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Relative angular momentum balances of quasi-geostrophic circulation models

by Patrick F. Cummins

ABSTRACT
Calculations of the local vertical component of relative angular momentum are considered for quasi-geostrophic circulation models. The angular momentum diagnostic measures the net horizontal spin of the flow within an ocean basin and complements traditional budgets of energy, vorticity and linear momentum. In particular, the diagnostic can help identify the role of bottom topographic stresses in driving ocean circulation. Calculations of torque balances are presented for several numerical experiments with single-layer and multi-layer models. Experiments with an idealized continental slope topography develop a cyclonic circulation in the abyssal layer which is driven by a significant topographic form torque.

1. Introduction
Numerical simulations of ocean circulation are routinely subjected to diagnostic analyses such as calculations of energy, vorticity or zonal momentum budgets. These analyses can provide important information on physical mechanisms. A case in point is the calculation of energy transfers in the quasi-geostrophic (QG) model introduced by Holland (1978), which has provided insight into eddy-mean flow interactions in mid-latitude basins. However, the subtle influences of bottom topography may not be readily determined through application of these standard diagnostics. In particular, while the potentially important role of bottom topographic stresses in driving ocean circulation has been discussed (Holloway and Müller, 1990), the
mechanism has yet to be identified unambiguously in closed-basin modelling studies (e.g., Barnier and Le Provost, 1993).

Angular momentum and torque balances may complement standard diagnostic calculations and help to elucidate, for example, the role of topography on ocean circulation. The component of angular momentum about the earth's polar axis is a well-studied quantity in meteorology (e.g., Oort and Peixoto, 1983) which occasionally has been applied in oceanography. The works of Munk and Palmén (1951) and Straub (1993) on the dynamics of the Antarctic Circumpolar Current are based, in part, on considerations of axial angular momentum. Ponte and Rosen (1994) recently computed the axial angular momentum and a partial torque balance from a simulation of the global ocean circulation.

The vertical component of relative angular momentum was proposed as a model diagnostic by Holloway and Rhines (1991; hereinafter HR), who considered nondivergent flow in f-plane basins. This component of angular momentum provides a measure of the net horizontal spin of the fluid. Conservation laws concerned with the vertical component of angular momentum were applied by Flierl et al. (1983) and Larichev (1984) to the study of eddies on the beta plane. Cummins and Holloway (1994) examined the evolution toward equilibrium of an unforced, inviscid, barotropic flow over topography and found that relative angular momentum provided a useful measure of the adjustment process.

The present paper extends the application of vertical angular momentum balances to the QG model of Holland (1978). In the following, a QG approximation to the relative angular momentum is given and a torque balance equation is derived. Applications of this balance equation are discussed in Section 4 to illustrate the use of angular momentum as a diagnostic quantity in numerical simulations. In particular, we discuss the role of topographic form torques in maintaining the relative angular momentum and in driving a basin-scale circulation.

2. Quasi-geostrophic relative angular momentum

We consider a variable depth, Cartesian ocean basin rotating at an angular frequency $f_0/2$ and filled with $N$ statically stable, immiscible fluid layers of constant density $\rho_k$. The layer index, $k$, varies from $k = 1$ for the upper layer to $k = N$ for the lowest layer. Undisturbed layer depths are denoted $H_k$ and the (positive upward) interfacial displacement between layers $k$ and $k + 1$ is $\eta_{k+1/2}(x, y, t)$. A horizontal moment arm $r = x\hat{x} + y\hat{y}$ is defined in the rotating reference frame with the origin placed for definiteness at the axis of rotation. Note that $dr/dt = u_k$, where $u_k$ is the horizontal velocity relative to the basin.

The vertical ($\hat{z}$) component of relative angular momentum of the fluid layer is defined as

$$L_k^z(t) = \rho_k \int \int \int \hat{z} \cdot (r \times u_k) dV, \quad (2.1)$$
where $dV \equiv dzdA$ represents a differential volume element. The vertical integration is over an interval given by $H_k + \eta_{k-1/2} - \eta_{k+1/2}$, hence (2.1) may be expanded as

$$L_k^R(t) = \rho_k H_k \int \int \hat{z} \cdot \mathbf{r} \times \mathbf{u}_k dA + \rho_k \int \int (\eta_{k-1/2} - \eta_{k+1/2})\hat{z} \cdot \mathbf{r} \times \mathbf{u}_k dA. \quad (2.2)$$

We assume a length scale, $l$, and a relative velocity scale $U$ such that the Rossby number, $R_0 = U/l$, is a small parameter. Letting $H_k$ be a depth scale and $d$ represent an interface perturbation scale, it is assumed that $d/H_k \sim O(R_0)$. An appropriate scaling for the relative angular momentum is then $L^* = \rho_k H_k U$. If (2.2) is nondimensionalized by $L^*$ we find that the first term on the right-hand side (rhs) is $O(1)$ while the second term is $O(d/H) = O(R_0)$. Contributions to the relative angular momentum due to interface displacements are thus of order Rossby number in comparison to the first term. A similar result holds for the lowest ($k = N$) layer, as the ratio of bottom depth variations to the lower layer depth is assumed to be of $O(R_0)$.

Under quasi-geostrophy, the horizontal velocity, $\mathbf{u}_k$ is expanded according to

$$\mathbf{u}_k = \mathbf{u}_k^{(0)} + R_0 \mathbf{u}_k^{(1)} + R_0^2 \mathbf{u}_k^{(2)} + \cdots, \quad (2.3)$$

where the $O(1)$ geostrophic flow, $\mathbf{u}_k^{(0)}$, is nondivergent and given in terms of a velocity streamfunction,

$$\mathbf{u}_k^{(0)} = \hat{z} \times \nabla \psi_k. \quad (2.4)$$

Denoting the first term of (2.2) as $L'$, the QG relative angular momentum, we have

$$L' = \rho_k H_k \int \int \hat{z} \cdot \mathbf{r} \times \mathbf{u}_k dA = \rho_k H_k \int \int \hat{z} \cdot (\mathbf{r} \times \hat{z} \times \nabla \psi_k) dA + O(R_0). \quad (2.5)$$

(2.5) may be simplified using vector identities and the divergence theorem to yield (HR)

$$L'(t) = 2\rho_k H_k \int \int (\psi_k^B - \psi_k) dA + O(R_0), \quad (2.6)$$

where $\psi_k^B$ is the (constant) value of the streamfunction on the boundary of the basin. Thus the relative angular momentum of the fluid layer to lowest order is related to the area integral of the geostrophic streamfunction. This quantity should be regarded as a measure of the total ‘spin’ of the geostrophic flow relative to the Cartesian basin. It is important to note that the relative angular momentum given in (2.6) is distinct from the vertical component of the relative vorticity, $\zeta_k = \hat{z} \cdot \nabla \times \mathbf{u}_k^{(0)} = \nabla^2 \psi_k$, a measure of the pointwise rotation of fluid layer, and also from the area-integrated vorticity, which is the circulation around the basin perimeter. It is rather
the distribution of the vorticity within the domain which determines the streamfunction field, and hence the relative angular momentum.\(^2\)

3. Relative angular momentum balance

Our interest is in obtaining an equation valid to lowest order for the rate of change of \(L'_k(t)\). This result is obtained below from the QG potential vorticity equation. As a preliminary, it is instructive to consider a balance relation derived from the horizontal momentum equation. The latter may be written for layer \(k\) as

\[
\rho_k \left( \frac{du_k}{dt} + 2\Omega \times u_k \right) = F_k,
\]

where \(F_k\) represents lateral pressure gradient and viscous forces and \(\Omega = 2f_0/2\). Taking the cross product of \(r\) with (3.1) and integrating over the volume of the fluid layer we obtain

\[
\frac{dL^R}{dt} + \rho_k \int \int \frac{d}{dt} (\eta_{k+1/2} - \eta_{k-1/2}) \, dA \quad (3.2)
\]

This may be reduced without approximation to

\[
\frac{d}{dt} (L'_k + \Omega_k) = H_k \int \int \hat{z} \cdot (r \times F_k) \, dV,
\]

where we identify

\[
L'_k(t) = \rho_k H_k \int \int \frac{f_0}{2} |r|^2 \, dA
\]

as the QG 'planetary' component of angular momentum. (3.3) is a balance equation relating the rate of change of the vertical component of the total QG angular momentum, \(L'_k + \Omega_k\), to the external torques acting at the lateral boundaries of the layer. The planetary component arises due to solid body rotation of the fluid and may be expressed as \(\Omega_k = \frac{1}{2}f_0\), where \(I = \rho_k H_k \int \int |r|^2 \, dA\) is the moment of inertia with respect to the \(\hat{z}\) axis. It may be noted that \(\Omega_k\) is much larger than \(L'_k\); that is \(\Omega_k/L'_k \sim O(R_0^{-1})\). However, \(\Omega_k\) is dominated by a large constant term and the derivation given in Appendix A shows that \(dL^\Omega/dt\) is the same order as \(dL'/dt\).

In (2.6), the QG relative angular momentum of a fluid layer is given to lowest order by the area integral of the geostrophic streamfunction. This is now exploited to

\[2. \text{As a concrete example, in a circular basin of radius } a, \text{ the relative angular momentum can be expressed in terms of the relative vorticity according to } \int \int \hat{z} \cdot r \times u_k^{(0)} \, dA = 0.5 \int \int (a^2 - r^2) \zeta_k \, dA, \text{ where } r \text{ is the radial coordinate.}\]
calculate the relative angular momentum balances of the QG model. To this end (3.3) is rewritten as,

\begin{equation}
\frac{dL_k}{dt} = T_k,
\end{equation}

where \( T_k = H_k \int \int \hat{z} \cdot (r \times F_k) dA - \frac{dL_k^0}{dt} \) represents lateral boundary torques less the rate of change of planetary angular momentum. The latter appears on the rhs of (3.5) as an 'apparent torque' on the relative angular momentum. The presence of such apparent torques represents the most significant difference between the QG model and the special case of nondivergent f-plane flow considered by HR, where \( \frac{dL_k^0}{dt} = 0 \). As indicated in Appendix A, the apparent torques associated with this term arise due to divergence of ageostrophic velocities and motion in the presence of a gradient in the rotation rate (the beta effect).

A suitable expansion of (3.5) in Rossby number is

\begin{equation}
L_k' = L_k^{(0)} + R_0 L_k^{(1)} + \cdots \\
T_k = R_0 T_k^{(1)} + \cdots \\
\frac{d}{dt} \rightarrow R_0 \frac{d}{dt},
\end{equation}

where the time derivative is order \( R_0 \), following the usual QG scaling (Pedlosky, 1987; Section 6.2). Substituting into (3.5) gives the \( O(R_0) \) relative angular momentum balance

\begin{equation}
\frac{dL_k^{(0)}}{dt} = T_k^{(1)}.
\end{equation}

It is possible to identify \( L_k^{(0)} \), the relative angular momentum of the geostrophic flow, as the streamfunction integral term of (2.6). Torques enter at order \( R_0 \) to change \( L_k^{(0)} \).

A balance equation for \( L_k^{(0)} \) can be obtained from (2.6) and (3.7) by using the QG potential vorticity equation. We now summarize details of the QG layer model (c.f., Holland, 1978) whose governing equation is put in the form

\begin{equation}
\frac{\partial (\psi_k^p - \psi_k)}{\partial t} = \nabla^{-2} J(\psi_k, \zeta_k + \beta y) + \frac{f_0}{H_k} (w_{k+1/2} - w_{k-1/2}) - A_p \nabla \cdot \zeta_k, \quad k = 1, N,
\end{equation}

where \( \nabla^{-2} \) is an inverse Laplacian operator defined by (B2) in Appendix B, and \( J(a,b) = (a_b - b_a) \) is the Jacobian operator. \( \beta \) is the gradient in the Coriolis parameter, \( f = f_0 + \beta y \). \( A_p \) is the viscosity for either harmonic \( (p = 2) \) or biharmonic \( (p = 4) \) dissipation.
The interfacial vertical velocities, \( w_{k+1/2} \), are given by the kinematic relation

\[
\frac{\partial \eta_{k+1/2}}{\partial t} + \nabla \cdot \mathbf{\nu} = 0,
\]

where \( \psi_{k+1/2} = \frac{H_{k+1} \psi_k + H_k \psi_{k+1}}{H_k + H_{k+1}} \) is a weighted average of the layer streamfunction. Interfacial displacements are expressed in terms of the streamfunction according to

\[
\eta_{k+1/2} = \frac{f_0}{g'_{k+1/2}} (\psi_{k+1} - \psi_k),
\]

with \( g'_{k+1/2} \) is the interfacial reduced gravity.

At the surface, a rigid lid is assumed, \( \eta_{1/2} = 0 \). Wind forcing, bottom topography and bottom friction enter conveniently as boundary conditions on the vertical velocity

\[
\int \int w_{k+1/2} \, dA = 0, \quad k - 1, N - 1,
\]

where the wind stress vector is \( \mathbf{\tau} \), \( h \) is the bottom topographic elevation and \( \epsilon \) is the coefficient of bottom friction. An auxiliary condition on the interfacial vertical velocities (McWilliams, 1977),

\[
\int \int (w_{k+1/2} - w_{k-1/2}) \, dA = 0, \quad k = 1, N - 1,
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\[
\int \int (w_{k+1/2} - w_{k-1/2}) \, dA = 0, \quad k = 1, N - 1,
\]
term in (3.13) is an apparent torque associated with motion in the presence of a gradient in the rotation rate (c.f. with (A3) of Appendix A). The last term on the rhs of (3.13) is the torque due to lateral dissipation and it is dependent on the specification of boundary conditions for the viscosity. HR derived an expression for the viscous torque due to harmonic dissipation with free-slip lateral boundaries. An extension of their result is given in Appendix B.

Torques associated with the interfacial vertical velocities \((w_{k+1/2}, k = 1, N - 1)\) effect the transfer of relative angular momentum between layers. Comparing with relations (A3) and (A4) of Appendix A, it is evident that these vertical stretching terms involve the rate of change of planetary angular momentum. Through boundary conditions (3.11), torques associated with the curl of the wind stress and with bottom topographic stresses are included. Substituting from (3.9), torques associated with vertical velocities may be written as:

\[
2\rho f_0 \int \int \nabla^{-2}w_{k+1/2} \, dA = 2\rho f_0 \left[ \int \int \nabla^{-2} \frac{d\eta_{k+1/2}}{dt} \, dA + \int \int \nabla^{-2}J(\psi_{k+1/2}, \eta_{k+1/2}) \, dA \right].
\]

The second group on the rhs of (3.14) is associated with form stresses at the interfaces. This is evident if we express the interfacial streamfunction as the geostrophic pressure: \(\rho f_0 \psi_{k+1/2} = p_{k+1/2}^{(0)}\). Then using the definition of \(\nabla^{-2}\) it is possible to write

\[
\rho f_0 \nabla^{-2}J(\psi_{k+1/2}, \eta_{k+1/2}) = \int \int \nabla \cdot (p_{k+1/2}^{(0)} \nabla \eta_{k+1/2} \times \nabla G) \, dx \, dy
\]

where \(G\) is the Green’s function defined in Appendix B. The transfer of horizontal momentum between layers is through a form stress mechanism and involves a correlation between the slope of the interface, \(\nabla \eta_{k+1/2}\) and the pressure at the interface, \(p_{k+1/2}^{(0)}\). Analogously, the transfer of momentum between the bottom topography and the overlying fluid layer occurs through a spatial correlation between the bottom pressure and the bottom slope. Such topographic stresses produce torques which play an important role in some of the numerical examples discussed in Section 4.

An angular momentum analysis of the QG model in a statistically steady state requires a time-averaged balance equation. From (3.13) we have

\[
\rho H_k \int \int \nabla^{-2} \left[ J(\psi_k, \xi_k) + J(\overline{\psi_k}, \beta y) + \frac{f_0}{H_k} \left( \overline{w_{k+1/2}} - \overline{w_{k-1/2}} \right) \right]

- A_p \nabla \bar{v}_{\xi_k} \, dA = 0, \quad k = 1, N,
\]
where the overbar denotes a time-averaging. (3.16) can be applied to calculate the balance of torques which maintain the relative angular momentum of the model to lowest order. A Reynolds decomposition of the vorticity advection and vertical velocity terms could be used to separate mean and eddy components, but this has been omitted for simplicity. It is worth noting that only the interfacial form stress component of the vertical velocity torque (3.15) is nonzero in the mean. In the next sections relative angular momentum balances based on (3.13) or (3.16) are computed for a number of cases.

4. Examples

a. Reduced-gravity circular basin

A simple example illustrating angular momentum conservation is the case of an unforced, inviscid, reduced-gravity fluid within a circular $f$-plane basin of radius, $a$. Here a single active layer overlies a deep motionless lower layer. The governing balance equation is obtained from (3.13) and (3.14) by considering a two-layer model ($N = 2$) and letting the lower layer depth tend to infinity. This yields

$$\frac{dL_r^{(0)}}{dt} = 2\rho_1 H_1 \int \int \nabla^{-2} \left[ J(\psi_1, \zeta_1) + \frac{f_0}{H_1} \partial_r \eta_{1/2} \right] dA. \quad (4.1)$$

Note that form stress momentum transfer between the active layer and the lower one tends to vanish as the lower layer depth increases to infinity.

For the circular domain it is convenient to use $M(r) = (r^2 - a^2)/4$, with $r$ a radial coordinate, to invert the Laplacian in place of $\nabla^{-2}$. $M$ is the solution of $\nabla^2 M = 1$ in a circular domain with boundary condition, $M(a) = 0$. Inside the area integrals $M(r)$ is equivalent to $\nabla^{-2}$, that is $\int \int \nabla^{-2} \psi dA = \int \int M \nabla^2 \psi dA = \int \int (\psi - \psi^0) dA$. Dropping the layer index and the expansion superscript, which is omitted henceforth, (4.1) becomes

$$\frac{dL_r'}{dt} = \frac{\rho f_0}{2} \int \int (r^2 - a^2) \partial_r \eta dA. \quad (4.2)$$

In this example, the torque associated with vorticity advection vanishes on integration, $\int \int MJ(\psi, \zeta) dA = 0$; the boundary pressure has no moment arm and does not exert a torque. Using (2.6), (3.10) and noting that mass conservation requires $\partial_t \int \int \eta dA = \partial_t \frac{f_0}{g} \int \int \psi dA = 0$, (4.2) further simplifies to

$$\frac{d}{dt} \rho H \left[ 2 \int \int \psi^0 dA + \frac{R_i^{-2}}{2} \int \int r^2 \psi^0 dA \right] = 0. \quad (4.3)$$

where $R_i = \sqrt{g/H/f_0}$ is the internal deformation radius. (4.3) is an expression of the conservation of total angular momentum of the fluid. Since no torque is exerted on the fluid, the sum of the relative and planetary components of angular momentum is
invariant. As the basic rotation rate, $f_0/2$, is constant, variations in the moment of inertia due to layer thickness fluctuations are compensated by variations in the relative angular momentum. (Due to (3.12), $\psi = \psi(t)$ and the first term is nonzero.) Larichev (1984) has derived a conservation law similar to (4.3) for isolated eddies.

It is interesting to relax the rigid lid assumption and include the additional term associated with surface displacements. (4.3) then becomes

$$\frac{d}{dt} \rho H \left[ 2 \int \int \psi dA + \frac{(R_i^{-2} + R_e^{-2})}{2} \int \int r^2 \psi dA \right] = 0,$$

(4.4)

where $R_e = \sqrt{gH/f_0}$ is an external deformation radius based on the upper layer thickness. Generally $R_e \gg R_i$ and the neglect of the surface displacement is warranted for the upper layer. There are situations, however, where neglect of the free surface displacement may not be a good approximation. For example, in the case of a barotropic fluid, fluctuations in $L^B$ due to free surface displacements may be comparable to fluctuations in $L^r$ if the basin scale is of the order of the external deformation radius (about 2000 km at midlatitudes with $H = 5000$ m). Similar remarks apply to the depth-integrated angular momentum in a multi-layer model. In the numerical experiments discussed below, the rigid lid is nevertheless retained because it is standard in the model formation. However, application of rigid lid models (QG or otherwise) to the study of depth-integrated angular momentum fluctuations in real ocean basins may be problematic.

b. Numerical simulations

Several numerical experiments were conducted to illustrate the application of angular momentum and the torque balance relations discussed in Section 3. These experiments may be classified as (1) barotropic cases with an unforced, inviscid model and with a stochastically forced-dissipative model, and (2) baroclinic cases with a steadily-forced, three-layer model. The idealized barotropic cases are intended to demonstrate simple processes affecting the rate of change of relative angular momentum. Experiments with the three-layer model are somewhat more realistically configured and may have greater relevance to the circulation of midlatitude basins. In all cases, the basin dimensions are $2560 \times 2560$ km with a uniform grid resolution of 20 km. Parameter choices are given in Table 1 for the single-layer cases, and in Table 2 for the multi-layer experiments. The forced-dissipative experiments employ a biharmonic viscosity with free-slip boundary conditions.

Some of the test cases discussed below were designed to illustrate the effects of the topographic form torque in driving a basin-scale circulation with angular momentum. A model topography, referred to as T1 and shown in Figure 1, was constructed for this purpose. With topography T1, a continental shelf/slope topography (limited to 500 m in amplitude) lies around the basin perimeter. Superposed on this large-scale topography is a mesoscale topography with an rms amplitude of 50 m and a $k^{-2}$
Table 1. Parameters of the barotropic model experiments. For all experiments, $f_0 = 1 \times 10^{-4}$ s$^{-1}$, $H = 5000$ m, $\rho = 1000$ kg m$^{-3}$ and topography $T_1$ is used. For the unforced inviscid experiments (UI1-UI3), $\beta = \epsilon = 0$. In the stochastically forced experiments (SF1, SF2) $A_4 = -2 \times 10^{10}$ m$^4$ s$^{-1}$, $\epsilon = 1 \times 10^{-7}$ s$^{-1}$ and the wind stress curl forcing is a Markov process with a white noise level of $5 \times 10^{-7}$ N m$^{-2}$ Hz$^{-1}$ and an integral time scale of 10 days. The stochastic forcing is isotropic and uniform between wavenumbers $k = 1$ (2560 km) and $k = 4$ (640 km).

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$U_{\text{mos}}$ (cm s$^{-1}$)</th>
<th>$\beta$ (10$^{-11}$ m$^{-1}$ s$^{-1}$)</th>
<th>Special</th>
</tr>
</thead>
<tbody>
<tr>
<td>UI1</td>
<td>5.0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>UI2</td>
<td>5.0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>UI3</td>
<td>5.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>SF1</td>
<td>3.6</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>SF2</td>
<td>3.4</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The wavenumber spectrum between $k = 4$ (640 km) and $k = 16$ (160 km). Topography $T_1$ is similar to the one used in Expt. 6 of Cummins and Holloway (1994), where it was shown that the shelf/slope topography induces a basin-scale cyclonic circulation in unforced, inviscid experiments.

Numerical integrations of the barotropic and multi-layer quasi-geostrophic models were carried out using standard, second-order accurate, finite difference techniques with Arakawa's method used to discretize Jacobian terms. For the unforced, inviscid experiments it is desirable that the discrete model maintain the low-order invariants of the system, as assumed by Salmon et al. (1976). Details on the integration of the unforced, inviscid cases are given in Cummins and Holloway (1994). The inviscid model maintains the potential vorticity invariant to machine precision, while the energy and potential enstrophy invariants are maintained to a slightly lower degree of accuracy. An extension of the method for two layers described by Holland (1978) was applied to integrate the multi-layer model. These experiments were first integrated to a state of statistical equilibrium, after which flow statistics were determined. Thus, in the multi-layer experiments an initial adjustment period of 18 years (c.f., Barnier and Le Provost, 1993) was allowed, and a similar period was used to compute the time-averaged vorticity terms within the integrals of (3.16).

Table 2. Parameters of the experiments with a three-layer model. In all cases, $f_0 = 1 \times 10^{-4}$ s$^{-1}$, $A_4 = -2 \times 10^{10}$ m$^4$ s$^{-1}$, $\epsilon = 1 \times 10^{-7}$ s$^{-1}$ and $\beta = 2 \times 10^{-11}$ m$^{-1}$ s$^{-1}$. Stratification parameters are $H_k = (300, 700, 4000)$ m and $g \kappa^2 = (3 \times 10^{-2}, 1.5 \times 10^{-2})$ m s$^{-2}$, yielding deformation radii of 39.5 and 22 km for first and second internal modes, respectively. The wind stress curl forcing is $\hat{z} \cdot \nabla \tau = -n \pi \tau_0 l^{-1} \sin(n \pi y/l)$, where $\tau_0 = 0.1$ N m$^{-2}$, $l = 2560$ km is the basin scale and $n = 1(2)$ for a single (double) gyre wind.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Wind Forcing</th>
<th>Topography</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC1</td>
<td>single gyre</td>
<td>Flat bottom</td>
</tr>
<tr>
<td>BC2</td>
<td>single gyre</td>
<td>T1</td>
</tr>
<tr>
<td>BC3</td>
<td>double gyre</td>
<td>T1</td>
</tr>
</tbody>
</table>
Figure 1. Bottom topography T1 used in several numerical experiments. The contour interval (CI) is 75 m. (b) Contours of the planetary vorticity, $\beta y + f_0 h / H$, with topography T1 and $\beta = 2 \times 10^{-11} \text{m}^{-1} \text{s}^{-1}$. CI = $3 \times 10^{-6} \text{s}^{-1}$.

i. Unforced, inviscid experiments. A number of barotropic, unforced, inviscid experiments with $f$-plane and beta-plane basins were carried out in which an initial field of random eddies is released and allowed to evolve to statistical equilibrium. These cases serve two purposes. The first is to examine the role of the torques associated with vorticity advection and topographic vortex stretching in driving the system to an equilibrium value of relative angular momentum. The second purpose is that the unforced, inviscid equilibria are a useful starting point for interpreting experiments with forcing and dissipation, since these equilibria indicate the state toward which nonlinearities acting alone tend to drive the flow (Griffa and Salmon, 1989).
Figure 2 shows the initial streamfunction field of experiment UII and the field averaged from $t = 0$ to $t = 5l/U_{rms}$, where $l/U_{rms}$ is an advection time scale with $l$ the basin scale (2560 km), and $U_{rms} = 5$ cm s$^{-1}$ the rms velocity. The initial eddy field is constructed from a wavenumber spectrum which is uniform between $k = 2$ (1280 km) and $k = 16$ (160 km). Upon release, the flow field evolves toward an absolute equilibrium state, which is characterized by a linear relation between streamfunction and potential vorticity, $q = \zeta + \hat{h}$ (Salmon et al., 1976), with $\hat{h} = f_0 h/H$. A large-scale cyclonic circulation over the continental slope develops during the adjustment to equilibrium (Cummins and Holloway, 1994). Eventually, the streamfunction field resembles a smoothed version of the topographic field.

For experiment UII, the balance equation (3.13) simplifies to

$$\frac{dL'}{dt} = P + T,$$

where $T = 2\rho H \int \int \nabla^{-2}J(\psi, \hat{h}) \, dA$ and $P = 2\rho H \int \int \nabla^{-2}J(\psi, \zeta) \, dA$ are the contributions to the torque due to topographic form stresses and boundary pressure. For convenience, the layer index subscript has been dropped in (4.5) and $H$ is taken as a representative depth of the barotropic model. The temporal variation of $L'$, $P$ and $T$ over the first few advective time units is given in Figure 3. Over the period shown, there is a net change in the relative angular momentum as it increases from its initial value to a value approaching that of the equilibrium state. This adjustment is fairly rapid with most of the increase in $L'$ occurring over the first advective time unit (about 1.6 years). The time series of the torque terms indicates that it is mainly the topographic form torque which produces the increase in $L'$. Over the period shown, the time series of $T$ shows large, rapid fluctuations about a small positive mean value ($\approx 1.3 \times 10^{15}$ J). This mean value of $T$ accounts for about 80% of the increase in angular momentum over the initial $5T_{rms}$.

In experiment UII, the topographic form torque is dominant in driving a circulation around the basin perimeter over the model continental shelf. The mechanism by which this takes place was described by Holloway (1987), and demonstrated by Treguier (1989) in a model with a zonally-reentrant channel geometry. Nonlinear interactions produce a time-averaged tendency for a negative spatial correlation between relative vorticity and mesoscale topographic irregularities. An increase in $C = -\int \int \hat{h}\hat{\zeta} \, dA$ is characteristic of the tendency toward statistical equilibrium (Griffa and Salmon, 1989). The continental slope provides a large-scale gradient in potential vorticity which induces a tendency for phase propagation of vorticity perturbations in a pseudo-westward direction, with shallow water to the right. This causes a systematic pseudo-westward displacement of pressure anomalies associated with the vorticity perturbations. In consequence, a spatially-averaged topographic form stress develops which acts on average to drive a circulation in the pseudo-westward direction. This stress, in UII, contributes to provide most of the net torque.
Figure 2. Initial streamfunction fields for the unforced, inviscid experiments (a) and time-averaged streamfunction fields from $t = 0$ to $t = 5L/U_{rms}$ from experiment UI1 (b) and UI3 (c). CI = 2000 m$^2$ s$^{-1}$. 
Figure 3. (a) Time series of $L'$ from experiments UI1 and UI2. (b) Time series of the torques $T$ and $P$ (Section 4b(i), Eq. (4.5)) from experiment UI1.

which ‘spins’ the circulation around the basin in a cyclonic sense. In the experiments of Treguier (1989), it is the β-effect that provides the large-scale potential vorticity gradient that induces an along-channel flow. As described below, the topographic forcing mechanism also operates in the forced-dissipative cases to drive a basin-scale circulation.

Two other unforced, inviscid experiments are listed in Table 1. Both are similar to UI1, except that in experiment UI2 the model is linearized, while in experiment UI3 the beta term is retained. In the linear experiment, the relative angular momentum (Fig. 3a) has no tendency to increase and it simply oscillates about its initial value. Although the boundary pressure torque associated with relative vorticity advection is not an important forcing, the absence of vorticity advection in the linearized model prevents the development of a mean topographic form torque. The reason for the different behavior is that, in the linearized case, $C$ is an invariant of motion, so that no net correlation between vorticity and topography is possible. Thus the mechanism given above for generating an average topographic form stress cannot occur. In experiment UI3, the inclusion of the beta term significantly modifies the $f/H$ contours (Fig. 1b) and large-scale potential vorticity gradient of the small-amplitude
topography, which has consequences for the time-averaged circulation (Fig. 2c). The statistical equilibrium flow of this case is useful to interpret results of the following sections.

**ii. Stochastically forced experiments.** Two experiments are included in Table 1 in which a single-layer basin is forced by a stochastic wind. These experiments are intended to demonstrate a simple situation in which the topographic form torque maintains the relative angular momentum of a basin in a statistically steady state. Bottom and lateral friction are retained to provide sinks for the dissipation of energy supplied by the wind. The wind stress curl forcing is specified according to a first-order Markov process as described by Treguier and Hua (1987). The stochastic forcing may be regarded as a means to force eddy energy in the model. However, it may also be considered a simple representation of the random direct forcing of the ocean by synoptic atmospheric disturbances (Frankignoul and Müller, 1979). The white noise level for the forcing ($5 \times 10^{-7}$ N m^-2 Hz^-1) is comparable to other studies (Treguier and Hua, 1987), but somewhat larger than observations from midlatitudes of the North Pacific (Chave et al., 1991).

Time-averaged streamfunction fields for experiments SF1 and SF2 are given in Figure 4. The shelf/slope topography in SF1 induces a rectified circulation around the basin perimeter which is significantly modified by the beta effect in SF2. The torques of Figure 5 indicate the sources and sinks which maintain the horizontal spin of the rectified flows of Figure 4. In both experiments, the topographic form torque is the main mechanism sustaining the relative angular momentum of the model. The topographic torque is balanced by the bottom friction torque in experiment SF1, whereas in experiment SF2 it is balanced by the beta torque. The wind forcing torque is not large in these experiments because the Markov process used to specify the stochastic wind has only a weak mean component.

The topographic form torque is due to the same mechanism as in the unforced, inviscid experiments, UI1 and UI3. Topographic stresses, which drive pseudowestward flow along the large-scale potential vorticity gradient, arise as the flow develops an anticorrelation between topography and vorticity. The latter is a consequence of the tendency for nonlinearity to drive the flow toward the statistical equilibrium state. Consequently, the pattern of the rectified flows of experiment SF1 and SF2 bear a similarity to equilibrium flows of experiments UI1 and UI3 (Fig. 2b and c), respectively.

Holloway (1992) has argued that a parameterization of the mean flows which develop in consequence of eddy-topography interactions is necessary for coarse resolution models. He has suggested that a suitable parameterization may involve incorporating a tendency toward a simplified form of the statistical equilibrium solution in which the streamfunction is proportional to the topographic variation, $\Psi \propto \hat{h}$. The similarity at large scales between the topography and streamfunction fields in
Figure 4. Mean streamfunction fields from the stochastically forced experiments SF1 (a) and SF2 (b). CI = 1000 and 500 m$^2$ s$^{-1}$ for (a) and (b), respectively.

experiment SF1 suggests that this representation may be a reasonable one for the transport properties of the rectified flows. However, systematic departures from $\bar{\psi} \propto \hat{h}$ must occur otherwise the topographic form torque in (3.6) would identically vanish. In experiment SF2, $\beta$ is $O(|\nabla \hat{h}|)$ and leads to significant modification of the time-averaged flow over the small-amplitude topography. However, neglect of the beta term in the parameterization of mean flows may be warranted in situations where the topography has a realistic finite amplitude. (For example, if topography T1
Figure 5. Relative angular momentum and torque balances computed from (3.16) for the stochastically forced experiments SF1 (a) and SF2 (b). Time-averaged torques associated with the curl of the wind stress, topographic form stress, bottom friction, boundary pressure and motion on the beta plane are denoted by the symbols WIND, T, BF, P and β, respectively. Units of $L^r$ are $10^{21}$ Joules $^{-1}$, while torques have units of $10^{15}$ Joules. Only torques that are significantly different than zero are indicated.

is rescaled to amplitudes that are more representative of oceanic topography, then the $f/H$ contours are virtually unaffected by the omission of $β$.)

### Steadily-forced multi-layer model

The baroclinic quasi-geostrophic model developed by Holland (1978) has been widely applied for process studies of ocean circulation. In this section, some examples of the application of the torque balance relation (3.16) to this model are discussed. Of the three experiments listed in Table 2, two are with a single-gyre wind forcing and a comparison is presented between a flat-bottom case and one with topography $T_1$. There is also a third experiment with a standard symmetric double-gyre wind and topography $T_1$.

The time-averaged circulation of experiment $BC_1$ given in Figure 6a shows an anticyclonic gyre with a northern boundary current in the upper two layers extending across the width of the basin. The gyre is composed of a highly inertial recirculating flow near the northern boundary and by a weaker Sverdrup flow which extends over most of the interior. The bottom layer has a relatively weak mean flow over the interior and Fofonoff-like gyres are found near the southern and northern boundaries which are associated with the tendency toward statistical equilibrium in flat bottom, beta-plane basins (Griffa and Castellari, 1991; Cummins, 1992).

Figure 7a shows results of application of the torque balance (3.16) to experiment $BC_1$. For each layer, the torque balance is closed to within about 1% and only those torques which are significantly greater than this residual are indicated. Despite the strongly inertial character of the circulation in $BC_1$, the time-averaged relative angular momentum of each layer is dominated by a balance between torques associated with motion on the beta plane and stretching of the interface through form stresses. Thus a geostrophic torque balance

$$
\frac{f_0}{H_k} \int \int \nabla^{-2}(w_{k-1/2} - w_{k+1/2}) \, dA = \beta \int \int \nabla^{-2} \frac{\partial \psi_k}{\partial x} \, dA
$$

(4.6)
Figure 6. (a) Mean streamfunction fields of the upper \((k = 1)\) and lower \((k = 3)\) layers from experiments BC1 (a), BC2 (b) and BC3 (c). CI = 10000 (2000) m² s⁻¹ for upper (lower) layer fields.

is obtained, analogous to the geostrophic vorticity balance. The wind stress curl applies an anticyclonic torque to the uppermost layer with a tendency to drive fluid southward across \(\beta\) contours to a region of lower rotation (lower \(f_0 + \beta y\)). The presence of \(\beta\) produces a tendency to resist this southward motion and hence an opposing cyclonic torque. Similar remarks apply to the middle layer where the
anticyclonic torque is supplied by interfacial form stresses. The vertical extent of the wind-forced relative angular momentum is largely confined to the two upper layers; the angular momentum content of the bottom layer is quite small in comparison. Dominance of the geostrophic torque balance is consistent with Harrison and Holland (1981), who found that the geostrophic vorticity balance was dominant over most of their QG model domain.
With experiment BC2, parameters remain unchanged from experiment BC1, except that topography T1 is included. As Figure 7b shows, the geostrophic torque balance is again obtained from the model. The relative angular momentum content of the upper layers for BC2 is not greatly different than those of BC1; the wind-driven anticyclonic gyre results in a net negative relative angular momentum for the two upper layers. However, the situation is different from the bottom layer where
a topographic form torque now produces a significant positive value of relative angular momentum for this layer. As in experiment SF2 of Section 4b(ii), the topographic form torque acts as a source driving a cyclonic circulation. It is notable that, even in comparison to a standard surface wind forcing, the topographic torque constitutes a significant transfer to the system. The mean circulation pattern in the southern half of the bottom layer in experiment BC2 (Fig. 6b) bears a qualitative resemblance to the unforced, inviscid equilibrium flow of experiment U13. The tendency toward the statistical equilibrium state is again evident in these steadily forced simulations.

The final experiment listed in Table 2 is one in which a symmetric, double-gyre wind forcing is specified. The response of the baroclinic model to such a forcing is a problem which has received considerable attention. In the flat-bottom case, the circulation in each layer is symmetric in the north-south coordinate and the time-averaged flow has zero net relative angular momentum. Thus the global relative angular momentum is not an interesting quantity in such a case. However, with the inclusion of topography T1, the symmetry of the response is broken, and a calculation of torque balances is again useful. The budget for experiment BC3 in Figure 7c indicates that bottom topography is an important source of relative angular momentum for the bottom layer. The circulation pattern over the southern part of the bottom layer (Fig. 6c) resembles that of experiment BC2 and the unforced, inviscid calculation, U13. In the upper layers, where the response is dominated by nearly
symmetric wind-driven gyres, there is relatively little net relative angular momentum in comparison to the single-gyre cases.

5. Summary

Calculations of the vertical component of relative angular momentum and torque balances for quasi-geostrophic models of ocean circulation have been discussed. The relative angular momentum provides a measure of the net horizontal spin of the circulation within a basin. To lowest order in Rossby number, this quantity is given by an area integral of the layer streamfunction field. Based on this relation, a balance equation (3.13) was derived which provides a diagnostic tool for process studies. Torque balances which sustain the relative angular momentum of a basin can be identified and, in particular, the role of topographic form torques in driving ocean circulation can be examined.

Use of the diagnostic was demonstrated in several experiments with single-layer and multi-layer models designed to illustrate the influence of topographic form torques. Topographic flows are generated as an anticorrelation develops between vorticity and topography due to tendencies toward the statistical mechanical equilibria. One of the interesting results of these experiments is that the topographic form torque is significant and can be of a magnitude comparable to the torque associated with a standard wind stress curl field. An additional result from multi-layer simulations is that the model response is dominated by a geostrophic torque balance, analogous to the familiar geostrophic vorticity balance.

Results show that calculation of the lowest order angular momentum and torque balances for quasi-geostrophic models is straightforward and provides insight into a dynamical balance of the system. The present approach based on integrals of the stream function field is limited to relatively simple ocean circulation models. Aside from the quasi-geostrophic models considered here, this approach could also be directly applied to a rigid lid, shallow water system for unstratified flow over finite topography.

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APPENDIX A

Rate of change of planetary angular momentum

Here we calculate an expression for the rate of change of planetary angular momentum to interpret torques on the relative angular momentum which appear on the rhs of (3.13). The derivation leading to (3.3) assumed a uniform rotation rate, \( f_0/2 \). QG models often include a beta effect so that the Coriolis parameter is \( f(y) = f_0 + \beta y \), which is equivalent to a variation of the rotation rate in the \( y \)-direction. The expression for the QG relative angular momentum (2.5) is unaffected if \( f(y) \) replaces
Replacing \( f_0 \) in (3.1) with \( f(y) \) we have \( dL^0_k/dt = \rho_k H_k \int \int f(y)r \cdot u_k dA \). Substituting (2.3) and expanding we obtain

\[
\frac{dL^0_k(t)}{dt} = \rho_k H_k \left[ \int \int f_0 r \cdot (\hat{z} \times \nabla \psi_k) \, dA + \int \int \beta y r \cdot (\hat{z} \times \nabla \psi_k) \, dA \right. \\
+ \int \int f_0 R_0 r \cdot u_k^{(1)} \, dA + \int \int \beta y R_0 r \cdot u_k^{(1)} \, dA \left. \right] + O(R_0^2)
\] (A1)

Although scaling analysis would suggest that the first term on the rhs of (A1) is dominant, this term actually integrates to zero. The third term may be simplified by use of the identity

\[
r \cdot u_k^{(1)} = \frac{1}{2} [\nabla \cdot (|r|^2 u_k^{(1)}) - |r|^2 \nabla \cdot u_k^{(1)}],
\]

and the expression for the divergence of the ageostrophic velocities

\[
R_0 \nabla \cdot u_k^{(1)} = - \frac{\partial \psi_k}{\partial z} = - \frac{(w_{k-1/2} - w_{k+1/2})}{H_k}.
\]

Substituting into (A1) and simplifying we obtain

\[
\frac{dL^0_k(t)}{dt} = - \frac{\rho_k H_k}{2} \left[ \beta \int \int |r|^2 \frac{\partial \psi_k}{\partial x} \, dA + f_0 \int \int |r|^2 \frac{(w_{k+1/2} - w_{k-1/2})}{H_k} \, dA \right. \\
+ \int \int \beta y R_0 r \cdot u_k^{(1)} \, dA \left. \right] + O(R_0^2).
\] (A2)

We retain the terms in (A2) which are of the same order as \( dL^0/dt \), that is terms of \( O(L^* U/l = \rho_k H_k U^2 l^2) \). The second term on the rhs of (A2) is of this order, while the first term is \( O(\beta^*) \) in comparison to the second term, where \( \beta^* = \beta l^2/U \) is the nondimensional beta. This term is retained under the assumption that \( \beta^* \) is \( O(1) \). The last term of (A2) is \( O(R_0 \beta^*) \) in comparison to the second term and neglected. Thus (A2) reduces to

\[
\frac{dL^0_k(t)}{dt} = - \frac{\rho_k H_k}{2} \left[ \int \int |r|^2 J(\psi_k, \beta y) \, dA \right. \\
+ \frac{f_0}{H_k} \int \int |r|^2 (w_{k+1/2} - w_{k-1/2}) \, dA \left. \right] + O(R_0^2).
\] (A3)

The terms that are retained on the rhs of (A3) are \( O(R_0) \), consistent with the Rossby number expansion given in (3.6). (A3) indicates two contributions to the rate of change of planetary angular momentum which are of the same order as the rate of change of relative angular momentum. These arise due to flow in the presence of a gradient in the rotation rate, and vortex stretching due to vertical motion at the interfaces.

Terms appearing on the rhs of the balance equation (3.13) include the \( O(R_0) \)
contributions to $dL_k^\Omega/dt$ given by (A3), with the $|r|^2/4$ spatial weighting implicit in the $\nabla^{-2}$ operator. To make this evident, we first note that the solution to $\nabla^2 \phi = 1$ with boundary condition $\phi = 0$ is given by $\phi = |r|^2/4 + \phi^H$, where $\nabla^2 \phi^H = 0$ and $\phi^H = -|r|^2/4$ on the boundary. Thus $\nabla^{-2}(1) = \phi$. We now write the vorticity equation (3.8) as $\partial_t \nabla^2 \psi = \nabla V T$, where $\nabla V T$ represents the sum of vorticity tendency terms, and the layer index $k$ is omitted. Multiplying by $-2\rho H \phi$ and integrating over the domain we get

$$
\frac{dL^{(0)}}{dt} = -2\rho H \int \int \left(\frac{|r|^2}{4} + \phi^H\right) \sum V T dA,
$$

where $L^{(0)} = 2\rho H \int \int (\psi^H - \psi) dA$. The term involving $\phi^H$ on the rhs involves a boundary circulation contribution, $\int \int \phi^H \Sigma V T dA = \int \int \phi^H \nabla^2 \psi, dA = -0.25 \phi |r|^2 \delta u \cdot ds$. In the circular basin example of Section 4a, $\phi^H = -a^2/4$, a constant.

**APPENDIX B**

Viscous decay of relative angular momentum

The study of HR elucidated the mechanism of viscous decay of angular momentum and inquired into the physical consistency of lateral boundary conditions. They noted that the relative angular momentum of a fluid in a rectangular domain decays to zero even when free-slip boundary conditions are imposed. (Free-slip is defined as zero viscous transfer of tangential stress across the boundary.) Their analysis for this case showed that the decay of angular momentum is due to the torque associated with the normal viscous stresses on the sidewalls. HR also remark that in a circular domain these stresses do not exert a torque.

We now consider the decay of relative angular momentum in a simply connected domain with the vorticity equation as a starting point. Suppressing for simplicity all terms involving advection, forcing and vertical transfer, the decay of vorticity is governed by

$$
\frac{\partial \zeta}{\partial t} = A_p \nabla^p \zeta,
$$

where $\zeta = \nabla^2 \psi, \psi = 0$ on the boundary, and $A_p$ is the viscosity for harmonic ($p = 2$) or biharmonic ($p = 4$) dissipation.

An inverse Laplacian operator, $\nabla^{-2}$, is required which is defined analytically as

$$
\nabla^{-2}(f) \equiv \int \int f(x_0, y_0)G(x, y; x_0, y_0) \, dx_0 \, dy_0.
$$

$G$ is the Green's function for the Poisson equation,

$$
\nabla^2 G = \delta(x - x_0, y - y_0),
$$

where $\delta$ is the Dirac delta function and the condition $G = 0$ is imposed at the
boundary. With this definition we may note that $\psi = \nabla^{-2}\zeta$, where Green's second identity and the appropriate boundary conditions have been applied. Thus the relation between the vorticity field and the streamfunction field, hence the relative angular momentum, is through the spatial correlation between the vorticity and the Green's function, $G$.

Taking $p = 2$, an equation governing the decay of angular momentum can be obtained by first multiplying (B1) by $G$ and integrating over $(x_0, y_0)$. Applying Green's second identity leads to

\[
\frac{\partial \psi}{\partial t} = A_2 \left[ \zeta - \oint \zeta^B \nabla G \cdot n \, ds_0 \right],
\tag{B4}
\]

where $\zeta^B$ is the boundary vorticity. $n$ is a unit outward normal vector and $s_0$ is a coordinate along the boundary. Integrating (B4) over the domain and multiplying by $-2\rho H$, we obtain the required balance equation governing the decay of relative angular momentum, $L' = -2\rho H \int \int \psi \, dA$,

\[
\frac{dL'}{dt} = -2\rho H A_2 \left[ \oint \mathbf{u} \cdot ds - \oint \oint \left( \oint \zeta^B \nabla G \cdot n \, ds_0 \right) \, dA \right].
\tag{B5}
\]

For no-slip boundaries, $\mathbf{u} \cdot ds = 0$, and the first term on the right side of (B3) vanishes. If the model domain has rectilinear boundaries, i.e. the radius of curvature of the walls is infinite, then a free-slip condition implies that $\zeta^B = 0$. (For the relation between tangential stress and boundary vorticity see HR and Batchelor (1967, Ch. 4).) In this case, the second term on the right side of (B5) vanishes, leaving $dL'/dt = -2\rho H A_2 \oint \mathbf{u} \cdot ds$, which relates the decay of relative angular momentum to the Kelvin circulation integral. The relation (B5) generalizes the corresponding equation (8) of HR to allow for no-slip boundaries and a nonrectilinear domain geometry.

In a similar fashion, the viscous decay of relative angular momentum with a biharmonic viscosity and free-slip, rectilinear walls can be also reduced to a boundary integral. The appropriate boundary conditions in this case are $\zeta^B = \nabla^2 \zeta^B = 0$ and the result is:

\[
\frac{dL'}{dt} = -2\rho H A_2 \oint \nabla \zeta^B \cdot n \, ds.
\tag{B6}
\]

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