investments (b, s) maximize the value available to these two agents, or

$$(b, s) \in \operatorname*{argmax}_{b' \in B, s' \in S} Z(b', s', \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)). \tag{26}$$

2.2.2 Ex Post Equilibrium

When markets open after investments have been chosen—so that the ex post equilibrium notion is applicable—buyer i (for example) faces sellers who are characterized not only by a schedule $\boldsymbol{v}$ of utility levels, but also by a schedule $\boldsymbol{s}$ of investments. The equilibrium incentive constraint for buyer i is that i's equilibrium payoff be at least the payoff i could obtain by matching with any seller j, given any exchange $(b, \boldsymbol{s}(j), t)$ that gives seller j at least her equilibrium utility. Unlike the case with an ex ante equilibrium, it is irrelevant whether player i could better his equilibrium payoff by matching with seller j with an exchange (b, s, t) that preserves player j's equilibrium payoff but for which $s \neq \boldsymbol{s}(j)$. As a first step to defining ex post equilibrium, we thus let

$$\begin{aligned}
\breve{\phi}(i, j, s, v) &= \max_{(b,t) \in B \times \mathbb{R}} U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) \quad s.t. \quad V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) \geq v \\
\breve{\psi}(j, i, b, u) &= \max_{(s,t) \in S \times \mathbb{R}} V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) \quad s.t. \quad U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) \geq u.
\end{aligned}$$

Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a feasible allocation. Analogously to our convention for ex ante equilibrium, if $\sup_{j \in N} \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \leq \underline{u}(i)$, we say that $\emptyset$ maximizes $\breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j))$ and that the maximum in that case is $\underline{u}(i)$, with a similar convention for sellers.

Definition 3. *An ex post equilibrium is a feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ satisfying, for all $i \in N$ and $j \in N$,*

$$J(i) \in \operatorname*{argmax}_{j \in N \cup \{\emptyset\}} \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \text{ and } \boldsymbol{u}(i) = \max_{j \in N \cup \{\emptyset\}} \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \tag{27}$$

$$I(j) \in \operatorname*{argmax}_{i \in N \cup \{\emptyset\}} \breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)) \text{ and } \boldsymbol{v}(j) = \max_{i \in N \cup \{\emptyset\}} \breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)). \tag{28}$$

Again, one of the requirements for equilibrium is that the maxima in (27)–(28) exist.

The incentive conditions (27)–(28) imply the individual rationality conditions, which are again given by (9). As we have noted, every ex ante equilibrium satisfies the exchange efficiency condition (23). Conditions (27)–(28) imply less, namely that every matched pair $(i, j) \in M$ satisfies

$$\boldsymbol{u}(i) = \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \qquad \text{and} \qquad \boldsymbol{v}(j) = \breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)). \tag{29}$$

We refer to a feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ satisfying (29) for all matched pairs as being *conditionally exchange efficient.* An exchange (b, s, t) is conditionally exchange efficient for a pair of agents $(i, j) \in M$ if it satisfies

$$U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) = \breve{\phi}(i, j, s, V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t)) \tag{30}$$

$$V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) = \breve{\psi}(j, i, b, U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t)). \tag{31}$$

Condition (30) indicates that conditional on a match between i and j, there is no possibility of increasing the buyer's utility without changing either the seller's investment or reducing her utility level. The interpretation of (31) is analogous.

In contrast to the equalities defining (unconditional) exchange efficiency, it is not the case that one of the conditions appearing in (29) is redundant. The utility possibilities created by buyer i contemplating a match with seller j are different from those created by seller j contemplating a match with buyer i, as it is j's investment $\boldsymbol{s}(j)$ that is held constant in the former instance and i's investment $\boldsymbol{b}(i)$ that is held constant in the latter. Hence, given a pair of agents (i, j) with exchange (b, s, t), it may be impossible for the buyer to choose an alternative investment b' (with the seller's investment remaining s) that (together with a suitably chosen transfer) increases the buyer's utility while meeting the seller's utility target, but at the same time it may be possible to increase the seller's utility if seller j chooses investment s' (and the buyer's investment remains b) while meeting the buyer's utility target. For similar reasons, it is not the case that one of the conditions (30)–(31) is redundant. This is most evident when utility is perfectly transferable, in which case the conditional exchange efficiency conditions (30)–(31) reduce to the requirement that both agents choose their own investment to maximize the value function Z while taking the investment of the other agent as given, that is, (b, s) satisfies

$$b \in \operatorname{argmax} Z(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)) \tag{32}$$

$$s \in \operatorname{argmax} Z(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j)), \tag{33}$$

while t is arbitrary. Dizdar (2012) has noted that these conditions can be interpreted as the requirement that (b, s) is a Nash equilibrium in the *full appropriation game* in which both i and j have the value function as the payoff function. When utility is imperfectly transferable, there is no analogous simplification. In particular, we can no longer evaluate conditional exchange efficiency of investments (b, s) independently of the transfer t.

Remark 5. Except for the requirement that unmatched agents choose an autarchy investment and receive their outside options, the specification of the utility functions $\underline{U}$ and $\underline{V}$ does not affect the ex post equilibrium conditions. Similarly, the ex ante equilibrium conditions only depend on the outside options. It thus suffices to specify the set of autarchy investments and the outside options for all agents when considering either ex ante or ex post equilibria. In our examples we typically find it convenient to proceed in this way rather than specifying the functions $\underline{U}$ and $\underline{V}$, and we typically also assume that autarchy investments are uniquely determined.

Remark 6. Problems in which agents must invest before trading are notorious for giving rise to hold-up problems (Grossman and Hart, 1986; Williamson, 1985). Felli and Roberts (2012) have studied the hold-up problem in a matching model in which agents first invest and thereafter engage in a bargaining process that prevents agents from capturing the full incremental return from a change in their investment. In contrast, the maximization problems appearing in (30) and (31) indicate that both agents in a partnership capture the incremental return from a change in their own investment. This precludes the existence of a hold-up problem in ex post equilibrium. As we see in Section 5.2 this observation does not apply in the nontransferable utility case, demonstrating that not only the competitive nature of the ex post equilibrium concept, but also (imperfect) transferability plays an important role in eliminating hold-up considerations. ■

2.2.3 Example

This section introduces an example that will serve as a building block in a number of our subsequent examples. Here we use it to illustrate the ex ante and ex post equilibrium concepts and to give a preview of Corollary 1, stating that ex ante equilibra are also ex post equilibria, while illustrating that inefficient ex post equilibria can also exist.

Example 1. Names, types, and investments are unidimensional, with $N = [0, 1]$; with $\mathfrak{B} = \mathfrak{S} = [\gamma, \gamma + \alpha]$, where $\gamma > 0$ and $\alpha > 0$; and with $B \times S = [0, \overline{b}] \times [0, \overline{s}]$, where $\overline{b}$ and $\overline{s}$ are assumed sufficiently large as not to pose constraints for the solutions of the maximization problems we consider below. Types are specified by $\boldsymbol{\beta}(i) = \gamma + \alpha i$ and $\boldsymbol{\sigma}(j) = \gamma + \alpha j$.

Utility is perfectly transferrable and preferences are additively separable with the cost functions appearing in (10)–(13) given by

$$\mathfrak{f}(b, \beta) = \frac{b^5}{5\beta} \quad \text{and} \quad \mathfrak{g}(s, \sigma) = \frac{s^5}{5\sigma}. \tag{34}$$

The return functions for unmatched agents satisfy $\underline{f}(b) = \underline{g}(s) = 0$ for all b and s, indicating that investments have no value outside a match. Autarchy investments are then zero for all agents, with resulting outside options $\underline{u}(i) = \underline{v}(j) = 0$, for all $i, j \in N$. The return functions for matched agents are given by $\hat{f}(b, s, t) = bs - t$ and $\hat{g}(s, b, t) = t - k$, where $k > 0$. The corresponding surplus and value functions are

$$z(b, s) = bs - k \quad \text{and} \quad Z(b, s, \beta, \sigma) = bs - \frac{b^5}{5\beta} - \frac{s^5}{5\sigma} - k. \tag{35}$$

As the surplus function z is supermodular and the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ are submodular, the value function Z is supermodular.

We might interpret bs as the value of a product that is purchased by the buyer, with the buyer and seller each bearing the costs of their value-enhancing investment, given by $\mathfrak{f}(b, \beta)$ and $\mathfrak{g}(s, \sigma)$. The buyer purchases the product by making a transfer t to the seller, who bears the additional cost k whenever trade occurs. With $k = 0$, this model is a special case of the model examined by Cole, Mailath, and Postlewaite (2001b, p. 338), featuring functional forms that serve as a key example in their paper.

The assumption $k > 0$ implies that the autarchy allocation is an ex post equilibrium: From (35), the highest payoff an agent can obtain from matching with an agent on the other side of the market who refrains from investing and must be provided with his or her outside option is $-k$, so that choosing autarchy is optimal. On the other hand, even though the investments $(b, s) = (0, 0)$ satisfy the conditional exchange efficiency conditions (32)–(33) for every pair (i, j), there can be no ex post equilibrium in which the agents in a matched pair choose these investments, because any such allocation violates the individual rationality constraints. The only other solution of (32)–(33) for the pair (i, j) coincides with the solution to the exchange efficiency condition (26) and is given by

$$b = (\boldsymbol{\beta}(i))^{\frac{4}{15}}(\boldsymbol{\sigma}(j))^{\frac{1}{15}} \text{ and } s = (\boldsymbol{\beta}(i))^{\frac{1}{15}}(\boldsymbol{\sigma}(j))^{\frac{4}{15}}. \tag{36}$$

If the pair of agents (i, j) is matched in an (ax ante or ex post) equilibrium it must choose these investments. In particular, every ex post equilibrium is exchange efficient.[6]

Let us consider fully matched allocations. The submodularity of the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ implies that higher types of agents will choose larger equilibrium investments, while the supermodularity of the surplus function z ensures that higher investments will be matched with higher investments. This allows us to conclude that any fully matched (ex ante or ex post) equilibrium will be positive assortative, that is, each buyer i is matched with seller $j = i$. (See Remark 9 and Corollary 5 in Section 4.3 for a formal development of this point.) From (36) we thus obtain that investments in any fully matched (ex ante or ex post) equilibrium are given by

$$\boldsymbol{b}(i) = (\boldsymbol{\beta}(i))^{\frac{1}{3}} \text{ and } \boldsymbol{s}(j) = (\boldsymbol{\sigma}(j))^{\frac{1}{3}}. \tag{37}$$

Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a positive assortative and exchange efficient allocation. Such an allocation will be an ex ante equilibrium if and only if it satisfies the individual rationality conditions (9), which here reduce to $\boldsymbol{u}(i) \geq 0$ and $\boldsymbol{v}(j) \geq 0$, and the incentive conditions (21) - (22). As the two incentive conditions are equivalent, we can focus on (21). Because $J(i) = i$, this can be rewritten as

$$\boldsymbol{u}(i) = \phi(i, i, \boldsymbol{v}(i)) = \max_{0 \leq j \leq 1} \phi(i, j, \boldsymbol{v}(j)). \tag{38}$$

Using (36) we can determine

$$\phi(i, j, v) = \frac{3}{5}(\boldsymbol{\beta}(i))^{\frac{1}{3}}(\boldsymbol{\sigma}(j))^{\frac{1}{3}} - k - v$$

and then use familiar incentive compatibility arguments to solve (38) for

$$\boldsymbol{u}(i) = \frac{3}{10}(\boldsymbol{\beta}(i))^{\frac{2}{3}} - k/2 - \theta \tag{39}$$

$$\boldsymbol{v}(j) = \frac{3}{10}(\boldsymbol{\sigma}(j))^{\frac{2}{3}} - k/2 + \theta, \tag{40}$$

where θ is a constant. Because these utility schedules are strictly increasing in names, the individual rationality condition is satisfied if and only if $\boldsymbol{u}(0) \geq 0$ and $\boldsymbol{v}(0) \geq 0$ holds. Recalling that we have defined $\boldsymbol{\beta}(0) = \boldsymbol{\sigma}(0) = \gamma$ individual rationality thus requires:

$$\frac{3}{5}\gamma^{\frac{2}{3}} + 2\theta \geq k \qquad \text{and} \qquad \frac{3}{5}\gamma^{\frac{2}{3}} - 2\theta \geq k. \tag{41}$$

In particular, a fully matched ex ante equilibrium exists if and only $\frac{3}{5}\gamma^{\frac{2}{3}} \geq k$ holds. If this inequality holds strictly, all ex ante equilibria are fully matched.[7]

[6] In the case $k = 0$ studied by Cole, Mailath, and Postlewaite (2001b, p. 338), there exists a collection of exchange inefficient ex post equilibria that are payoff equivalent to the autarchic allocation, in which some agents match but choose zero investments.

[7] In the case $\frac{3}{5}\gamma^{\frac{2}{3}} = k$ the fully matched ex ante equilibrium is unique, but there exists one additional ex ante equilibrium. This differs from the fully matched one only in that agents $i = j = 0$ choose to stay unmatched.

When is the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ with $J(i) = i$ and investments given by (37) an ex post equilibrium? Given that the allocation is positive assortative (and thus fully matched), we can rewrite the incentive conditions for ex post equilibrium in a manner analogous to (21) - (22) to obtain

$$\boldsymbol{u}(i) = \breve{\phi}(i, i, \boldsymbol{s}(i), \boldsymbol{v}(i)) = \max_{0 \le j \le 1} \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \tag{42}$$

$$\boldsymbol{v}(j) = \breve{\psi}(j, j, \boldsymbol{b}(j), \boldsymbol{u}(j)) = \max_{0 \le i \le 1} \breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)). \tag{43}$$

Solving the maximization problem embedded in the definition of the functions $\breve{\phi}$ and $\breve{\psi}$ for the investments given by (37), delivers

$$\breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) = \frac{4}{5}(\boldsymbol{\beta}(i))^{\frac{1}{4}}(\boldsymbol{\sigma}(j))^{\frac{5}{12}} - \frac{1}{5}(\boldsymbol{\sigma}(j))^{\frac{2}{3}} - k - \boldsymbol{v}(j)$$
$$\breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)) = \frac{4}{5}(\boldsymbol{\beta}(i))^{\frac{5}{12}}(\boldsymbol{\sigma}(j))^{\frac{1}{4}} - \frac{1}{5}(\boldsymbol{\beta}(i))^{\frac{2}{3}} - k - \boldsymbol{u}(j).$$

Using these expressions to solve (42)–(43) shows that these conditions are satisfied if and only if (39)–(40) hold. We can thus conclude that the set of fully matched ex post equilibria coincides with the set of fully matched ex ante equilibria. As we have noted above, there exists an additional, Pareto inefficient, ex post equilibrium, namely the autarchy allocation. ■

Remark 7. When condition (41) holds as a strict inequality a continuum of fully matched equilibria arises out of the ability to split the value $\frac{3}{5}\gamma^{\frac{2}{3}} - k > 0$ between the two bottom types in any way that respects their individual rationality conditions. This multiplicity arises from our assumption that there are equal masses of buyers and sellers. If we generalized the model to allow there to be more sellers than buyers (for example), then the shortage of buyers would push surplus toward buyers, and there would be a unique ex ante equilibrium in which θ is determined by the condition $\boldsymbol{v}(0) = 0$. We are thus dealing with a nongeneric case, but nothing in our analysis exploits this nongenericity. ■

3 Efficiency

3.1 Equilibrium and Efficiency

Are equilibria efficient? Section 3.1.2 shows that a feasible allocation is an ex ante equilibrium if and only if it is satisfies *pairwise efficiency*, a refinement of Pareto efficiency that we define in Section 3.1.1. This gives us the counterparts of the standard welfare theorems for ex ante equilibrium, as one would expect of a competitive economy with complete markets. Section 3.1.3 shows that a feasible allocation is an ex post equilibrium if and only if it satisfies *pairwise conditional efficiency.* As the name suggests, pairwise conditional efficiency is weaker than pairwise efficiency, with the difference between the two concepts reflecting the possibility of coordination failures in the choice of investments. Section 3.1.5 shows that the failure of pairwise efficiency in ex post equilibria can alternatively be interpreted as reflecting the existence of too few prices.

3.1.1 Pareto and Pairwise Efficiency

Our point of departure is a notion of Pareto efficiency, requiring that it is not possible to construct a Pareto improvement by changing the allocation for a finite set of agents.

Definition 4. *A feasible allocation* $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ *is a* finite Pareto improvement *on the feasible allocation* $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *if both allocations agree except for a finite set of agents and*

$$\begin{aligned} \boldsymbol{u}'(i) &\geq \boldsymbol{u}(i) \quad \forall i \in N \\ \boldsymbol{v}'(i) &\geq \boldsymbol{v}(j) \quad \forall j \in N, \end{aligned}$$

with a strict inequality for at least one i or j. A feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is Pareto efficient if it allows no finite Pareto improvements.*

It is immediate from this definition that Pareto efficient allocations are exchange efficient.

If the sets of buyers and sellers are finite, the restriction to allocations that differ only for finitely many agents has no effect and Pareto efficiency as defined here is the standard definition. One might consider simply applying the standard definition of Pareto efficiency—without the finiteness restriction—to cases with infinitely many buyers and sellers, but doing so can lead to counterintuitive results. Appendix B provides an example. An alternative approach to Pareto efficiency with infinite sets of agents is to follow Aumann (1964) in requiring a Pareto superior allocation to make a positive measure of agents better off. Our restriction to finite improvements plays an analogous role.[8]

There may exist Pareto efficient, individually rational allocations in which the matching differs from the matching of any ex ante equilibrium. For example, suppose that half of the buyers have high types and half have low types, with a similar division for sellers. There are no investments, utility is perfectly transferable, and outside options are zero. A match between two low agents produces a zero value, a match between a low and a high agent produces value 1, and a match between two high agents produces value 4. Then the allocation in which low buyers are matched with high sellers and high buyers with low sellers, with the value shared equally within each partnership, is Pareto efficient.[9] However, ex ante equilibrium requires that high buyers match with high sellers.

We examine the following:

Definition 5. *A feasible allocation* $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is* pairwise efficient *if it is individually rational and*

$$\boldsymbol{u}(i) \geq \phi(i, j, \boldsymbol{v}(j)) \quad \forall (i,j) \in N^2 \tag{44}$$

$$\boldsymbol{v}(j) \geq \psi(j, i, \boldsymbol{u}(i)) \quad \forall (i,j) \in N^2. \tag{45}$$

Pairwise efficiency again obviously implies exchange efficiency, and we can view pairwise efficiency as augmenting the conditions for exchange efficiency (placing restrictions on the payoffs of matched pairs of agents) with a stability requirement (imposing restrictions on the

[8] Our formulation is similar in spirit to notions examined by Kaneko and Wooders, who explore core concepts for economics with an infinite number of agents based on finite blocking coalitions (e.g., Kaneko and Wooders (1996)).

[9] The Pareto efficiency of this allocation hinges on our assumptions that only transfers within a match are feasible. If unrestricted transfers were possible, then every Pareto efficient allocation features the same matching as the ex ante equilibrium.

payoffs attainable from matching with some other agent) that is familiar from the literature on matching problems without investments (Gale and Shapley, 1962; Roth and Sotomayor, 1990).

Remark 8. Pairwise efficiency is a refinement of Pareto efficiency (with the discussion preceding Definition 5 indicating that this refinement is strict).[10] To show this, suppose that the individually rational feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is not Pareto efficient. Then there exists an alternative feasible allocation $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ with $\boldsymbol{u}'(i) \geq \boldsymbol{u}(i)$ and $\boldsymbol{v}'(j) \geq \boldsymbol{v}(j)$ for all i and j and (we can assume, with the case of a seller being analogous) a buyer i' such that $\boldsymbol{u}'(i') > \boldsymbol{u}(i')$. Because $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is individually rational, buyer i' is matched in allocation $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$. Let $j' = J'(i')$. Then we have

$$\phi(i', j', \boldsymbol{v}(j')) \geq \phi(i', j', \boldsymbol{v}'(j')) \geq \boldsymbol{u}'(i') > \boldsymbol{u}(i'),$$

where the first inequality follows from Lemma 1 and $\boldsymbol{v}'(j') \geq \boldsymbol{v}(y')$, and the second inequality holds because $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ is feasible. We thus have $\phi(i', j', \boldsymbol{v}(j')) > \boldsymbol{u}(i')$, ensuring that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ fails (44) and hence is not pairwise efficient. ■

3.1.2 Ex Ante Equilibrium and Pairwise Efficiency

The following is straightforward:

Proposition 1. *Let Assumption 1 hold. Then a feasible allocation is pairwise efficient if and only if it is an ex ante equilibrium.*

Proof. First, let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be an ex ante equilibrium. Then the second component of (19) implies (44) and the second component of (20) implies (45). As ex ante equilibrium are feasible and individually rational, it follows that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is pairwise efficient.

Conversely, let the feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be pairwise efficient. Then individual rationality holds by definition, while conditions (44)–(45) give

$$\begin{aligned} \boldsymbol{u}(i) &\geq \sup_{j \in N} \phi(i, j, \boldsymbol{v}(j)) \\ \boldsymbol{v}(j) &\geq \sup_{i \in N} \psi(j, i, \boldsymbol{u}(j)). \end{aligned}$$

Conditions (5)–(8) in the definition of feasibility ensure for that each of these inequalities either (i) the supremum is attained and the condition holds with equality, or (ii) the supremum is not attained and the agent in question is unmatched. This implies the incentive constraints (19)–(20), ensuring that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is an ex ante equilibrium. ■

3.1.3 Ex Post Equilibrium and Pairwise Conditional Efficiency

We will link ex post equilibria to the following efficiency notion.

[10] Goldman and Starr (1982) define an allocation in an exchange economy to be pairwise efficient if there are no two agents who can make themselves (weakly, with at least one strictly) better off by trading goods with one another, and show that this notion of pairwise efficiency in this context is strictly weaker than Pareto efficiency. The contrasting results reflect the fact that in an exchange economy, two agents can engage in bilateral trade without affecting the utility of other agents in the economy, whereas in our case agent i can match with agent j only if withdrawing from his current match with agent $J(i)$.

Definition 6. *A feasible allocation* $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is pairwise conditionally efficient if it is individually rational and*

$$\begin{aligned} \boldsymbol{u}(i) &\geq \breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) \quad \forall (i, j) \in N^2, & (46) \\ \boldsymbol{v}(j) &\geq \breve{\psi}(j, i, \boldsymbol{b}(i), \boldsymbol{u}(i)) \quad \forall (i, j) \in N^2. & (47) \end{aligned}$$

The modifier "conditional" captures the idea that each agent's payoff satisfies an efficiency criterion *given* the investments of the agents on the other side of the market. We can view pairwise conditional efficiency as the coupling of conditional exchange efficiency with a stability requirement. The (omitted) proof of the following is analogous to the proof of Proposition 1.

Proposition 2. *Let Assumption 1 hold. Then a feasible allocation is pairwise conditionally efficient if and only if it is an ex post equilibrium.*

Upon observing that conditions (44)–(45) in the definition of pairwise efficiency can be rewritten as

$$\begin{aligned} \boldsymbol{u}(i) &\geq \breve{\phi}(i, j, s, \boldsymbol{v}(j)) \quad \forall s \in S, \ (i, j) \in N^2, & (48) \\ \boldsymbol{v}(j) &\geq \breve{\psi}(j, i, b, \boldsymbol{u}(j)) \quad \forall b \in B, \ (i, j) \in N^2, & (49) \end{aligned}$$

it is immediate that pairwise efficiency implies pairwise conditional efficiency. Combining this observation with Propositions 1 and 2 we obtain:

Corollary 1. *Let Assumption 1 hold. Then:*
[1.1] Every pairwise efficient allocation is also pairwise conditionally efficient.
[1.2] Every ex ante equilibrium $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is also an ex post equilibrium.*

Corollary 1 gives us the counterpart of one of the welfare theorems for the relationship between pairwise efficient allocations and ex post equilibria, namely that pairwise efficient allocations are ex post equilibria. This ensures that whenever a pairwise efficient allocation exists, then a pairwise efficient ex post equilibrium also exists. Combining Corollary 1 with Remark 8, we see that incomplete markets, arising here out of the fact that investments must be chosen before matches are formed, preclude neither pairwise nor Pareto efficiency.

3.1.4 Ex Post Equilibrium Coordination Failures

Corollary 1 does not preclude the existence of pairwise inefficient equilibria. We may view such equilibia as arising from coordination failures in the choice of investments: If pairwise efficiency fails, there is a pair (i, j), perhaps matched to each other or perhaps not, and an exchange (b, s, t) that would make both i and j better off. However, realizing the increased payoffs promised by the exchange (b, s, t) requires that *both* agents choose different investments.

The interpretation of pairwise inefficiencies as coordination failures can be vividly illustrated by considering the case of one-sided investment. Suppose that S is a singleton and hence only buyers make investments (the argument similarly applies to the case in which B is a singleton). Then ϕ and $\breve{\phi}$ are identical, in the sense that for all (i, j, v), we have $\phi(i, j, v) = \breve{\phi}(i, j, s, v)$, where s is the sole element of S. Consequently, (44) and (46) are equivalent. As (44) is in turn equivalent to (45) it follows that every ex post equilibrium is an ex ante equilibrium. There is no coordination to be done in this case, and hence no coordination failures. We thus have the following:

Corollary 2. *Let Assumption 1 hold. If either B or S is a singleton, then every ex post equilibrium is pairwise efficient.*

More generally, because ex post equilibria are individually rational, an ex post equilibrium can only fail pairwise efficiency if (44)–(45) are violated, which means that there exist agents i and j and investments $b \neq \boldsymbol{b}(i)$ and $s \neq \boldsymbol{s}(j)$ that (when accompanied by an appropriate transfer t) would make both agents strictly better off when matching with each other. With ex ante contracting, there is no difficulty for the agents to coordinate on such investments, as either buyer i or seller j can contemplate the exchange (b, s, t). In the case of an ex post equilibrium, however, neither i nor j can count on the other agent to abandon their equilibrium investment choice, precluding coordination on the joint deviation.

Example 1, in which the inefficient autarchy allocation is an ex post equilibrium, has already provided an illustration of the kind of coordination failure in investment choices that can arise in ex post equilibrium. The following less degenerate example, a slight adaptation of an example from Cole, Mailath, and Postlewaite (2001b, p. 339), shows that coordination failures may lead to both underinvestment and overinvestment, in each case resulting in a fully matched ex post equilibrium that is Pareto inefficient.

Example 2. We use the specification from Example 1, but let the surplus function be given by

$$z(b,s) = \max\left\{bs - k,\ 2\sqrt{bs} - k\right\},$$

where $k > 0$. Taking the cost functions in (34) into account, the value function is

$$Z(b,s,\beta,\sigma) = \max\left\{bs - k,\ 2\sqrt{bs} - k\right\} - \frac{b^5}{5\beta} - \frac{s^5}{5\sigma}.$$

One interpretation of the above specification is that the agents have two technologies available, with the surplus given by whichever of these is the most productive. We refer to these as the high technology, with surplus function $z_1(b,s) = bs - k$ and the low technology, with surplus function $z_2(b,s) = 2\sqrt{bs} - k$.

Suppose, first that the agents have only the high technology available and let $k < \frac{3}{5}\gamma^{\frac{2}{3}}$ As we have seen in Example 1, pairwise efficiency then implies positive assortment and investments given by

$$\boldsymbol{b}_1(i) = (\boldsymbol{\beta}(i))^{\frac{1}{3}}, \qquad \boldsymbol{s}_1(j) = (\boldsymbol{\sigma}(j))^{\frac{1}{3}}. \tag{50}$$

Payoffs satisfy, for a matched pair $i = j$,

$$\boldsymbol{u}_1(i) + \boldsymbol{v}_1(j) \;=\; [\boldsymbol{\beta}(i)\boldsymbol{\sigma}(j)]^{\frac{1}{3}} - \frac{(\boldsymbol{\beta}(i))^{\frac{5}{3}}}{5\boldsymbol{\beta}(i)} - \frac{(\boldsymbol{\sigma}(j))^{\frac{5}{3}}}{5\boldsymbol{\sigma}(j)} - k \;=\; \frac{3}{5}(\boldsymbol{\beta}(i))^{\frac{2}{3}} - k > 0. \tag{51}$$

Suppose, second, that only the low technology is available and let $k < \frac{8}{5}\gamma^{\frac{1}{4}}$. As the corresponding value function is again supermodular, we can follow the same steps as in the analysis of Example 1 to obtain analogous results: pairwise efficiency calls for positive assortment with investment functions

$$\boldsymbol{b}_2(i) = (\boldsymbol{\beta}(i))^{\frac{1}{4}}, \qquad \boldsymbol{s}_2(j) = (\boldsymbol{\sigma}(j))^{\frac{1}{4}} \tag{52}$$

and payoffs satisfying, for a matched pair $i = j$,

$$\boldsymbol{u}_2(i) + \boldsymbol{v}_2(j) \;=\; 2[(\boldsymbol{\beta}(i))^{\frac{1}{4}}(\boldsymbol{\sigma}(j))^{\frac{1}{4}}]^{\frac{1}{2}} - \frac{(\boldsymbol{\beta}(i))^{\frac{5}{4}}}{5\boldsymbol{\beta}(i)} - \frac{(\boldsymbol{\sigma}(j))^{\frac{5}{4}}}{5\boldsymbol{\sigma}(j)} - k \;=\; \frac{8}{5}(\boldsymbol{\beta}(i))^{\frac{1}{4}} - k > 0. \tag{53}$$

Now suppose that both technologies are available, so that the surplus function is given by $z(b,s)$ as specified above, and assume $k < \min\{\frac{3}{5}\gamma^{\frac{2}{3}}, \frac{8}{5}\gamma^{\frac{1}{4}}\}$, implying that pairwise efficient allocations are fully matched. Appendix C.1 confirms that the surplus function z is again supermodular, ensuring the supermodularity of the value function Z and, thus, that pairwise efficient allocations are positive assortative (cf. Remark 9 in Section 4.3.2). Let $\gamma = 9$ and $\alpha = 3$, so that

$$\begin{array}{rclll} \boldsymbol{\beta}(i) & = & 9+3i, & \boldsymbol{\beta}(0)=9, & \boldsymbol{\beta}(1)=12 \\ \boldsymbol{\sigma}(j) & = & 9+3i, & \boldsymbol{\sigma}(0)=9, & \boldsymbol{\sigma}(1)=12 \end{array} .$$

We begin by constructing a pairwise efficient ex post equilibrium. Let $\beta^* \approx 10.53$ be the value of β solving

$$\frac{3}{5}\beta^{\frac{2}{3}} = \frac{8}{5}\beta^{\frac{1}{4}}.$$

Any matched pair with types $\beta = \sigma < \beta^*$ can earn higher payoffs with the low technology than with the high technology. Any matched pair with types $\beta = \sigma > \beta^*$ can earn higher payoffs with the high technology than with the low technology. In a pairwise efficient allocation any pair of matched agents (i,j) for whom $\boldsymbol{\beta}(i) = \boldsymbol{\sigma}(j) < \beta^*$ thus uses the low technology (with investments given by (52)) and any pair of matched agents (i,j) for whom $\boldsymbol{\beta}(i) = \boldsymbol{\sigma}(j) > \beta^*$ uses the high technology (with investments given by (50)). Assuming that the resulting value (as given by (51) for low pairs and (53) for high types) is split equally between the two agents in each match then gives a pairwise efficient allocation. At β^*, investments take a jump from $(\beta^*)^{\frac{1}{4}} \approx 1.80$ to $(\beta^*)^{\frac{1}{3}} \approx 2.19$. Figure 1 illustrates the investments. The equilibrium investments take a jump at β^*, but equilibrium utilities do not. Indeed, β^* is determined in order that the jump from one technology to the other causes no jump in utilities. This is a necessary condition for equilibrium, since otherwise some agents on one side of the a jump in utilities would prefer altering their actions to move to the other side (cf. Lemma 2 in Section 4.3).

Now we construct two Pareto inefficient ex post equilibria, with the first featuring underinvestment and the second featuring overinvestment. As in the pairwise efficient equilibrium constructed above, both of these equilibria are positive assortative and feature strictly increasing investment functions. Further, we continue to assume that matched pairs split the value resulting from their investments equally.

In the first of these equilibria, investments duplicate the pairwise efficient investments in an economy in which only the low technology is available. That is, $\boldsymbol{b}(i) = \boldsymbol{b}_2(i)$ and $\boldsymbol{s}(j) = \boldsymbol{s}_2(j)$ for all i and j, where the investment schedules $\boldsymbol{b}_2$ and $\boldsymbol{s}_2$ are defined in (52) and are illustrated by the lower investment schedule in Figure 1. Agents with types above β^* are investing too little in this equilibrium, in the sense that any pair of matched agents with types above β^* could both be made better off by both increasing their investments in order to exploit the high technology. Appendix C.2 confirms that despite this inefficiency the incentive constraints for an ex post equilibrium are satisfied.

In the second of these equilibria, investments duplicate the pairwise efficient investments in an economy in which only the high technology is available. That is, $\boldsymbol{b}(i) = \boldsymbol{b}_1(i)$ and $\boldsymbol{s}(j) = \boldsymbol{s}_1(j)$ for all i and j, where the investment schedules $\boldsymbol{b}_1$ and $\boldsymbol{s}_1$ are defined in (50) and illustrated by the higher investment schedule in Figure 1. Agents with types below β^* are investing too much in this equilibrium, in the sense that any pair of matched agents with types below β^* could both be made better off by both decreasing their investments in

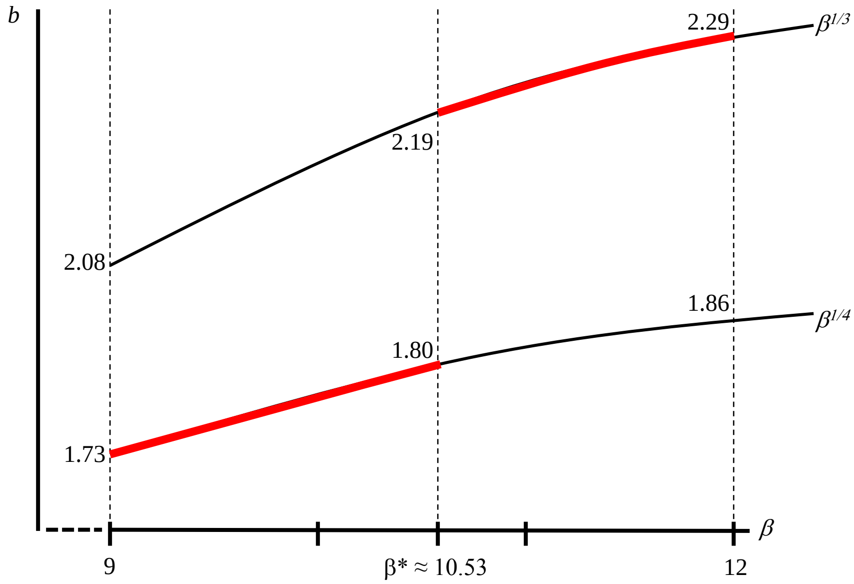

Figure 1: Illustration of the equilibrium investment functions from Example 2. The upper curve shows the buyer's investment function (with the seller's investment function being analogous) in an ex post equilibrium when only the high technology is used. The lower curve shows the buyer's investment function (with the seller's investment function again being analogous) in an ex post equilibrium when only the low technology is used. Pairwise efficient allocations use both technologies, with investments following the lower curve up to β^* and then jumping to the upper curve, as shown by the heavy (or, if in color, red) line.

order to exploit the low technology. Appendix C.3 confirms that despite this inefficiency the incentive constraints for an ex post equilibrium are satisfied.

We note that as long as $k > 0$ is not too large, any ex post equilibrium is either the autarchy allocation (which is an ex post equilibrium here, just as in Example 1) or fully matched. In particular, individual rationality implies that every pair of matched agents chooses strictly positive, conditionally efficient investments, since the strictly negative surplus that accompanies matching with zero investments otherwise would imply that at least one of the agents in the pair would rather stay unmatched. Pairwise conditional efficiency then implies that if any agents are matched, all must be, since otherwise we could find buyers (for example) arbitrarily close to one another, one unmatched with a payoff of zero and one matched with a payoff bounded away from zero, yielding a discontinuity that is precluded by pairwise conditional efficiency. ■

3.1.5 Prices

The equilibrium utility schedule $\boldsymbol{u}(i)$ can be viewed as identifying the price, in utility terms, that a seller must provide in order to match with buyer i, with a similar interpretation for $\boldsymbol{v}(j)$. This section reformulates our equilibrium notions along the lines suggested by the literature on hedonic pricing.[11] Agents face prices, specifying transfers, and an equilibrium

[11] The literature on hedonic pricing, with early contributions by Becker (1965), Houthakker (1952), Lancaster (1966) and Muth (1966) and a classic exposition by Rosen (1974), is centered around the idea that goods can

price function causes the quantity demanded for each possible match to equal the quantity supplied.This allows us to provide an alternative interpretation of the coordination failures that lie behind ex post equilibria that are not pairwise efficient, this time as a reflection of incomplete markets. For convenience, we focus on fully matched allocations.

Prices are given by a function $t(b,\beta,s,\sigma)$, with the interpretation that buyer i with investment b and type $\boldsymbol{\beta}(i)$ can buy any match $(b,\boldsymbol{\beta}(i),s,\sigma)$ by paying $t(b,\boldsymbol{\beta}(i),s,\sigma)$, with a similar provision for sellers. We make no assumptions about the sign of $t(b,\beta,s,\sigma)$.

We say that a feasible allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ can be supported by prices $t(b,\beta,s,\sigma)$ if an auctioneer or market maker could post such prices, offering to buy or sell a match to any agent at the posted price, and have the resulting optimizations on the part of the agents yield the allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$. A hedonic equilibrium is a feasible allocation that can be supported by prices, together with its supporting prices.

We must pay some attention to the domain of the price function $t(b,s,\beta,\sigma)$. Following Mailath, Postlewaite, and Samuelson (2013b, p. 547), we say that prices are *complete* if t is defined on the domain $B\times\mathfrak{B}\times S\times\mathfrak{S}$. For a given allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ we say that a function $t(b,\beta,s,\sigma)$ is a specification of *ex post* prices if the domain of this function is $(B\times\mathfrak{B}\times\mathbb{S})\cup(\mathbb{B}\times S\times\mathfrak{S})$, where

$$\begin{aligned}\mathbb{S} &= \{(s,\sigma)\in S\times\mathfrak{S} : s=\boldsymbol{s}(j),\sigma=\boldsymbol{\sigma}(j) \text{ for some } j\in N\}\\ \mathbb{B} &= \{(b,\beta)\in B\times\mathfrak{B} : b=\boldsymbol{b}(i),\beta=\boldsymbol{\beta}(i) \text{ for some } i\in N\}.\end{aligned}$$

Then $\mathbb{S}$ identifies the (s,σ) pairs that appear in the ex post market, and $\mathbb{B}$ does the same for sellers.

Definition 7. *A fully matched feasible allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ is supported by complete prices $t: B\times\mathfrak{B}\times S\times\mathfrak{S}\to\mathbb{R}$ if for all buyers $i\in N$*

$$(\boldsymbol{b}(i),\boldsymbol{s}(J(i)),\sigma(J(i))) \in \operatorname*{argmax}_{(b,s,\sigma)\in B\times S\times\mathfrak{S}} U(b,s,\boldsymbol{\beta}(i),\sigma,t(b,\boldsymbol{\beta}(i),s,\sigma)) \tag{54}$$

$$\boldsymbol{u}(i) = \max_{(b,s,\sigma)\in B\times S\times\mathfrak{S}} U(b,s,\boldsymbol{\beta}(i),\sigma,t(b,\boldsymbol{\beta}(i),s,\sigma)) \geq \underline{u}(i), \tag{55}$$

with an analogous condition holding for all sellers $j\in N$.

A fully matched feasible allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ is supported by ex post prices t : $(B\times\mathfrak{B}\times\mathbb{S})\cup(\mathbb{B}\times S\times\mathfrak{S})\to\mathbb{R}$ if, for all buyers $i\in N$,

$$(\boldsymbol{b}(i),\boldsymbol{s}(J(i)),\boldsymbol{\sigma}(J(i))) \in \operatorname*{argmax}_{(b,s,\sigma)\in B\times\mathbb{S}} U(b,s,\boldsymbol{\beta}(i),\sigma,t(b,\boldsymbol{\beta}(i),s,\sigma))$$

$$\boldsymbol{u}(i) = \max_{(b,s,\sigma)\in B\times\mathbb{S}} U(b,s,\boldsymbol{\beta}(i),\sigma,t(b,\boldsymbol{\beta}(i),s,\sigma)) \geq \underline{u}(i),$$

with an analogous condition holding for all sellers $j\in N$.

We can in general expect an allocation supported by prices to be supported by a variety of price functions. The individual rationality constraints identify the bounds placed on such functions by the option of not participating in the market.

Complete prices attach a price to every possible combination (b,β,s,σ) of investments and types. A price function defined on the restricted domain $(B\times\mathfrak{B}\times\mathbb{S})\cup(\mathbb{B}\times S\times\mathfrak{S})$ gives

be defined as bundles of attributes. Hedonic equilibria in competitive matching models with multidimensional types and perfectly transferable utility have been studied by Ekeland (2010a).

us just enough prices to evaluate the maximization problems that appear in the definition of ex post equilibrium. For example, given a candidate equilibrium $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$, a buyer of type β can consider any investment $b \in B$, but can consider matches only with seller investments s and types σ satisfying $(s, \sigma) \in \mathbb{S}$. In order to attach prices to such choices, we need prices defined on $(B \times \mathfrak{B} \times \mathbb{S}) \cup (\mathbb{B} \times S \times \mathfrak{S})$.

Proposition 3. *Let Assumption 1 hold.*

[3.1] A fully matched feasible allocation can be supported by complete prices if and only if it is a fully matched ex ante equilibrium.

[3.2] A fully matched feasible allocation can be supported by ex post prices if and only if it is a fully matched ex post equilibrium.

Proof. [3.1]. Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a fully matched ex ante equilibrium. Then for every (b, β, s, σ) with the property that there exists a buyer i' with $\boldsymbol{\beta}(i') = \beta$, we let the price $t(b, \beta, s, \sigma) = t(b, \boldsymbol{\beta}(i'), s, \sigma)$ satisfy (the existence of a solution to the following equation is implied by Assumption 1):

$$U(b, s, \boldsymbol{\beta}(i'), \sigma, t(b, \boldsymbol{\beta}(i'), s, \sigma)) = \boldsymbol{u}(i').$$

This price is well defined: if there are buyers i and i' with $\beta = \boldsymbol{\beta}(i) = \boldsymbol{\beta}(i')$, then the incentive constraints imply that in equilibrium we must have $\boldsymbol{u}(i) = \boldsymbol{u}(i')$. For those (b, β, s, σ) for which there exists no i with $\boldsymbol{\beta}(i) = \beta$, let $t(b, \beta, s, \sigma)$ satisfy

$$V(s, b, \sigma, \beta, t(b, \beta, s, \sigma)) < \underline{V}(s, \sigma).$$

The existence of such a price is again ensured by Assumption 1.

This formulation of prices ensures that every buyer i receives payoff $\boldsymbol{u}(i)$ no matter what (b, s, σ) he chooses, which in turn ensures that (54)–(55) hold. Next, every seller can choose any (s, b, β) with the property that $\beta = \boldsymbol{\beta}(i')$ for some i' at a price that gives buyer i' a utility of $\boldsymbol{u}(i')$, whereas choosing any other (s, b, β) results in less than the seller's outside option. Hence, the optimization problem faced by seller j is equivalent to

$$\max_{i \in N} \psi(j, i, \boldsymbol{u}(i)),$$

which duplicates the incentive constraint (22), ensuring that the optimal choice of seller j is $(\boldsymbol{s}(j), \boldsymbol{b}(I(j)), \boldsymbol{\beta}(I(j)))$.

Conversely, let the fully matched feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be supported by complete prices. Then (54)–(55) and the corresponding seller conditions immediately give the individual rationality constraint (9). Suppose that one of the incentive constraints (19)–(20) fails, say (19), so that there exist i' and j' with

$$\boldsymbol{u}(i') < \phi(i', j', \boldsymbol{v}(j')).$$

This implies that there exist (b', s', t') for which

$$\begin{aligned} \boldsymbol{u}(i') &< U(b', s', \boldsymbol{\beta}(i'), \boldsymbol{\sigma}(j'), t') \\ \boldsymbol{v}(j') &\leq V(s', b', \boldsymbol{\sigma}(j'), \boldsymbol{\beta}(i'), t'). \end{aligned}$$

This in turn ensures that there is no $t(b', \boldsymbol{\beta}(i'), s', \boldsymbol{\sigma}(j'))$ at which both (55) and the corresponding seller condition can be satisfied, contradicting the assumption that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is supported by complete prices.

[3.2] The proof for ex post equilibria is identical, except that $\phi(i,j,\boldsymbol{v}(j))$ is replaced by $\breve{\phi}(i,j,\boldsymbol{s}(j),\boldsymbol{v}(j))$ and $\psi(j,i,\boldsymbol{u}(i))$ is replaced by $\breve{\psi}(j,i,\boldsymbol{b}(i),\boldsymbol{u}(i))$. ■

The inability to support an ex post equilibrium which is not pairwise efficient with complete prices arises out of the fact that if a pair of agents with types (β',σ') strictly prefers an exchange (b',s',t') to what they obtain an equilibrium, then there is no price one could post that would discourage both sides of the market from trying to demand (resp. supply) (b',s',β',σ'). A seller of type σ' will be willing to choose investment s' and sell to buyer type β' with investment b' at a high price, while a buyer of type β' would like to choose b' and buy from seller σ' with investment s' at a low price. We can thus interpret a failure of pairwise efficiency in an ex post equilibrium as a problem of missing markets. Markets are "complete enough" only to ensure pairwise conditional efficiency.

We offer two interpretations of ex post prices. First, we might think of a market maker who posts prices, standing ready to trade any good at the posted price. The market maker must post prices for the goods that actually trade in equilibrium, meaning those (b,β,s,σ) for which there exists a pair $(i,j)\in M$ with $(b,\beta,s,\sigma)=(\boldsymbol{b}(i),\boldsymbol{\beta}(i),\boldsymbol{s}(j),\boldsymbol{\sigma}(j))$. We can think of a price for a good in $B\times\mathfrak{B}\times\mathbb{S}$ as the result of the market maker's standing ready to answer questions from buyers of the form, "what if I bring b to the market, am of type β and attempt to buy (b,β,s,σ)?" We can think of prices for goods in $\mathbb{B}\times S\times\mathfrak{S}$ as responses to similar enquiries from sellers. Notice, however, that a good outside the set $(B\times\mathfrak{B}\times\mathbb{S})\cup(\mathbb{B}\times S\times\mathfrak{S})$ requires a doubly counterfactual inquiry, and hence might be viewed as less likely to occur. Hence, one case of interest will be that in which prices are defined only on the set $(B\times\mathfrak{B}\times\mathbb{S})\cup(\mathbb{B}\times S\times\mathfrak{S})$, giving ex post prices.

Alternatively, we can interpret prices as a description of the terms at which trade on a decentralized market occurs. Prices attached to untraded goods would be interpreted as expectations as to what those prices would be if the corresponding goods appeared in the market. Our interpretation of ex post prices would then be that trades involving a departure from equilibrium behavior on the part of only a single player are salient enough or happen often enough in the process leading to equilibrium as to generate common price expectations, but that the same is not true for doubly counterfactual goods.

The coordination-failure and missing-prices interpretations of inefficient ex post equilibria are related. For an allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ to fail pairwise efficiency, there must be a pair (i,j) and an exchange (b,s,t) that makes both better off than under the allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$. The coordination difficulty is that buyer i can entertain exchange $(b,\boldsymbol{s}(j),t)$ and seller j can entertain $(\boldsymbol{b}(i),s,t)$, but there is no way (under the ex post equilibrium concept) for them to *coordinate* on the exchange (b,s,t). The allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ can then be an ex post equilibrium but fail pairwise efficiency if neither of the exchanges $(b,\boldsymbol{s}(j),t)$ or $(\boldsymbol{b}(i),s,t)$ can make buyer i and seller j both better off, even though (b,s,t) does so.

If both agents in the pair (i,j) would be better off making the exchange (b,s,t) then they are under the allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$, then the allocation $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ cannot be supported by any collection of prices that includes the price $t(b,\boldsymbol{\beta}(i),s,\boldsymbol{\sigma}(j))$, and hence cannot be supported by complete prices. In effect, the existence of the price $t(b,\boldsymbol{\beta}(i),s,\boldsymbol{\sigma}(j))$ solves the coordination problem for agents i and j be allowing either one of them to demand the coordinated deviation to the exchange $(b,s,t(b,\boldsymbol{\beta}(i),s,\boldsymbol{\sigma}(j)))$. However, $(J,I,\boldsymbol{b},\boldsymbol{s},\boldsymbol{u},\boldsymbol{v})$ might be supported by ex post prices, because such prices specify a price $t(b,\boldsymbol{\beta}(i),\boldsymbol{s}(j),\boldsymbol{\sigma}(j))$ and specify a price $t(\boldsymbol{b}(i),\boldsymbol{\beta}(i),s,\boldsymbol{\sigma}(j))$, but do not specify a price $t(b,\boldsymbol{\beta}(i),s,\boldsymbol{\sigma}(j))$. Because the latter price is missing from the market, the "coordinated" exchange (b, s, t) is out of the agents' reach.

3.2 Pairwise Constrained Efficiency and Separability

This section introduces a *pairwise constrained efficiency* notion, stronger than pairwise conditional efficiency but weaker than pairwise efficiency, and a property of the agents' preferences that we refer to as separability. Section 3.2.3 presents one of our main results: if preferences are separable, then every ex post equilibrium is pairwise constrained efficient. This generalizes a corresponding result of (Cole, Mailath, and Postlewaite, 2001b, Lemma 2), showing that pairwise constrained efficiency requires virtually nothing beyond separability. As we discuss in Section 3.2.4, constrained efficiency links the inefficiencies that can arise in ex post equilibrium to the (lack of) heterogeneity of equilibrium investment choices. We return to this point in Section 4.1. Pairwise constrained efficiency also plays an important role in Section 4.3, where it provides the foundation for establishing conditions under which ex post equilibria exhibit positive assortative matching.

3.2.1 Pairwise Constrained Efficiency

Pairwise efficiency and pairwise conditional efficiency both require that there be no pair of agents who could match and improve their payoffs. The notions differ in terms of the sets of investments for the agents on the other side of the market that an agent can contemplate when calculating the payoffs from a match. As indicated by condition (48), pairwise efficiency allows buyer i to consider any seller investment $s \in S$ when assessing the payoff from a match with seller j, whereas condition (46) indicates that under pairwise conditional efficiency buyer i can only consider investment $\boldsymbol{s}(j)$. Our next efficiency concept lies between those two notions. Condition (56) in the following definition requires that buyer i cannot gain by matching with seller j, given that the seller's investment must be drawn from the set of investments $\boldsymbol{S}$ that are chosen by some seller and hence are "in the market." Cole, Mailath, and Postlewaite (2001b, p. 356) refer to equilibria with this property as "efficient in a constrained sense," and so we refer to this notion as pairwise constrained efficiency.[12]

Definition 8. *A feasible allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is pairwise constrained efficient if it is individually rational and*

$$\begin{aligned} \boldsymbol{u}(i) &\geq \breve{\phi}(i, j, s, \boldsymbol{v}(j)) \quad \forall s \in \boldsymbol{S},\ (i,j) \in N^2 && (56) \\ \boldsymbol{v}(j) &\geq \breve{\psi}(j, i, b, \boldsymbol{u}(j)) \quad \forall b \in \boldsymbol{B},\ (i,j) \in N^2, && (57) \end{aligned}$$

where $\boldsymbol{B}$ is the image of N under $\boldsymbol{b}$ and $\boldsymbol{S}$ is the image of N under $\boldsymbol{s}$.

The following is immediate from the definitions:

Corollary 3.
[3.1] Pairwise efficient allocations are pairwise constrained efficient.
[3.2] Pairwise constrained efficient allocations are pairwise conditionally efficient.

[12] Felli and Roberts (2012) say that an investment is "constrained efficient" if it maximizes the value available in a match, conditional on holding fixed the identities of the agents in the match and the investment of the other agent. The counterpart of this notion in our terminology is conditional exchange efficiency.

$$
\begin{aligned}
\text{E}x\ ante\ equilibrium \quad &\Longleftrightarrow \quad \text{P}airwise\ efficiency \\
&\Longrightarrow \quad \text{P}airwise\ constrained\ efficiency \\
&\Longrightarrow \quad \text{P}airwise\ conditional\ efficiency \\
&\Longleftrightarrow \quad \text{E}x\ post\ equilibrium.
\end{aligned}
$$

Figure 2: Summary of Propositions 1–2 and Corollaries 1 and 3.

We can summarize Propositions 1–2 and Corollaries 1 and 3 in Figure 2.

Ex post equilibria can be pairwise constrained efficient without being pairwise efficient. The pairwise inefficient autarchy equilibrium in Example 1 provides an illustration. It is constrained pairwise efficient because all buyers and all sellers make identical investments, in which case pairwise conditional efficiency implies pairwise constrained efficiency. Less obvious, but implied by Proposition 4 below, is that the Pareto inefficient ex post equilibria appearing in Example 2 are also pairwise constrained efficient. These examples thus demonstrate that pairwise constrained efficiency is strictly weaker than pairwise efficiency. The following example shows that without any restriction on preferences ex post equilibria need not be pairwise constrained efficient, so that pairwise constrained efficiency lies strictly between pairwise conditional efficiency and pairwise efficiency—both converses that are not asserted by Figure 2 fail.

Example 3. Let $N = \mathfrak{B} = \mathfrak{S} = [0,1]$, let $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ be identity functions, and let $B = [0, \overline{b}]$ and $S = [0, \overline{s}]$ for sufficiently large $\overline{b}$ and $\overline{s}$. Utility is perfectly transferable. Outside options and autarchy investments are zero.

We work with the value function

$$Z(b, s, \beta, \sigma) = \min\{\beta, \sigma\} \min\{b, s, 1\}.$$

It is straightforward to confirm that this value function is supermodular in every pair of variables. Consequently, as has been the case for the value functions in our previous examples, Z is supermodular. In any pairwise efficient allocation, buyer i matches with seller $j = i$ and investments satisfy $\boldsymbol{b}(i) = \boldsymbol{s}(j) = 1$ for all $i > 0$ and $j > 0$. (The agents $i = j = 0$ receive their outside options, either by staying unmatched or by matching with each other.)

Now consider the feasible allocation in which matching is positive assortative, the value in each match is split equally between the two agents in the match, and the investments are given by

$$
\begin{aligned}
b(i) &= \begin{cases} e^i - c & i < \frac{1}{2} \\ 1 & i \geq \frac{1}{2} \end{cases} \\
s(j) &= \begin{cases} e^j - c & j < \frac{1}{2} \\ 1 & j \geq \frac{1}{2} \end{cases},
\end{aligned}
$$

where $c = e^{\frac{1}{2}} - 1 \approx .65$. Figure 3 illustrates these investment functions. Investments are strictly increasing in index for agents below 1/2. At 1/2, the investments hit their exchange efficient value of 1, and thereafter remain constant. The constant c is chosen so as to paste together the investment schedules and hence also the utility schedules for agents above and

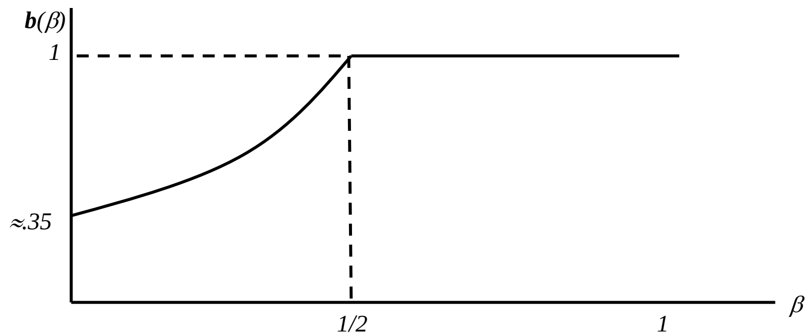

Figure 3: Illustration of the ex post equilibrium investment functions in Example 3. The solid line gives the ex post equilibrium investments, while the dotted line shows the exchange efficient investments for agents below 1/2.

below 1/2, ensuring that the latter are continuous (as they must be in ex post equilibrium, cf. Lemma 2.2 in Section 4.2.2).

Agents above 1/2 are investing as they would in a pairwise efficient allocation and receive the corresponding utilities, while those below 1/2 are investing too little and thus fail to realize the maximal possible value from their match. Moreover, the outcome is not pairwise constrained efficient, as those agents with indices less than 1/2 would do better to each choose an investment of 1, and such an investment is present on both sides of the market.

To show that the above allocation is an ex post equilibrium, we must show that no buyer $i \in [0, 1/2)$ would prefer to match with some seller $j \in (i, 1/2]$.[13] Deviating from the proposed allocation to a such a higher seller poses a trade-off. The fact that seller j has a higher investment allows buyer i to participate in the production of a higher value with j than does buyer i in i's current match. However, this value is not as large as that generated in j's current match, and so seller j must receive more than half of the value when matching with i, in order to be willing to participate. Appendix D confirms that these deviations are unprofitable. ■

3.2.2 Separability

Example 3 differs from Examples 1–2 in that it features preferences that are not separable in the sense of the following definition.

Definition 9. *Preferences are separable if there exist continuous functions* $\hat{f} : B \times S \times \mathbb{R} \to \mathbb{R}$*,*

[13]It is immediate that there are no gains from deviating to a match with a seller lower than i, and matching with seller 1/2 is equivalent to matching with any higher seller. Moreover, symmetry ensures that if no buyer has an incentive to deviate, then the same holds for sellers.

$\hat{g} : S \times B \times \mathbb{R} \to \mathbb{R}$, $\underline{f} : B \to \mathbb{R}$, $\underline{g} : S \to \mathbb{R}$, $\hat{U} : \mathbb{R} \times B \times \mathfrak{B} \to \mathbb{R}$ and $\hat{V} : \mathbb{R} \times S \times \mathfrak{S}$ such that:

$$U(b, s, \beta, \sigma, t) = \hat{U}(\hat{f}(b, s, t), b, \beta) \tag{58}$$

$$V(s, b, \sigma, \beta, t) = \hat{V}(\hat{g}(s, b, t), s, \sigma) \tag{59}$$

$$\underline{U}(b, \beta) = \hat{U}(\underline{f}(b), b, \beta) \tag{60}$$

$$\underline{V}(s, \sigma) = \hat{V}(\underline{g}(s), s, \sigma), \tag{61}$$

where $\hat{U}$ and $\hat{V}$ are strictly increasing in their first arguments.

The buyer condition (58) (for example) indicates that (i) the buyer's utility does not depend on the seller's type σ, and (ii) if we can find one buyer who prefers matching with a seller on terms (b, s', t') to matching on terms (b, s, t), then every buyer has this preference, i.e., writing buyer β's preferences as $\succsim_\beta$, we have that for all $\beta, \beta' \in \mathfrak{B}$,[14]

$$(b, s, t) \succsim_\beta (b, s', t') \iff (b, s, t) \succsim_{\beta'} (b, s', t'). \tag{62}$$

Notice that the buyer's investment is the same across these two pairs, so the important content of this property is that the buyer's trade-off between s and t does not depend on the buyer's type.

It is trivial to verify that additively separable preferences are indeed separable. Further, when utility is perfectly transferable, separability implies additive separability: the representation given by (12)–(15) is not only sufficient but also necessary for separability.[15]

We indicated in Section 2.1.4 that the existing literature on investment-and-matching problems has focussed on models with separable preferences. In particular, Iyigun and Walsh (2007) work with the special case of the additively separable preferences given by (10)–(13).[16] Cole, Mailath, and Postlewaite (2001a,b), Dizdar (2012), and Acemoglu (1996) specialize further to the case of perfectly transferable utility, working with utility functions for matched agents of the form given by (14)–(15), typically captured by a value function of the form given in (16). Preferences are also additively separable in the models with perfectly transferable utility considered in Mailath, Postlewaite, and Samuelson (2013a,b). Separability is less evident in Felli and Roberts (2012), but holds in an (equivalent) version of their model in which what they call the "quality" of an agent is interpreted as the agent's investment choice.

[14]Condition (58) obviously implies (62). To see the converse, suppose (62) holds. Then we can omit σ as an argument of U, and can choose an arbitrary $\beta^* \in \mathfrak{B}$ and define $\hat{f}(b, s, t) = U(b, s, \beta^*, t)$. Now for any triple (b, s', t'), let $y' = \hat{f}(b, s', t')$ and then define $\hat{U}(y', b, \beta) := U(b, s', \beta, t')$. To confirm that this construction is well defined, we note that if $\hat{f}(b, s', t') = \hat{f}(b, s'', t'')$, then by definition $(b, s', t') \sim_{\beta^*} (b, s'', t'')$, with (62) then ensuring that $(b, s', t') \sim_\beta (b, s'', t'')$ for any $\beta \in \mathfrak{B}$, and hence $U(b, s', \beta, t') = U(b, s'', \beta, t'')$.

[15]To see this, consider the buyers. Suppose that (58) holds and that utility is perfectly transferable. We can then omit σ as an argument of $\tilde{U}$. Choose some $s^* \in S$ and let $\tilde{U}(b, s^*, \beta) =: -\mathfrak{f}(b, \beta)$. Then choose some $\beta^* \in \mathfrak{B}$ and define $\tilde{f}(b, s) := \tilde{U}(b, s, \beta^*) - \tilde{U}(b, s^*, \beta^*)$. Using separability for the second of the following equalities we then have

$$\begin{aligned} \tilde{U}(b', s', \beta') &= \tilde{U}(b', s', \beta') - \tilde{U}(b', s^*, \beta') + \tilde{U}(b', s^*, \beta') \\ &= \tilde{U}(b', s', \beta) - \tilde{U}(b', s^*, \beta) - \mathfrak{f}(b', \beta') \\ &= \tilde{f}(b', s') - \mathfrak{f}(b', \beta'), \end{aligned}$$

yielding (14). Defining $\underline{f}(b) := \underline{U}(b, \beta^*) - \tilde{U}(b, s^*, \beta^*)$ and using an analogous argument gives (12).

[16]Preferences in Peters and Siow (2002) also satisfy (10)–(13) (with neither $\hat{f}$ nor $\hat{g}$ depending on t) and are thus separable. See Section 5 for further discussion.

Our definition of separability does not impose the additive structure appearing in (10)–(13). In contrast, Han (2002) works with perfectly transferable utility and preferences that can be written as

$$U(b,s,\beta,\sigma,t) = f^{\dagger}(b,s,\beta) - \mathfrak{f}(b) - t \tag{63}$$

$$V(s,b,\sigma,\beta,t) = \tilde{g}(s,b) - \mathfrak{g}(s,\sigma) + t. \tag{64}$$

Here separability fails because the buyer's tradeoff between b and s depends on the buyer's type.

3.2.3 Separability and Pairwise Constrained Efficiency

Separability of preferences implies the pairwise constrained efficiency of ex post equilibria. The proof of the following result also shows that for fully matched ex post equilibria, we need only the first part of the definition of separability, namely (58)–(59), to obtain this conclusion.

Proposition 4. *Let Assumption 1 hold and let preferences be separable. Then ex post equilibria are pairwise constrained efficient.*

Proof. Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be an ex post equilibrium, and suppose that it is not pairwise constrained efficient. Then there exists a pair of agents (i, j) for whom (56)–(57) fail. Suppose it is (56) that fails (with the case in which (57) fails being analogous). Then there exists a pair of investments (b, s) with $s = \boldsymbol{s}(j')$ for some j' and a transfer t such that

$$\hat{U}(\hat{f}(b,s,t), b, \boldsymbol{\beta}(i)) > \boldsymbol{u}(i) \tag{65}$$

$$\hat{V}(\hat{g}(s,b,t), s, \boldsymbol{\sigma}(j)) \geq \boldsymbol{v}(j). \tag{66}$$

Suppose first that seller j' is matched, and let i' be the buyer matched with seller j' and let their exchange be (b', s, t'). One of the possibilities buyer i can contemplate is to match with seller j', with exchange (b, s, t). Condition (65) ensures that the exchange (b, s, t) with seller j provides buyer i with more than his equilibrium utility, and so separability ensures that such a match with seller j' does likewise. The incentive constraints (27)-(28) for ex post equilibrium ensure that the exchange (b, s, t) decreases j''s utility, or

$$\hat{V}(\hat{g}(s,b,t), s, \boldsymbol{\sigma}(j')) \;<\; \boldsymbol{v}(j') \;=\; \hat{V}(\hat{g}(s,b',t'), s, \boldsymbol{\sigma}(j')). \tag{67}$$

Next, by separability, the fact that buyer i' is willing to consummate an equilibrium match featuring exchange (b', s, t') with seller j' ensures that buyer i' would also be willing to make this exchange with seller j. The incentive constraints (27)-(28) for ex post equilibrium ensure that this does not increase j's utility, or

$$\hat{V}(\hat{g}(s,b',t'), s, \boldsymbol{\sigma}(j)) \leq \boldsymbol{v}(j). \tag{68}$$

From (66) and (68), we have

$$\hat{V}(\hat{g}(s,b,t), s, \boldsymbol{\sigma}(j)) \geq \hat{V}(\hat{g}(s,b',t'), s, \boldsymbol{\sigma}(j)),$$

whereas (67) together with separability implies the reverse strict inequality. Hence, we have obtained a contradiction to the assumption that (65)–(66) hold.

Now suppose that seller j' is not matched. Then

$$\underline{V}(s, \boldsymbol{\sigma}(j')) = \boldsymbol{v}(j') \; > \; \hat{V}(\hat{g}(s,b,t), s, \boldsymbol{\sigma}(j')) \tag{69}$$

holds, where the equality is from feasibility and the strict inequality follows from separability: if it failed, buyer i and seller j' could match with exchange (b,s,t) with seller j' receiving at least her equilibrium utility $\boldsymbol{v}(j')$ and buyer i receiving more than his equilibrium utility (from (65)), contradicting the incentive constraints for ex post equilibrium. By (61), the outer inequality in (69) implies

$$\underline{V}(s, \boldsymbol{\sigma}(j)) \; > \; \hat{V}(\hat{g}(s,b,t), s, \boldsymbol{\sigma}(j)),$$

whereas (66) in conjunction with the incentive constraint $\boldsymbol{v}(j) \geq \underline{V}(s, \boldsymbol{\sigma}(j))$ implies the reverse weak inequality. This contradiction finishes the proof. ∎

Figure 4 illustrates the argument proving Proposition 4. The proposed deviation involves agents (i,j) and investments (b,s) at some transfer t. Investment s is already in the market, chosen by seller j'. Focussing on the case in which seller j' is matched, we note that (because of separability) agent i could also make himself better off if he could make the exchange (b,s,t) with agent j'. The equilibrium hypothesis is that agent i has no such opportunity to improve his payoff, meaning that engaging in the exchange (b,s,t) with buyer i must be less attractive for seller j' than j''s equilibrium exchange (b',s',t') with buyer i'. Because of separability, the identity of j''s partners in these exchange does not affect his utility, so that we can write this condition as

$$(s,b',t') \succ_{j'} (s,b,t).$$

Similarly, we can exploit separability to argue that seller j could have used investment s to buy investment b' from buyer i' at transfer t' and to infer from the fact that she chooses not to, that

$$(s,b,t) \succ_{j} (s,b',t').$$

Sellers j and j' thus rank the options (b,s,t) and (b',s,t') differently, implying that (as in our Example 3) preferences are not separable.

3.2.4 Separability, Coordination Failures and Heterogeneity

The link between separability and pairwise constrained efficiency is important for two reasons. We postpone one of these to Section 4.3, where separability and pairwise constrained efficiency play a central role in establishing conditions for positive assortative matching. This section highlights the second reason, the role of separability in limiting the scope of coordination failures.

Section 3.1.4 observed that failures of pairwise efficiency of ex post equilibria can be interpreted as coordination failures in investment choices—an ex post equilibrium can only fail pairwise efficiency if (44)–(45) are violated, which means that there exist agents i and j and investments $b \neq \boldsymbol{b}(i)$ and $s \neq \boldsymbol{s}(j)$ that (when accompanied by an appropriate transfer t) would make both agents strictly better off when matching with each other. When preferences are separable, the pairwise constrained efficiency conditions (56)–(57) imply that the only coordination failures that can arise are those in which both agents in a pair (i,j) could be made better off by choosing a pair of investments (b',s') (and an appropriate transfer t') with the property that neither b' nor s' is in the market. Formally, Proposition 4 leads immediately

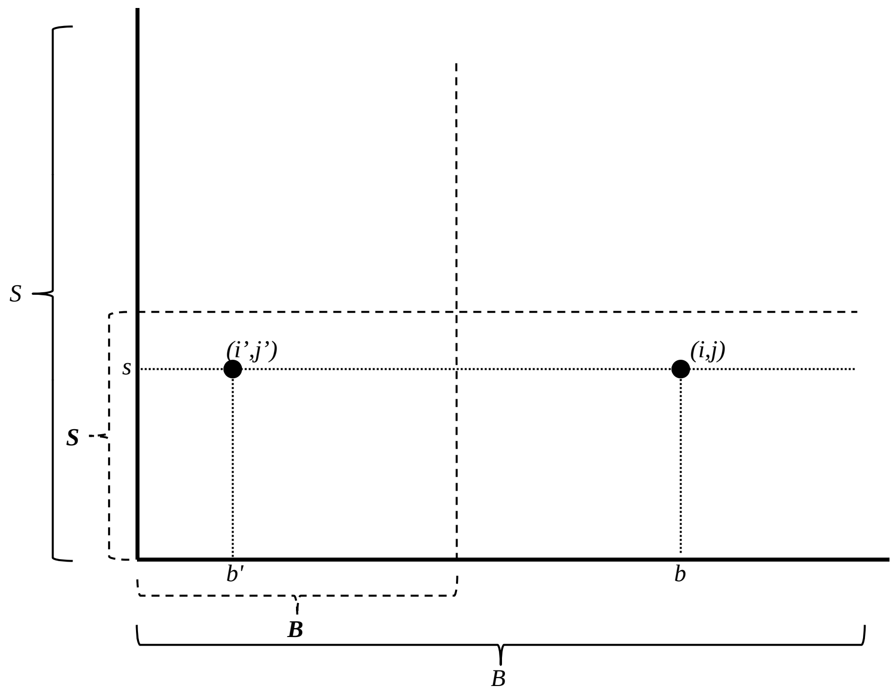

Figure 4: Illustration of the argument that ex post equilibria are pairwise constrained efficient. The horizontal axis shows the set B of possible buyer investments and the set $\boldsymbol{B}$ of investments that are in the market. The vertical axis similarly shows the set S of possible seller investments and the set $\boldsymbol{S}$ of investments that are in the market. The hypothesis is that a pair of agents (i, j) could gain by matching with one another and exchanging (b, s, t). Investment b is not contained in $\boldsymbol{B}$, but investment s is contained in $\boldsymbol{S}$, and hence is chosen by some seller j'. If j' is matched, then there is a pair (i', j') who are matched in equilibrium and execute exchange (b', s, t').

to the following result, generalizing a corresponding result for perfectly transferable utility in Cole, Mailath, and Postlewaite (2001b, Proposition 4).

Corollary 4. *Let Assumption 1 hold and let preferences be separable. Suppose* $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ *is an ex post equilibrium and there exist agents* i *and* j *and an exchange* (b', s', t') *such that*

$$\begin{aligned} U(b', s', \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t') &\geq \boldsymbol{u}(i) \\ V(s', b', \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t') &\geq \boldsymbol{v}(j), \end{aligned}$$

with at least one equality strict. Then there exists no i' *for which* $\boldsymbol{b}(i') = b'$ *and no* j' *with* $\boldsymbol{s}(j') = s'$.

The scope for coordination failures in ex post equilibrium is thus limited by the heterogeneity of investments that are actually chosen in equilibrium. The richer the sets $\boldsymbol{B}$ and $\boldsymbol{S}$

of equilibrium investments, the fewer exchanges (b', s', t') there are with $b' \notin \boldsymbol{B}$ and $s' \notin \boldsymbol{S}$, and hence the fewer opportunities for a failure of pairwise efficiency. In particular, if the sets $\boldsymbol{B}$ and $\boldsymbol{S}$ include every investment that is chosen by some agent in a pairwise efficient allocation, the ex post equilibrium in question must be pairwise efficient. In essence, it is enough to ensure the right investments are in the market, at which point the market will ensure that they are chosen by the right agents.

For example, it is immediate from Corollary 4 that with separable preferences, any ex post equilibrium satisfying $\boldsymbol{B} = B$ (or $\boldsymbol{S} = S$, with the following discussion focusing on the first of these cases) is pairwise efficient. When might this condition be satisfied? Suppose that utility is perfectly transferable. Then fully matched ex post equilibria will satisfy $\boldsymbol{B} = B$ if for every $b' \in B$ there exists a buyer i for whom choosing that investment is a dominant strategy in the full appropriation game, that is, for all $s \in S$ the investment b' is the unique investment satisfying (32). If b' is also the unique autarchy investment of buyer i or all ex post equilibria are fully matched (conditions ensuring this are discussed in Section 4.2), the pairwise efficiency of ex post equilibria follows. This dominant strategy condition is stringent, but is satisfied, for example, in Chiappori, Iyigun, and Weiss (2009).

Whether the dominant strategy condition of the previous paragraph holds can depend upon whether the agents in the economy are sufficiently heterogeneous. The following example illustrates.

Example 4. Let $N = [a, 1]$ with $0 < a < 1/6$ and $\mathfrak{B} = \mathfrak{S} = [a, 3]$. Utility is perfectly transferable and preferences are separable. Let there be two possible investments on each side of the market, so that $B = \{L, H\}$ and $S = \{L, H\}$. The return functions $\tilde{f}(b, s)$ and $\tilde{g}(s, b)$ in (14)–(15) are given by

	L	H
L	1	2
H	2	5

,

while investment costs are 0 for an L investment and $\frac{1}{\beta}$ or $\frac{1}{\sigma}$ in the case of an H investment. The return functions for unmatched agents satisfy $\underline{f}(b) = \underline{g}(s) = 0$, so that for all types outside options are zero and L is the autarchy investment.

Suppose first that $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are the identity functions, so that the sets of buyer and seller types in the economy are both $[a, 1]$. Pairwise efficiency for this economy calls for all agents with types less than 1/2 to choose L investments, and all agents with types greater than 1/2 to choose H investments. Agents who choose L match with one another, as do agents who choose H, with the matching being arbitrary within these constraints. However, there is also an ex post equilibrium in which every agent chooses L. Agents with names (and hence types) greater than 1/2 are not choosing exchange efficient investments, but the equilibrium is pairwise constrained efficient. Pairwise constrained efficiency does not imply exchange efficiency in this case because the sets of investments in the market, $\boldsymbol{B} = \boldsymbol{S} = \{L\}$, are too sparse.

Suppose now that $\boldsymbol{\beta}(i) = 2i$ and $\boldsymbol{\sigma}(j) = 2j$, and so the set of buyer and seller types in the economy are both $[2a, 2]$. As before, pairwise efficiency for this economy calls for all agents with types less than 1/2 to choose L investments, and all agents with types greater than 1/2 to choose H investments. Every ex post equilibrium must be fully matched (as the existence of an unmatched pair of agents leads to an immediate contradiction of the pairwise conditional efficiency of ex post equilibria). For buyers and sellers with names above 1/2 (and thus types above 1), the ex post exchange efficiency conditions (32)–(33) now imply

$\boldsymbol{b}(i) = \boldsymbol{s}(j) = H$, irrespectively of the partner they are matched with and the investment chosen by that partner. Similarly, for matched buyers and sellers with names below 1/6 (and thus types below 1/3), every ex post equilibrium satisfies $\boldsymbol{b}(i) = \boldsymbol{s}(j) = L$. Consequently, in every ex post equilibrium all investments are in the market, i.e., $\boldsymbol{B} = B$ and $\boldsymbol{S} = S$, ensuring that every pairwise constrained efficient allocation is pairwise efficient. Hence, Proposition 4 implies that every ex post equilibrium is pairwise efficient. ■

3.2.5 Prices

Section 3.1.5 showed that we can alternatively formulate the notions of ex ante and ex post equilibria in terms of prices attached to quadruples (b, β, s, σ) of investments and types. If preferences are separable, then we can write prices simply as a function $t(b, s)$, as do Mailath, Postlewaite, and Samuelson (2013b). To see this, suppose we have a set of (possibly complete) ex post prices $t(b, \beta, s, \sigma)$ that support an ex post equilibrium, and let preferences be separable. Suppose there exist values (b, β, s) and distinct values σ and σ' such that $t(b, \beta, s, \sigma) > t(b, \beta, s, \sigma')$. Then the good (b, β, s, σ) does not trade in equilibrium, because buyer preferences are independent of σ and hence no buyer will buy the more expensive good (b, β, s, σ) when the equivalent but cheaper good (b, β, s, σ') is available. We can thus reduce the price of (b, β, s, σ) to $t(b, \beta, s, \sigma')$ without disrupting the equilibrium. In particular, the fact that buyers still have the good (b, β, s, σ') available at the original price ensures that it is still optimal for no buyer to demand (b, β, s, σ), even at its new, lower price. In addition, no sellers were offering this good for sale at the previous price (since otherwise the market would not clear), and lowering the price of a good cannot make it more attractive for sellers to offer for sale. We can thus assume that prices take the form $t(b, \beta, s)$. An analogous argument now ensures that prices also need not depend on β.

Mailath, Postlewaite, and Samuelson (2013a,b), continuing with separable preferences, explore the circumstances under which prices in the ex post market can be written as a function of the seller's investment s only. This is not an implication of separability, and requires additional conditions.

Can prices be written as functions of just (b, s) even when preferences are not separable? The following example makes it clear that we cannot expect to do so in general.

Example 5. Let $N = \mathfrak{B} = \mathfrak{S} = B = S = [0, 1]$ and let $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ be identity functions. Let utility be perfectly transferable. Outside options are zero with autarchy investments of zero for all types. The value function is given by

$$Z(b, s, \beta, \sigma) = \beta\sigma(b - b^2)(s - s^2).$$

Then the pairwise efficient outcome calls for every buyer to choose $b = 1/2$ and every seller to choose $s = 1/2$, and for agents to match positive assortatively. However, there is no way to support such an outcome with prices of the form $t(b, s)$. ■

4 Characterization of Ex Post Equilibria

This section develops conditions under which we can refine the characterization of ex post equilibria we have obtained in Propositions 2 and 4. Our goal is to establish conditions under which ex post equilibria will be Pareto efficient, a task we complete in Section 4.4, and so we organize our discussion around three obvious sources of inefficiency.

First, exchange efficiency is a necessary condition for both pairwise efficiency and Pareto efficiency. Every ex ante equilibrium is thus exchange efficient. In contrast, Examples 2 and 4 each exhibit ex post equilibria in which some (and in some examples all) pairs of agents choose investments that are exchange inefficient, i.e., that place them strictly inside their utility frontiers. Section 4.1 establishes conditions under which an ex post equilibrium will be exchange efficient.

Second, as in the ex post equilibrium of Example 1 in which all agents choose their autarchy investments and remain unmatched, there may be too few agents participating in the market. Section 4.2 identifies conditions ensuring that every ex post equilibrium is fully matched. As demonstrated by Example 8 these conditions are stronger than the ones required to ensure that ex ante equilibria are fully matched, but they are quite straightforward. For fully matched equilibria, assuming continuity of the maps from names into types has important implications that we also record here.

Third, as we illustrate in in Examples 8 and 9 of Section 4.1.3, agents may be matched with the "wrong" partners. Section 4.3 identifies conditions under which such mismatch cannot arise. In doing so we focus on the case which has been most prominent in the literature, in which all ex ante equilibria are positive assortative (e.g., Cole, Mailath, and Postlewaite (2001b), Iyigun and Walsh (2007), and Peters and Siow (2002)), and establish conditions under which the same holds for all ex post equilibria.[17] These conditions ensure that one of the key questions addressed in the matching literature since Becker (1973), namely whether competitive matching leads to positive assortment, will continue to have a positive answer despite the potential coordination failures that may arise when investments are chosen before agents enter the matching market.

4.1 Exchange Efficiency of Ex Post Equilibria

In models with perfect transferability and separable preferences, two approaches to establishing exchange efficiency have been considered. The first, suggested by Dizdar (2012), is to seek conditions under which, for any pair of types (i, j), conditional efficiency of an exchange for that pair implies (unconditional) exchange efficiency for that pair. We develop this approach, assuming neither perfect transferability nor separability, in Section 4.1.1. Doing so requires utility functions to be quasiconcave in exchanges, but has the advantage that no assumptions about the distribution of types in the economy and no assumptions about how the preferences of various types are related to one another are needed. The second approach was first pursued by Cole, Mailath, and Postlewaite (2001b). The idea is to "leverage" the full set of pairwise conditional efficiency conditions (46)–(47) to infer exchange efficiency of ex post equilibria, even when conditional exchange efficiency for a given pair of types does not imply exchange efficiency for that pair. We comment on this approach in Section 4.1.2.

Exchange efficiency is of course only one step toward (pairwise or Pareto) efficiency. It will be little solace to learn that each pair of matched agents lies on their utility frontier if some agents, who could strictly gain from doing so, are not matched at all or if we have the "wrong" agents matched to each other. Examples 8 and 9 in Section 4.1.3 illustrate these possibilities.

[17] Positive assortment rather than negative assortment is not critical to our argument. Replacing Definition 10 appearing in Section 4.3 by the corresponding generalized condition for negative assortative matching from Legros and Newman (2007b) will give results for negative assortative matching equivalent to the ones we obtain here.

4.1.1 From Conditional Exchange Efficiency to Exchange Efficiency

Suppose we are given a pair of types (β, σ) and an exchange (b, s, t) solving

$$(b,t) \in \operatorname*{argmax}_{(b',t') \in B \times \mathbb{R}} U(b', s, \beta, \sigma, t') \text{ s.t. } V(s, b', \sigma, \beta, t') \geq V(s, b, \sigma, \beta, t), \tag{70}$$

$$(s,t) \in \operatorname*{argmax}_{(s',t') \in S \times \mathbb{R}} V(s', b, \sigma, \beta, t') \text{ s.t. } U(b, s', \beta, \sigma, t') \geq U(b, s, \beta, \sigma, t). \tag{71}$$

Does it follow that (b, s, t) also solves

$$(b,s,t) \in \operatorname*{argmax}_{(b',s',t') \in B \times S \times \mathbb{R}} U(b', s', \beta, \sigma, t') \text{ s.t. } V(s', b', \sigma, \beta, t') \geq V(s, b, \sigma, \beta, t)? \tag{72}$$

If the answer is positive for all $(\beta, \sigma) \in \mathfrak{B} \times \mathfrak{S}$, it follows from the definitions of the utility frontier functions $\breve{\phi}$, $\breve{\psi}$, and ϕ that (30)–(31) imply (24) for all $(i, j) \in N^2$. Because (24) and (25) are equivalent, it follows that the conditional exchange efficiency of an allocation implies its (unconditional) exchange efficiency, ensuring the exchange efficiency of every ex post equilibrium.

The natural approach to establishing a connection between (70)–(71) and (72) is to consider the Kuhn-Tucker conditions for the solutions to these problems. This requires differentiability assumptions which strengthen the continuity and monotonicity requirements from Assumption 1. Convexity of the set of feasible investments in conjunction with quasiconcavity of the utility functions then implies that the Kuhn-Tucker conditions developed in Arrow and Enthoven (1961) are applicable and hence that conditional exchange efficiency implies exchange efficiency.[18]

Proposition 5. *Let Assumption 1 hold. Let B and S be convex and let U and V be quasiconcave and differentiable in (b, s, t) for all $(\beta, \sigma) \in \mathfrak{B} \times \mathfrak{S}$, with the partial derivatives with respect to t satisfying $U_t < 0$ and $V_t > 0$. Then every ex post equilibrium is exchange efficient.*

Proof. As explained above, it suffices to show that (70)–(71) imply (72).

Using the strict Pareto property, we can exchange the role of the objective function and the constraint in (71) to obtain that an exchange (b, s, t) satisfies (71) if and only if and (s, t) solves

$$\max_{(s',t') \in S \times \mathbb{R}} U(b, s', \beta, \sigma, t') \text{ s.t. } V(s', b, \sigma, \beta, t') \geq V(s, b, \sigma, \boldsymbol{\beta}, t). \tag{73}$$

Using U_b, U_s, V_b, and V_s to denote the vectors of partial derivatives of the utility functions with respect to the corresponding variables, the Kuhn-Tucker-Lagrange conditions for (70) are (Arrow and Enthoven, 1961, p. 790) that there exists $\lambda \geq 0$ satisfying, for all $(b', t') \in B \times \mathbb{R}$,

$$(U_b(b, s, \beta, \sigma, t) + \lambda V_b(s, b, \sigma, \beta, t)) \cdot (b' - b) + (U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t))\,(t' - t) \leq 0. \tag{74}$$

Similarly, the Kuhn-Tucker-Lagrange conditions for (73) are that there exists $\mu \geq 0$ satisfying, for all $(s', t') \in S \times \mathbb{R}$,

$$(U_s(b, s, \beta, \sigma, t) + \mu V_s(s, b, \sigma, \beta, t)) \cdot (s' - s) + (U_t(b, s, \beta, \sigma, t) + \mu V_t(s, b, \sigma, \beta, t))\,(t' - t) \leq 0. \tag{75}$$

[18] Quasiconcavity does not depend upon the sign convention we adopt for transfers, so the assumptions that U and V are quasiconcave are symmetric, despite the fact that transfers enter these functions with different signs.

Because (i) both U and V are quasiconcave in (b, s, t), (ii) t is unconstrained, and (iii) $V_t > 0$ holds, these Kuhn-Tucker-Lagrange conditions are necessary for (70) and (73) (Arrow and Enthoven, 1961, p. 791). Further, setting $b' = b$ in (74) and $s' = s$ in (75) we obtain

$$U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t) = 0,$$
$$U_t(b, s, \beta, \sigma, t) + \mu V_t(s, b, \sigma, \beta, t) = 0.$$

Because $U_t < 0$ and $V_t > 0$ holds, these equalities imply $\mu = \lambda > 0$, so that (74) and (75) imply the existence of $\lambda \geq 0$ such that

$$\begin{aligned}(U_b(b, s, \beta, \sigma, t) + \lambda V_b(s, b, \sigma, \beta, t)) \cdot (b' - b) \; + \\ (U_s(b, s, \beta, \sigma, t) + \lambda V_s(s, b, \sigma, \beta, t)) \cdot (s' - s) \; + \\ (U_t(b, s, \beta, \sigma, t) + \lambda V_t(s, b, \sigma, \beta, t))\,(t' - t) \;\; \leq \;\; 0\end{aligned}$$

holds for all $(b', s', t') \in B \times S \times \mathbb{R}$. These are the Kuhn-Tucker-Lagrange conditions for (72). Because t is unconstrained and $U_t < 0$ holds, condition (a) in Theorem 3 from Arrow and Enthoven (1961) is satisfied and these conditions are then sufficient for (72). Hence, (b, s, t) solves (72). ■

As indicated by (30)–(31), conditional exchange efficiency for a pair may be understood as the requirement that the "conditional utility frontiers" $\breve{\phi}$ and $\breve{\psi}$ both pass through the point (u, v) in utility space induced by the exchange (b, s, t). The first part of the proof of Proposition 5 establishes that the two conditional utility frontiers must have the same slope in such a point of intersection. The second part then shows that this equal slope condition is sufficient to imply that (u, v) lies on the unconditional utility frontier. The convexity and differentiability assumptions imposed in Proposition 5 play an essential role in this argument, by ensuring that local considerations suffice to evaluate whether there is any scope to increase both agent's utilities by adjusting their exchange. Further, the fact that either agent in a match can contemplate an adjustment of the transfer when considering a change in her or his investment plays a crucial role in the proof of Proposition 5. In particular, as we will see in Section 5, there is no counterpart to Proposition 5 in the nontransferable utility case.

When utility is perfectly transferable, the question we address in this section reduces to the question of whether conditions (32)–(33) imply condition (26). Recall that (32)–(33) are the conditions for a pair of investments (b, s) to be a Nash equilibrium in the full appropriation game in which buyer i chooses $b \in B$, seller j chooses $s \in S$, and both agents have the value $Z(b, s, \boldsymbol{\beta}(i)), \boldsymbol{\sigma}(j))$ as a payoff function, whereas (26) states that (b, s) maximizes this value. Hence, in the perfectly transferable case we are asking for conditions under which all Nash equilibria of the full appropriation game solve the value maximization problem. As Dizdar (2012) has noted, any solution to the value maximization problem is a Nash equilibrium in the full appropriation game, so assuming the existence of a unique equilibrium in the full appropriation game is clearly sufficient for such a result. Proposition 5 provides a complementary result, showing that all Nash equilibria in the full appropriation game solve the value maximization problem whenever the value function is differentiable and concave in (b, s) on the convex domain $B \times S$.[19] While our approach generalizes to the

[19] Concavity and differentiability of Z implies that both $U(b, s, \beta, \sigma) = Z(b, s, \beta, \sigma) - t$ and $V(s, b, \sigma, \beta) = t$ are quasiconcave and differentiable. Applying Proposition 5 to this specification of utility functions, yields the result. The same result could be obtained by noting that the full appropriation game is a potential game with the potential $Z(b, s, \beta, \sigma)$ and applying the observation from Footnote 4 in Monderer and Shapley (1996).

case of imperfectly transferable utility, Dizdar's observation has no natural counterpart with imperfectly transferable utility as, in general, different solutions to the exchange efficiency problems (24)–(25) feature distinct investments.

A value function Z can be both supermodular in (b, s) and concave in (b, s), so that the conditions appearing in Proposition 5 are applicable in the case of a supermodular value function. When investments are unidimensional and Z twice differentiable, we simply need the supermodularity requirement $Z_{bs}(b, s, \beta, \sigma) > 0$, along with standard concavity conditions $Z_{bb}(b, s, \beta, \sigma) \leq 0$ and $Z_{bs}(b, s, \beta, \sigma)^2 \leq Z_{bb}(b, s, \beta, \sigma)Z_{ss}(b, s, \beta, \sigma)$, with the last condition ensuring that the complementarities giving rise to $Z_{bs} > 0$ are not so strong as to overwhelm the "partial concavity" of the value function in each of b and s (as they do in Examples 1 and 2).

Example 6. Let $k = 0$ in Example 1. The resulting value function $Z(b, s, \beta, \sigma) = bs - b^5/5\beta - s^5/5\sigma$ is supermodular but is not concave (it is convex in a neighborhood of the origin). Proposition 5 thus does not apply. Indeed, as we have noted in Footnote 6, for $k = 0$ Example 1 admits a fully-matched zero-investment ex post equilibrium that is not exchange efficient. We could replace $\mathfrak{f}$ and $\mathfrak{g}$ with functions that are increasing in b and s (with positive derivatives at zero) for which the value function Z would be concave. However, the zero-investment equilibrium would remain, and so Proposition 5 would then imply that zero investments are exchange efficient. ■

4.1.2 Leveraging Pairwise Conditional Efficiency

This section examines an approach to exchange efficiency, pioneered by Cole, Mailath, and Postlewaite (2001b, Section 6), that we refer to as the "leveraging approach."

Every allocation satisfying the pairwise conditional efficiency conditions (46)–(47) is conditionally exchange efficient. This imposes restrictions on the investments of any matched pair $(i, j) \in M$, and Section 4.1.1 exploited (only) these implications of pairwise conditional efficiency. However, pairwise conditional efficiency imposes incentive constraints on all pairs $(i, j) \in N^2$, including those that are not matched to each other. The leveraging approach exploits these latter restrictions by using the information that *all* matched pairs engage in conditionally efficient exchanges, to conclude that these exchanges must be (unconditionally) efficient.

The following example illustrates the basic idea of the leveraging approach.

Example 7. Consider the economy given in Example 2. There we assumed $\gamma = \boldsymbol{\beta}(0) = \boldsymbol{\sigma}(0) = 9$ and $\alpha + \gamma = \boldsymbol{\beta}(1) = \boldsymbol{\sigma}(1) = 12$. We constructed two kinds of exchange inefficient ex post equilibria. In the first kind all agents choose the low technology, even though exchange efficiency dictates that sufficiently high pairs of matched agents choose the high technology. In the second kind, all agents choose the high technology, even though exchange efficiency dictates that sufficiently low pairs of matched agents choose the low technology. All other ex post equilibria in this example were exchange efficient.

Suppose now that $\gamma = 8$ and $\alpha = 6$ holds, so that buyer and seller types are uniformly distributed on $[8, 14]$. In this case, the two kinds of exchange inefficient ex post equilibria do not exist, ensuring the exchange efficiency of all ex post equilibria. There are two steps to the argument.

First, we argue that in every ex post equilibrium other than the autarchy allocation (which is exchange efficient) there is both a matched pair of agents choosing the high technology and a matched pair of agents choosing the low technology. Appendix C calculated that for matched pairs with types $\beta = \sigma$ that exceed $\overline{\beta} \approx 13.4$, the only conditionally efficient investments compatible with the individual rationality constraints are the high-technology investments $(\beta^{\frac{1}{3}}, \sigma^{\frac{1}{3}})$. For these agents, the high technology is sufficiently lucrative that if one agent chooses the investment optimal for the low technology, the other agent will *not* find it a best response to do so. Matched agents with relatively high types must then choose their efficient investments, appropriate for the high technology. A similar argument shows that matched pairs with types $\beta = \sigma$ that fall short of $\underline{\beta} \approx 8.93$ must choose their efficient investments, appropriate for the low technology. As every ex post equilibrium other than the autarchy allocation is fully matched with positive assortative matching, our claim follows.

Second, pairwise conditional efficiency implies that the equilibrium utility schedules $\boldsymbol{u}$ and $\boldsymbol{v}$ must be continuous. Should $\boldsymbol{u}$ (for example) take a jump at some i^*, then buyers very close to but on the low-utility side of i^* would prefer to match with nearby sellers on the high side, precluding pairwise conditional efficiency. (Lemma 2 in Section 4.2.2 gives the formal argument.) Agents can then jump between the low-technology investments to the high-technology investments only at type $\beta^* \approx 10.53$, the type at which a pair of matched agents of identical type produce precisely the same maximal value under the low technology and under the high technology. This ensures that all agents with types above β^* must choose the optimal high-technology investments and agents below β^* must choose the optimal low-technology investments. Hence, no matched pair of agents can choose inefficient investments.

A similar argument shows that if $\boldsymbol{\beta}(i) = 9 + 5i$ and $\boldsymbol{\sigma}(j) = 9 + 5j$, so that the sets of types of buyers and sellers appearing in the economy are both $[9, 14]$, then there is no ex post equilibrium in which a pair of matched agents inefficiently invests in the low technology. However, in the absence of sufficiently low types from the economy, the equilibrium in which matched agents inefficiently overinvest cannot be eliminated. The reverse statement holds if the set of types is $[8, 12]$. In general, if the sets of buyers and sellers are identical intervals that include β^*, there will be equilibria in which some agents inefficiently underinvest if the interval of types does not include $\overline{\beta}$, and equilibria in which some agents inefficiently overinvest if the interval does not include $\underline{\beta}$. For the leveraging approach to work, we must have a rich enough set of types in the economy to ensure that both low-technology and high-technology investments are chosen by some pairs of matched agents. ■

The first step of the argument in Example 7 resembles the arguments from Section 4.1.1 in that matched pairs of agents are considered in isolation. It is only in the second step of the argument that the additional constraints implied by pairwise conditional efficiency, namely the continuity of the equilibrium utility schedules, are "leveraged" to obtain exchange efficiency.

Appendix E continues the discussion of the leveraging approach. Example 11 shows that when preferences are separable, pairwise constrained efficiency opens additional possibilities for leveraging pairwise conditional efficiency. In particular, the argument exploits pairwise constrained efficiency to establish the counterpart to the result from the first step in Example 7. To do so it draws inferences about the choices of agents who need not be close.[20] Similar forces appear in Section 3.2.4.

[20]The preferences in Example 7 are separable, but the arguments of Example 7 did not exploit the resulting pairwise constrained efficiency. Appendix E explains why the separability-based argument illustrated in Example 11 has no force in Example 7.

In general, the implications of pairwise constrained efficiency and hence the leveraging approach become more powerful when separability and hence pairwise constrained efficiency holds. Nevertheless, there is little hope of obtaining reasonably general conditions under which the leveraging approach can be used to infer exchange efficiency. Appendix E illustrates this with Example 12, adapted from Dizdar (2012).

4.1.3 Is Exchange Efficiency Enough?

This section presents two examples of exchange efficient ex post equilibria that are Pareto inefficient (and hence not pairwise efficient). If every pair of matched agents is on their payoff frontier and yet the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is not Pareto efficient, then the Pareto dominating allocation must feature some different matching than that specified by I and J. We explore these matching issues in two steps.

First, it may be that not enough agents are matched. In particular, every ex post equilibrium in which all agents are unmatched and choose their autarchy investments is trivially exchange efficient, but may fail to be Pareto efficient. Example 1 has already provided an example of such an ex post equilibrium. However, for each pair (i, j), the autarchy investments in Example 1 are also a pair of conditionally efficient (but exchange inefficient) investments for a match between i and j. As a result, this example has the appearance of simply having relabeled a failure of exchange efficiency to be nonparticipation. In Example 8 below, conditional efficiency implies exchange efficiency for any pair of matched agents and Pareto efficiency requires that all agents match. Nonetheless, we exhibit a Pareto (and hence pairwise) inefficient ex post equilibrium in which no agents match. Section 4.2 introduces an assumption sufficient to ensure that all ex post equilibria are fully matched.

Second, it may be that all agents are matched, but are matched with the "wrong" partners. Example 9 below satisfies Section 4.2's condition to ensure all ex post equilibria are fully matched. Nevertheless, there exists an exchange efficient ex post equilibrium which is Pareto inefficient because agents choose their investments in anticipation of matching with the wrong partner. In this example, utility is perfectly transferable, preferences are separable, and any matched pair of agents has a unique conditionally efficient pair of investments. The culprit in the failure of efficiency is the lack of structure on the value function Z, which we address in Section 4.3 by introducing the single crossing conditions in Definition 10.

Example 8. Let $N = [0, 1]$, $\mathfrak{B} = \mathfrak{S} = [3, 4]$, let $\boldsymbol{\beta}(i) = i + 3$ and $\boldsymbol{\sigma}(j) = j + 3$, and let $B = [0, \bar{b}]$ and $S = [0, \bar{s}]$ for sufficiently large $\bar{b}$ and $\bar{s}$, Utility is perfectly transferable and additively separable, with the cost functions appearing in (10)–(13) given by

$$\mathfrak{f}(b, \beta) = \frac{(b+1)^3}{3\beta} \quad \text{and} \quad \mathfrak{g}(s, \sigma) = \frac{(s+1)^3}{3\sigma}.$$

The return functions for matched agents are given by $\hat{f}(b, s, t) = (b + 1)(s + 1) - t$ and $\hat{g}(s, b, t) = t$, with the corresponding surplus and value functions

$$z(b, s) = (b + 1)(s + 1) \quad \text{and} \quad Z(b, s, \beta, \sigma) = (b + 1)(s + 1) - \frac{(b+1)^3}{3\beta} - \frac{(s+1)^3}{3\sigma}.$$

The return functions for unmatched agents are given by $\underline{f}(b) \equiv 1 \equiv \underline{g}(s)$, giving

$$\begin{aligned} \underline{U}(b,\beta) &= 1 - \frac{(b+1)^3}{3\beta} \\ \underline{V}(s,\sigma) &= 1 - \frac{(s+1)^3}{3\sigma}. \end{aligned}$$

We first note that every ex ante equilibrium is fully matched. In particular, for any pair of types $(\beta,\sigma) \in \mathfrak{B} \times \mathfrak{S}$, it is a simple calculation that investments $b = s = 1$ provide a strictly higher total payoff than remaining unmatched. The supermodularity of the functions z, $-\mathfrak{f}$ and $-\mathfrak{g}$ ensures that matching in ex ante equilibrium must be positive assortative. Solving the value maximization problem for agents $i = j$ allows us to calculate that the investment functions in any ex ante equilibrium are given by

$$\begin{aligned} \boldsymbol{b}(i) = \boldsymbol{\beta}(i) - 1 &= i + 2 \\ \boldsymbol{s}(j) = \boldsymbol{\sigma}(j) - 1 &= j + 2. \end{aligned}$$

Now consider ex post equilibria. The full appropriation game has a unique Nash equilibrium, ensuring that any fully matched ex post equilibrium satisfies exchange efficiency. We next argue that it is an ex post equilibrium for each agent to remain unmatched, while choosing investment zero. The investment choices in this allocation are clearly optimal given that agents are unmatched, and confirming that the allocation is an ex post equilibrium requires showing that there is no pair of agents who could profitably match when one of the agents' investments is fixed at zero. A seller of type σ can initiate a match with a buyer of type β whose investment is fixed at zero, with the seller choosing investment s and generating value

$$s + 1 - \frac{(s+1)^3}{3\sigma}.$$

The first-order condition for maximizing this quantity is $1 - (s+1)^2/\sigma = 0$, which we can solve for $(s+1) = \sigma^{\frac{1}{2}}$, allowing a total value of $\frac{2}{3}\sigma^{\frac{1}{2}}$. We then have an ex post equilibrium if for all $(\beta,\sigma) \in [3,4] \times [3,4]$, we have

$$\frac{2}{3}\sigma^{\frac{1}{2}} \leq 2 - \frac{1}{3\beta} - \frac{1}{3\sigma},$$

where the left side is the value created in the candidate match and the right side is the sum of payoffs the agents earn while unmatched. This will clearly be least likely to hold when $\beta = 3$, and it is then straightforward that the resulting inequality holds for $\sigma \in [3,4]$.

It is immediate that this ex post equilibrium is Pareto inefficient. We have noted that any allocation matching all agents with $b = s = 1$ gives every agent a strictly higher payoff, which suffices for the conclusion. Obviously, an ex ante equilibrium in which the buyer and seller in each match receive an equal payoff also makes every agent strictly better off. ■

Example 9. Suppose that utility is perfectly transferable and preferences are additively separable. There are two types of buyer, $\underline{\beta}$ and $\overline{\beta}$, with equal masses of each type. There are two types of seller, $\underline{\sigma}$ and $\overline{\sigma}$, with equal masses of each types. There are eight buyer investments, $\{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4, \overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_4\}$, and eight seller investments, $\{\underline{s}_1, \underline{s}_2, \underline{s}_3, \underline{s}_4, \overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4\}$.

	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\underline{b}_1$	20	8	3	2
$\underline{b}_2$	10	9	4	3
$\underline{b}_3$	8	8	7	6
$\underline{b}_4$	8	8	6	0

	$\overline{s}_1$	$\overline{s}_2$	$\overline{s}_3$	$\overline{s}_4$
$\underline{b}_1$	5	5	6	6
$\underline{b}_2$	6	0	7	7
$\underline{b}_3$	6	7	15	10
$\underline{b}_4$	6	7	9	8

	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\overline{b}_1$	9	8	3	2
$\overline{b}_2$	10	15	4	3
$\overline{b}_3$	8	8	0	5
$\overline{b}_4$	8	8	6	6

	$\overline{s}_1$	$\overline{s}_2$	$\overline{s}_3$	$\overline{s}_4$
$\overline{b}_1$	0	5	6	6
$\overline{b}_2$	6	6	7	7
$\overline{b}_3$	6	7	8	9
$\overline{b}_4$	6	9	9	20

Figure 5: Functions $\tilde{\ }(b, s) = \tilde{b}(s, b)$ for Example 9.

	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\underline{b}_1$	$40, 40$	$16, 16$	$6, 6$	$4, 4$
$\underline{b}_2$	$20, 20$	$18, 18$	$8, 8$	$6, 6$
$\underline{b}_3$	$16, 16$	$16, 16$	$14, 14$	$12, 12$
$\underline{b}_4$	$16, 16$	$16, 16$	$12, 12$	$0, 0$

Buyer $\underline{\beta}$, Seller $\underline{\sigma}$

	$\overline{s}_1$	$\overline{s}_2$	$\overline{s}_3$	$\overline{s}_4$
$\underline{b}_1$	$10, 10$	$10, 10$	$12, 12$	$12, 12$
$\underline{b}_2$	$12, 12$	$0, 0$	$14, 14$	$14, 14$
$\underline{b}_3$	$12, 12$	$14, 14$	$30, 30$	$20, 20$
$\underline{b}_4$	$12, 12$	$14, 14$	$18, 18$	$16, 16$

Buyer $\underline{\beta}$, Seller $\overline{\sigma}$

	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\overline{b}_1$	$18, 18$	$16, 16$	$6, 6$	$4, 4$
$\overline{b}_2$	$20, 20$	$30, 30$	$8, 8$	$6, 6$
$\overline{b}_3$	$16, 16$	$16, 16$	$0, 0$	$10, 10$
$\overline{b}_4$	$16, 16$	$16, 16$	$12, 12$	$12, 12$

Buyer $\overline{\beta}$, Seller $\underline{\sigma}$

	$\overline{s}_1$	$\overline{s}_2$	$\overline{s}_3$	$\overline{s}_4$
$\overline{b}_1$	$0, 0$	$10, 10$	$12, 12$	$12, 12$
$\overline{b}_2$	$12, 12$	$12, 12$	$14, 14$	$14, 14$
$\overline{b}_3$	$12, 12$	$14, 14$	$16, 16$	$18, 18$
$\overline{b}_4$	$12, 12$	$18, 18$	$18, 18$	$40, 40$

Buyer $\overline{\beta}$, Seller $\overline{\sigma}$

Figure 6: Full appropriation games for Example 9.

Preferences are given by (14)–(15), where $\tilde{f}(b, s) = \tilde{g}(s, b)$ and where these functions are given in Figure 5. The cost functions are given by, for $i = 1, 2, 3, 4$:

$$\mathfrak{f}(\underline{b}_i, \underline{\beta}) = \mathfrak{f}(\overline{b}_i, \overline{\beta}) = \mathfrak{g}(\underline{s}_i, \underline{\sigma}) = \mathfrak{f}(\overline{s}_i, \overline{\sigma}) = 0$$
$$\mathfrak{f}(\underline{b}_i, \overline{\beta}) = \mathfrak{f}(\overline{b}_i, \underline{\beta}) = \mathfrak{g}(\underline{s}_i, \overline{\sigma}) = \mathfrak{g}(\overline{s}_i, \underline{\sigma}) = 45.$$

We have $\underline{f} \equiv 0 \equiv \underline{g}$. Hence, outside options are zero. Autarchy investments are any of $\{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4\}$ for a buyer of type $\underline{\beta}$ and any of $\{\overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_4\}$ for a buyer of type $\overline{\beta}$, with sellers being analogous.

The cost functions ensure that it is a strictly dominated strategy for agent $\underline{\beta}$ to choose an investment $\overline{b}_k$ (for any k), or for $\overline{\beta}$ to choose $\underline{b}_k$, for $\underline{\sigma}$ to choose $\overline{s}_k$, or for $\overline{\sigma}$ to choose $\underline{s}_k$. The associated full appropriation games, omitting these obviously strictly dominated strategies, are then given in Figure 6. Each full appropriation game has a unique Nash equilibrium that is also the unique outcome of the iterated elimination of strictly dominated strategies. The unique pairwise efficient allocation forms matches $(\underline{\beta}, \underline{\sigma})$ (with investments (b_1, s_1), for a total payoff of 40) and matches $(\overline{\beta}, \overline{\sigma})$ (with investments (b_4, s_4), for a total payoff of 40). However, there is an ex post equilibrium matching $(\overline{\beta}, \underline{\sigma})$ (with investments (b_2, s_2), for a total payoff of 30) and matching $(\underline{\beta}, \overline{\sigma})$ (with investments (b_3, s_3), for a total payoff of 30). The latter equilibrium is Pareto inefficient. ■

4.2 Full Matching

4.2.1 Sufficient Conditions for Full Matching

A simple sufficient condition to ensure that all pairwise efficient allocations and, hence all ex ante equilibria, are fully matched is to assume that for every $(i, j) \in N^2$, there exists some exchange (b, s, t) with

$$U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) > \underline{u}(i) \quad \textit{and} \quad V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) > \underline{v}(j). \tag{76}$$

Any feasible allocation in which there exists a pair of unmatched agents (i, j) is then Pareto dominated by an otherwise unchanged allocation in which these two agents match with an exchange (b, s, t) satisfying (76).[21] This condition can be interpreted as the requirement that *all* possible matches are productive, allowing the matching partners to achieve utilities strictly higher than their outside options. The vast majority of our examples satisfy this condition.[22]

Assuming all matches to be productive, however, does not suffice to ensure that all ex post equilibria are fully matched. This is evident from Examples 1 and 8 in which unmatched agents choose their autarchy investments of zero and no match involving an agent who has chosen an investment of zero can generate any strictly positive surplus. A condition sufficient to ensure that all ex post equilibria are fully matched is that matches are productive even when one of the agents in the match has chosen an autarchy investment. The following is immediate from the pairwise conditional efficiency of ex post equilibria:

Proposition 6. *Let Assumption 1 hold. Suppose that for all $(i, j) \in N^2$, either (i) for all autarchy investments b of buyer i there exists (s, t) with $U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) > \underline{u}(i)$ and $V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) > \underline{v}(j)$, or (ii) an analogous condition holds for the autarchy investments of seller j. Then every ex post equilibrium $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is fully matched.*

The sufficient conditions of Proposition 6 will hold only if autarkic investments are also of value within a match. We have built this malleability of investments into our model. An alternative would be to assume that matched and unmatched agents have access to fundamentally different technologies, with incompatible types of investments. We discuss this issue further in Appendix E.

Suppose that preferences are separable. Then the conditions appearing in Proposition 6 will hold if we have

$$\hat{f}(b, s, 0) \geq \underline{f}(b) \text{ and } \hat{g}(s, b, 0) \geq \underline{g}(s), \tag{77}$$

[21] In the extension of our model to the case in which the masses of buyers and sellers may differ, (76) suffices to ensure that the short side of the market is fully matched.

[22] In some of our examples (e.g., Examples 1, 3, and 5) the strict inequalities from (76) hold for all matches but possibly those involving either the lowest possible type of buyer or lowest possibly type of seller, with equality holding for such matches. In these cases the bottom types only may remain unmatched (cf. the discussion in footnote 7) with the resulting ex ante equilibria being payoff equivalent to fully matched ex ante equilibria. For our purposes it is without loss of generality to treat these equilibria as being fully matched.

with at least one inequality strict, whenever b is an autarchy investment for some type of buyer and s an autarchy investment for some type of seller. These conditions will in turn be satisfied if all autarchy investments are strictly positive, the functions $\hat{f}$ and $\hat{g}$ are strictly increasing in the partner's investment, and being unmatched is equivalent to being matched to a partner with a zero investment. The conditions appearing in (77) are also satisfied in Iyigun and Walsh (2007). In contrast, the conditions appearing in Proposition 6 fail in many of our examples, which are designed to highlight the difference between ex ante and ex post equilibria and use convenient but special functional forms.

4.2.2 Full Matching, Continuity, and Separability

So far we have made no assumptions on the functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$. To make further progress, we require some regularity of the map from names into types to ensure that we can link assumptions on the agents' utility functions (which are expressed in terms of types) to properties of the utility frontiers (which are expressed in terns of names).

Assumption 2. *The functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are continuous.*

Assumptions 1 and 2 imply continuity of the utility frontiers and, as a consequence, the continuity of equilibrium utility schedules for fully matched equilibria. Appendix F proves:

Lemma 2. *Let Assumptions 1 and 2 hold. Then*

[2.1] The functions ϕ, ψ, $\breve{\phi}$, and $\breve{\psi}$ are continuous.

[2.2] In any fully matched ex ante or ex post equilibrium $(J, I, b, s, \boldsymbol{u}, \boldsymbol{v})$ the functions $\boldsymbol{u}$ and $\boldsymbol{v}$ are continuous.

The intuition for the second part of this result is standard: if the utility schedule $\boldsymbol{u}$ (for example) took a jump at i^*, then some buyer with a name very close to i^* and with a utility on the lower side of the jump could increase his utility by matching with seller $J(i)$ currently matched with another buyer i who is also close to i^* but on the upper side of the jump. Of course, Lemma 2 trivially holds in the finite case.

Our next result exploits separability to show that fully matched ex post equilibria are ex ante equilibria in an economy in which the investment opportunities are restricted in a particular way. In light of the constrained efficiency result from Proposition 4 this is not surprising. To state the result, we introduce some notation and terminology that we also require in Section 4.3.

For any pair of nonempty closed sets $\tilde{B} \subseteq B$ and $\tilde{S} \subseteq S$, define $\phi_{\tilde{B},\tilde{S}} : N \times N \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi_{\tilde{B},\tilde{S}}(i, j, v) = \max_{b \in \tilde{B}, s \in \tilde{S}, t \in \mathbb{R}} U(b, s, \beta, \sigma, t) \quad s.t. \quad V(s, b, \sigma, \beta, u) \geq v.$$

Define $\psi_{\tilde{S},\tilde{B}}$ analogously. We assume that $\tilde{B}$ and $\tilde{S}$ are nonempty and closed to ensure that the utility frontiers $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ are well defined and Lemmas 1 and 2 are applicable. Given such sets $\tilde{B}$ and $\tilde{S}$, consider an allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ satisfying $\boldsymbol{b}(i) \in \tilde{B}$ and $\boldsymbol{s}(j) \in \tilde{S}$. We say that this allocation is *individually rational on* $(\tilde{B}, \tilde{S})$ if $\boldsymbol{u}(i) \geq \max_{b \in \tilde{B}} \underline{U}(b, \boldsymbol{\beta}(i))$ and $\boldsymbol{v}(j) \geq \max_{s \in \tilde{S}} \underline{V}(s, \sigma(j))$ hold for all i and j. If, in addition, the pairwise efficiency conditions (44)–(45) from Definition 5 hold for $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$, then $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is *pairwise efficient on* $(\tilde{B}, \tilde{S})$. If we let $\tilde{B} = B$ and $\tilde{S} = S$, we have $\phi_{B,S} = \phi$ and $\psi_{S,B} = \psi$ and recover the standard definition of pairwise efficiency. Note that we may apply Proposition 1 to conclude that an allocation which is pairwise efficient on some sets $\tilde{B}$ and $\tilde{S}$ is an ex ante equilibrium in an economy in which $\tilde{B}$ and $\tilde{S}$ are the sets of available investments.

Lemma 3. *Let Assumptions 1 and 2 hold, let preferences be separable, let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a fully matched ex post equilibrium, and let $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{S}}$ be the closures of the sets $\boldsymbol{B}$ and $\boldsymbol{S}$ of investments chosen by buyers and sellers. Then $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is pairwise efficient on $(\overline{\boldsymbol{B}}, \overline{\boldsymbol{S}})$.*

Proof. Individual rationality of $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ on $(\overline{\boldsymbol{B}}, \overline{\boldsymbol{S}})$ is immediate. Applying the definition of pairwise efficiency on $(\overline{\boldsymbol{B}}, \overline{\boldsymbol{S}})$, it thus suffices to show that, for all i and j,

$$\boldsymbol{u}(i) \geq \sup_{s \in \boldsymbol{S}} \breve{\phi}(i, j, s, \boldsymbol{v}(j)) = \max_{s \in \overline{\boldsymbol{S}}} \breve{\phi}(i, j, s, \boldsymbol{v}(j)) = \phi_{B, \overline{\boldsymbol{S}}}(i, j, \boldsymbol{v}(j)) \geq \phi_{\overline{\boldsymbol{B}}, \overline{\boldsymbol{S}}}(i, j, \boldsymbol{v}(j))$$

$$\boldsymbol{v}(j) \geq \sup_{b \in \boldsymbol{B}} \breve{\psi}(j, i, b, \boldsymbol{u}(i)) = \max_{b \in \overline{\boldsymbol{B}}} \breve{\psi}(j, i, b, \boldsymbol{u}(i)) = \psi_{\overline{\boldsymbol{B}}, S}(j, i, \boldsymbol{u}(i)) \geq \psi_{\overline{\boldsymbol{S}}, \overline{\boldsymbol{B}}}(j, i, \boldsymbol{u}(i)).$$

The first inequality in each case follows from Proposition 4 and the definition of pairwise constrained efficiency (cf. (56)–(57)), the subsequent equality is implied by continuity of $\breve{\phi}$ and $\breve{\psi}$ (and the continuity of $\boldsymbol{u}$ and $\boldsymbol{v}$) established in Lemma 2, the second equality follows from the relationship between $\breve{\phi}$ and ϕ and between $\breve{\psi}$ and ψ (cf. (48)–(49)), and the final inequality follows from the observation that restricting agents to a smaller set of investments cannot increase the utility possibilities open to them. ∎

4.3 Positive Assortative Matching

Throughout this section we consider fully matched allocations. We seek conditions under which all fully matched ex post equilibria are payoff equivalent to positive assortative allocations. Assuming unidimensional names is a prerequisite for such an investigation. The following assumption directs our attention to the two most commonly studied cases.

Assumption 3. *The sets $N \subset \mathbb{R}$ is either a finite or an interval.*

Section 4.3.1 shows that familiar single crossing conditions on the restricted utility frontiers $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ introduced in Section 4.2.2 ensures payoff equivalence to positive assortative equilibria. Section 4.3.2 considers assumptions on the underlying utility functions U and V (and set of types and investments) that, when coupled with the natural monotonicity requirements on the maps from names to types, imply the requisite single crossing properties of the utility frontiers.

4.3.1 A Single Crossing Condition for Positive Assortative Matching

In their study of matching models with imperfectly transferable utility Legros and Newman (2007b) have introduced the concept of generalized increasing differences. For an economy with a finite number of agents they show that generalized increasing differences ensures payoff equivalence of equilibrium matchings to positive assortative matching, and that a strict version of this property yields positive assortative equilibrium matchings. Generalized increasing differences is a property on the functions describing the utility frontiers. As noted by Legros and Newman (2007b) the property of (strict) generalized increasing differences is equivalent to the (strict) single crossing condition that appears in the following definition. The interpretation is that higher buyers have a comparative advantage in matching with higher sellers, and vice versa.

Definition 10. *Let $\tilde{B} \subseteq B$ and $\tilde{S} \subseteq S$ be closed sets. Then $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ satisfy single crossing if for all $\bar{\imath} > \underline{i}$ and $\bar{\jmath} > \underline{j}$,*

$$\phi_{\tilde{B},\tilde{S}}(\underline{i}, \bar{\jmath}, v_1) \geq \phi_{\tilde{B},\tilde{S}}(\underline{i}, \underline{j}, v_2) \implies \phi_{\tilde{B},\tilde{S}}(\bar{\imath}, \bar{\jmath}, v_1) \geq \phi_{\tilde{B},\tilde{S}}(\bar{\imath}, \underline{j}, v_2) \tag{78}$$

$$\psi_{\tilde{S},\tilde{B}}(\underline{j}, \bar{\imath}, u_1) \geq \psi_{\tilde{S},\tilde{B}}(\underline{j}, \underline{i}, u_2) \implies \psi_{\tilde{S},\tilde{B}}(\bar{\jmath}, \bar{\imath}, u_1) \geq \psi_{\tilde{S},\tilde{B}}(\bar{\jmath}, \underline{i}, u_2). \tag{79}$$

If the inequalities in the consequents of (78)–(79) *are strict, then single crossing is said to be strict.*

By Lemma 1.2, the functions $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ appearing in Definition 10 are inverse. This implies that conditions (78)–(79) are not independent, but equivalent to each other.

The following key lemma asserts the payoff equivalence of fully matched ex ante equilibria to positive assortative ex ante equilibria when the utility frontiers ϕ and ψ satisfy single crossing. Further, with strict single crossing the equivalence is exact. For the finite case, this result is the counterpart to Proposition 1 in Legros and Newman (2007b). Extending the result to infinite sets of agents is straightforward when the single crossing is strict, but raises a number of technical issues otherwise. We resolve these with the help of the continuity result in Lemma 2. The proof is in Appendix G.2.

Lemma 4. *Let Assumptions 1–3 hold and assume that ϕ and ψ satisfy single crossing. Then every fully matched ex ante equilibrium is payoff equivalent to a positive assortative ex ante equilibrium. If ϕ and ψ satisfy strict single crossing, then every fully matched ex ante equilibrium is positive assortative.*

In general, the result in Lemma 4 has no obvious counterpart for ex post equilibria, as there is no natural generalization of the single crossing conditions to the conditional utility frontiers $\breve{\phi}$ and $\breve{\psi}$. However, when preferences are separable we can use Lemma 3 to infer that a fully matched ex post equilibrium is an ex ante equilibrium in an economy in which buyers are restricted to choose investments in $\overline{\boldsymbol{B}}$ and sellers are restricted to choose investments in $\overline{\boldsymbol{S}}$. Provided that the functions $\phi_{\overline{\boldsymbol{B}},\overline{\boldsymbol{S}}}$ and $\psi_{\overline{\boldsymbol{S}},\overline{\boldsymbol{B}}}$ satisfy single crossing, we can then apply Lemma 4 to obtain a positive assortment result. However, as we explain in Remark 10 below, strict single crossing must fail when $\boldsymbol{B}$ or $\boldsymbol{S}$ is a singleton and preferences are separable. Consequently, the following result does not contain a counterpart of the strict single crossing result from Lemma 4.

Proposition 7. *Let Assumptions 1–3 hold, let preferences be separable, and assume that $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ satisfy single crossing for all nonempty closed $\tilde{B} \subseteq B$ and $\tilde{S} \subseteq S$. Then every fully matched ex-post equilibrium is payoff equivalent to a positive assortative ex post equilibrium.*

Proof. Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a fully matched ex post equilibrium. By Lemma 3 the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is pairwise efficient on $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{S}}$ and thus a fully matched ex ante equilibrium in the corresponding economy in which the sets of feasible investments are given by $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{S}}$. Because these sets are compact, Assumptions 1–3 hold in the restricted economy.

We can then apply Lemma 4 to infer the existence of a payoff equivalent, positive assortative ex ante equilibrium $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ in the restricted economy.

It remains to show that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ is an ex post equilibrium in the original economy. First, $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ is clearly feasible in the original economy. Second, the allocations $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ and $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ have identical payoffs and the only new investments we have possibly added when moving from $\boldsymbol{b}$ and $\boldsymbol{s}$ to $\boldsymbol{b}'$ and $\boldsymbol{s}'$ are contained in the closures of the sets $\boldsymbol{B}$ and $\boldsymbol{S}$. Noting that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is individually rational and satisfies the pairwise constrained efficiency conditions (56)–(57) in the original economy, we can then conclude that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ also has these properties. From Corollary 3 and Proposition 2 this implies that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}, \boldsymbol{v})$ is an ex post equilibrium in the original economy, giving the result. ■

The sufficient conditions in Proposition 7 require that single crossing not only holds for B and S, but for arbitrary closed subsets of B and S. Example 13 in Appendix G.3 illustrates the importance of this stronger condition by providing an example adapted from Dizdar (2012). In this example, preferences are separable and (78)–(79) hold for ϕ and ψ, but there exist ex post equilibria which are not payoff equivalent to allocations in which matching is positive assortative.

4.3.2 Sufficient Conditions for Single Crossing

The single crossing properties in Definition 10 are not written in terms of the primitives of the problem. We have formulated Proposition 7 in terms of this single crossing property for two reasons. First, Definition 10 succinctly and intuitively identifies what is needed to ensure positive assortative matching, namely a single crossing condition on utility frontiers. Second, as Legros and Newman (2007b) discuss, it is difficult to find general sufficient conditions, ensuring that single crossing is satisfied. In this section we exploit separability to identify conditions on utility functions guaranteing single crossing of $\phi_{\tilde{B}\tilde{S}}$ and $\psi_{\tilde{B}\tilde{S}}$ for all closed subsets of B and S, ensuring the applicability of Proposition 7.[23] As it is common in the literature (cf. Section 2.1.4) we restrict attention to the case of unidimensional types and investments. In addition, we assume that agents with higher names have higher types.

Assumption 4.

[4.1] The sets $\mathfrak{B}$, $\mathfrak{S}$, B, and S are subsets of $\mathbb{R}$.

[4.2] The functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are strictly increasing.

The results in this section continue to hold of $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are only weakly increasing, but we will need the full strength of Assumption 4.2 in Section 5.

Recall that with separable preferences we have (cf. (58)–(59))

$$U(b, s, \beta, \sigma, t) = \hat{U}(\hat{f}(b, s, t), b, \beta)$$
$$V(s, b, \sigma, \beta, t) = \hat{V}(\hat{g}(s, b, t), s, \sigma).$$

We say that separable preferences satisfy *outer single crossing* if

$$\hat{U}(x_1, \overline{b}, \underline{\beta}) \geq \hat{U}(x_2, \underline{b}, \underline{\beta}) \quad \Longrightarrow \quad \hat{U}(x_1, \overline{b}, \overline{\beta}) \geq \hat{U}(x_2, \underline{b}, \overline{\beta}) \tag{80}$$
$$\hat{V}(y_1, \overline{s}, \underline{\sigma}) \geq \hat{V}(y_2, \underline{s}, \underline{\sigma}) \quad \Longrightarrow \quad \hat{V}(y_1, \overline{s}, \overline{\sigma}) \geq \hat{V}(y_2, \underline{s}, \overline{\sigma}) \tag{81}$$

[23] Han (2002) provides conditions for ex ante equilibria to exhibit positive assortative matching when preferences are given by (63)–(64), so that utility is perfectly transferable but not separable.

hold whenever $\bar{b} > \underline{b}$, $\overline{\beta} > \underline{\beta}$, $\bar{s} > \underline{s}$ and $\overline{\sigma} > \underline{\sigma}$. The interpretation of the outer single crossing properties is obvious: given the returns associated with the different investments, higher types are (weakly) more inclined to chose higher investments.

Now define

$$\begin{aligned}\rho(b,s,y) &= \max_{t\in\mathbb{R}} \hat{f}(b,s,t) \ s.t. \ \hat{g}(s,b,t) \geq y \\ \sigma(s,b,x) &= \max_{t\in\mathbb{R}} \hat{g}(s,b,t) \ s.t. \ \hat{f}(b,s,t) \geq x\end{aligned}$$

for all b, s, y, and x. The functions ρ and σ are the utility frontiers for an economy in which pairs of agents are described by their investments (b,s) and the utility functions for a match between such agents with a transfer t are given by the return functions $\hat{f}$ and $\hat{g}$.

We say that separable preferences satisfy *inner single crossing* if

$$\rho(\underline{b},\bar{s},y_1) \geq \rho(\underline{b},\underline{s},y_2) \implies \rho(\bar{b},\bar{s},y_1) \geq \rho(\bar{b},\underline{s},y_2) \tag{82}$$

$$\sigma(\underline{s},\bar{b},x_1) \geq \sigma(\underline{s},\underline{b},x_2) \implies \sigma(\bar{s},\bar{b},x_1) \geq \sigma(\bar{s},\underline{b},x_2) \tag{83}$$

hold whenever $\bar{b} > \underline{b}$ and $\bar{s} > \underline{s}$. The interpretation of inner single crossing is again obvious: agents which have chosen higher investments are more eager to match with agents who have chosen high investments.

Remark 9. In the additively separable case outer single crossing holds if and only if the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ are submodular. To see this, consider the case of (80). With additively separable preferences for the buyer we can rewrite this as

$$x_1 - \mathfrak{f}(\bar{b},\underline{\beta}) \geq x_2 - \mathfrak{f}(\underline{b},\underline{\beta}) \implies x_1 - \mathfrak{f}(\bar{b},\overline{\beta}) \geq x_2 - \mathfrak{f}(\underline{b},\overline{\beta}).$$

This holds for all $x_1, x_2 \in \mathbb{R}$ if and only if

$$\mathfrak{f}(\bar{b},\underline{\beta}) - \mathfrak{f}(\underline{b},\underline{\beta}) \geq \mathfrak{f}(\bar{b},\overline{\beta}) - \mathfrak{f}(\underline{b},\overline{\beta}).$$

As we require (80) for all $\bar{b} > \underline{b}$ and $\overline{\beta} > \underline{\beta}$ this is submodularity of $\mathfrak{f}$.

An analogous argument shows that with perfectly transferable utility and separable preferences the inner single crossing conditions (82)–(83) hold if and only if the surplus function z is supermodular. In the case of separable preferences with perfectly transferable utility the slightly more general result that supermodularity of Z suffices for the single crossing conditions in Proposition 7 is immediate from Theorem 2.7.6 in Topkis (1998). ■

Outer single crossing ensures that in equilibrium agents with higher types choose higher investments, whereas inner single crossing implies positive assortment of investments. Because we have assumed that types are increasing in names this suffices to imply positive assortment in ex post equilibrium. Appendix H proves:

Corollary 5. *Let Assumptions 1–4 hold. Suppose preferences are separable and satisfy outer and inner single crossing. Then every fully matched ex post equilibrium is payoff equivalent to a fully matched allocation satisfying positive assortative matching.*

The proof proceeds by showing that outer and inner single crossing of preferences imply the single crossing conditions appearing in Proposition 7, and then applying this proposition.

Corollary 5 still leaves us with the task of determining when the outer and inner single crossing conditions (80)–(83) hold. We focus on additively separable preferences. In this case outer single crossing is equivalent to the submodularity of the cost functions $\mathfrak{f}$ and $\mathfrak{g}$ (cf. Remark 9). It remains to identify conditions on the return functions $\hat{f}$ and $\hat{g}$ ensuring the inner single crossing conditions (82)–(83) for the case of imperfectly transferable utility. (With transferable utility, supermodularity of the surplus function z is necessary and sufficient. Again, see Remark 9.) The following result, proven in Appendix H.3, does so.

Corollary 6. *Let Assumptions 1–4 hold, and let preferences be additively separable with submodular cost functions $\mathfrak{f}$ and $\mathfrak{g}$. Suppose further that there exists continuous functions $F : \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$, $f : B \times S \to \mathbb{R}$, $g : S \times B \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that*

$$\hat{f}(b, s, t) = F(f(b, s) - t) \tag{84}$$

$$\hat{g}(s, b, t) = G(g(s, b) + h(t)), \tag{85}$$

where F and G are strictly increasing, f and g are supermodular and increasing in their second argument, and h is increasing and concave. Then every fully matched ex post equilibrium is payoff equivalent to a positive assortative ex post equilibrium.

The proof shows that under the stated assumptions, (84)–(85) imply inner single crossing at which point the result follows from Corollary 5.

The assumptions on the return functions in the statement of Corollary 6 are patterned after the ones in the example studied by Iyigun and Walsh (2007) that we have discussed in Section 2.1.4. We can think of investments as determining an amount of a second period consumption good, given by $f(b, s) + g(s, b)$, and a baseline division of this consumption good across the two agents, given by $(f(b, s), g(s, b))$. When $h(t) = t$ is the identity function (as in Iyigun and Walsh, 2007) the division of the consumption good can be changed without cost; the case of concave h allows for the possibility that there are increasing costs in transferring the consumption good from one agent to the other.

Remark 10. Throughout this section we have focussed on establishing conditions under which $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ satisfy the single crossing condition, rather than the strict single crossing condition from Definition 10. It is an immediate consequence of the separability of preferences that $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ must fail the strict single crossing condition when $\tilde{B}$ or $\tilde{S}$ is a singleton. To see this suppose, for instance, that $\tilde{S}$ is a singleton, so that all sellers choose the same investment. Separability then implies that if one buyer is indifferent between matching with seller j or seller j', then all buyers will be indifferent. More generally, as long as we cannot exclude the possibility that in an ex post equilibrium different agents choose the same investment, even assuming strict versions of inner and outer single crossing does not imply strict single crossing of the corresponding frontiers $\phi_{\overline{\boldsymbol{B}},\overline{\boldsymbol{S}}}$ and $\psi_{\overline{\boldsymbol{S}},\overline{\boldsymbol{B}}}$. However, if conditional exchange efficiency implies that all matched agents choose different investments (see Examples 1, 2, 7, 8, 11, and 12) then strict inner single crossing and strict outer single crossing imply strict single crossing of the relevant utility frontier, and then Lemma 4 can be applied to infer that fully matched ex post equilibria are positive assortative. ■

4.4 Efficient Ex Post Equilibria

Sections 4.2.1 and 4.3.1 have identified conditions, namely condition (76) and the conditions appearing in Lemma 4, under which all ex ante equilibria are positive assortative. It is clear that under these conditions, being positive assortative and exchange efficient is necessary for the Pareto (and pairwise) efficiency of ex post equilibria. The following result shows that having the correct, positive assortative matching and being exchange efficient are then also sufficient for Pareto efficiency of ex post equilibria.

Proposition 8. *Let Assumptions 1 and 3 hold, let condition* (76) *hold and assume ϕ and ψ satisfy single crossing. Then every positive assortative ex post equilibrium that is exchange efficient is also Pareto efficient.*

Proof. Let $(J, I, b, s, \boldsymbol{u}, \boldsymbol{v})$ be positive assortative and exchange efficient. We show that there exists no $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ which is a finite Pareto improvement on $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$.

Suppose, contrariwise, that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ is a finite Pareto improvement on the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$. As $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is fully matched, the allocation $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ has at most a finite number of unmatched agents, with identical numbers of unmatched buyers and sellers. From condition (76), these agents can be matched with each other in a way that is still a finite improvement on $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$. Hence, we may assume without loss of generality that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ is fully matched. Let n be the cardinality of the set $\{i \mid J'(i) \neq J(i)\} = \{j \mid I'(j) = I(j)\}$ (where the equality of these sets is from the fact that $J(i) \neq i$ is equivalent to $I(i) \neq i$). Because $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is exchange efficient, we must have $n > 0$, ensuring that there is a lowest type (cf. Assumption 3), $\underline{i}$, such that $J'(i) \neq i$ holds. Let $\bar{j} = J'(\underline{i}) > \underline{i}$ and $\bar{i} = I'(\underline{i}) > \underline{i}$ (where the strict inequalities hold because $\underline{i}$ is also the lowest type for whom $I'(j) \neq j$ holds).

Because both allocations are fully matched, if one buyer has a different partner then there must be at least one other buyer with a different partner, and hence we cannot have $n = 1$. Next, consider the case $n = 2$. Then we have $\bar{j} = \bar{i}$. From exchange efficiency of $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$, we have:

$$\begin{aligned} \boldsymbol{u}(\underline{i}) &= \phi(\underline{i}, \underline{i}, \boldsymbol{v}(\underline{i})), \\ \boldsymbol{u}(\bar{i}) &= \phi(\bar{i}, \bar{i}, \boldsymbol{v}(\bar{i})). \end{aligned}$$

From feasibility of $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ we have:

$$\begin{aligned} \boldsymbol{u}'(\underline{i}) &\leq \phi(\underline{i}, \bar{i}, \boldsymbol{v}'(\bar{i})), \\ \boldsymbol{u}'(\bar{i}) &\leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}'(\underline{i})). \end{aligned}$$

Because $\boldsymbol{v}'(\bar{i}) \geq \boldsymbol{v}(\bar{i})$ and $\boldsymbol{v}'(\underline{i}) \geq \boldsymbol{v}(\underline{i})$ holds, the latter two inequalities imply

$$\begin{aligned} \boldsymbol{u}'(\underline{i}) &\leq \phi(\underline{i}, \bar{i}, \boldsymbol{v}(\bar{i})), \\ \boldsymbol{u}'(\bar{i}) &\leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}(\underline{i})). \end{aligned}$$

Because $\boldsymbol{u}'(\underline{i}) \geq \boldsymbol{u}(\underline{i})$ and $\boldsymbol{u}'(\bar{i}) \geq \boldsymbol{u}(\bar{i})$ holds, the exchange efficiency equalities then yield:

$$\begin{aligned} \phi(\underline{i}, \underline{i}, \boldsymbol{v}(\underline{i})) &\leq \phi(\underline{i}, \bar{i}, \boldsymbol{v}(\bar{i})), \\ \phi(\bar{i}, \bar{i}, \boldsymbol{v}(\bar{i})) &\leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}(\underline{i})). \end{aligned}$$

These inequalities contradict the single crossing property unless they both hold with equality. But equality in both of these inequalities can only hold if $\boldsymbol{u}'(i) = \boldsymbol{u}(i)$ and $\boldsymbol{v}'(i) = \boldsymbol{v}(i)$

holds for $i = \underline{i}, \bar{i}$, contradicting the assumption that $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ is a finite Pareto improvement on $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$.

Now consider the case $n > 2$. We argue that if there exists such a finite Pareto improvement, then there also exists a finite Pareto improvement with $n' < n$. Repeating this argument a finite number of times then yields the existence of a finite Pareto improvement with $n = 2$, which we have already shown to be impossible. We consider two cases, namely $\bar{j} = \bar{i}$ and $\bar{j} \neq \bar{i}$.

In the first of these case, we can apply the argument from the case $n = 2$ to conclude that $\boldsymbol{u}'(i) = \boldsymbol{u}(i)$ and $\boldsymbol{v}'(i) = \boldsymbol{v}(i)$ holds for $i = \underline{i}, \bar{i}$. Consequently, if $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ is a finite Pareto improvement on $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$, so will be the allocation which coincides with $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ except that the buyers and sellers with types $\underline{i}$ and $\bar{i}$ are assigned their original partners and exchanges from the allocation $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$. This new finite Pareto improvement has cardinality $n' = n - 2$.

Suppose now that we have $\bar{j} \neq \bar{i}$. As in the case $n = 2$ exchange efficiency gives us:

$$\boldsymbol{u}(\underline{i}) = \phi(\underline{i}, \underline{i}, \boldsymbol{v}(\underline{i})),$$
$$\boldsymbol{u}(\bar{i}) = \phi(\bar{i}, \bar{i}, \boldsymbol{v}(\bar{i})).$$

Feasibility gives us:

$$\boldsymbol{u}'(\underline{i}) \leq \phi(\underline{i}, \bar{j}, \boldsymbol{v}'(\bar{j})),$$
$$\boldsymbol{u}'(\bar{i}) \leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}'(\underline{i})).$$

Using $\boldsymbol{v}'(\bar{j}) \geq \boldsymbol{v}(\bar{j})$ and $\boldsymbol{v}'(\underline{i}) \geq \boldsymbol{v}(\underline{i})$ this yields

$$\boldsymbol{u}'(\underline{i}) \leq \phi(\underline{i}, \bar{j}, \boldsymbol{v}(\bar{j})),$$
$$\boldsymbol{u}'(\bar{i}) \leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}(\underline{i})).$$

Combining the first of these with the exchange efficiency condition and $\boldsymbol{u}'(\underline{i}) \geq \boldsymbol{u}(\underline{i})$, yields

$$\phi(\underline{i}, \underline{i}, \boldsymbol{v}(\underline{i})) \leq \phi(\underline{i}, \bar{j}, \boldsymbol{v}(\bar{j})).$$

Because $\bar{i} > \underline{i}$ and $\bar{j} > \underline{i}$, the single crossing property implies

$$\phi(\bar{i}, \underline{i}, \boldsymbol{v}(\underline{i})) \leq \phi(\bar{i}, \bar{j}, \boldsymbol{v}(\bar{j})).$$

From previous inequalities, we have

$$\phi(\bar{i}, \bar{i}, \boldsymbol{v}(\bar{i})) \leq \phi(\bar{i}, \underline{i}, \boldsymbol{v}(\underline{i})),$$

so that we can infer

$$\boldsymbol{u}(\bar{i}) \leq \phi(\bar{i}, \bar{j}, \boldsymbol{v}(\bar{j})).$$

If this last inequality is strict, we can change $(J', I', \boldsymbol{b}', \boldsymbol{s}', \boldsymbol{u}', \boldsymbol{v}')$ by (i) "rematching" buyer and seller $\underline{i}$ with each other and having them make the exchange from the original allocation and (ii) matching buyer $\bar{i}$ and seller $\bar{j}$ with each other and fixing an exchange for them such that both of them strictly improve on their utility in the original allocation. Hence, we have found a finite Pareto improvement with cardinality $n' = n - 1$. If, on the other hand, we have

$$\boldsymbol{u}(\bar{i}) = \phi(\bar{i}, \bar{j}, \boldsymbol{v}(\bar{j})),$$

then it must have been the case that $\boldsymbol{u}'(i) = \boldsymbol{u}(i)$ and $\boldsymbol{v}'(j) = \boldsymbol{v}(j)$ must have held for $i = \underline{i}, \overline{i}$ and $j = \underline{i}, \overline{j}$, so that performing the same rematching as described above again generates a finite Pareto improvement with cardinality $n' = n - 1$. ■

We can combine Proposition 8 with previous results to obtain conditions under which all ex post equilibria are Pareto efficient. In particular, Proposition 5 provides sufficient conditions for ex post equilibria to be exchange efficient, Proposition 6 offers sufficient conditions for ex post equilibria to be fully matched, and Proposition 7 gives sufficient conditions for positive assortment of fully matched ex post equilibria. Together, the conditions appearing in Propositions 5–7, which imply the conditions from Proposition 8, thus preclude coordination failures in ex post equilibrium:

Corollary 7. *Let the conditions from Propositions 5–7 hold. Then every ex post equilibrium is Pareto efficient.*

One may wonder whether the conclusion in Propositions 8 and Corollary 7, can be strengthened to obtain the pairwise rather than just Pareto efficiency of all ex post equilibria. From Proposition 1 such a result would imply that all ex post equilibria are ex ante equilibria. The following example provides a counterexample to the conjecture that the conditions in Proposition 8 imply pairwise efficiency of positive assortative ex post equilibria.

Example 10. Let $N = \{0, 1\}$, $\mathfrak{B} = \{\underline{\beta}, \overline{\beta}\}$ and $\mathfrak{S} = \{\underline{\sigma}, \overline{\sigma}\}$. Buyer 0 has type $\underline{\beta}$ and buyer 1 has type $\overline{\beta} > \underline{\beta}$. Sellers are similarly either $\underline{\sigma}$ or $\overline{\sigma} > \underline{\sigma}$. For simplicity, we will refer to agents by their types rather than names throughout the following.

There are three possible buyer investments, $\{b_1, b_2, b_3\}$ and three possible seller investments, $\{s_1, s_2, s_3\}$. Preferences are separable and utility is perfectly transferable with surplus function $z(b, s)$ given by

	s_1	s_2	s_3
b_1	10	0	0
b_2	0	8	0
b_3	0	0	12

.

The cost-of-investment functions $\mathfrak{f}$ and $\mathfrak{g}$ are given by

$$\begin{aligned}
&\mathfrak{f}(b_1, \underline{\beta}) = \mathfrak{f}(b_2, \underline{\beta}) = 0, && \mathfrak{f}(b_3, \underline{\beta}) = 20 \\
&\mathfrak{f}(b_1, \overline{\beta}) = 20, && \mathfrak{f}(b_2, \overline{\beta}) = \mathfrak{f}(b_3, \overline{\beta}) = 0 \\
&\mathfrak{g}(s_1, \underline{\sigma}) = \mathfrak{g}(s_2, \underline{\sigma}) = 0, && \mathfrak{g}(s_3, \underline{\sigma}) = 20 \\
&\mathfrak{g}(s_1, \overline{\sigma}) = 20, && \mathfrak{g}(s_2, \overline{\sigma}) = \mathfrak{g}(s_3, \overline{\sigma}) = 0.
\end{aligned}$$

Outside options are zero. As we only consider fully matched equilibria, autarchy investments are irrelevant. This example satisfies the conditions of Proposition 8.

There is a continuum of ex ante equilibria in this economy, featuring identical matching and investments but differing in payoffs. In every ex ante efficient equilibrium, buyer $\underline{\beta}$ and seller $\underline{\sigma}$ match and choose investments (b_1, s_1), while buyer $\overline{\beta}$ and seller $\overline{\sigma}$ match and choose investments (b_3, s_3). Let $\underline{u}$ denote the equilibrium payoff earned buyer $\underline{\beta}$, with $\overline{u}$, $\underline{v}$, and $\overline{v}$

similarly defined. The set of equilibrium payoffs is the set of quadruples $(\underline{u}, \overline{u}, \underline{v}, \overline{v})$ satisfying:

$$\begin{aligned}
\underline{u} &\in [0, 10] \\
\underline{v} &\in 10 - \underline{v} \\
\overline{u} &\geq \max\{\underline{u} - 2, 0\} \\
\overline{v} &\geq \max\{\underline{v} - 2, 0\} \\
\overline{u} + \overline{v} &= 12.
\end{aligned}$$

The second condition indicates that the pair $(\underline{\beta}, \underline{\sigma})$ is on its utility frontier, while the last condition indicates the same for the pair $(\overline{\beta}, \overline{\sigma})$. The first condition indicates that the pair $(\underline{\beta}, \underline{\sigma})$ can achieve payoffs anywhere on its utility frontier. The remaining two conditions indicate that the pair $(\overline{\beta}, \overline{\sigma})$ must secure a payoff on its frontier that allows both agents in this match a payoff at least as high as they can achieve by matching with partner $\underline{\sigma}$ or $\underline{\beta}$.

We now consider an allocation with the identical matching and investments, but in which

$$\begin{aligned}
\underline{u} = \underline{v} &= 5 \\
\overline{u} &= 10 \\
\overline{v} &= 2.
\end{aligned}$$

This is an ex post equilibrium. The only ex post equilibrium conditions that are not immediate are the requirements that buyer $\underline{\beta}$ and seller $\overline{\sigma}$ are earning higher payoffs than they could earn by matching with each other. But given that seller $\overline{\sigma}$ is choosing investment s_3, there is no investment that buyer $\underline{\beta}$ can choose that yields him a positive payoff with such a seller. For similar reasons, there is no profitable deviation for seller $\overline{\sigma}$. We thus have an ex post equilibrium that fails pairwise efficiency. ■

The point illustrated by Example 10 is that ex ante and ex post equilibria give rise to fundamentally different incentive constraints. Buyer and seller investments are both up for grabs when agents consider the alternative matchings that give rise to the pairwise efficiency conditions, and this imposes tighter incentive constraints on equilibrium payoffs than do the corresponding pairwise conditional efficiency considerations. We thus cannot in general expect ex post equilibria to be pairwise efficient, even if they are exchange efficient and the matching is unambiguously "correct."

Remark 11. Fully matched ex ante and ex post equilibria are equivalent in Example 1, but are not equivalent in Example 10. The continuity built into Example 1 ensures that the local incentive constraints determining whether a positive assortative and exchange efficient allocation is pairwise efficient are identical to the local incentive constraints for pairwise conditional efficiency. The finiteness of Example 10 precludes such a continuity-based argument. ■

5 Nontransferable Utility

The most obvious case excluded by our model is that in which utility is nontransferable, as exemplified by Peters and Siow (2002). Following the "marriage market" literature arising out of Gale and Shapley (1962), Peters and Siow (2002) assume that there is no possibility for altering the utilities of a matched pair of agents once their investments are sunk. This section extends our analysis to accommodate nontransferable utility. Many of our results carry over with no or relatively small changes, while others require significant modifications. Most importantly, nontransferable utility leads to a rather dramatic failure of exchange efficiency, thus undermining the possibility to obtain general conditions guaranteeing the Pareto efficiency of ex post equilibria.

5.1 The Model

We continue to work with the model introduced in Section 2, but now replace Assumptions 1.2–1.3. with the assumptions that neither U nor V depend on t:

Assumption 5.
[5.1] The functions U, $\underline{U}$, V, $\underline{V}$ are continuous.
[5.2] $U(b,s,\beta,\sigma,t) = U(b,s,\beta,\sigma,t')$ holds for all (b,s,β,σ), t and t'.
[5.3] $V(s,b,\sigma,\beta,t) = V(s,b,\sigma,\beta,t')$ holds for all (s,b,σ,β), t and t'.

Assumption 5 is maintained throughout the following. We simplify the notation by writing $U(b,s,\beta,\sigma)$ and $V(s,b,\sigma,\beta)$ for the utilities agents obtain from engaging in an exchange (b,s,t) (which, by Assumptions 5.2–5.3, are independent of t). We similarly simply refer to (b,s) as an exchange. As a hint of difficulties to come, we observe that non-transferability of utility implies a failure of the strict Pareto property (cf. equations (1)–(2)).

The definition of a feasible allocation, Definition 1, remains unchanged.

5.2 Equilibrium and Efficiency with Nontransferable Utility

As in the transferable utility case, our first step in defining ex ante and ex post equilibria is to define functions ϕ, ψ, $\breve{\phi}$, and $\breve{\psi}$, describing the utility possibilities available to a matched pair of agents. In the absence of effective transfers, we must accommodate the possibility that the set of exchanges satisfying the constraint appearing in the definition of these functions is empty. We do so by assigning $-\infty$ as the value of the relevant function in such a case. Formally, we let

$$\begin{aligned}
\breve{u}(i,j,b) &= \max_{s\in S} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)),\\
\breve{v}(j,i,s) &= \max_{b\in B} V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)),\\
\bar{u}(i,j) &= \max_{b\in B} \breve{u}(i,j,b),\\
\bar{v}(j,i) &= \max_{s\in S} \breve{v}(j,i,s)
\end{aligned}$$

and define

$$\phi(i,j,v) = \begin{cases} \max_{b,s} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \ \text{ s. t. } \ V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \geq v & \text{if } v \leq \bar{v}(j,i) \\ -\infty & \text{otherwise,} \end{cases}$$

$$\psi(j,i,u) = \begin{cases} \max_{b,s} V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \ \text{ s. t. } \ U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \geq u & \text{if } u \leq \bar{u}(i,j) \\ -\infty & \text{otherwise,} \end{cases}$$

$$\breve{\phi}(i,j,s,v) = \begin{cases} \max_{b} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \ \text{ s. t. } \ V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \geq v & \text{if } v \leq \breve{v}(j,i,s) \\ -\infty & \text{otherwise,} \end{cases}$$

$$\breve{\psi}(j,i,b,u) = \begin{cases} \max_{s} V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \ \text{ s. t. } \ U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \geq u & \text{if } u \leq \breve{u}(i,j,b) \\ -\infty & \text{otherwise.} \end{cases}$$

The existence of all the maxima appearing in these definitions is assured by Assumption 5.1.

With these modifications, the definitions of ex ante and ex post equilibria (Definitions 2–3 in Section 2.2) as well as the definitions of Pareto efficiency, pairwise efficiency, pairwise conditional efficiency, and pairwise constrained efficiency (Definitions 4–8 in Section 3.1) carry over without any further changes to the case of nontransferable utility. Ex ante equilibria continue to satisfy the exchange efficiency condition (23) and ex post equilibria continue to satisfy the conditionally exchange efficiency condition (29). It is also easily verified that Propositions 1 and 2 as well as the associated Corollaries 1, 2 and 3 continue to hold when Assumption 1 is replaced by Assumption 5 in the statement of these results.[24] In particular, the result that every ex ante equilibrium is an ex post equilibrium holds with nontransferable utility, ensuring that pairwise efficient allocations are ex post equilibria, and we again have the relationships summarized in Figure 2.

It is less obvious that Proposition 4 continues to hold with non-transferable utility. However, neither the definition of separability, Definition 9, nor our proof of Proposition 4 makes use of the existence of effective transfers. Hence, even with non-transferable utility separability of preferences implies the pairwise constrained efficiency of ex post equilibria. Further, Corollary 4 can be inferred from Proposition 4 without having to resort to the strict Pareto property and thus continues to hold, too. As a consequence under separability, the "leveraging approach" to bounding the inefficiencies that may arise in an ex-post equilibrium discussed in Section 4.1.2 remains applicable with non-transferable utility and imposes significant constraints in the construction of ex-post equilibria. The example considered in Section 5.4 illustrates this.

In contrast to the results discussed in the preceding paragraph, the exchange efficiency result in Proposition 5 makes essential use the (imperfect) transferability of utility. To investigate the effects of assuming non-transferable utility on this result, we begin by observing that under nontransferability an exchange (b,s) is conditionally efficient for the pair (i,j) if and only if

$$b \in \arg\max_{b' \in B} U(b',s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \text{ s.t. } V(s,b',\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \geq V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)), \tag{86}$$

$$s \in \arg\max_{s' \in S} V(s',b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \text{ s.t. } U(b,s',\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \geq U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)). \tag{87}$$

[24] Because there are no effective transfers, we obviously cannot establish a counterpart to Proposition 3 concerning prices.

The implications of (86)–(87) are most easily understood when investments are unidimensional and utility functions satisfy an additional monotonicity property. We thus strengthen Assumption 4 to:

Assumption 6.

[6.1] The sets $\mathfrak{B}$, $\mathfrak{S}$, B, *and* S *are subsets of* $\mathbb{R}$.

[6.2] The functions $\boldsymbol{\beta}$ *and* $\boldsymbol{\sigma}$ *are strictly increasing.*

[6.3] The function $U(b,s,\beta,\sigma)$ *is strictly increasing in* s *and* $V(s,b,\sigma,\beta)$ *is strictly increasing in* b.

Assumption 6 states that, as in the model considered in Peters and Siow (2002), each agent in a matched pair strictly prefers that his or her partner chooses a higher investment level. As a consequence, the constraint appearing in (86) reduces to $b' \geq b$ and the constraint in (87) reduces to $s' \geq s$. It follows that every exchange (b,s) with the property that neither the buyer nor the seller strictly prefers a unilateral increase in his or her own investment is conditionally efficient for the pair under consideration. The following result, which imposes the counterparts to the differentiability and convexity requirements from Proposition 5, is then immediate. As before, we use subscripts to denote partial derivatives.

Proposition 9. *Let Assumptions 5–6 hold, let* $B = [\underline{b}, \overline{b}]$ *and* $S = [\underline{s}, \overline{s}]$, *and* U *and* V *quasiconcave and differentiable in* (b,s). *Then an exchange* (b,s) *is conditionally efficient for the pair* (i,j) *if and only if*

$$\begin{aligned} U_b(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j)) \cdot (\overline{b}-b) &\leq 0, \\ V_s(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i)) \cdot (\overline{s}-s) &\leq 0. \end{aligned}$$

Proposition 9 indicates that the only exchanges excluded by conditional efficiency are ones in which the buyer (for example, with a similar statement for the seller) underinvests to such an extent that the buyer could increase his own utility by increasing his investment for the given investment of the seller. Even with the quasiconcavity conditions of Proposition 5, pairwise conditional efficiency thus does not preclude exchanges in which both agents underinvest (in the sense that both agents' utilities could be increased by an increase in both investments). Similarly, pairwise conditional efficiency does not preclude cases in which both agents overinvest (in the sense that both agents' utilities could be increased by a decrease in both investments).

Ex post equilibria with underinvestment can be interpreted as reflecting a hold-up problem, arising from the fact that in the absence of effective transfers the gain from increasing an investment is completely captured by the agent on the other side in the market. Overinvestment can be interpreted in terms of an investment arms race, in which agents chose wastefully high investments because they cannot use transfers to provide the agent on the other side of the market with an appropriate utility level.

5.3 Matching in Ex Post Equilibrium

It is immediate that Proposition 6 continues to hold with nontransferable utility, providing conditions under which all ex post equilibria are fully matched. Here we establish conditions ensuring that fully matched ex post equilibria are payoff equivalent to positive assortative allocations.

Section 5.1 noted that the strict Pareto property fails under nontransferable utility. In the absence of the strict Pareto property, Lemma 1 does not hold. Rather than seeking a counterpart to Proposition 7, the proof of which relied on Lemma 1, we provide sufficient conditions for positive assortment by giving a direct proof for a counterpart to Corollary 5.

In the following result we consider the case of separable preferences (cf. Definition 9), so that utility functions are given by

$$\begin{aligned} U(b, s, \beta, \sigma) &= \hat{U}(\hat{f}(b, s), b, \beta) \\ V(s, b, \sigma, \beta) &= \hat{V}(\hat{g}(s, b), s, \sigma) \end{aligned}$$

for functions $\hat{U}$ and $\hat{V}$ that are strictly increasing in their first arguments. We say that preferences satisfy *strict outer single crossing* if the inequalities in the consequents of conditions (80)–(81) from Section 4.3.2 hold strictly. The proof of the following is in Appendix I.

Proposition 10. *Let Assumptions 3 and 5–6 hold, let preferences be separable and satisfy strict outer single crossing. Then every fully matched ex post equilibrium is payoff equivalent to a positive assortative ex post equilibrium.*

There are two key differences between the conditions appearing in the statement of Corollary 5 and Proposition 10. First, there is no counterpart to the requirement that the functions $\hat{f}$ and $\hat{g}$ satisfy a single crossing condition in the statement of Proposition 10. The role of this single crossing requirement in the proof of Corollary 5 is to ensure that the matching of investments induced by a matching of agents can be taken to be positive assortative. In the absence of transfers, the requirement in Assumption 6 that U and V are strictly increasing in the partner's investment suffices for this step of the argument. Second, in Proposition 10 strict outer single crossing rather than (weak) outer single crossing is imposed. In the absence of effective transfers, this strengthening of the single crossing condition is required to ensure that investments are increasing in names.

Proposition 10 isolates exchange inefficiency as the key stumbling block in extending our efficiency result in Proposition 8 for the case of (imperfectly) transferable utility to the case of nontransferable utility.

5.4 Example

To illustrate our results, we consider an example patterned after the model in Peters and Siow (2002).[25] The set of names, types, and investments are compact intervals in $\mathbb{R}$ with $N = \mathfrak{B} = \mathfrak{S} = [0, 1]$ and $B = S = [0, \overline{x}]$ with $\overline{x} > 4$. The functions $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are the identity functions. Utility functions are given by

$$\begin{aligned} U(b, s, \beta, \sigma) &= b + s - \frac{b^2}{2(\beta + 1)}, \qquad \underline{U}(b, \beta) = b - \frac{b^2}{2(\beta + 1)} \\ V(s, b, \sigma, \beta) &= b + s - \frac{s^2}{2(\sigma + 1)}, \qquad \underline{V}(s, \sigma) = s - \frac{s^2}{2(\sigma + 1)}. \end{aligned}$$

[25] The main difference between this example and the scenario considered in Peters and Siow (2002) is that the set of feasible investments in Peters and Siow (2002) is type dependent. Our model and results can be extended to cover this possibility, but for the sake of clarity we prefer not to do so here.

These utility functions represent additively separable preferences with $\hat{f}(b,s) = \hat{g}(s,b) = b+s$, $\underline{f}(b) = \hat{f}(b,0)$, $\underline{g}(s) = \hat{g}(s,0)$, and

$$\hat{U}(y,b,\beta) = y - \frac{b^2}{2(\beta+1)} \quad \text{and} \quad \hat{V}(y,s,\sigma) = y - \frac{s^2}{2(\sigma+1)}.$$

The autarchy investments $\underline{b}(i)$ and $\underline{s}(j)$ chosen by unmatched agents are given by the solution to the first order conditions $\underline{U}_b(b,i) = 0$ and $\underline{V}_s(s,j) = 0$, implying

$$\underline{b}(i) = 1 + i > 0 \quad \text{and} \quad \underline{s}(j) = 1 + j > 0,$$

resulting in the outside options

$$\underline{u}(i) = \frac{i+1}{2} \quad \text{and} \quad \underline{v}(j) = \frac{j+1}{2}.$$

Because autarchy investments are strictly positive, condition (77) holds with strict inequalities, ensuring that all ex post equilibria are fully matched. Because Assumptions 3–6 hold and $\hat{U}$ and $\hat{V}$ satisfy strict single crossing, Proposition 10 implies that every ex post equilibrium is payoff equivalent to an allocation with positive assortative matching. We thus restrict attention to positive assortative allocations in the following. Further, we simplify the exposition by considering symmetric allocations in which matched agents choose identical investments, that is, $i = j$ implies $\boldsymbol{b}(i) = \boldsymbol{s}(j)$. Notice, however, that asymmetric ex ante and ex post equilibria also exist, despite the symmetry of the model.

The investment functions in the symmetric ex ante equilibrium are given by

$$\boldsymbol{b}^*(i) = 2(i+1) \quad \text{and} \quad \boldsymbol{s}^*(j) = 2(j+1),$$

resulting in equilibrium utility levels

$$\boldsymbol{u}^*(i) = 2(i+1) \quad \text{and} \quad \boldsymbol{v}^*(j) = 2(j+1).$$

The corresponding allocation with positive assortative matching is also an ex post equilibrium. In the following we exhibit additional ex post equilibria failing exchange (and hence pairwise and Pareto) efficiency, thus illustrating the scope of possible inefficiencies indicated by Proposition 9.

First we construct an underinvestment equilibrium. Consider the investment functions $\boldsymbol{b}$ and $\boldsymbol{s}$ given by $\boldsymbol{b}(i) = \boldsymbol{s}(j) = 2$ for all i and j. In this allocation almost all agents underinvest relative to the symmetric ex ante equilibrium. The resulting utility levels in the corresponding allocation with positive assortative matching are

$$\boldsymbol{u}(i) = 4 - \frac{2}{(i+1)} \geq \underline{u}(i) \quad \text{and} \quad \boldsymbol{v}(j) = 4 - \frac{2}{(j+1)} \geq \underline{v}(j),$$

ensuring that the individual rationality constraints are satisfied. As the conditions from Proposition 9 are satisfied, this allocation is an ex post equilibrium. Every agent fares strictly better in this equilibrium than under autarchy except for agents $i = j = 1$, who are choosing investments (and earning utilities) just equal to their autarchy levels. The latter equality implies that we could not construct a similar equilibrium with an investment level lower than 2.

Next we consider an overinvestment equilibrium. Consider the investment functions $\boldsymbol{b}$ and $\boldsymbol{s}$ given by

$$\boldsymbol{b}(i) = \begin{cases} 3 & \text{if } i < 1/2 \\ \boldsymbol{b}^*(i) & \text{otherwise} \end{cases}$$
$$\boldsymbol{s}(j) = \begin{cases} 3 & \text{if } j < 1/2 \\ \boldsymbol{s}^*(j) & \text{otherwise.} \end{cases}$$

Relative to the symmetric ex ante equilibrium this allocation features overinvestment for all agents $i < 1/2$ and $j < 1/2$. Given positive assortative matching, the resulting utility levels are

$$\boldsymbol{u}(i) = \begin{cases} 6 - \frac{9}{2(i+1)} & \text{if } i < 1/2 \\ \boldsymbol{u}^*(i) & \text{otherwise} \end{cases}$$
$$\boldsymbol{v}(j) = \begin{cases} 6 - \frac{9}{2(j+1)} & \text{if } j < 1/2 \\ \boldsymbol{u}^*(j) & \text{otherwise.} \end{cases}$$

The allocation is individually rational and is conditionally exchange efficiency. To show that it is an ex post equilibrium, we show that it is pairwise constrained efficient. We show that (46) holds for buyers (with (47) for sellers being analogous). For any pair (i, j), we have

$$\breve{\phi}(i, j, \boldsymbol{s}(j), \boldsymbol{v}(j)) = 2\boldsymbol{s}(j) - \frac{(\boldsymbol{s}(j))^2}{2(\beta + 1)},$$

which follows from noting that (i) agent i can match with j and provide j with utility $\boldsymbol{b}(j)$ only if $b \geq \boldsymbol{b}(I(j)) = \boldsymbol{s}(j)$, and (ii) agent i will then choose $b = \boldsymbol{s}(j)$, since $\boldsymbol{s}(j)$ exceeds agent i's autarchy investment. We then need only note that $2s - (s^2)/(2(\beta + 1))$ is concave and maximized by $s = 2(\beta + 1)$, ensuring that no agent i can do better than to match with agent $j = 1$.

Remark 12. The existence of ex post equilibria failing Pareto efficiency in this example appears to be in conflict with the results of Peters and Siow (2002), who also examine a nontransferable utility model in which matches are arranged after investments are sunk, but conclude that all of the equilibria in this model are efficient.

This conflict is only apparent, as the equilibrium notion employed by Peters and Siow (2002) is more demanding than our notion of ex post equilibrium.[26] Peters and Siow (2002) define a "return function," identifying for each buyer investment b the seller investment s with which a buyer who chooses b would be matched (note that preferences are separable), with the inverse of this function similarly identifying a buyer investment b to be matched with each value of s chosen by a seller. Their equilibrium concept requires that every agent chooses a utility-maximizing investment, subject to the matching possibilities specified by the return function, and the resulting demand-for-investment functions clear the market.

The return function in Peters and Siow (2002) is the functional equivalent of a *complete* set of prices in our model. Their argument that the resulting equilibria are efficient is analogous

[26] Bhaskar and Hopkins (2011) apply a counterpart of the ex post equilibrium concept to the model in Peters and Siow (2002), finding inefficient equilibria resembling the ones we describe above.

to the argument used to establish our Proposition 3, showing that ex post equilibria supported by complete prices are pairwise efficient. The complete prices in our model and the return function in Peters and Siow (2002) each solve the coordination problems that can give rise to inefficient ex post equilibria. We are not sanguine that such coordination problems are easily solved in markets where matches are formed only after investments are sunk, and so we have not built such a solution to the coordination problem into our ex post equilibrium concept. ■

Remark 13. Bhaskar and Hopkins (2011) examine a variant of the model with separable preferences and nontransferable utility in which agents first choose their expenditure on investment, then receive a realization of a random investment level drawn from a distribution whose specification depends on the cost devoted to investment, and then match after realized investments are drawn. In this model, Bhaskar and Hopkins (2011) find that, except for knife-edge cases, (their counterpart to) ex post equilibria are inefficient. Their interpretation is that one should be cautious in interpreting Peters and Siow (2002)'s observation that equilibria are efficient, since this efficiency breaks down once one moves beyond the case of deterministic investments.[27]

The presence of stochastic investments in Bhaskar and Hopkins (2011) implies that one of the key results in our paper (and the papers discussed in Section 2.1.4) does not hold, namely that any ex ante equilibrium is also an ex post equilibrium. Bhaskar and Hopkins (2011) focus exclusively on the case in which markets are available only ex post, i. e., matches are determined only after investments are sunk. The preceding statement can be made precise only upon extending their model to also encompass ex ante markets, i.e., markets that allow investments and matches to be simultaneously determined. However, we can readily identify the source of difficulty without going too deeply into details.

Define an allocation as specifying, for each agent, an expenditure on investment, a probability distribution over the partner with whom the agent matches, and a probability distribution over the accompanying pairs of realized investments. Suppose we have a model in which all buyers are the same type (and similarly all sellers are of the same type) and suppose first that markets are available ex ante. Consider a candidate equilibrium in which all buyers (and similarly sellers) choose the same expenditure on investment and hence induce the same distribution over realized investments. The equilibrium matching will then be arbitrary, and the distribution over the pair of investments that a buyer will realize in his match is the product of the buyer's and seller's investment distribution. Now suppose, in contrast, that markets are only available ex post, and so matching occurs after investments are realized, and suppose that a supermodular function translates investments into payoffs. Then the ex post market will match investments positive assortatively, introducing correlation into the realized distribution of investments that a buyer will realize in his match, an impossibility in the ex ante case. Hence, the two settings give rise to different sets of feasible allocations. One then cannot simply argue that ex ante equilibria are also ex post equilibria, and it is less surprising that investment levels might differ across the two cases. We would expect the result that ex ante equilibria are also ex post equilibria to hold if, under ex post markets, matching takes place after expenditures on investments have been made, but before realized

[27] One interpretation offered by Bhaskar and Hopkins (2011) for their interest in stochastic investments is that the latter serve as a perturbed model which can be used to choose between the multiple ex post equilibria that appear in the model with deterministic investments. They leave open the question of whether, as the perturbation become small, the equilibrium converges to a (pairwise efficient) allocation satisfying Peters and Siow (2002)'s equilibrium definition.

investments are drawn.

Once investment realizations are stochastic, another problem of missing markets appears even when ex ante markets are available, since the agents are exposed to uninsurable risk. This poses no difficulty in Bhaskar and Hopkins (2011) because the agent's utilities are linear in their investment realizations, ensuring that the investigation of ex ante equilibrium can proceed just as it does in the case of deterministic investments, with expected investments replacing investments in the agents' payoff functions. In general, however, the presence of uninsurable risk will introduce inefficiencies that could be mitigated by a complete set of markets. ■

6 Discussion

6.1 Existence of Equilibrium

The existence of ex post equilibria is implied by the existence of ex ante equilibria (Corollary 1), but we have not addressed the question of when the latter exist. As we have noted, the pairwise efficiency conditions characterizing ex ante equilibria are equivalent to the stability conditions from the literature on matching and assignment models, which contains a number of existence results. As long as the functions ϕ and ψ emerging from our investment-choice problem satisfy the conditions from these results, we can apply them to infer existence of ex ante equilibria in our model.

In the finite case, our Assumption 1 ensures that the continuity and monotonicity assumptions of Alkan and Gale (1990, Theorem 1) are met (cf. our Lemma 1). As long as the agents' outside options are feasible within each match (i.e., the full matching condition (76) holds), this suffices for the existence of an ex ante equilibrium in our model. With nontransferable utility, existence of ex ante equilibria is implied by Gale and Shapley (1962) even without the additional condition on outside options.

With an infinite number of agents, the case most commonly considered in the literature is that in which types are continuously distributed and utility is perfectly transferable. Conditions ensuring the existence of pairwise efficient allocations are provided by Chiappori, McCann, and Nesheim (2010) and Ekeland (2010b)). These results allow for multidimensional types, but require restrictions on utility functions reminiscent of the supermodularity conditions in Cole, Mailath, and Postlewaite (2001b), who prove existence for their unidimensional model. Legros and Newman (2007a) study matching models with imperfectly transferable utility and a continuum of types under assumptions akin to the ones we impose in Section 4.3. They identify conditions (including the continuous differentiability of ϕ and ψ, which we could obtain from an appropriate strengthening of Assumption 2) under which the existence of equilibrium follows from the existence of the solution to a differential equation.

The literature has obtained more general existence results for models with an infinite number of agents than the ones cited above, but these results use a notion of feasibility different from the one we employ. Kaneko and Wooders (1996) present a general existence result for stable allocations in a model with either perfectly or imperfectly transferable utility, but their notion of an f-core considers any allocation to be feasible which lies in the closure of our set of feasible allocations. To make their result applicable to our setting would then require the identification of additional conditions ensuring feasibility of a stable outcome. That this is a non-trivial task becomes clear when considering the case of perfectly transferable utility in which stable allocations coincide with the solutions of an optimal transport problem (e.g. Gretsky, Ostroy, and Zame, 1992; Ekeland, 2010b). In particular, our existence problem is analogous to the existence of solutions to the so-called Monge problem, which is a notoriously difficult problem, whereas general existence results have been obtained for the so-called Kantorovich problem which considers an enlarged set of feasible allocations (Villani, 2009).[28]

6.2 Foundations for Competitive Matching

We have focussed on investment decisions in competitive matching environments by building the assumption that agents behave competitively into our equilibrium notions. In particular all agents solve a maximization problem that takes prices (whether in monetary or utility terms) as fixed at the candidate equilibrium level. Cole, Mailath, and Postlewaite (2001b), Dizdar (2012) and Peters and Siow (2002) adopt a similar approach. The advantage of this approach is that it takes the hold-up problem and positional externalities off the table, allowing us to isolate the role of coordination problems.

Makowski (2004, pp. 19-20), building on work by Gretsky, Ostroy, and Zame (1992, 1999), argues that one should be leery of simply assuming the matching market to be competitive, even when dealing with a continuum of agents, because by "accepting this point of view, one runs the danger of making continuum analysis totally unconnected with the analysis of large but finite economies....."[29] Cole, Mailath, and Postlewaite (2001a) show that allocations in a finite model satisfying a "double overlap" condition will satisfy a constrained efficiency condition analogous to the constrained efficiency condition that characterizes ex post equilibria (when preferences are separable) in our model. It would be important to investigate similar conditions in our setting. Bhaskar and Hopkins (2011, Appendix B) show that their competitive matching market, with a continuum of agents, is the limit of a sequence of models with finite numbers of agents. Hadfield (1999) also offers such a limiting analysis. However, Peters (2007, 2011) examines models whose equilibria do not exhibit convergence to competitive equilibrium as the number of agents grows arbitrarily large. Investigating the conditions under which matching markets with a large numbers of agents will be competitive remains an important area for further work.

[28] The problem of finding a solution to the pairwise efficiency conditions in our model is a Monge problem because we specify a matching as a map from names on one side of the market into names on the other side. In the Kantorovich problem the set of feasible matchings is identified with a joint probability measure over $N \times N$, with the constraint that the induced marginal distributions, over the sets of buyers and sellers, match the distributions of buyer and seller names. The interpretation is that the probability attached to any subset of $N \times N$ as the probability that agents from this subset are drawn to match. Again, see Villani (2009).

[29] Makowski (2004) defines the post-investment matching market as being perfectly competitive if the equilibrium price vector is a continuous function of the measures describing the investments present in the ex post market, so that an investment deviation by a small group of agents can have only a small effect on equilibrium prices. An individual member of the continuum is then "viewed as the limit of a small group of individuals," and may or may not have market power.

Appendices

A Proof of Lemma 1 (Section 2.2.1)

We first confirm that ϕ (and similarly ψ) is well defined on $N \times N \times \mathbb{R}$. Fix a pair $(i,j) \in N \times N$. Then for any $v \in \mathbb{R}$, we can fix a pair (b,s) and then use Assumption 1.3 to infer that there is some t for which $V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t) \geq v$. This ensures that the maximization problem in (17) is feasible, and the existence of the maximum then follows from the continuity assumed in Assumption 1.1, the fact that B and S are compact, and the fact that U and V move in opposite dictions in t (Assumptions 1.2–1.3).

[Lemma 1.1] We provide the proof for the function ϕ, with the case of ψ being similar. It is immediate from (17) that ϕ is weakly decreasing in v. To see that it is strictly decreasing, fix (i,j) and let $\overline{v} > \underline{v}$. Then there exists an exchange $(\overline{b},\overline{s},\overline{t})$ with $\phi(i,j,\overline{v}) = U(\overline{b},\overline{s},\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),\overline{t})$ and $V(\overline{s},\overline{b},\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),\overline{t}) \geq \overline{v}$. By Assumptions 1.1 and Assumption 1.3, there exists $\varepsilon > 0$ such that $V(\overline{s},\overline{b},\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),\overline{t}-\varepsilon) \geq \underline{v}$. Using Assumption 1.2, we then have $\phi(i,j,\underline{v}) \geq U(\overline{b},\overline{s},\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),\overline{t}-\varepsilon) > U(\overline{b},\overline{s},\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),\overline{t}) = \phi(i,j,\overline{v})$, giving the result.

[Lemma 1.2] We establish that $u = \phi(i,j,\psi(j,i,u))$. Fix $(i,j) \in N \times N$ and $u \in \mathbb{R}$. Then $\psi(j,i,u)$ exists (as established in our opening remarks) and we can let $v := \psi(j,i,u)$. The definition of ψ (cf. (18)) ensures that there exist b, s and t such that

$$\begin{aligned} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t) &\geq u \\ V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t) &= v. \end{aligned}$$

This implies that $\phi(i,j,v) \geq u$. To complete the argument by showing that this is in fact an equality, suppose $\phi(i,j,v) > u$. Then there exists b, s and t with

$$\begin{aligned} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t) &> u \\ V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t) &\geq v. \end{aligned}$$

From the strict Pareto property (cf. (1)–(2)) this in turn ensures that there exists t' for which

$$\begin{aligned} U(b,s,\boldsymbol{\beta}(i),\boldsymbol{\sigma}(j),t') &> u \\ V(s,b,\boldsymbol{\sigma}(j),\boldsymbol{\beta}(i),t') &> v, \end{aligned}$$

contradicting the definition $v := \psi(j,i,u)$.

[Lemma 1.3] As an implication of Lemma 1.2, ϕ has full range as a function of v and, from Lemma 1.1, is strictly decreasing in v. Hence, ϕ is continuous in v. The same argument gives continuity of ψ in u. ■

B Pareto Efficiency (Section 3.1.1)

We present an example motivating the restriction to finite Pareto improvements in our definition of Pareto efficiency.

Let $N = \mathfrak{B} = \mathfrak{S} = [0,1]$, let the sets $B = \{b\}$ and $S = \{s\}$ be singletons, and let $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ be identity functions. Utility is perfectly transferable, with

$$\begin{aligned} U(b,s,\beta,\sigma,t) &= 1 + \beta + \sigma - t \\ V(s,b,\sigma,\beta,t) &= t, \end{aligned}$$

and with $\underline{U}$ and $\underline{V}$ being identically zero. Now consider a fully matched allocation in which J and I are identity functions and payoffs are given by

$$\boldsymbol{u}(i) = 2i + \frac{1}{2} \text{ and } \boldsymbol{v}(j) = \frac{1}{2}.$$

This allocation is feasible and is easily seen to be Pareto efficient in the sense of Definition 4. However, we can construct a Pareto improvement by changing the allocation for a countable set of agents. Let

$$Z = \{\ldots,\ \frac{1}{2}-\frac{15}{32},\ \frac{1}{2}-\frac{7}{16},\ \frac{1}{2}-\frac{3}{8},\ \frac{1}{2}-\frac{1}{4},\ \frac{1}{2},\ \frac{1}{2}+\frac{1}{4},\ \frac{1}{2}+\frac{3}{8},\ \frac{1}{2}+\frac{7}{16},\ \frac{1}{2}+\frac{15}{32},\ \ldots\}.$$

All agents whose name does not fall in the set Z are matched with their current partners. In the set Z, the matching is as follows:

$$\begin{array}{cc}
\text{buyer} & \text{seller} \\
\hline
\vdots & \vdots \\
\frac{1}{2}-\frac{15}{32} & \frac{1}{2}-\frac{7}{16} \\
\frac{1}{2}-\frac{7}{16} & \frac{1}{2}-\frac{3}{8} \\
\frac{1}{2}-\frac{3}{8} & \frac{1}{2}-\frac{1}{4} \\
\frac{1}{2}-\frac{1}{4} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}+\frac{1}{4} \\
\frac{1}{2}+\frac{1}{4} & \frac{1}{2}+\frac{3}{8} \\
\frac{1}{2}+\frac{3}{8} & \frac{1}{2}+\frac{7}{16} \\
\frac{1}{2}+\frac{7}{16} & \frac{1}{2}+\frac{15}{32} \\
\vdots & \vdots
\end{array}.$$

We again set $\boldsymbol{v}(j) = 1/2$ for all j. Then every buyer in the set Z has been moved up to a match with a higher seller, with the seller commanding no higher a payoff, and hence every buyer in Z is strictly better off, whereas every seller and every buyer not in the set Z receives the same utility as before.

C Calculations for Example 2 (Section 3.1.4)

C.1 Supermodularity of the Surplus Function

We verify that the function

$$z(b, s) = \max\{2\sqrt{bs}, bs\}$$

is supermodular, which implies the corresponding result for any value of k. Because $2\sqrt{bs}$ and bs are both supermodular and $bs \geq 2\sqrt{bs}$ holds if and only if $bs \geq 4$, it suffices to show that at $b = 4/s$ the partial derivative of bs with respect to s is larger than the partial derivative of $2\sqrt{bs}$ with respect to s. As these partial derivatives are $4/s$ and $2/s$, the result follows.

C.2 Underinvestment Equilibrium

We consider a candidate equilibrium in which the allocation splits the value equally between any two agents in a match. Our calculations for the economy in which only the low investment is available confirm some of the ex post equilibrium incentive constraints for this allocation. In particular, it is clear that these investments are optimal if only the low technology is available, and clear that the presence of the high technology does not obviate this optimality for agents with $\boldsymbol{\beta}(i) < \beta^*$ and $\boldsymbol{\sigma}(j) < \beta^*$.

Completing the argument that this allocation is an ex post equilibrium requires confirming that no buyer or seller whose type is above β^* would prefer to increase their investment and make use of the high technology. It will be unprofitable for a buyer of type β (the case of a seller is analogous) to match with some seller of type σ and hence investment $\sigma^{\frac{1}{4}}$, and choose an investment b large enough that the high technology is applicable, if

$$\max_{b,\beta,\sigma} b\sigma^{\frac{1}{4}} - \frac{b^5}{5\beta} - \frac{\sigma^{\frac{5}{4}}}{5\sigma} - \frac{4}{5}\beta^{\frac{1}{4}} - \frac{4}{5}\sigma^{\frac{1}{4}} < 0.$$

The first three terms in this expression are the value produced by the deviating buyer and seller, given that the buyer can choose investment b and the seller is fixed at her equilibrium investment $\sigma^{\frac{1}{4}}$, while the final two terms are the equilibrium payoffs of the buyer and seller. Taking a derivative of this expression with respect to σ gives

$$\frac{1}{4}b\sigma^{-\frac{3}{4}} - \frac{1}{4}\sigma^{-\frac{3}{4}},$$

which is positive as long as $b > 1$, which we can assume without loss of generality.[30] We can accordingly maximize the left side by setting $\sigma = \boldsymbol{\beta}(1) =: \overline{\beta}$ and write the problem as

$$\max_{b,\beta} b\overline{\beta}^{\frac{1}{4}} - \frac{b^5}{5\beta} - \frac{4}{5}\beta^{\frac{1}{4}} - \overline{\beta}^{\frac{1}{4}}.$$

The first order conditions for an interior solution for b and β are now

$$\overline{\beta}^{\frac{1}{4}} - \frac{b^4}{\beta} = 0$$
$$\frac{b^5}{5\beta^2} - \frac{1}{5}\beta^{-\frac{3}{4}} = 0.$$

Given that $\overline{\beta} = 12$, the first of these implies that $b > \beta^{\frac{1}{4}}$, which in turn implies that the second of these derivatives is positive for every value of β. We can thus also set $\beta = \overline{\beta}$ and solve for $b = \overline{\beta}^{\frac{5}{16}}$. The inequality we need to establish is then

$$\overline{\beta}^{\frac{5}{16}}\overline{\beta}^{\frac{1}{4}} - \frac{\overline{\beta}^{\frac{25}{16}}}{5\overline{\beta}} - \frac{4}{5}\overline{\beta}^{\frac{1}{4}} - \overline{\beta}^{\frac{1}{4}} < 0.$$

This is equivalent to $\overline{\beta}^{\frac{5}{16}} < 9/4$, which holds for all values of $\overline{\beta}$ below approximately 13.40.

C.3 Overinvestment Equilibrium

We examine a candidate equilibrium in which the allocation splits the value equally between any two agents in a match. Our calculations for the economy in which only the high

[30] Given that $\boldsymbol{s}_2(j)$ is maximized by $\boldsymbol{s}_2(1) = 12^{\frac{1}{4}} \approx 1.86$, we can have $bs > 2\sqrt{bs}$ only if $b > 1$.

investment is available confirm some of the ex post equilibrium incentive constraints for this allocation. In particular, it is clear that these investments are optimal if only the high technology is available, and clear that the presence of the low technology does not obviate this optimality for agents with $\boldsymbol{\beta}(i) > \beta^*$ and $\boldsymbol{\sigma}(j) > \beta^*$.

Completing the argument that this allocation is an ex post equilibrium requires confirming that no buyer or seller whose type is below β^* would prefer to decrease their investment and make use of the low technology. It will be unprofitable for a buyer of type β (the case of a seller is analogous) to match with some seller of type σ and hence investment $\sigma^{\frac{1}{3}}$, and choose an investment b small enough that the low technology is applicable, if

$$\max_{b,\beta,\sigma} 2\sqrt{b\sigma^{\frac{1}{3}}} - \frac{b^5}{5\beta} - \frac{\sigma^{\frac{5}{3}}}{5\sigma} - \frac{3}{10}\beta^{\frac{2}{3}} - \frac{3}{10}\sigma^{\frac{2}{3}} < 0.$$

Taking a derivative with respect to σ gives

$$\frac{2}{6}b^{\frac{1}{2}}\sigma^{-\frac{5}{6}} - \frac{2}{6}\sigma^{-\frac{1}{3}} = \frac{\sigma^{-\frac{1}{3}}}{3}\left[\frac{b^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} - 1\right] < 0,$$

where the inequality follows from the fact that values of $b \geq \sigma$ (and hence $b \geq 9$) will be so expensive as to obviously be suboptimal. Hence, we can maximize the left side by setting $\sigma = \boldsymbol{\beta}(0) =: \underline{\beta}$. We thus have the inequality

$$\max_{b,\beta} 2\sqrt{b\underline{\beta}^{\frac{1}{3}}} - \frac{b^5}{5\beta} - \frac{3}{10}\beta^{\frac{2}{3}} - \frac{1}{2}\underline{\beta}^{\frac{2}{3}} < 0,$$

which we can differentiate in b to find the first-order condition $b^{-\frac{1}{2}}\underline{\beta}^{\frac{1}{6}} - \frac{b^4}{\beta} = 0$, and then solve for $b = \underline{\beta}^{\frac{1}{27}}\beta^{\frac{2}{9}}$ and insert into the desired inequality to obtain

$$\max_{\beta} 2\sqrt{\underline{\beta}^{\frac{1}{27}}\beta^{\frac{2}{9}}\underline{\beta}^{\frac{1}{3}}} - \frac{1}{5}\underline{\beta}^{\frac{5}{27}}\beta^{\frac{1}{9}} - \frac{3}{10}\beta^{\frac{2}{3}} - \frac{1}{2}\underline{\beta}^{\frac{2}{3}} < 0.$$

A derivative in β gives

$$\frac{2}{9}\beta^{-\frac{8}{9}}\underline{\beta}^{\frac{5}{27}} - \frac{1}{45}\underline{\beta}^{\frac{5}{27}}\beta^{-\frac{8}{9}} - \frac{2}{10}\beta^{-\frac{1}{3}}$$

which is negative if

$$10\beta^{-\frac{8}{9}}\underline{\beta}^{\frac{5}{27}} - \underline{\beta}^{\frac{5}{27}}\beta^{-\frac{8}{9}} - 9\beta^{-\frac{1}{3}} < 0,$$

which is $\underline{\beta}^{\frac{1}{3}} < \beta$, which is obvious. Hence we can set $\beta = \underline{\beta}$ and our inequality becomes

$$2\underline{\beta}^{\frac{1}{9}}\underline{\beta}^{\frac{5}{27}} - \frac{1}{5}\underline{\beta}^{\frac{5}{27}}\underline{\beta}^{\frac{1}{9}} - \frac{3}{10}\underline{\beta}^{\frac{2}{3}} - \frac{1}{2}\underline{\beta}^{\frac{2}{3}} < 0$$

and hence

$$\underline{\beta} > \left(\frac{9}{4}\right)^{\frac{27}{10}} \approx 8.93.$$

D Calculations for Example 3 (Section 3.2.1)

We have to confirm that in the candidate equilibrium, no buyer $i \in [0, 1/2)$ would prefer to match with some seller $j \in (i, 1/2]$. Such a match will be undesirable if

$$\frac{1}{2}i(e^i - c) > i(e^j - c) - \frac{1}{2}j(e^j - c).$$

The left side is buyer i's equilibrium payoff. The first term on the right is the total value produced when buyer i matches with seller j, in the process choosing the investment $e^j - c$ that is optimal for such a match. The final term on the right is seller j's equilibrium payoff, consisting of half the value produced in seller j's equilibrium match. This expression holds as an equality when $i = j$, so it suffices to show that the derivative of the right side with respect to j is negative, or (multiplying the result by 2)

$$0 > (2i - j)e^j - e^j + c.$$

The term $2i - j$ can be no larger than j, so (substituting j for $2i - j$, inserting the value of c, and rearranging) it suffices to show that $(1 - j)e^j > e^{\frac{1}{2}} - 1$. Because $2 > e^{\frac{1}{2}}$ this inequality hold for $j = 0$ and $j = 1/2$. As $(1 - j)e^j$ is concave, the argument is complete.

E The Leveraging Approach (Section 4.1.2)

This appendix continues the discussion of the leveraging approach to exchange efficiency with an example patterned after the ones considered in Cole, Mailath, and Postlewaite (2001b). In particular, as in Examples 1, 2, and 7 names, types, and investments are unidimensional, utility is perfectly transferable, preferences are separable, value functions are supermodular, and types are continuously distributed.

In both of the following examples the autarchy allocation, which is trivially exchange efficient, is an ex post equilibrium. Our focus is on the remaining ex post equilibria, which, by arguments similar to the ones discussed in Examples 1, 2, and 7, are positive assortative, feature strictly positive investments for all types, and have continuous equilibrium utility schedules. These properties are taken for granted.

Example 11. We consider the same specification as in Example 1, but with the surplus function now given by

$$z(b, s) = \max\left\{bs - k,\ 2(bs)^2 - k\right\}.$$

Cole, Mailath, and Postlewaite (2001b) have studied the resulting supermodular value function,

$$Z(b, s, \beta, \sigma) = \max\left\{bs - k,\ 2(bs)^2 - k\right\} - \frac{b^5}{5\beta} - \frac{s^5}{5\sigma}$$

for the case $k = 0$. We assume that k is strictly positive, but sufficiently small for all allocations considered in the following to satisfy the individual rationality constraints. Using the terminology introduced in Example 2, we refer to $z_1(b, s) = bs - k$ as the low technology and to $z_2(b, s) = 2(bs)^2 - k$ as the high technology. Recall that the functions assigning types to names are assumed to be given by $\boldsymbol{\beta}(i) = \gamma + \alpha i$ and $\boldsymbol{\sigma}(j) = \gamma + \alpha j$ with $\alpha > 0$ and $\gamma > 0$.

If only the low technology is available, then exchange efficiency calls for the investments

$$\boldsymbol{b}_1(i) = (\boldsymbol{\beta}(i))^{\frac{1}{3}}, \qquad \boldsymbol{s}_1(j) = (\boldsymbol{\sigma}(j))^{\frac{1}{3}}.$$

In the case of the high technology, the exchange efficient investments are

$$\boldsymbol{b}_2(i) = 4\boldsymbol{\beta}(i), \qquad \boldsymbol{s}_2(j) = 4\boldsymbol{\sigma}(j).$$

When both technologies are available, exchange efficiency calls for all agents with types below $\beta^* = \sigma^* = (3/2^9)^{\frac{3}{10}} \approx .21$ to use the low technology with investments $\boldsymbol{b}_1(i)$ and $\boldsymbol{s}_1(j)$, while those above this type choose the high technology with investments $\boldsymbol{b}_2(i)$ and $\boldsymbol{s}_2(j)$. When $\gamma < \beta^* < \gamma + \alpha$, as we assume throughout the following, the efficient investments take a jump at type β^*.

As in Examples 2 and 7 there are two candidates for exchange inefficient ex post equilibria, namely one in which all buyers choose the investments specified by $\boldsymbol{b}_1$ and all sellers choose the investments specified by $\boldsymbol{s}_1$ and another one in which agents choose the investments specified by $\boldsymbol{s}_2$ and $\boldsymbol{b}_2$. Straightforward but tedious calculations show that there are values $\underline{\beta}$ and $\overline{\beta}$, satisfying $0 < \underline{\beta} < \beta^* < \overline{\beta} \approx .71$, such that for matched pairs (i, i) with types below $\underline{\beta}$ conditional exchange efficiency dictates that they choose investments $\boldsymbol{b}_1(i) = \boldsymbol{s}_1(i)$, whereas for matched pairs (i, i) with types above $\overline{\beta}$ conditional exchange efficiency dictates that they choose investments $\boldsymbol{b}_2(i) = \boldsymbol{s}_2(i)$. For $\gamma < \underline{\beta} < \overline{\beta} < \gamma + \alpha$ this provides the counterpart to the first step in Example 7. By a continuity argument analogous to the one given in the second step of Example 7, this ensures the exchange efficiency of ex post equilibria.

But what if $\gamma < \underline{\beta}$ but $\gamma + \alpha = 0.7 < \overline{\beta}$? Then the efficient investments for the low technology are conditionally exchange efficient for all agents and the argument in the first step of Example 7 is no longer applicable to provide a starting point for the subsequent leveraging step. However, we can now use the pairwise constrained efficiency of ex post equilibria to obtain the result from the first step in Example 7. In particular, consider a pair of agents $i = j$ whose types are just above the cutoff β^*. Exchange efficiency calls for these agents to choose investments slightly above 0.81 and operate the high technology (see Figure 7). These investments are in the market, currently chosen by agents with types near 0.7. It then follows from Corollary 4 that there are no ex post equilibria in which these agents inefficiently choose the low-technology investments. From this point on, the exchange efficiency of ex post equilibria follows exactly as in the case $\overline{\beta} < \gamma + \alpha$.

The leveraging approach here exploits forces analogous to those in Example 4. In contrast, suppose that the sets of types in Example 7 does not contain the counterpart to the value $\overline{\beta}$ applicable in that example, which Appendix C.2 calculates as approximately 13.40. Then the market contains none of investments that would be exchange efficient for any pair of agents for whom exchange efficiency calls for the high technology, and the type of argument developed in this example does not apply. ■

On the basis of Examples 11, one might reason as follows. Suppose we have two technologies, one requiring rather low and one rather high investments. Suppose the optimal investments conditional on using either of the technologies is strictly increasing in type. Suppose further that pairwise efficiency requires some agents to use the low technology and some to use the high technology. Then if the set of types is sufficiently diverse, it should be impossible to support an exchange inefficient equilibrium in which either every agent chooses the low technology or every agent chooses the high technology. The key would be to consider a candidate equilibrium in which all agents use (say) the low technology. If the set of types is sufficiently rich, some of these agents must then be choosing investments that others would use to operate the high technology in the pairwise efficient allocation, at which point Corollary 4 gives us a contradiction.

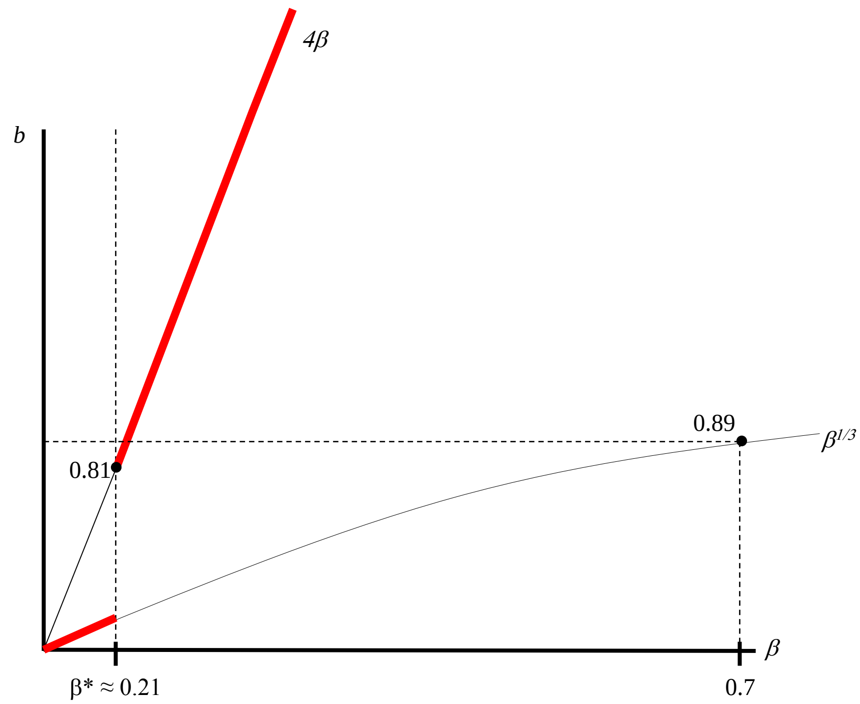

Figure 7: Illustration of the leveraging argument for Example 11. The heavy (red) line illustrates the pairwise efficient equilibrium investments. We are interested in a candidate inefficient equilibrium in which every pair of agents chooses the low-technology investment, given by $\beta^{\frac{1}{3}}$. Corollary 4 ensures that this is not an equilibrium. Instead, pairwise conditional efficiency fails for agents with types just above β^*. A pair of matched such agents could both increase their payoffs by choosing (exchange efficient) investments slightly above 0.81 and operating the high technology, and these investments are in the market, currently chosen by agents with types near 0.7. Corollary 4 then ensures that the allocation is not an ex post equilibrium.

Two issues arise in pursuing this argument (in addition to a collection of details that have to be filled in to make it precise). First, for this argument to be compelling, investments must be malleable, in the sense that an investment chosen for use in the low technology can be used in the high technology. We have been content to take malleability for granted throughout, but become more cautious as malleability becomes a critical ingredient rather than a convenience. Second, we present an example, modeled after a similar example from Dizdar (2012)), showing that there is no obvious extension of this argument to cases in which are there more than two technologies.

Example 12. We consider the same specification as in Example 11, but with the surplus function now given by

$$z(b,s) = \max\left\{2\sqrt{bs},\ bs,\ 2(bs)^2 - k\right\}.$$

We now think of there being three technologies, the low technology $z_1(b,s) = 2\sqrt{bs}$, the medium technology $z_2(b,s) = bs$, and the high technology $z_3(b,s) = 2(bs)^2 - k$. The value function Z is supermodular.

Example 2 calculated that the low technology generates a higher value than the medium technology, conditional on matching positive assortatively and choosing exchange efficient investments, for types $\beta < \beta^* \approx 10.53$. We can calculate that for a pair of matched agents with $\boldsymbol{\beta}(i) = \boldsymbol{\sigma}(j)$, the high technology gives equilibrium payoffs $(307.2)\boldsymbol{\beta}(i)^4 - k$. Let k solve $(307.2)12^4 - k = \frac{8}{5}12^{\frac{1}{4}}$. Then the low and the high technologies provide equal values at $\beta = \sigma = 12$ (cf. (53)). Let β^{**}, slightly larger than 12, be the value of $\beta = \sigma$ at which the medium and high technologies give the same value.

Let the set of types be $[8, 14]$. Then pairwise efficiency calls for agents in $[8, \beta^*)$ to use the low technology and choose investments given by (52), for agents in (β^*, β^{**}) to use the medium technology and choose investments given by (50), and for agents in $(\beta^{**}, 14]$ to use the high technology and choose investments $\boldsymbol{b}(i) = 4\boldsymbol{\beta}(i)$ and $\boldsymbol{s}(j) = 4\boldsymbol{\sigma}(j)$. However, there is an ex post equilibrium in which all agents in $[8, 12)$ use the low technology (and investments (52)) and agents in $(12, 14)$ use the high technology (and investments ($\boldsymbol{b}(i) = 4\boldsymbol{\beta}(i)$ and $\boldsymbol{s}(j) = 4\boldsymbol{\sigma}(j)$)). Notice that investments take a jump at 12, while equilibrium utility is continuous. Figure 8 illustrates.

This equilibrium fails exchange efficiency, as agents in $(\beta^*, 12)$ are inefficiently using the low rather than medium technology, and agents in $(12, \beta^{**})$ are inefficiently using the high rather medium technology. However, the leveraging argument does not preclude this equilibrium, and will not do so no matter how the interval $[8, 14]$ of types is enlarged. The difficulty is that equilibrium investments take a jump from about 1.86 to 48 as agents switch from the low to the high technology at type 12. The investments needed to make deviations to the medium technology profitable lie within this interval. Making the interval of types larger will bring more low investments into the market as well as more high investments, but will never fill in the required middle interval. ∎

F Proof of Lemma 2 (Section 4.2.2)

[Lemma 2.1] We show that $\breve{\phi}$ is continuous. Because S is compact this implies the continuity of $\phi(i, j, v) = \max_{s \in S} \breve{\phi}(i, j, s, v)$. The argument for the continuity of $\breve{\psi}$ and ψ is analogous.

Define the function $\tau : S \times B \times \mathfrak{S} \times \mathfrak{V} \times \mathbb{R} \to \mathbb{R}$ by

$$V(s, b, \sigma, \beta, \tau(s, b, \sigma, \beta, v)) = v.$$

To confirm that the function τ is well defined, we note that for each (s, b, σ, β), the function V has $\mathbb{R}$ as its range (Assumption 1.3), ensuring that there exists a value t satisfying $V(s, b, \sigma, \beta, t) = v$, and the fact that V is strictly increasing in t ensures that this value is unique. Moreover, because V is continuous and strictly increasing in its last argument, τ is continuous. Now define

$$\begin{aligned} \overline{\tau}(i, j, s, v) &= \max_{b \in B} \tau(s, b, \boldsymbol{\sigma}(i), \boldsymbol{\beta}(j), v) \\ \underline{\tau}(i, j, s, v) &= \min_{b \in B} \tau(s, b, \boldsymbol{\sigma}(i), \boldsymbol{\beta}(j), v). \end{aligned}$$

Berge's maximum theorem (Ok, 2007, p. 306) ensures that $\overline{\tau}$ and $\underline{\tau}$ are continuous. Then we have

$$\breve{\phi}(i, j, s, v) = \max_{(b,t) \in B \times [\underline{\tau}(i,j,s,v), \overline{\tau}(i,j,s,v)]} U(b, s, \boldsymbol{\beta}(i), \boldsymbol{\sigma}(j), t) \quad s.t. \quad V(s, b, \boldsymbol{\sigma}(j), \boldsymbol{\beta}(i), t) \geq v.$$

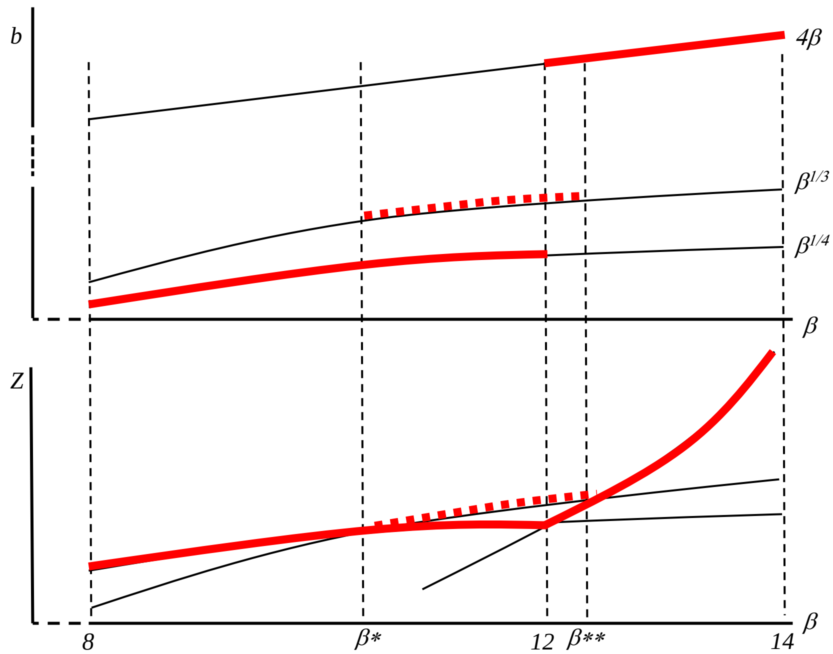

Figure 8: Illustration of the ex post equilibrium failing pairwise efficient in Example 12. The top panel shows the buyer's investment as a function of β (with the seller's investment being analogous) for the low technology ($\beta^{1/4}$), medium technology ($\beta^{1/3}$) and high technology (4β) (not drawn to scale). The ex post equilibrium investments follow the heavy (red) line in the top panel, using only the low and high technology, while pairwise efficiency calls for investments to be given by the dashed red line, using the medium technology, for $\beta \in (\beta^*, \beta^{**})$. The bottom panel shows the value Z generated by a pair of matched agents of identical type, as a function of β. The values realized by the ex post equilibrium matches are given by the heavy (red) line, while the dashed line indicates the value realized for $\beta \in (\beta^*, \beta^{**})$ under the pairwise efficient allocation.

This maximization problem again satisfies the conditions of Berge's maximum theorem, giving the result.

[Lemma 2.2] Suppose $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is a fully matched ex post equilibrium and $\boldsymbol{u}$ is not continuous (the case of $\boldsymbol{v}$ is similar). Then there exists a value $\delta > 0$ and sequences $\{\underline{i}_n\}_{n=1}^{\infty}$ and $\{\bar{i}_n\}_{n=1}^{\infty}$ such that

$$\boldsymbol{u}(\bar{i}_n) - \delta > \boldsymbol{u}(\underline{i}_n)$$
$$|\bar{i}_n - \underline{i}_n| < \frac{1}{n}.$$

The conditional exchange efficiency condition (29) for $\bar{i}_n$ gives us

$$\boldsymbol{u}(\bar{i}_n) = \breve{\phi}(\bar{i}_n, J(\bar{i}_n), \boldsymbol{s}(J(\bar{i}_n)), \boldsymbol{v}(J(\bar{i}_n))).$$

The fact that $\breve{\phi}(i, j, s, v)$ is continuous in i on the compact set N (and hence uniformly continuous) then enures that for sufficiently large n,

$$\boldsymbol{u}(\underline{i}_n) < \boldsymbol{u}(\bar{i}_n) - \delta = \breve{\phi}(\bar{i}_n, J(\bar{i}_n), \boldsymbol{s}(J(\bar{i}_n)), \boldsymbol{v}(J(\bar{i}_n))) - \delta < \left[\breve{\phi}(\underline{i}_n, J(\bar{i}_n), \boldsymbol{s}(J(\bar{i}_n)), \boldsymbol{v}(J(\bar{i}_n))) + \frac{\delta}{2}\right] - \delta,$$

with the outside two terms then giving

$$\boldsymbol{u}(\underline{i}_n) < \breve{\phi}(\underline{i}_n, J(\bar{i}_n), \boldsymbol{s}(J(\bar{i}_n)), \boldsymbol{v}(J(\bar{i}_n))) - \frac{\delta}{2},$$

contradicting the incentive constraint (27) for $\underline{i}_n$. ■

G Appendix for Section 4.3.1

Section G.1 provides simple necessary and sufficient conditions for (strict) single crossing of $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$, stated as Lemma 5. A similar result appears in Legros and Newman (2007b). Building on Lemma 5, Section G.2 gives the proof of Lemma 4. Lemma 5 is also used in the proof of Corollary 5 in Appendix H.

G.1 Cross Matched Agents

Let Assumption 1 hold. The functions $\phi_{\tilde{B},\tilde{S}}$ and $\psi_{\tilde{S},\tilde{B}}$ then satisfy the properties noted in Lemma 1 for any choice of nonempty closed sets $\tilde{B} \subset B$ and $\tilde{S} \subset S$. As the following argument only uses these properties, we may then simplify notation by considering the case $\phi = \phi_{B,S}$ and $\psi = \psi_{S,B}$.

Let $\underline{i} < \bar{i} \in N$ and $\underline{j} < \bar{j} \in N$. If there exists utility levels $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in \mathbb{R}$ such that

$$\underline{u} = \phi(\underline{i}, \bar{j}, \overline{v}) \geq \phi(\underline{i}, \underline{j}, \underline{v}) \tag{88}$$

$$\overline{u} = \phi(\bar{i}, \underline{j}, \underline{v}) \geq \phi(\bar{i}, \bar{j}, \overline{v}) \tag{89}$$

then we say that the pairs $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ are a *cross match*. We may apply the inverse and monotonicity relationships in Lemma 1 to obtain that $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ are a cross match if and only if there exists utility levels $\underline{u}, \overline{u}, \underline{u}, \overline{u} \in \mathbb{R}$ such that

$$\underline{v} = \psi(\underline{j}, \bar{i}, \overline{u}) \geq \psi(\underline{j}, \underline{i}, \underline{u}) \tag{90}$$

$$\overline{v} = \psi(\bar{j}, \underline{i}, \underline{u}) \geq \psi(\bar{j}, \bar{i}, \overline{u}). \tag{91}$$

To motivate the terminology of a cross match observe, first, that the equalities in the above conditions indicate that the utility levels are chosen in such a way that they are consistent with the agents in the pairs $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ matching with each other and choosing exchange efficient exchanges. Second, the inequalities indicate that if the agents under consideration were matched in this way, then no agent has an incentive to switch partners.

We say that a cross match can be *uncrossed* if the inequalities in (88)–(89) (or, equivalently, the inequalities in (90)–(91)) can only hold as equalities, indicating that the agents in the cross match can be reassigned to form matches $(\underline{i}, \underline{j})$ and $(\bar{i}, \bar{j})$ without changing their payoffs. If a cross match cannot be uncrossed, then the strict Pareto property (or, more formally, Lemma 1) implies that the pairs $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ are a *strict cross match*, meaning that u_1, u_2, v_1, v_2 can be chosen such that the inequalities in (88)–(91) hold strictly.

Lemma 5. *Let Assumption 1 hold. Then the functions ϕ and ψ satisfy strict single crossing if and only if there exist no cross matches. They satisfy single crossing if and only if every cross match can be uncrossed.*

Proof. The result for strict single crossing is immediate from the definitions.

Suppose there exists a cross match that cannot be uncrossed. Then, as noted above, there exists a strict cross match $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ with

$$\phi(\underline{i}, \overline{j}, \overline{v}) > \phi(\underline{i}, \underline{j}, \underline{v})$$
$$\phi(\overline{i}, \underline{j}, \underline{v}) > \phi(\overline{i}, \overline{j}, \overline{v}),$$

contradicting the single crossing condition (78). Hence, if single crossing holds, then every cross match can be uncrossed. To prove the reverse implication, suppose the single crossing condition (78) fails. Then there exist $\underline{i} < \overline{i}$, $\underline{j} < \overline{j}$ and $\underline{v}$, $\overline{v}$ such that

$$\phi(\underline{i}, \overline{j}, \overline{v}) \geq \phi(\underline{i}, \underline{j}, \underline{v})$$
$$\phi(\overline{i}, \overline{j}, \overline{v}) < \phi(\overline{i}, \underline{j}, \underline{v}).$$

Upon setting $\underline{u} = \phi(\underline{i}, \overline{j}, \overline{v})$ and $\overline{u} = \phi(\overline{i}, \underline{j}, \underline{v})$ we then have a cross match that cannot be uncrossed. ■

G.2 Proof of Lemma 4

Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a fully matched ex ante equilibrium. Suppose J is strictly increasing. Because J is a measure preserving bijection this implies that J is the identity function, ensuring that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is positive assortative.

We may thus suppose that J is not strictly increasing or, equivalently, that there exists $\underline{i} < \overline{i}$ and $\underline{j} < \overline{j}$ such that $(\underline{i}, \overline{j}) \in M$ and $(\overline{i}, \underline{j}) \in M$ hold. Using the incentive constraints (21) every such pair of matches satisfies

$$\boldsymbol{u}(\underline{i}) \;=\; \phi(\underline{i}, \overline{j}, \boldsymbol{v}(\overline{j})) \;\geq\; \phi(\underline{i}, \underline{j}, \boldsymbol{v}(\underline{j})) \tag{92}$$
$$\boldsymbol{u}(\overline{i}) \;=\; \phi(\overline{i}, \underline{j}, \boldsymbol{v}(\underline{j})) \;\geq\; \phi(\overline{i}, \overline{j}, \boldsymbol{v}(\overline{j})), \tag{93}$$

so that $(\underline{i}, \overline{j})$ and $\overline{i}, \underline{j})$ are a cross match. We refer to a cross match in which (92) – (93) holds as an *equilibrium cross match.*

From Lemma 5 the existence of an equilibrium cross match contradicts strict single crossing. Hence, if strict single crossing holds $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is positive assortative and the proof of the strict single crossing result in Lemma 4 is finished.

Suppose N is finite. Then the conclusion of Lemma 4 is immediate from the ability to uncross any given cross match asserted in Lemma 5: We can start with the lowest buyer-seller pair and proceed upward until we find a pair that is not matched to each other. This pair must then be part of an equilibrium cross match, which we can uncross. We can repeat this exercise, doing so at most finitely many times, until arriving at a payoff-equivalent ex ante equilibrium featuring the identity matching.

To finish the proof, it remains to consider the case in which N is an interval and show that $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is payoff equivalent to a positive assortative allocation when single crossing holds. Without loss of generality we let $N = [0, 1]$. The incentive constraints (21)–(22) imply

$$\boldsymbol{u}(i) \geq \phi(i, i, \boldsymbol{v}(i))$$
$$\boldsymbol{v}(j) \geq \psi(j, j, \boldsymbol{u}(j))$$

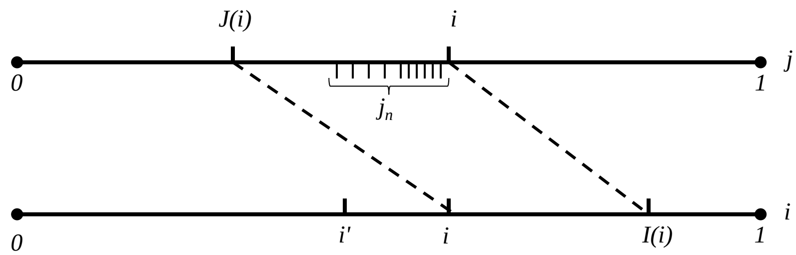

Figure 9: Illustration for the proof of Lemma 4 when $N = [0, 1]$. We hypothesize the existence of a buyer i matched with seller $J(i) < i$, and suppose that buyer i and seller i cannot achieve their equilibrium payoffs when matched with one another. We first invoke the cross match argument of Section G.1 to conclude that every buyer $i' < i$ is matched with a seller $J(i') < i$, and hence feasibility requires that (almost) all sellers $j < i$ must be matched with buyers less than i. This allows us to consider a sequence $\{j_n\}$ of sellers whose types converge to i. Each such seller must be matched with a buyer less than i. We use these converging sequences of matched pairs and the continuity of ϕ to derive a contradiction.

for all i and j. If all these inequalities hold as equalities, then it is clear that the equilibrium is payoff equivalent to an equilibrium satisfying positive assortative matching. We accordingly suppose there exists $i \in [0, 1]$ such that buyer i and seller i cannot achieve their equilibrium payoffs when matched to each other, that is

$$\boldsymbol{u}(i) > \phi(i, i, \boldsymbol{v}(i)). \tag{94}$$

We show that this leads to a contradiction.

The inequality in (94) implies $J(i) \neq i$ and $I(i) \neq i$. If i were part of an equilibrium cross match, then Lemma 5 implies that this cross match could be uncrossed, contradicting (94). Hence, we must either have $J(i) < i < I(i)$ or the reverse chain of inequalities. We focus on the first of these cases throughout the following (with the case $I(i) < i < J(i)$ following from an analogous argument, swapping the roles of buyers and sellers throughout the following). This gives us the configuration illustrated in Figure 9.

If $J(i') > i$ holds for some $i' < i$, then, because $i > J(i)$ holds, we have an equilibrium cross match with pairs $(\underline{i}, \overline{j}) = (i', J(i'))$ and $(\overline{i}, \underline{j}) = (i, J(i)))$. We can uncross to match i with $J(i')$ while preserving payoffs. This gives us an equilibrium cross match with pairs $(\underline{i}, \overline{j}) = (i, J(i'))$ and $(\overline{i}, \underline{j}) = (I(i), i)$ which we can uncross to obtain a contradiction to (94). Hence, we have that $i' < i$ implies $J(i') < i$.

As the equilibrium matching is measure preserving, $J(i') < i$ for all $i' < i$ implies that $I(j) < i$ holds for almost all sellers $j < i$. We can thus choose a sequence $\{j_n\}_{n=1}^{\infty}$ of sellers with $j_n > J(i)$ and $j_n \nearrow i$ and $i_n = I(j_n) \leq i$ for all n. As (i_n, j_n) are matched, the equilibrium feasibility conditions (23) and the incentive constraints (21)–(22) imply

$$\boldsymbol{u}(i_n) = \phi(i_n, j_n, \boldsymbol{v}(j_n)) \geq \phi(i_n, J(i), \boldsymbol{v}(J(i))).$$

Because $i \geq i_n$, and $j_n > J(i)$ holds, the single crossing property (78) implies that the above weak inequality also holds for i. We can then use $\boldsymbol{u}(i) = \phi(i, J(i), \boldsymbol{v}(J(i)))$ to obtain $\boldsymbol{u}(i) \leq \phi(i, j_n, \boldsymbol{v}(j_n))$, and the equilibrium incentive constraints then imply

$$\boldsymbol{u}(i) = \phi(i, j_n, \boldsymbol{v}(j_n))$$

for all n. The continuity of ϕ and $\boldsymbol{v}$, established in Lemma 2, along with $j_n \nearrow i$, then ensures

$$\lim_{n\to\infty} \phi(i, j_n, \boldsymbol{v}(j_n)) = \phi(i, i, \boldsymbol{v}(i)) = \boldsymbol{u}(i).$$

The second of these equalities contradicts (94), finishing the proof. ■

G.3 Mismatch Example

In the following example utility is perfectly transferable and preferences are separable. All ex post equilibria are fully matched. Assumptions 1–3 hold and the frontiers ϕ and ψ satisfy single crossing. Lemma 4 thus ensures that all ex ante equilibria are payoff equivalent to positive assortative allocations. Nevertheless, there exists an ex post equilibria that is negative assortative and not payoff equivalent to a positive assortative allocation. This shows that the requirement in Proposition 7 that single crossing not only holds for B and S but for all closed subsets of these cannot be simply discarded.

A related example is given in Dizdar (2012, Section 5.3). Dizdar exploits multidimensional investments in a setting similar to that examined by Cole, Mailath, and Postlewaite (2001b) to construct value functions that have different single crossing properties when restricted to different subsets of investments. As in our example, this opens the possibility for ex post equilibria featuring matchings distinct from the ones that obtain in all ex ante equilibria.

Example 13. Let $N = \{0, 1\}$, $\mathfrak{B} = \{\beta_1, \beta_2\}$, $\mathfrak{S} = \{\underline{\sigma}, \overline{\sigma}\}$ and $B = S = \{L, H_1, H_2\}$. Buyer types are given by $\boldsymbol{\beta}(0) = \beta_1 < \beta_2 = \boldsymbol{\beta}(1)$. Seller types are given by $\boldsymbol{\sigma}(0) = \underline{\sigma} < \overline{\sigma} = \boldsymbol{\sigma}(1)$. Utility is perfectly transferable and preferences are additively separable (cf. (14)–(15)). The cost functions $\mathfrak{f}$ and $\mathfrak{g}$ are given by

	L	H_1	H_2
$\underline{\sigma}$	0	12	12
$\overline{\sigma}$	0	1	1
β_1	0	1	12
β_2	0	12	1

.

The surplus function is $z(b, s)$ is symmetric and given by

	L	H_1	H_2
L	1	1	1
H_1	1	8	4
H_2	1	4	10

.

As the notation suggests, we think of L as a low investment and H_1 and H_2 as alternative types of high investments, with H_1 being more productive. Seller 0 with type $\underline{\sigma}$ has L as a dominant strategy in the full appropriation game, while there are circumstances under which seller 1 with type $\overline{\sigma}$ will find it advantageous to choose H_1 or H_2. The buyer with type β_i has a cost advantage in choosing investments H_i.

We assume that autarchy investments are L and outside options are zero, ensuring that all ex post equilibria are fully matched (cf. Proposition 6).

It is clear that the above specification satisfies Assumptions 1–3. The maximal value $W(i,j)$ that can be obtained in a match between a buyer with type i and a seller with type j is given by

$$W(1,1)=8, \quad W(0,1)=6, \quad W(1,0)=W(0,0)=1,$$

which is a strictly supermodular function, implying that ϕ and ψ satisfy the strict single crossing property. Consequently, Lemma 4 implies that every pairwise efficient allocation is positive assortative. Indeed, it is easy to check that in any such allocation buyer 0 (with type β_1) matches with seller 0 (with type $\underline{\sigma}$), choosing investments (L,L) and sharing a value of 1, while buyer 1 (with type β_2) matches with seller 1 (with type $\overline{\sigma}$), choosing investments (H_2,H_2) and sharing a value of 8.

However, there is also a negative assortative ex post equilibrium in which buyer 1 matches with seller 0, choosing investments (L,L) and sharing a value of 1 equally, while buyer 0 matches with seller 1, choosing investments (H_1,H_1) and sharing a value of 6 equally. To see that such an allocation is an ex post equilibrium it suffices to note that the investment choices are equilibria in the relevant full appropriation games and that the only potentially profitable deviation involves matching buyer 1 with seller 1 for a value of 8. Alas, to achieve this value both buyer 1 and seller 1 would have to change their investments. We can relate the existence of this equilibrium to a failure of single crossing by considering the maximal values $\breve{W}(i,j)$ that can be realized in the various matches given the sets $\boldsymbol{B}=\{L,H_1\}$ and $\boldsymbol{S}=\{L,H_1\}$ of investments that are in the market. These values are

$$\breve{W}(1,1)=1, \quad \breve{W}(0,1)=6, \quad \breve{W}(1,0)=\breve{W}(0,0)=1,$$

which is strictly submodular rather than supermodular, as would be required for the frontiers $\phi_{\boldsymbol{B},\boldsymbol{S}}$ and $\psi_{\boldsymbol{S},\boldsymbol{B}}$ to satisfy single crossing. ■

H Appendix for Section 4.3.2

We begin by stating the counterpart to Lemma G.1 for the functions ρ and σ and then turn to the proofs of Corollaries 5 and 6.

H.1 Cross Matched Investments

Let Assumption 1 hold and let preferences be separable. Then the functions ρ and σ appearing in (82)–(83) satisfy the counterparts to the properties established for ϕ and ψ in Lemma 1.

We define a *cross match in investments* in analogy to the cross matches introduced in Appendix G.1, namely as a pair of investment choices $(\underline{b},\overline{s})$ and $(\overline{b},\underline{s})$ with $\underline{b}<\overline{b}$ and $\underline{s}<\overline{s}$ such that there exists $\underline{f},\overline{f},\underline{g},\overline{g}\in\mathbb{R}$ satisfying

$$\underline{f}=\rho(\underline{b},\overline{s},\overline{g})\geq\rho(\underline{b},\underline{s},\underline{g}) \tag{95}$$

$$\overline{f}=\rho(\overline{b},\underline{s},\underline{g})\geq\rho(\overline{b},\overline{s},\overline{g}). \tag{96}$$

As in Appendix G.1 we say that a cross match in investments can be uncrossed if the inequalities in (95)–(96) can only hold as equalities and observe that a cross match in investments cannot be uncrossed if and only if it is strict, that is, there exists $\underline{f}$, $\overline{f}$, $\underline{g}$, and $\overline{f}$

such that both inequalities in (95)–(96) hold strictly. As the proof of Lemma 5 relied solely on Lemma 1 the following is then immediate:

Lemma 6. *Let Assumption 1 hold and let preferences be separable. Then preferences satisfy inner single crossing if and only if every cross match in investments can be uncrossed.*

H.2 Proof of Corollary 5

We fix a pair of nonempty closed sets $\tilde{B} \subset B$ and $\tilde{S} \subset S$ and, to simplify notation, let $\phi = \phi_{\tilde{B},\tilde{S}}$ and $\psi = \psi_{\tilde{B},\tilde{S}}$. From Lemma 5 in Appendix G.1 it suffices to show that for these utility frontiers every cross match can be uncrossed or, equivalently, that the existence of a strict cross match leads to a contradiction.

Suppose that the pairs $(\underline{i}, \overline{j})$ and $(\overline{i}, \underline{j})$ with $\underline{i} < \overline{i}$ and $\underline{j} < \overline{j}$ are a strict cross match. That is, there exists $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in \mathbb{R}$ such that

$$\underline{u} = \phi(\underline{i}, \overline{j}, \overline{v}) > \phi(\underline{i}, \underline{j}, \underline{v}) \tag{97}$$

$$\overline{u} = \phi(\overline{i}, \underline{j}, \underline{v}) > \phi(\overline{i}, \overline{j}, \overline{v}). \tag{98}$$

Let $\underline{\beta} = \boldsymbol{\beta}(i)$, $\overline{\beta} = \boldsymbol{\beta}(i)$, $\underline{\sigma} = \boldsymbol{\sigma}(\underline{j})$, and $\overline{\sigma} = \boldsymbol{\sigma}(\overline{j})$. From Assumption 4.2, we have $\underline{\beta} < \overline{\beta}$ and $\underline{\sigma} < \overline{\sigma}$. Consider any pair of exchanges $(\underline{b}, \overline{s}, t_1)$ and $(\overline{b}, \underline{s}, t_2)$ such that

$$\underline{u} = \hat{U}(\hat{f}(\underline{b}, \overline{s}, t_1), \underline{b}, \underline{\beta}), \qquad \overline{u} = \hat{U}(\hat{f}(\overline{b}, \underline{s}, t_2), \overline{b}, \overline{\beta})$$

$$\underline{v} = \hat{V}(\hat{g}(\underline{s}, \overline{b}, t_2), \underline{s}, \underline{\sigma}), \qquad \overline{v} = \hat{V}(\hat{g}(\overline{s}, \underline{b}, t_1), \overline{s}, \overline{\sigma})$$

hold (the existence of such exchanges is assured by the definition of ϕ). Let $\underline{f} = \hat{f}(\underline{b}, \overline{s}, t_1)$, $\overline{f} = \hat{f}(\underline{b}, \overline{s}, t_2)$, $\underline{g} = \hat{g}(\underline{s}, \overline{b}, t_2)$, and $\overline{g} = \hat{g}(\overline{s}, \underline{b}, t_1)$. By definition of ρ, we have

$$\underline{f} = \rho(\underline{b}, \overline{s}, \overline{g}) \text{ and } \overline{f} = \rho(\overline{b}, \underline{s}, \underline{g}).$$

Using separability of the sellers' preferences we have the inequalities

$$\phi(\underline{i}, \underline{j}, \underline{v}) \geq \hat{U}(\rho(\overline{b}, \underline{s}, \underline{g}), \overline{b}, \underline{\beta}) \tag{99}$$

$$\phi(\overline{i}, \overline{j}, \overline{v}) \geq \hat{U}(\rho(\underline{b}, \overline{s}, \overline{g}), \overline{b}, \overline{\beta}) \tag{100}$$

$$\phi(\underline{i}, \underline{j}, \underline{v}) \geq \hat{U}(\rho(\underline{b}, \underline{s}, \underline{g}), \underline{b}, \underline{\beta}) \tag{101}$$

$$\phi(\overline{i}, \overline{j}, \overline{v}) \geq \hat{U}(\rho(\overline{b}, \overline{s}, \overline{g}), \overline{b}, \overline{\beta}). \tag{102}$$

Combining the strict inequalities in (97)–(98) with (99)–(100) we obtain

$$\hat{U}(\underline{f}, \underline{b}, \underline{\beta}) > \hat{U}(\overline{f}, \overline{b}, \underline{\beta})$$

$$\hat{U}(\overline{f}, \overline{b}, \overline{\beta}) > \hat{U}(\underline{f}, \underline{b}, \overline{\beta}),$$

so that the outer single crossing property (80) implies $\underline{b} \leq \overline{b}$. Because $\hat{U}$ is strictly increasing in its first argument, $\underline{b} = \overline{b}$ is inconsistent with the above two inequalities holding simultaneously. We thus have $\underline{b} < \overline{b}$. We can repeat this argument using the equivalent restatement of (97)–(98) for ψ and the outer single crossing condition (81) for the seller to obtain the inequality $\underline{s} < \overline{s}$.

Combining the strict inequalities in (97)–(98) with (101)–(102) we obtain

$$\hat{U}(\rho(\underline{b},\overline{s},\overline{g}),\underline{b},\underline{\beta}) > \hat{U}(\rho(\underline{b},\underline{s},\underline{g}),\underline{b},\underline{\beta})$$
$$\hat{U}(\rho(\overline{b},\underline{s},\underline{g}),\overline{b},\overline{\beta}) > \hat{U}(\rho(\overline{b},\overline{s},\overline{g}),\overline{b},\overline{\beta}).$$

Because $\hat{U}$ is strictly increasing in its first argument this implies

$$\rho(\underline{b},\overline{s},\overline{g}) > \rho(\underline{b},\underline{s},\underline{g})$$
$$\rho(\overline{b},\underline{s},\underline{g}) > \rho(\overline{b},\overline{s},\overline{g}).$$

Hence, $(\underline{s},\overline{b})$ and $(\overline{b},\underline{s})$ are a strict cross match in investments. From Lemma 6 this contradicts the inner single crossing condition (82). ■

H.3 Proof of Corollary 6

It suffices to show that the inner single crossing condition (83) holds. As single crossing is an ordinal property and F and G are strictly increasing, we may assume that F and G are the identity functions. We then have

$$\sigma(s,b,x) = g(s,b) + h(f(b,s) - x).$$

Let $\underline{s} < \overline{s}$, $\underline{b} < \overline{b}$, and $x_1, x_2 \in \mathbb{R}$ satisfy

$$g(\underline{s},\overline{b}) + h(f(\overline{b},\underline{s}) - x_1) = g(\underline{s},\underline{b}) + h(f(\underline{b},\underline{s}) - x_2). \tag{103}$$

We show that this implies

$$g(\overline{s},\overline{b}) + h(f(\overline{b},\overline{s}) - x_1) \geq g(\overline{s},\underline{b}) + h(f(\underline{b},\overline{s}) - x_2), \tag{104}$$

which (because of continuity and monotonicity in x) suffices for σ as given above to satisfy the inner single crossing condition (83).

From (103) we have

$$g(\underline{s},\overline{b}) - g(\underline{s},\underline{b}) = h(f(\underline{b},\underline{s}) - x_2) - h(f(\overline{b},\underline{s}) - x_1) \geq 0, \tag{105}$$

where the inequality holds because g is increasing in b. As h is increasing this implies

$$f(\underline{b},\underline{s}) - x_2 \;\geq\; f(\overline{b},\underline{s}) - x_1.$$

Because f is supermodular, we have

$$[f(\underline{b},\underline{s}) - x_2] - \left[f(\overline{b},\underline{s}) - x_1\right] \geq [f(\underline{b},\overline{s}) - x_2] - \left[f(\overline{b},\overline{s}) - x_1\right]$$

and because f is increasing in s we have

$$f(\overline{b},\overline{s}) - x_1 \geq f(\overline{b},\underline{s}) - x_1$$
$$f(\underline{b},\overline{s}) - x_2 \geq f(\underline{b},\underline{s}) - x_2.$$

From the concavity of h these inequalities imply

$$h(f(\underline{b},\underline{s}) - x_2) - h(f(\overline{b},\underline{s}) - x_1) \geq h(f(\underline{b},\overline{s}) - x_2) - h(f(\overline{b},\overline{s}) - x_1)$$

Using supermodularity of g and (105) this suffices to give (104), finishing the proof. ■

I Proof of Proposition 10 (Section 5.3)

In the absence of effective transfers, the strict Pareto property fails and Lemma 2 does not hold. Thus, we cannot apply Lemmas 5 and 6 or the arguments from the end of the proof of Lemma 4 to infer monotonicity properties of the equilibrium, and instead we offer a direct proof.

Let $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ be a fully matched ex post equilibrium. We first show that $\boldsymbol{b}$ and $\boldsymbol{s}$ are increasing.

Consider $\boldsymbol{b}$ and suppose it is not increasing (the argument for $\boldsymbol{s}$ is analogous). Then there exist buyers $\underline{i} < \bar{i}$ with types $\underline{\beta} = \boldsymbol{\beta}(\underline{i})$ and $\overline{\beta} = \boldsymbol{\beta}(\bar{i})$ choosing investments $\underline{b} = \boldsymbol{b}(\underline{i})$ and $\bar{b} = \boldsymbol{b}(\bar{i})$ such that $\underline{\beta} < \overline{\beta}$ and $\underline{b} > \bar{b}$ holds (where the first of these two inequalities is from Assumption 6.2). Let $\underline{s} = \boldsymbol{s}(J(\underline{i}))$ and $\bar{s} = \boldsymbol{s}(J(\bar{i}))$. We must have

$$\hat{U}(\hat{f}(\underline{b}, \underline{s}), \underline{b}, \underline{\beta}) \geq \hat{U}(\hat{f}(\bar{b}, \bar{s})), \bar{b}, \underline{\beta}),$$

because otherwise $\underline{i}$ would prefer to match with $J(\bar{i})$ with investments $(\bar{b}, \bar{s})$ and, by separability this option is available to $\underline{i}$. Applying strict single crossing of $\hat{U}$ to the previous inequality implies

$$\hat{U}(\hat{f}(\underline{b}, \underline{s}), \underline{b}, \overline{\beta}) > \hat{U}(\hat{f}(\bar{b}, \bar{s})), \bar{b}, \overline{\beta}).$$

Using separability, this contradicts the hypothesis that $\bar{b}$ is the equilibrium investment of $\bar{i}$. Hence $\boldsymbol{b}$ and $\boldsymbol{s}$ are increasing.[31]

If $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ has no equilibrium cross match, then J is increasing and $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$ is positive assortative (cf. the first paragraph of Appendix G.2).

Let $(\underline{i}, \bar{j})$ and $(\bar{i}, \underline{j})$ with $\underline{i} < \bar{i}$ and $\underline{j} < \bar{j}$ be an equilibrium cross match. Let $\underline{\beta} < \overline{\beta}$ and $\underline{\sigma} < \overline{\sigma}$ denote the types of the agents in such a cross match, where the strict inequalities are from Assumption 6.2. Similarly, let $\underline{b}, \bar{b}, \underline{s}, \bar{s}$ denote their investments. We show

$$\underline{b} = \bar{b} \text{ and } \underline{s} = \bar{s} \tag{106}$$

which, from separability, implies

$$\boldsymbol{u}(i) = \breve{\phi}(i, \underline{j}, \boldsymbol{s}(\underline{j}), \boldsymbol{v}(\underline{j})) = \breve{\phi}(i, \bar{j}, \boldsymbol{s}(\bar{j}), \boldsymbol{v}(\bar{j})) \tag{107}$$

$$\boldsymbol{v}(j) = \breve{\psi}(j, \underline{i}, \boldsymbol{b}(\underline{i}), \boldsymbol{u}(\underline{i})) = \breve{\psi}(j, \bar{i}, \boldsymbol{b}(\bar{i}), \boldsymbol{u}(\bar{i})) \tag{108}$$

for $i \in \{\underline{i}, \bar{i}\}$ and $j \in \{\underline{j}, \bar{j}\}$.

Because $\boldsymbol{b}$ and $\boldsymbol{s}$ are increasing, we have $\bar{b} \geq \underline{b}$ and $\bar{s} \geq \underline{s}$. Assuming that one of these inequalities holds as an equality but that the other one is strict, results in an immediate contradiction because in that case (using Assumption 6.3 and separability) the two agents choosing the same investment would both strictly prefer to match with the higher of the two agents on the other side of the market. It thus remains to exclude the possibility that $\bar{b} > \underline{b}$ and $\bar{s} > \underline{s}$ hold. If this were the case, separability and Assumption 6.3 imply that agents $\bar{i}$ and $\bar{j}$ would both strictly prefer to match with each other rather than with their assigned partner. Hence, (106)–(108) hold in all cross matches.

[31] Notice that we need the strict single crossing of $\hat{U}$ for this, since otherwise we could have a range of values of β that are indifferent over a range of values of b, and hence could not so readily conclude that $\boldsymbol{b}$ and $\boldsymbol{s}$ are increasing.

As in the proof of Lemma 4 in Appendix G.2, conditions (107)-(108) imply that any equilibrium cross match can be uncrossed by matching the pair $(\underline{i}, \underline{j})$ and the pair $(\bar{i}, \bar{j})$ without changing their investments or utility levels, resulting in an ex post equilibrium payoff equivalent to $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$. When N is finite, an induction argument, again as in Appendix G.2, then shows that every fully matched ex post equilibrium is payoff equivalent to a positive assortative ex post equilibrium.

By Assumption 3, it remains to consider the case in which N is an interval. If

$$\boldsymbol{u}(i) = \breve{\phi}(i, i, \boldsymbol{s}(i), \boldsymbol{v}(i))$$
$$\boldsymbol{v}(j) = \breve{\psi}(j, j, \boldsymbol{b}(j), \boldsymbol{u}(j))$$

holds for all $i, j \in N$, then it is clear that changing J and I to the identity functions, while leaving $\boldsymbol{b}$, $\boldsymbol{s}$, $\boldsymbol{u}$, and $\boldsymbol{v}$ unchanged, results in an ex post equilibrium payoff equivalent to $(J, I, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v})$. Suppose then that there exists $i^* \in N$ such that

$$\boldsymbol{u}(i^*) > \breve{\phi}(i^*, i^*, \boldsymbol{s}(i^*), \boldsymbol{v}(i^*)) \tag{109}$$

holds (with the case in which the corresponding inequality for some $j^* \in N$ holds being analogous). We show that this results in a contradiction.

Inequality (109) implies $J(i^*) \neq i^*$ and, thus, $I(i^*) \neq i^*$. We assume $J(i^*) < i^*$ with the case $J(i^*) > i^*$ being analogous. We then have $J(i^*) < i^* < I(i^*)$: otherwise the pairs $(I(i^*), i^*)$ and $(i^*, J(i^*))$ form a cross match, with (109) then contradicting (107).

Throughout the following, let $b^* = \boldsymbol{b}(i^*)$, $s^* = \boldsymbol{s}(J(i^*))$, $\tilde{b} = \boldsymbol{b}(I(i^*))$ and $\tilde{s} = \boldsymbol{s}(i^*)$. Because $\boldsymbol{b}$ and $\boldsymbol{s}$ are increasing, we must have $\tilde{b} > b^*$ and $\tilde{s} > s^*$ as the equalities $\tilde{b} = b^*$ and $\tilde{s} = \tilde{s}^*$, which must hold otherwise, imply a violation of (109).

Suppose there exists $i' < i^*$ such that $J(i') > i^*$. We then have a cross-match with $\underline{i} = i'$ $\bar{i} = i^*$, $\underline{j} = J(i^*)$ and $\bar{j} = J(i')$). From (106) this implies $\boldsymbol{s}(J(i')) = s^*$. Because $\boldsymbol{s}$ is increasing, this implies $\boldsymbol{s}(i^*) = s^*$, a contradiction. We thus have that $J(i') < i^*$ holds for all buyers $i' < i^*$. Because the equilibrium matching is measure preserving, it follows that $I(j) < i^*$ holds for almost all sellers $j < i^*$. We can thus choose a sequence $\{j_n\}_{n=1}^{\infty}$ of sellers with $j_n > J(i^*)$ and $j_n \nearrow i^*$, with $i_n = I(j_n) < i^*$ for all n. Observe that all the matchings (i_n, j_n) generate a cross-match with the matching $(i^*, J(i^*))$. From (106) this implies $\boldsymbol{s}(j) = s^*$ for all $j \in [J(i), i^*)$. Using continuity of $\hat{V}$ and $\hat{g}$ and separability of preferences, it follows that

$$\hat{V}(g(\tilde{s}, \tilde{b}), \tilde{s}, \boldsymbol{\sigma}(i^*)) = \hat{V}(g(s^*, b^*), s^*, \boldsymbol{\sigma}(i^*)) \tag{110}$$

holds.

By an analogous argument we can construct a sequence of cross matches, involving buyers $i_n \searrow i^*$ matched with sellers $j_n > j$, leading to the conclusion that

$$\hat{U}(f(b^*, s^*), b^*, \boldsymbol{\beta}(i^*)) = \hat{U}(f(\tilde{b}, \tilde{s}), \tilde{b}, \boldsymbol{\beta}(i^*)) \tag{111}$$

holds.

Because $\boldsymbol{u}(i^*) = \hat{U}(f(b^*, s^*), b^*, \boldsymbol{\beta}(i^*))$ and $\boldsymbol{v}(i^*) = \hat{V}(g(\tilde{s}, \tilde{b}), \tilde{s}, \boldsymbol{\sigma}(i^*))$ and $\tilde{s} = \boldsymbol{s}(i^*)$ hold, the equalities in (110)–(111) then imply

$$\boldsymbol{u}(i^*) \leq \breve{\phi}(i^*, i^*, (i^*), \boldsymbol{v}(i^*)),$$

contradicting (109) and thus finishing the proof. ■

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