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On the influence of the Continental Slope on the Western Boundary Layer: The enhanced transport and recirculation

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ABSTRACT

Quasi-geostrophic theory is used to study the effect of a continental slope on the Western Boundary Layer. The compression of the vortex tubes by the slope results in a strong northward boundary current called the Continental Slope Boundary Current (hereafter CSBC). On the $\beta$-plane, for a reasonably high slope, we find a strong barotropic recirculation which enhances the total transport of the Western Boundary Current significantly. The two-layer model further shows that the CSBC is trapped in the lower layer. In oceans with very deep lower layers, the CSBC transport increases dramatically. Consequently, even for a very weak lower layer incoming flow, we can still have a very strong barotropic CSBC transport compared to the Inertial Boundary Current. Additionally, for an ocean with a very deep lower layer, we can always have comparable total transport in both layers even when the lower layer incoming flow is very weak.

1. Introduction

One of the most important challenges facing theories of the large scale ocean circulation is to explain the very large transport measured in the Western Boundary Currents. Recent observations show that the total transport of the Gulf Stream is much larger than that needed to balance the Sverdrup transport. In fact, the total Sverdrup transport is only about 30 Sv ($1 \text{ Sv} = 10^6 \text{m}^3/\text{sec}$), while the observed total transport in the Western Boundary Current can reach more than 100 Sv downstream (Holland, 1973; Stommel et al., 1978). The transport increases rapidly from Miami downstream about 2000 km. Furthermore, by analyzing the North Atlantic charts, Stommel et al. (1978) point out that the southward transport of the Sverdrup interior flow is very well balanced by the Gulf Stream in layers above the main thermocline, but this balance breaks down in layers below the main thermocline. This indicates that the imbalance mainly occurs in the deep ocean.

Classical theories such as Stommel's bottom friction model and Munk's lateral friction model fail to explain this imbalance. One very obvious reason is that it is not

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appropriate to use a linear equation to describe this very strong current. A nonlinear inertial boundary layer model cannot explain it either, if the interior flow has a transport of the same order of the wind-driven transport, although this model can obtain a more reasonable width of the Western Boundary Current which is independent of the estimate of the turbulent viscosity (Charney, 1955; Morgan, 1956; Pedlosky, 1979).

One alternative is the existence of strong recirculations which close near the Western Boundary Layer instead of in the interior of the basin. Bryan (1963) and Veronis (1966) studied the recirculation induced by nonlinear advection and friction which leads to an enhanced transport. Holland (1973) also studied the baroclinic-topography torque mechanism which (along with the lateral friction) can produce a strong recirculation and therefore result in an enhanced transport. But, as Holland notes, this effect is essentially linear. Stommel et al. (1978) argue that given the difference in thermocline thickness between the eastern and western boundaries, mass balance can be achieved by invoking an additional barotropic flow.

In this paper, we present another theory which offers a possible mechanism for the enhanced transport. Since the continental slope is very sharp and has a width of the same order as that of the Western Boundary Current, it may play an important role for the Western Boundary Currents. The great intensity of the Western Boundary Current also suggests that the nonlinear effect must be significant. Although the narrowness of the boundary layer suggests that friction might be important, yet as a first step in studying the combined effect of nonlinearity and the slope, we introduce a frictionless model. For simplicity, we present a quasi-geostrophic layered model with a slope-like topography. We will find that for reasonably high continental slopes, the transport increases dramatically. In addition, the continental slope can force a very strong recirculation in the lower layer. In fact, the boundary current is composed of two parts: one is the Inertial Boundary Current (hereafter called IBC) studied by Charney, Morgan, etc., and the other is a boundary current forced by the strong continental slope referred to as the Continental Slope Boundary Current in this paper (hereafter called CSBC) whose transport can be of the same magnitude as that of the IBC.

In Section 2, we use a one-layer QG model on a $\beta$-plane to study the CSBC. We will find that a realistically high slope can force a very strong CSBC transport which then increases the total transport significantly. As an improvement, Section 3 presents a two-layer QG model which uses a more realistic incoming flow. Surprisingly, we will find that a very weak lower layer incoming flow can still substantially enhance the total transport in the case of a very deep lower layer, as is the case in the real ocean. Finally, we discuss some further results and summarize the principal conclusions in Section 4.

The major question which we will address is: Can a realistically high continental slope and a very weak lower layer inflow enhance the total western boundary transport significantly?
2. The barotropic CSBC on a $\beta$-plane

In this section, we determine the structure of the boundary current in the presence of a continental slope. It will be found that the boundary current is now composed of two parts: the well known IBC, which is independent of the continental slope, and the CSBC.

a. The boundary layer problem (CSBC + IBC)

In the absence of forcing and dissipation, the steady one-layer QG model gives the potential vorticity conservation

$$J(\psi, \nabla^2 \psi + \beta y + \eta^*) = 0$$

where

$$\eta^* = f_0 \frac{h(x)}{H}.$$ 

Here, $H$ is the depth of the interior ocean, $h(x)$ is the height of the slope and $f_0$ is the Coriolis parameter. Assuming a uniform inflow $\psi \to U y$ in the interior, we obtain

$$\nabla^2 \psi - \frac{\beta}{U} \psi = -\beta y - \eta^*.$$ (2.2a)

We now nondimensionalize (2.2a) by

$$(x, y) = L(x', y'), \psi = U L \psi', \eta^* = \frac{L}{U} \eta'^* = \frac{h(x)}{H} R_o^{-1}, R_o = \frac{U}{fL}$$ (2.2b)

where $L$ and $U$ are the space and velocity scales of the interior flow; then by omitting the primes, (2.2a) becomes

$$\psi_{xx} + \psi_{yy} - \frac{L^2}{U/\beta} \psi = -\frac{L^2}{U/\beta} y - \eta^*(x).$$ (2.3a)

Supposing that the boundary layer width $L_\beta$ is of the same order as that of the continental slope, which is much narrower than the interior meridional length scale, we have

$$l = \frac{L_\beta}{L} \ll 1, \quad \xi = \frac{x}{l}, \eta^*(x) = \eta^*(\xi).$$ (2.3b)

After choosing

$$\eta^*(\xi) = \eta_0 e^{-\xi}, (h = h_0 e^{-\xi}),$$ (2.3c)
we derive the nondimensional equation in the boundary layer

\[ \psi_{\xi\xi} - F_B \psi = -F_B y - \eta(\xi) \]  

(2.4a)

where

\[ F_B = \frac{l^2 L^2}{U/\beta} \equiv \frac{L_B^2}{U/\beta} \]  

(2.4b)

\[ \eta(\xi) = l^2 \eta^*(\xi) = \eta_o e^{-\xi}, \quad \eta_o = l^2 \eta_o^*. \]  

(2.4c)

Choosing \( L_B^2 = |U/\beta| \), we have \( F_B = 1 \) in the case of a westward inflow. Then, we have

\[ \eta_o = \frac{f h_o}{\beta L H} = \frac{\delta H/H}{\delta f/f}, \]

which is independent of \( U \). (2.4c) then gives a characteristic scale of the slope width.

\[ L_H = \sqrt{\frac{U}{\beta}}/s \]  

(2.4d)

The boundary conditions are

\[ \psi|_{\xi \to \infty} \to y, \quad \psi|_{\xi = 0} = 0. \]

Since (2.4a) is a linear equation, we can separate the solution into two parts: \( \psi = \psi_I + \psi_C \) where \( \psi_I \) and \( \psi_C \) satisfy, respectively,

\[ \frac{d^2 \psi_I}{d\xi^2} - \psi_I = -y \]  

(2.5a)

\[ \psi_I|_{\xi \to \infty} \to y, \quad \psi_I|_{\xi = 0} = 0 \]  

(2.5b)

and

\[ \frac{d^2 \psi_C}{d\xi^2} - \psi_C = -\eta(\xi) \]  

(2.6a)

\[ \psi_C|_{\xi \to \infty} = \psi_C|_{\xi = 0} = 0. \]  

(2.6b)

It should be noted that this separation is only for convenience. In fact, neither \( \psi_I \) nor \( \psi_C \) satisfies the nonlinear equation (2.1). This artificial separation is possible only in the present uniform incoming flow case. The solution of (2.5) gives the well-known IBC (Pedlosky, 1979):

\[ \psi_I = y(1 - e^{-\xi}). \]  

(2.7)
The velocity and transport are always northward at the western boundary for \( y > 0 \), and the magnitude increases linearly with \( y \). The width of the boundary current is 1 in this stretched coordinate. On the other hand, (2.6) yields the contribution due to the continental slope (2.4c):

\[
\psi_C = \frac{\eta_0}{1 - s^2} (e^{-s\xi} - e^{-\xi}).
\]  

This solution describes the CSBC on a \( \beta \)-plane. Since the CSBC has no net meridional transport, it is purely a recirculation flow. This is the principal result of the present work and some details about the form of the streamfunction are given in Appendix A. It is worth noting that although this solution appears to be singular when \( s = 1 \), it is not. As \( s \to 1 \), \( \psi_C \) approaches the limiting solution \( \psi_C = \frac{\eta_0}{2} e^{-\xi/2} \).

Hence we end up with the full solution:

\[
\psi = y(1 - e^{-\xi}) + \frac{\eta_0}{1 - s^2} (e^{-s\xi} - e^{-\xi}).
\]

Now we will discuss some properties of the CSBC:

1. **Velocity profile.** The meridional velocity of the CSBC is

\[
\nu_C = \psi_{Cy} = \frac{1}{l} \psi_C = \frac{\eta_0}{l(1 - s^2)} (e^{-\xi} - se^{-s\xi}),
\]

so at the wall

\[
\nu_C|_{\xi=0} = \frac{\eta_0}{l(1 + s)} = \nu_m > 0.
\]

This indicates that the velocity of the CSBC at the western wall is always northward. In Appendix A, we demonstrate that the flow field is composed of two parts: a very narrow northward jet on the inshore side with anticyclonic vorticity \( (0 < \xi < \xi_1) \), and a relatively wider southward countercurrent on the offshore side \( (\xi_1 < \xi < \infty) \). Both have the same transports but in the opposite directions (Fig. 1).

The width of the northward jet \( \xi_1 \) is 1 when \( s = 1 \). When \( s \) varies from 0.2 to 3, \( \xi_1 \) changes from 2.01 to 0.54, so it is usually of the order of the width of the IBC. The width of the southward current is usually several times that of the inshore jet. In fact, from (2.10), we can calculate that, as long as \( s = 0(1) \), 75% of the transport of the southward current is within the width \( 3\xi_1 \) and 90% is within \( 4\xi_1 \). As a result, we may consider the entire CSBC to be a recirculation gyre confined near the western wall.

2. **Transport.** Since the total transport now is composed of two parts, it is interesting to compare the magnitude of these two parts. First of all, noticing that the IBC depends
linearly on $y$ while the CSBC is independent of $y$, the CSBC should always be stronger than the IBC for small $y$. (But this solution does not satisfy the boundary condition $v = 0$ at $y = 0$ near the boundary $x = 0$. This means that an arbitrary northward flow can be added there. We will discuss this problem in Section 2b.) However, as $y$ gets larger, the transport of the IBC increases and eventually surpasses that of the CSBC. Quantitively, we have from (3.7) that the transport per unit depth of the IBC is $T_I \sim y$ while the northward transport per unit depth of the CSBC is $T_C \sim \eta_o/(1 + s)^2$ (see Appendix A (A.12)), so

$$\frac{T_I}{T_C} \sim \frac{y}{y_o} \quad \text{where} \quad y_o = \frac{\eta_o}{(1 + s)^2}. \quad (2.12)$$

Then we have

$$T_C > T_I \quad \text{for} \quad y < y_o. \quad (2.13)$$

For typical continental slopes, we have $y_o > 1$ (see discussion regarding (2.16) and note $y_o \sim T_C$ there), thus the CSBC can transport more than the IBC all the way along the boundary. This means that the continental slope can intensify the Western Boundary Current significantly.

Figures 2 and 3 give an example of the velocity profiles and streamfunctions for $\eta_o = 1$, and $s = 0.5$. We set the boundary layer parameter $l = 0.1$ in all the figures. We see from Figure 2 that $V_C(0)$ is always greater than $V_I(0)$ until about $y = 0.7$. According to Figures 3a and b, the transports in the boundary layer ($x < l$) of the pure IBC and the total flow field at $y = 0.8$ are respectively 0.50 and 0.82. Subtraction of the two transports gives the transport of the CSBC as 0.32, which accounts for a relatively increased transport of $0.32/0.50 = 64%$. Therefore, the CSBC is really very strong compared with the pure IBC.

Before we give some dimensional estimates, we should point out that in all the nondimensional results and plottings of this article, we use the value $l = 0.1,$
Figure 2. Velocity profiles in different sections in the case of a wider slope \((s = 0.5)\). Notations are the same as in the text. \(V_T\) is the full solution \((V_T = V_I + V_C + V_O)\). All cases have \(\eta = 1\).

(a) \(y = 0\), (b) \(y = 0.1\), (c) \(y = 0.3\), (d) \(y = 0.7\). "\(\cdot\)" is the width of the IBC, "X-X" is the width of the slope. Following figures are the same.

corresponding to a velocity of 20 cm/s with \(L = 1000\) km. This value of \(l\) is too large for the real ocean. It is only for the convenience of the clarity of the plottings that we choose this value in all the calculations and figures. As a matter of fact, the value of \(l\) does not influence the transport of the boundary layer and it affects all the velocities by a common factor. This means that different \(l\)'s would not change the discussion in this paper which mainly examines the transport or the relative magnitudes between the IBC and the CSBC.

For the real ocean, we take

\[ L \sim 1000\ \text{km}, \quad U \sim 1.5\ \text{cm s}^{-1}, \quad \beta \sim 1.5 \times 10^{-13}\ \text{cm}^{-1}\ \text{s}^{-1}, \quad H \sim 4\ \text{km}, \quad f \sim 10^{-4}\ \text{s}^{-1}. \]

This gives

\[ L_B \sim 33\ \text{km}, \quad l \sim 0.033. \]

Correspondingly, the slope in Figure 3 of \(\eta_0 = 1, s = 0.5\) has its dimensional height and width of 600 m and 66 km respectively. Thus, the dimensional transport value of the IBC in Figure 3 becomes \(0.5 \times U L H = 30\) Sv at \(x = 33\) km from the western wall and \(y = 800\) km from the beginning of the Western Boundary Current, while the dimensional transport value of the CSBC becomes about \(0.32 \times U H L = 19\) Sv. The total transport then increases to 49 Sv.
Figure 3. Streamfunctions of the same parameters as Figure 2. (a) $IBC$, (b) $IBC + CSBC$, (c) $IBC + CBSC + CSCO$. (a) and (c) have the same $y$ scale, but (b) is of half the scale. The solid line is the width of the $IBC$ and the condot line is the width of the slope. Following figures are the same. A 0.1 interval in the nondimensional transport represents a dimensional transport of 6 Sv in section 2.a.

iii. The total flow field and recirculation. The total flow field is given by (2.9), which is plotted in Figure 3b. Several points are noteworthy. First, notice the greatly intensified boundary current compared to the pure $IBC$, which has been discussed above. Second, notice that a large amount of incoming flow must first go southward for a long distance before turning northward. This occurs because the $IBC$ has no meridional velocity near $y = 0$, while the CSBC is always first southward then northward approaching the boundary. So the total flow near $y = 0$ is dominated by the CSBC. We can find the stagnation point $(0, y^*)$ at the boundary by virtue of (2.9). Setting $\psi_x = 0$, and $\psi_y = 0$, we have

$$\xi = 0, y^* = -\frac{\eta_0}{(1 + s)}. \quad (2.14)$$

For $s \ll 1$, compared with (2.12), we have $y^* \sim -y_\phi$. This is obvious since only that far along the boundary can the southward $IBC$ cancel the northward CSBC in the region
\( y < 0. \) Hence the IBC + CSBC total flow field goes very far southward, then turns back toward the north. This kind of boundary layer flow pattern suggests that there is a strong recirculation near the boundary. However, we can't close the recirculation on the northern part without perhaps adding frictional effects. Probably the topography distorts and enhances the recirculation already produced by other effects studied by previous authors (Bryan, 1963; Veronis, 1966; Holland, 1973). The flow pattern here strongly resembles the southern part of the recirculation patterns in Holland's (1973) work.

*iv. The geometrical influence.* According to (2.7), at \( y \sim 1 \), we have the transport of the IBC, \( T_I \sim 1 \). On the other hand, from (2.12), we know that both the width \((1/s)\) and the height of the slope \( (\eta) \) can affect the CSBC transport significantly. The condition for the CSBC to have a transport comparable with that of the IBC at \( y \sim 1 \) is

\[
T_C \sim \frac{\eta_0}{(1 + s)^2} \sim 1. \tag{2.15}
\]

For a fixed height, a flatter slope whose width is larger than that of the IBC \((s \ll 1)\) can force a very strong CSBC and recirculation. But a very steep slope would make the CSBC contribute very little which implies that the vertical wall approximation is very good. As a matter of fact, the calculation for the same parameters as in Figures 2 and 3, except with \( s = 2 \), shows that \( V_I(0) \) would be larger than the \( V_C(0) \) after about \( y > 0.3 \) (compared with 0.7 in the case of \( s = 0.5 \) there).

On the other hand, for a fixed slope width \( 1/s \), the higher the slope, the stronger the CSBC is. The critical height can be derived in the following. For a shallower slope \((s \ll 1)\), after using (2.4a), we can rewrite (2.15) as

\[
T_C \sim \frac{\eta_0}{1} \sim \frac{h}{H} \frac{f}{\beta L} \sim \frac{h}{H} \frac{a}{L}, \tag{2.16}
\]

where \( a \) is the radius of the earth. Then from (2.15), the critical height \( h_c \) for the CSBC transport to be comparable to the IBC is

\[
\frac{h_c}{H} \sim \frac{L}{a} \quad \text{or} \quad \delta f \sim f \frac{\delta H}{H}. \tag{2.17}
\]

For typical values of \( L \sim 1000 \) km, \( a \sim 7000 \) km, \( H \sim 4 \) km, we have

\[
h_c \sim 600 \text{ m},
\]

which is of course very common for the continental slope or continental rise.
Hence, we come to the conclusion that for a realistic slope, the CSBC could be very important compared with the IBC. A significantly enhanced Western Boundary Current can be achieved by including the continental slope.

v. Physical mechanism. The potential vorticity conservation is now

$$\frac{d}{dt} \left( \frac{\beta y + \nabla^2 \psi}{H - h} \right) = 0.$$ 

We can easily find that for the upslope motion toward the western boundary, the $\beta$ effect and relative vorticity effect give opposite directions of motion. In fact, as a particle moves southward, planetary vorticity decreases while relative vorticity increases. Conversely as a particle flows northward, $\beta y$ increases while relative vorticity decreases. So the trajectory of a particle depends on the competition between these two effects. For a fluid column just entering the far offshore slope where relative vorticity is weak, the $\beta$ effect is dominant. Therefore, the fluid column will move southward in order to compensate for the decrease in depth. As the fluid column moves southward, positive relative vorticity is produced until it becomes as important as planetary vorticity and eventually dominates. Then, the relative vorticity results in a strong northward flow on the inshore side of the continental slope.

The criterion (2.17) making the CSBC comparable to the IBC also has a clear physical explanation. It means that the relative vorticity produced by the planetary vorticity deviation ($\delta f$) is comparable to that yielded by the compression of the bottom slope on the vortex tube ($f + \xi (\delta H/H - f H/H)$).

For the geometric effect, it is also clear that a slope width much narrower than that of the IBC (or a very steep slope) will not make the IBC feel the effect of the slope. So it is just like a vertical wall. However, a very wide slope would influence the flow before the interior incoming flow is affected by the boundary.

Finally, we should note that the solution here can only exist in the westward incoming flow case, just like the IBC. The case of a linear slope shows similar results. One difference is that the maximum northward velocity is not directly at the wall. This occurs because the exponential form of the slope has its maximum topographical effect (very steep slope) on the wall.

b. The corner problem ($IBC + CSBC + CSCO$)

In the discussion above, we studied the half plane boundary layer problem which can't satisfy the $\psi = 0$ condition at $y = 0$ within the boundary layer. This means that we have flow coming into the boundary layer from the south of $y = 0$. Will this inflow affect the boundary layer structure? To understand this, we need to study the solution which can satisfy $\psi = 0$ at $y = 0$. This is a corner problem. We use

$$\psi_{xx} + i^2 \psi_{yy} - \psi = -y - \eta_0 e^{-st}$$
to replace (2.3). It is easy to see that we also need a boundary layer near \( y = 0 \). If the stretched coordinate is \( \lambda = y/l \), separating \( \psi \) into the IBC, the CSBC and the CSCO (the Continental Slope Corner solution) such that

\[
\psi = \psi_I + \psi_C + \psi_o
\]  

(2.18)

where the IBC and the CSBC are the same as (2.7) and (2.8). The CSCO satisfies

\[
\psi_o|_{\xi = 0} = \psi_o|_{\xi = \infty} = 0, \psi_o|_{\lambda = 0} = -\psi_C(\xi), \psi_o|_{\lambda = \infty} \to 0.
\]

A Fourier sine transformation yields

\[
\psi_o = -\frac{2}{\pi} \eta_o \int_0^{\infty} \frac{k}{(k^2 + 1)(s^2 + k^2)} e^{-\sqrt{k^2 + 1}\lambda} \sin k\xi \, dk. \tag{2.19}
\]

Therefore, it turns out that the corner effect decays so rapidly with \( y \) that it becomes negligible after one boundary layer thickness \( l \). One example of velocity profiles is shown in Figure 2. It is obvious that after \( y > l \), the velocity produced by \( \psi_o \) almost vanishes. Hence, we know that

\[
\psi_o, \psi_C \text{ dominate for } 0 < y < l \ll 1
\]

\[
\psi_C, \psi_I \text{ dominate for } l < y < y_o \sim 1
\]

\[
\psi_I \text{ dominates for } y_o < y.
\]

Figure 3c displays one example of the full solution (2.18), which illustrates very clearly that the corner effect affects only the corner region without changing the flow field far from the \( y = 0 \) boundary. So we know that the CSBC will always be very important for the enhanced northward transport in the western boundary layer, whether in the half plane case or in the corner problem.

Hence, our analysis of the one-layer model gives these conclusions: (1) For a slope of reasonable height, we have a very strong CSBC which will increase the transport of the Western Boundary Current significantly. (2) For a given slope height, a shallower slope produces a strong CSBC while a very steep slope can suppress the CSBC severely. (3) For the half plane case, a very strong recirculation appears. (4) For the corner problem, it is shown that the corner could only affect the flow field near the corner. However, it does affect the southward flow very much in the figures.

3. Two-layer CSBC

Although the CSBC discussed above can account for a large amount of transport for a reasonable continental slope, the one layer model, as we know, overestimates the effect of topography. Therefore, we adopt a two-layer model to provide a better understanding of the influences due to the baroclinic incoming flow and the baroclinic structure.
Figure 4. The geometry of the two-layer model.

A QG model with a continental slope beneath the lower layer is used

\begin{align*}
J(q_1, q_1) &= 0 \\
J(q_2, q_2) &= 0
\end{align*}

with

\begin{align*}
q_1 &= \beta y + \nabla^2 \psi_1 + \frac{f^2}{g' H_1} (\psi_2 - \psi_1) \\
q_2 &= \beta y + \nabla^2 \psi_2 + \frac{f^2}{g' H_2} (\psi_1 - \psi_2) + \frac{f}{H_2} h(x).
\end{align*}

The boundary conditions are now

\begin{align*}
\psi_1 &\rightarrow U_y \psi_2 \rightarrow A U_y \text{ in the interior} \quad (3.1) \\
\psi_1|_{x=0} &= \psi_2|_{x=0} = 0. \quad (3.2)
\end{align*}

Here, we have written the reduced gravity as $g' = g(\Delta \rho)/\rho_0$, the upper layer incoming flow $U_1$ as $U$, and the lower layer ones $U_2$ as $A U$. Also, we assume in the interior $h(x) \rightarrow 0$. $H_1$ and $H_2$ are the mean depths of the two layers in the interior. The geometry of the two-layer model is shown in Figure 4.
For the inflow (3.1), we can easily get the equations

\[ \nabla^2 \psi_1 - \left( A \frac{f^2}{g' H_1} + \frac{\beta}{U} \right) \psi_1 + \frac{f^2}{g' H_1} \psi_2 = -\beta_y \]

\[ \nabla^2 \psi_2 - \frac{1}{A} \left( \frac{f^2}{g' H_2} + \frac{\beta}{U} \right) \psi_2 + \frac{f^2}{g' H_2} \psi_1 = -\beta_y - \frac{f}{H_2} h(x). \]

Using the nondimensional quantities \((x, y) = L(x', y'), \psi_n = U_L \psi'_n\) and introducing the boundary layer coordinate as in (2.3a), we get the nondimensional boundary layer equations

\[ L_1 (\psi_1, \psi_2) = \psi_{1\xi\xi} - (AF_1 + 1) \psi_1 + F_1 \psi_2 = -y \] (3.3a)

\[ L_2 (\psi_1, \psi_2) = \psi_{2\xi\xi} - \frac{1}{A} (F_2 + 1) \psi_2 + F_2 \psi_1 = -y - \eta(\xi) \] (3.3b)

with boundary conditions

\[ \psi_1 \rvert_{\xi \rightarrow -\infty} \rightarrow y, \psi_2 \rvert_{\xi \rightarrow -\infty} \rightarrow A_y, \psi_1 \rvert_{\xi = 0} = \psi_2 \rvert_{\xi = 0} = 0, \] (3.4)

where

\[ F_n = \left( \frac{L_B}{L_{Dn}} \right)^2, \eta(\xi) = \frac{h_0 f L}{H_2 U} l^2 h(x) = \eta_0 e^{-sx/l} = \eta_0 e^{-\xi} \] (3.5a)

\[ L_{Dn} = \sqrt{g' H_n / f}, \sqrt{U/\beta} = L_B = \frac{U}{\beta} \ll L \] (3.5b)

Other parameters are the same as in Section 2. The slope scale is still \(L_H = L_B/s\). We still separate the streamfunction in each layer into two parts, \(\psi_{in}\) and \(\psi_{cn}\) which satisfy \(\psi_n = \psi_{in} + \psi_{cn}\) where \(n = 1, 2\). The IBCs are determined by

\[ L_1 (\psi_{i1}, \psi_{i2}) = -y \] (3.6a)

\[ L_2 (\psi_{i1}, \psi_{i2}) = -y \] (3.6b)

\[ \psi_{i1} \rvert_{\xi \rightarrow -\infty} \rightarrow y, \psi_{i2} \rvert_{\xi \rightarrow -\infty} \rightarrow A_y, \psi_{in} \rvert_{\xi = 0} = 0 \quad n = 1, 2, \] (3.7)

and the CSBCs by

\[ L_1 (\psi_{c1}, \psi_{c2}) = 0 \] (3.8a)

\[ L_2 (\psi_{c1}, \psi_{c2}) = -\eta_0 e^{-\xi} \] (3.8b)

\[ \psi_{c} \rvert_{\xi = 0} = \psi_{cn} \rvert_{\xi = 0} = 0 \quad n = 1, 2. \] (3.9)

These linear ordinary differential equations can be solved exactly, as done in Appendix B. However, since the solution is very complicated, we begin with some simpler cases in order to understand the basic mechanism.
a. $A = 1$ case (the barotropic incoming flow case)

First, to concentrate on the effect of stratification, we study the case where $A = 1$ (barotropic inflow). Since the baroclinic effect is introduced through the topography in the lower layer, we should expect that the baroclinic part of the solution is due solely to the topography.

With $A = 1$ and (3.6), (3.7), we obtain the IBC solutions

$$\psi_{12} = y(1 - e^{-\xi}).$$

(3.10)

Therefore, we get only the barotropic mode.

However, the CSBC parts are different. Subtracting (3.8a,b) and (3.9), we get

$$\frac{(\psi_{C1} - \psi_{C2})_t}{(F_1 + F_2 + 1)(\psi_{C1} - \psi_{C2})} = \frac{\eta_0e^{-s\xi}}{2(F_1 + F_2 + 1 - s^2)} (e^{-s\xi} - e^{-\sqrt{F_1 + F_2 + 1}\xi}).$$

(3.11)

This is an inhomogeneous equation, yielding the baroclinic solution

$$\psi_{CC} = (\psi_{C1} - \psi_{C2})/2 = - \frac{\eta_0}{2(F_1 + F_2 + 1 - s^2)} (e^{-s\xi} - e^{-\sqrt{F_1 + F_2 + 1}\xi}).$$

(3.12)

This streamfunction is of the same form as the one-layer model solution (2.8) except for: (a) the decay scale is the baroclinic mode decay scale $\sqrt{F_1 + F_2 + 1}$ instead of the barotropic mode scale 1, (b) a negative sign instead of a positive one. Therefore, we have a southward "jet" in the vertical shear on the inshore side which means that the northward velocity in the lower layer is always larger than that in the upper layer at the western boundary. As for the barotropic component, we add (3.8a) $F_1^{-1}$ and (3.8b) $F_2^{-1}$ to yield

$$(\psi_{C1}/F_1 + \psi_{C2}/F_2)_t - (\psi_{C1}/F_1 + \psi_{C2}/F_2) = - \frac{\eta_0}{F_2} e^{-s\xi}$$

with boundary conditions $\psi_{C1}/F_1 + \psi_{C2}/F_2 = 0$ at both $\xi = 0$ and $\infty$,

which gives the barotropic solution

$$\psi_{CB} = \frac{\psi_{C1}/F_1 + \psi_{C2}/F_2}{F_1^{-1} + F_2^{-1}} = \frac{\eta_0}{1 - s^2} (e^{-s\xi} - e^{-\xi}) \frac{F_1}{F_1 + F_2}.$$ 

(3.12)

Note here $\eta$ is normalized by $H_2$ (see (3.5)), while in (2.8), it is normalized by the total depth $H$. For the same total depth, (3.12) is exactly the same as in the one-layer case (2.8). According to (3.11) and (3.12), we obtain

$$\nu_{1C} \big|_{\xi=0} = \frac{F_1\eta_0}{F_1 + F_2} \left( \frac{1}{1 + s} - \frac{1}{\sqrt{F_1 + F_2 + 1 + s}} \right) > 0.$$
Hence, we know that in both layers, the inshore side jets are northward, and the lower layer jet is stronger than that in the upper layer. This indicates that the slope effect is trapped in the lower layer. Physically, this results because in the case of a westward inflow, the only standing wave is a topographic Rossby wave. In the stratified fluid, this wave is trapped near the bottom (Rhines, 1970, Charney and Flierl, 1981).

Also worth noting is the effect of the baroclinic structure of the ocean. In the real ocean, to the first order approximation, we have \( H_2^2 = 4H_1 \) or \( F_1 = 4F_2 \). In the barotropic inflow case, for different baroclinic structure (i.e. different depth ratio \( H_1/H_2 \)), (3.10) tells us that the IBC streamfunctions in both layers would not change at all. But (3.11) and (3.12) imply that the CSBC must change in each layer. However, the barotropic part of the CSBC is very different. Noting (3.5) and (3.12), we have \( \eta_0 \sim h_o/H_2 \sim h_o F_2 \). Hence

\[
\psi_{CB} \sim \frac{F_1 F_2}{F_1 + F_2} \sim H_1 + H_2 = H.
\]

We then come to the conclusion that the barotropic CSBC is independent of the baroclinic structure. This fact that different baroclinic structures will not seriously affect the barotropic CSBC is very important and will be studied later. For the baroclinic part of the CSBC, we can show that increasing the lower layer depth produces a weak lower layer transport and a stronger upper layer transport. Thus, in the presence of stratification and a barotropic inflow, we know that: (1) The topographic effect is confined in the lower layer. (2) The barotropic CSBC transport does not depend on the depth ratio \( H_1/H_2 \).

b. \( A \ll 1 \) case

Now, we proceed to study another limiting case \( A \ll 1 \), i.e., the case in which the lower layer incoming flow is much weaker than that in the upper layer, as in the real ocean. In this situation, both the IBC and the CSBC have baroclinic structures (i.e. the vertical sheared velocities). To understand the mechanism for the \( A \ll 1 \) and to avoid unnecessary algebra, we find the leading order solutions by a perturbation method rather than deriving them from the general solutions. All the leading order solutions can be derived from the general solution formed in Appendix B.

First of all, we derive the asymptotic solution for the IBC. Applying naive perturbation methods to (3.6a,b) shows that we have another even smaller boundary layer

\[
\lambda = \xi/\sqrt{A}.
\]

Hence, we set \( \psi_{in} = \psi_{in}(\xi, \lambda) \) to make the lower layer satisfy the boundary conditions. We use the expansion

\[
\psi_{in} = \psi_{in}^{(0)} + A\psi_{in}^{(1)} + A^2\psi_{in}^{(2)} + \ldots n = 1, 2.
\]
From (3.13) and (3.6), the $O(1)$ problem becomes

$$
\frac{d^2 \psi_{11}^{(0)}}{d \xi^2} - \psi_{11}^{(0)} + F_1 \psi_{12}^{(0)} = -y \\
\frac{d^2 \psi_{12}^{(0)}}{d \lambda^2} - (F_2 + 1) \psi_{12}^{(0)} = 0
$$

which yields the solutions

$$
\psi_{11}^{(0)} = y(1 - e^{-\xi}) \quad (3.14a) \\
\psi_{12}^{(0)} = 0. \quad (3.14b)
$$

(3.14a) gives the leading order solution for the upper layer. For the lower layer, we need to include the higher order dynamics. At $O(A)$, we obtain

$$
\frac{d^2 \psi_{12}^{(1)}}{d \lambda^2} - (F_2 + 1) \psi_{12}^{(1)} = -F_2 \psi_{11}^{(0)} - y
$$

with

$$
\psi_{12}^{(1)} \big|_{\xi=\infty} \to y, \psi_{12}^{(1)} \big|_{\xi=0} = 0.
$$

Then substituting from (3.14a), we can get the leading order solution for the lower layer

$$
\psi_{12}^{(1)} = y \left[ 1 - e^{-\sqrt{F_2 + 1} \lambda} + \frac{F_2}{F_2 + 1} (e^{-\sqrt{F_2 + 1} \lambda} - e^{-\xi}) \right], \quad (3.15)
$$

which is composed of two parts: the first part is of the form of the IBC and the second part has the form of the CSBC. This looks like an IBC + CSBC solution determined by the upper layer interface "slope." Therefore we get the leading order solution for the IBCs as:

$$
\psi_{11} = y(1 - e^{-\xi}) + O(A) \quad (3.16a) \\
\psi_{12} = Ay \left[ 1 - e^{-\sqrt{F_2 + 1/A} \xi} + \frac{F_2}{F_2 + 1} (e^{-\sqrt{F_2 + 1/A} \xi} - e^{-\xi}) \right] + O(A^2). \quad (3.16b)
$$

This in turn gives

$$
\nu_{11} \sim ye^{-\xi}, \nu_{12} \sim \sqrt{\frac{A}{F_2 + 1}} ye^{-\sqrt{F_2 + 1/A} \xi}.
$$
Figure 5. Schematic figures of the two-layer IBC and CSBC velocity profiles. All notations are the same as in the text.

Thus, the upper layer IBC has a strong $O(1)$ velocity and transport with the decay scale 1. It interacts very little with the lower layer. In contrast, the lower layer has an $O(\sqrt{A})$ velocity and an $O(A)$ transport with the baroclinic mode scale $(F_2 + 1/A)^{-1/2}$, which is much smaller than the scale of the upper layer IBC, as shown in Figure 5.

Now we turn to the study of the CSBC. As with the case of the IBC, we need a narrower boundary layer (3.13) for the lower layer. But in the present case, since $\psi_{c1}$ and $\psi_{c2}$ are of the same order, the narrower scale will also influence the leading order solution for the upper layer. Hence we use the expansions

$$\psi_{c1} = A \psi_{c1}^{(0)} + A^{3/2} \psi_{c1}^{(1)} + A^{2} \psi_{c1}^{(2)} + \ldots, \quad n = 1, 2$$

Substituting these expansions into (3.8a,b), and applying the boundary conditions (3.9), we obtain the leading order problem

$$\frac{d^2 \psi_{c2}^{(0)}}{d\lambda^2} - (F_2 + 1) \psi_{c2}^{(0)} = -\eta, \text{ for } \psi_{c2}^{(0)}|_{\lambda=0,\infty} = 0.$$  

Rewriting the slope in (3.5a) as $\eta = \eta_0 e^{-s\xi} = e^{-s\sqrt{A\lambda}}$, we can get the leading solution

$$\psi_{c2}^{(0)} = \frac{\eta_0}{F_2 + 1} (e^{-s\xi} - e^{-s\sqrt{F_2+1}\lambda}). \quad (3.17)$$

In equations for $\psi_{c1}^{(n)}$, the low order problem results in the trivial solutions $\psi_{c1}^{(0)} = \Psi_{c1}^{(0)}(\xi)$, $\psi_{c1}^{(1)} = \Psi_{c1}^{(1)}(\xi)$. To determine the structure of the leading order, we need to invoke higher order dynamics. The $O(A)$ order dynamics requires that

$$\frac{d^2}{d\lambda^2} \psi_{c1}^{(2)} = -F_1 \psi_{c2}^{(0)} + \Psi_{c1}(0) - \frac{d^2 \psi_{c1}^{(0)}}{d\xi^2}.$$  

To avoid secularity, we must insist that the projection of the RHS on $\xi$ vanishes, which
in turn gives
\[ \frac{d^2 \Psi^{(0)}_{C1}}{d \xi^2} - \Psi^{(0)}_{C1} = -\frac{F_1}{F_2 + 1} \eta_o e^{-s\xi}, \]

where we have used (3.17). This, in turn, gives
\[ \Psi^{(0)}_{C1} = \frac{\eta_o F_1}{(1 - s^2)(F_2 + 1)} (e^{-s\xi} - e^{-\xi}). \] (3.18)

Finally, we have the leading order solution for the CSBC
\[ \psi_{C1} \sim \frac{AF_1 \eta_o}{(1 - s^2)(F_2 + 1)} (e^{-s\xi} - e^{-\xi}) + O(A^{3/2}) \] (3.19a)
\[ \psi_{C2} \sim \frac{A \eta_o}{F_2 + 1} (e^{-s\xi} - e^{-\sqrt{F_2 + 1}/At}) + O(A^{3/2}). \] (3.19b)

At the boundary, we have the leading order velocities
\[ \nu_{C1}|_{\xi=0} \sim \frac{AF_1 \eta_o}{(1 + s)(F_2 + 1)}, \nu_{C2}|_{\xi=0} \sim \frac{A \eta_o}{\sqrt{F_2 + 1}}. \]

Also, from Appendix A (A.9), we can find the widths of the CSBC as \( \xi_{(C1)1} \sim O(1), \)
\( \xi_{(C2)1} \sim \sqrt{A} \ln 1/\sqrt{A} \ll O(1). \) This is shown in Figure 5.

(3.16) and (3.19) give the order of transport per unit depth as
\[ T_{C1} \sim y \]
\[ T_{C2} \sim Ay \]
\[ T_{C1} \sim \frac{A \eta_o F_1}{(F_2 + 1)(1 + s)^2} \] (3.20)
\[ T_{C2} \sim \frac{A \eta_o}{F_2 + 1}. \]

For a flat slope \( s \ll 1 \) and \( \eta_o \sim F_1 \sim F_2 \sim 1, \) we have \( T_{C1} \sim T_{C2} \sim O(A). \) Comparing the CSBCs to the IBCs in (3.20), we know that in the upper layer the IBC is always dominant. But in the lower layer, the IBC is of the same order as the CSBC. Furthermore, from Section 3, the southward part of the recirculation is determined mainly by the relative magnitude between the IBC and the CSBC. It follows that the upper layer has a very weak recirculation as far south as \( y_1^* \sim -A \ll O(1) \) while the lower layer has a very strong recirculation as far south as \( y_2^* \sim -1. \) The qualitative flow pattern is shown in Figure 6. A strong IBC in the upper layer and a strong recirculation in the lower layer are very clear. This is similar to Holland's (1973) result in the sense
that a continental slope can force a very strong recirculation in the lower layer. We are going to explore some of the details in the following.

i. The barotropic transport (or total transport). The total transports of the IBC and the CSBC can be written respectively as

\[ T_I = T_{i1}/F_1 + T_{i2}/F_2, \]
\[ T_C = T_{c1}/F_1 + T_{c2}/F_2, \]

where a common factor is neglected. From (3.20), we can write (3.21) as

\[ T_I \sim \frac{y(A + F_2/F_1)}{F(1 + F_2/F_1)} = \frac{y(A + H_1/H_2)}{F(1 + H_1/H_2)}, \quad T_C \sim \frac{A\eta_o}{F_2} = A \frac{g'h_o}{fLU}. \] (3.22)

Here we have used \( F^{-1} = (F_1 + F_2)/F_1 F_2 \sim H_1 + H_2 \) and \( s \ll 1 \), the latter requiring a shallow slope. We used (3.5a,b). If \( A = 1 \), (3.22) becomes the same as in the \( A = 1 \) case discussed in Section 3a. In the case of \( A < 1 \), \( T_I \) is very different from \( T_C \) in the sense that \( T_I \) depends on the ratio between the depths of different layers, while \( T_C \) does not (this is consistent with the discussion about (3.12)). When the lower layer gets deeper \((H_1/H_2 \rightarrow 0)\), \( T_I \) decreases, while \( T_C \) does not. So, for a deeper lower layer, the CSBC becomes stronger relative to the IBC. This is the major effect of the baroclinic structure, and it will be addressed in more detail later.

Similar to Section 2, at \( y \sim 1 \), (3.5) and (3.22) yield

\[ \frac{T_C}{T_I} = \left(\frac{1 + F_2/F_1}{A + F_2/F_1}\right) A \frac{h_o a}{H L}. \] (3.23)
Therefore, the barotropic IBC can be comparable to the CSBC. The critical height of the slope is

\[
\frac{h_{oc}}{H} \sim \frac{1 + F_2/(F_1 A)}{1 + F_2/F_1} \cdot \frac{L}{a}.
\]  

(3.24)

Here we see that if the lower layer is so thick that \(F_2/F_1\) becomes small enough to be comparable to the lower layer incoming flow i.e. \(F_2/F_1 \sim A \ll 1\), we can obtain a simpler criterion

\[
\frac{h_{oc}}{H} \sim 2 \frac{L}{a}.
\]  

(3.24a)

For the real ocean, \(F_2/F_1 \sim 1/4\) and \(A \sim 1/5\), so (3.24a) is a very good estimate. Even for the \(A = 0.1\) case, we can still have

\[
\frac{h_{oc}}{H} \sim 2.6 \frac{L}{a}.
\]  

(3.24b)

This is an important result. It says that even if the lower layer incoming flow is very weak, we can still have a barotropic CSBC transport strong enough to be comparable to that of the IBC and the height of relief needed is only twice that of the one-layer model (comparing (2.17)). As discussed regarding (2.17), (3.24a) is also a reasonable height for the real continental slope. For typical values, \(h_{oc}\) is about 1200 m.

It is important to note that the deep layer structure enhances the effect of a small incoming flow. Indeed, for layers of equal depth (\(F_2/F_1 \sim 1 \gg A\)), the CSBC transport is comparable to the IBC transport if

\[
\frac{h_{oc}}{H} \sim \frac{1}{2A} \frac{L}{a} \gg 2 \frac{L}{a}.
\]  

(3.24c)

That is, a much higher slope is needed. If, furthermore, the upper layer becomes very deep, i.e. \(F_2/F_1 \gg 1\) and \(A \ll 1\), we would obtain an even higher slope

\[
\frac{h_{oc}}{H} \sim \frac{1}{A} \frac{L}{a}.
\]  

(3.24d)

ii. The influences of a deep lower layer. Eq. (3.22) clearly shows that a deep lower layer will reduce the total IBC transport while maintaining the CSBC transport, which then implies that the CSBC transport is relatively stronger. As a matter of fact, a deep lower layer has two opposite effects on the total transport of a boundary current. The
first effect is to decrease the total incoming flow transport (for $A < 1$, where the incoming flow in the lower layer is weaker than that in the upper layer), which in turn decreases the transport of the boundary current. The second effect, which is related to the bottom slope, is to enhance the boundary layer transport, since the influence of the bottom topography is confined to the lower layer, a deep lower layer implies that the bottom slope can influence a larger fraction of fluid. This will increase the total transport.

Since the IBC is not related to the bottom topography and has the same transport as the incoming flow (see (3.20)), it will obviously decrease as the lower layer gets deeper. However, the CSBC is influenced by both of these opposing effects. As a consequence, the CSBC can maintain its transport due to a balance between the two effects.

As for the second effect, we should point out that the increase of the transport in one layer does not necessarily imply an increase of the velocity in that layer, because the transport in that layer is the product of the volume and the velocity. In fact, using (3.20), we can prove the following (if the lower layer is not very shallow): a deep lower layer will increase the CSBC velocity in the upper layer, but decrease its transport; in the lower layer, the opposite occurs. Since the IBC velocity field in each layer remains the same for different baroclinic structures (see (3.20)), we then have a weak lower layer recirculation and a strong upper layer one in the case of a deep lower layer.

iii. The influence of the horizontal scale of the continental slope. From (3.20), we know that $T_{J1}$ and $T_{J2}$ do not depend on the geometry of the topography while $T_{C1}$ and $T_{C2}$ are very sensitive to the geometry. If the slope is so steep that $s \gg 1$, we may have $T_{C1} \ll A \eta_o \sim O(A)$. Indeed, $s \gg 1$ means the characteristic scale of the slope $L_H$ is much less than the barotropic mode scale. We could choose the topography as $\eta = \eta_o e^{-s_o \lambda}$ instead of (3.5) with $s_o \sim 1$. A similar perturbation method gives the solutions

$$\psi_{C1} \sim 0(A^2)$$

$$\psi_{C2} \sim \frac{A \eta_o}{F_2 + 1 - s_o^2} (e^{-s_o \lambda} - e^{-\sqrt{F_1 + 1}/\lambda}).$$

This shows that the upper layer CSBC will decrease very rapidly if the topography scale is less than that of the barotropic IBC mode. However, even if the topographic scale is very small, the lower layer still has a large transport due to the CSBC. The total CSBC transport then decreases about 50% and is totally trapped in the lower layer. Furthermore, if the slope becomes even steeper ($s_o \gg 1$), $\psi_{C2}$ will also decrease dramatically, so that the total CSBC transport becomes negligible compared with the IBC. In other words, the vertical wall approximation is now very good. Hence, in the
dimensional form, we can have three different slope regimes:

\[
\text{Shallow slope} \left( L_H \geq \sqrt{\frac{U_1}{\beta}} \right): T_{C1} \sim T_{C2} \sim 0(A \eta_o); \\
\text{Medium slope} \left( \sqrt{\frac{U_1}{\beta}} \gg L_H \geq \sqrt{\frac{U_2}{\beta}} \right): T_{C1} \sim 0(A^2 \eta_o) \ll T_{C2} \sim 0(A \eta_o); \\
\text{Steep slope} \left( L_H \ll \sqrt{\frac{U_2}{\beta}} \right): T_{C1} \sim 0(A^2 \eta_o) \sim T_{C2}.
\]

iv. Physical mechanism. Physically, we can interpret the effects of the vertical structure and the horizontal scale of the slope through a discussion of topographic Rossby waves. In the presence of a westward basic flow and stratification, the only standing wave is the bottom trapped topographic Rossby wave, whose vertical trapping scale is the Rossby height (Charney and Flierl, 1981),

\[ H_s = L_H f/N, \]

where \( L_H \sim 1/s \) is the horizontal scale of the slope and \( N \) is the buoyancy frequency. Therefore, a larger horizontal slope scale means a larger vertical trapping scale of the excited wave. Similarly, a deeper lower layer allows a smaller buoyancy frequency at depth, which also yields a larger vertical trapping scale. A larger vertical trapping scale in turn implies that the topographic effect can influence more fluid. In this way, the CSBC transport is increased (relative to the IBC). The physical idea here seems to suggest that this phenomenon might occur in a variety of physical settings.

So in the \( A \ll 1 \) case, we reach the following conclusions: (1) As the lower layer gets deeper, the barotropic IBC transport decreases while the barotropic CSBC transport remains the same. Thus the CSBC transport becomes more important. As a result, the bottom slope will be very important even for a very weak lower layer inflow. A strong barotropic CSBC (compared with the IBC) can be forced for the slope height of about twice that of the one-layer case. (2) For a very steep slope, the CSBC will be trapped severely in the lower layer and turns out to be less important. Both are similar to the \( A = 1 \) case.

It should be pointed out that we would not have a trapped CSBC if the stratification was very weak. As a matter of fact, in the limiting case of \( F_1, F_2 \rightarrow \infty \), we would obtain an equivalent barotropic CSBC, having a stronger upper layer CSBC for the case with \( A < 1 \).

c. General cases

For general \( A \), we can derive the exact solutions (see (B.6)). We can prove the existence of the two-layer IBC and CSBC (in Appendix B) for any westward inflow in either layer. Conversely the boundary layer solutions (B.6) do not exist for eastward flows in both layers. The case in which the flow enters the boundary layer in one layer
and leaves the boundary layer in the other layer has been discussed by Spiegel and Robinson (1968).

Not only in the case of a small \( A \), but also in the case with \( A \sim 1 \), numerical calculations of the exact solution confirm the results derived from the asymptotic solution above. Furthermore, the calculations suggest that (3.24a) is a very modest estimate for the required slope height in the case of both a very weak lower layer incoming flow and a very deep lower layer. For instance, in light of (3.24a), we take the slope height as \( h_o = 2h_1 \), where \( h_1 = 1 \) is the slope height in the one-layer case in Figure 3a, b. Setting \( A = 0.2, H_2/H_1 = 4, H_1 + H_2 = H, s = 0.5, l = 0.1 \), where \( H \) is the same as the total depth in Figure 3a, b (implying \( F_1 = 5/2, F_2 = 5/8, \eta_o = 5/2 \)), we have the total transports for the IBC and the CSBC as

\[
\psi_{IB} \big|_{x=l, y=0.8} = 0.21 \psi_{CB} \big|_{x=l, y=0.8} = (\psi_B - \psi_{IB}) \big|_{x=l, y=0.8} = 0.19. \tag{3.25a}
\]

This gives the increased total transport relative to the IBC as

\[
R = \frac{\psi_{CB}}{\psi_{IB}} \big|_{x=l, y=0.8} = 0.9. \tag{3.25b}
\]

Here, the corresponding dimensional quantities of \( x = 1 \) and \( y = 0.8 \) are \( x = L_B = \delta_I = \sqrt{U/\beta} \) and \( y = 0.8L \sim L \). Thus (3.25b) gives the relatively increased transport approximately within the boundary layer and at the end of the boundary current. It should be noticed that \( \delta_I \) is not \( \xi_{IB} \) (the point of maximum \( \psi_{CB} \), as denoted by \( \xi_1 \) in Figure 1), and \( \xi_{IB} \) changes with \( A, F_1 \) and \( F_2 \). However, from the study of the barotropic case, we should expect that \( \xi_{IB} \) is close to \( \delta_I \) for the parameters calculated here. This means that the total increased transport into the boundary layer is almost the same as the IBC itself. Consequently, we have a very strong barotropic recirculation (Fig. 7). In dimensional quantities, in accord with both the one-layer case in Figure 3 and the two-layer case in Figure 7, we choose the barotropic quantities as

\[
L \sim 1000 \text{ km}, \beta \sim 1.5 \times 10^{-12} \text{ cm}^{-1} \text{s}^{-1}, f \sim 10^{-4} \text{ s}^{-1}, H \sim 4 \text{ km}
\]

and the baroclinic structure as

\[
H_1 \sim 800 \text{ m}, H_2 \sim 3200 \text{ m}, U_1 = U \sim 4 \text{ cm s}^{-1}, U_2 = AU \sim 0.8 \text{ cm s}^{-1},
\]

implying

\[
L_B \sim 52 \text{ km}, l \sim 0.052.
\]

The slope in Figure 7 has its dimensional height and width of 1200 m and 104 km respectively. Then, at \( y = 800 \text{ km} \) and \( x = 52 \text{ km} \), the dimensional total transports of (3.25a) are

\[
\psi_{IB} = 0.21 \times H \times U \times L = 33.6 \text{ Sv}, \psi_{CB} = 0.19 \times H \times U \times L = 30.4 \text{ Sv}.
\]

Thus we have the total transport of 64 Sv.
Figure 7. Streamfunctions of the two-layer IBC + CSBC in the case of a deeper lower layer $H_2 = 4H_1 = 4H/5$, $h_o = 2$ (or $F_1 = 5/2$, $F_2 = 5/8$, $\eta_o = 5/2$) and $A = 0.2$. The streamline interval is 0.1. (a) barotropic fields, (b) lower layer fields, (c) upper layer fields. Solid lines are for the CSBC + IBC, and the dashed lines are for the corresponding IBC (see text). A 0.1 interval in the nondimensional transport represents a dimensional transport of 16 Sv in section 3.c.

In comparison, the barotropic incoming flow case ($A = 1$) with $h_o = h_1$, which yields the same barotropic transport as the one-layer case in Figure 3a, b as proved before, gives only $R = 0.65$. Thus the barotropic recirculation is weaker. In fact, the dimensional transports for the barotropic IBCs are the same for both cases, while the CSBC, and thus the total transport, increase by about 10 Sv in the deep lower layer.
In addition, in Figure 7, the ratios between the lower layer and the upper layer transports for the IBC + CSBC and IBC are respectively

\[ p_T = \frac{F^{-1}_2(T_{12} + T_{C2})}{F^{-1}_1(T_n + T_{Cl})} \bigg|_{x=l,y=0.8} = 1.7, \quad p_I = \frac{F^{-1}_2(T_{12})}{F^{-1}_1(T_n)} \bigg|_{x=l,y=0.8} = 0.95. \]

Reducing \( A \) further to 0.1, we still have \( p_T = 1, p_I = 0.5. \) These suggest that even in the cases of very weak lower layer incoming flow, the slope is still able to increase the lower layer transport so much that the lower layer can have a transport comparable or larger than the upper layer one while the lower layer incoming transport is smaller than the upper layer one.

The numerical calculations here support the conclusions derived earlier. Furthermore, it can be seen that (3.24a) is a very modest estimate for the slope height in the case of a very weak lower layer incoming flow and a very deep lower layer.

4. Further discussion and conclusions

In this article, by adopting frictionless QG models, we studied the effect of a continental slope on the Western Boundary Current. The relationships among the present model and other previously studied models are shown in Table 1.

The compression of vortex tubes by the slope forces a strong northward boundary current over the slope which is called the Continental Slope Boundary Current or the CSBC.

In the \( \beta \)-plane case, the CSBC will exist together with the Inertial Boundary Current (or the IBC) in the region of westward inflows. The total flow field suggests a strong recirculation pattern and a greatly enhanced total transport in the boundary layer for the slope of a width wider than the IBC boundary layer thickness. When the critical height satisfies

\[ \frac{h_o}{H} \sim \frac{L}{a} \]

the forced CSBC transport can be comparable to the IBC at \( y \sim L \). Here, \( L \) is the scale of the interior flow and \( a \) is the radius of the earth. If the slope is very steep, the CSBC will be severely suppressed, suggesting that the vertical wall approximation is very good.

The two-layer model further shows that the topographic effect is trapped in the lower layer if the stratification is not very weak. Consequently, as compared with the IBC in each layer, the CSBC transport is relatively important at the lower layer. A deep lower layer would maintain the barotropic CSBC transport while decreasing the barotropic IBC transport. As a result, for the deep layer case of \( H_2 \sim 4H_1 \), the critical slope height
(the height of relief for the CSBC to have the comparable transport to the IBC) is only twice that of the one-layer case, i.e.

$$\frac{h_o}{H} \sim 2 \frac{L}{a}$$

which is of course still very reasonable for the real ocean.

As for the baroclinic structure, we find a very strong recirculation trapped in the lower layer and a very weak recirculation on the top layer, which can account for the observation that the interior Sverdrup flow is very well balanced in the upper ocean while the balance is broken in the deep ocean.

In addition, the slope can increase the total transport of the lower layer much more than that of the upper layer for a large depth ratio $H_2/H_1$. Consequently, the lower layer can have a total transport comparable to the upper layer even for a very weak lower layer incoming flow.

Since the models here are highly idealized, much further work is needed. First of all, in the narrow boundary region, friction can be as important as the relative vorticity due to the strong velocity shear. Especially in the region of an eastward flow where frictionless IBC and CSBC cannot exist, friction allows the circulation to be closed. In the second, because of either the narrow boundary layer or the very high continental slope, an ageostrophic (or semigeostrophic) model should be used, especially in the two-layer case. However, we should point out that from the critical height of the slope,
it seems possible to apply the studies here to the continental rise, instead of the continental slope.

Acknowledgments. The author would like to acknowledge his indebtedness to Drs. J. Pedlosky and R. X. Huang. Dr. Pedlosky suggested the barotropic QG model as well as the corner problem in Section 2, and discussed and corrected the manuscript many times. Dr. Huang gave many useful suggestions and corrected the first manuscript. This work was supported in part by a grant (ATM 84-13515) from the Division of Atmospheric Sciences of the National Science Foundation. This is Woods Hole Oceanographic Institution Contribution Number 7095.

APPENDIX

A. The properties of the CSBC streamfunction

The barotropic CSBC streamfunction is of the form

\[ \psi(\xi) = \eta(e^{-b\xi} - e^{-a\xi})/(a^2 - b^2), \]  

(A.1)

where \( \eta > 0 \), and \( a, b > 0 \).

Since \( \psi = 0 \) at both \( \xi = 0 \) and \( \infty \), the net meridional transport is zero. So if we have \( v > 0 \) somewhere, we must also have \( v < 0 \) somewhere else. The meridional velocity profile is

\[ v \sim \psi_\xi = \eta( ae^{-a\xi} - be^{-b\xi} )/(a^2 - b^2). \]  

(A.2)

At \( \xi = 0 \)

\[ v\big|_{\xi=0} = \eta/(a + b) = v_m > 0, \]  

(A.3)

so \( v \) is always northward right at the wall. Furthermore, we have the vorticity

\[ \nu_\xi = \eta(b^2e^{-b\xi} - a^2e^{-a\xi})/(a^2 - b^2), \]  

(A.4)

which has the zero point at \( \xi_0 \) when

\[ b^2e^{-b\xi_0} = a^2e^{-a\xi_0}, \]  

(A.5)

or at

\[ \xi_0 = \frac{2}{b-a} \ln \frac{b}{a} > 0 \]  

(A.6)

which is always greater then zero. In the limiting case \( a \to b \), it becomes

\[ \xi_0 \to \frac{2}{a} = \frac{2}{b}. \]  

(A.7)

With (A.5), it is easy to check that \( v\big|_{\xi_0} = -ae^{a\xi_0}/b(a + b) < 0 \). This gives the negative extreme velocity. As a result, we have the velocity profile as shown in Figure 1. The velocity field is composed of two parts. On the inshore side is a very strong and narrow
northward jet; on the offshore part a relatively weaker and wider southward countercurrent exists. The velocity reaches maximum $v_m$ right at the boundary $\xi = 0$.

To calculate the total net northward transport, we should first find the width of the jet or the point $\xi_1$ at which $\nu(\xi) = 0$, which is given by

$$be^{-b\xi_i} - ae^{-a\xi_i} = 0,$$

(A.8)

then

$$\xi_1 = \frac{1}{b - a} \ln \frac{b}{a} = \frac{\xi_o}{2} > 0.$$  

(A.9)

Substituting $\xi_1$ into (A.1), we get the transport of the CSBC

$$T_c = \psi(\xi_1) = \frac{\eta}{a^2 - b^2} \left[ e^{-b\xi_1} - e^{-a\xi_1} \right] = \nu_m \frac{1}{a} \left( \frac{b}{a} \right)^{b/a-b}$$

(A.10)

The factor here $1/a (b/a)^{b/a-b}$ can be proved having the limits

$$\frac{1}{a} \text{ as } \frac{b}{a} \to 0, \quad \frac{1}{ae} \text{ as } \frac{b}{a} \to 1, \quad \frac{1}{b} \text{ as } \frac{b}{a} \to \infty.$$  

(A.11)

So approximately, we have

$$T_c \sim \nu_m/c_m - \eta_o/(a + b)^2,$$

(A.12)

where $c_m = \max\{a, b\}$.

**B. The exact solution of IBC + CSBC in a two layer QG model**

1. The two-layer IBC and CSBC

Here, we write (3.3a,b) as

$$L_1(\psi_1, \psi_2) = \psi_{12} - (AF_1 + F_B)\psi_1 + F_1\psi_2 = -F_B Y$$

$$L_2(\psi_1, \psi_2) = \psi_{22} - \frac{1}{A} (F_2 + F_B)\psi_2 + F_2\psi_1 = -F_B y - \eta(\xi),$$

where $F_B = 1$ for the westward inflow $U > 0$ and $F_B = -1$ for the eastward inflow $U < 0$.

Similar to Section 3, for the IBC parts, we obtain the solutions:

$$\psi_{I1} = y(1 + F_1 e^{-\xi} + B_2 e^{-\lambda \xi} + 1)$$

(A.1a)

$$\psi_{I2} = A y(1 + B_1 e^{-\xi} + B_2 e^{-\lambda \xi} + 1).$$

(A.1b)

Here

$$E = (B_2 - 1)/(B_1 - B_2)$$

$$F = (1 - B_1)/(B_1 - B_2)$$

$$B_n = (AF_1 + F_B - \lambda_n^2)/AF_1 = F_2/(F_2 + F_B - A\lambda_n^2)$$
\[ \lambda_1^2 = (D_1 - D_2)/2A \]  
\[ \lambda_2^2 = (D_1 - D_2)/2A \]  
\[ D_1 = F_2 + A^2F_1 + (1 + A)F_B \]  
\[ D_2 = \left[ [F_2 + A^2F_1 + (1 + A)F_B]^2 - 4AF_B(AF_1 + F_2 + F_B) \right]^{1/2}, \]

where \( \lambda_n \) are solutions of the eigenvalue equation,

\[ J(\lambda) = A\lambda^4 - [F_2 + A^2F_1 + (1 + A)F_B]\lambda^2 + F_B(AF_1 + F_2 + F_B) = 0. \]  

Additionally, after some manipulation, \( D_2 \) in (B.2) becomes

\[ D_2 = \left[ [F_2 - A^2F_1 + (1 - A)F_B]^2 + 4A^2F_1F_2 \right]^{1/2}. \]  

From (B.2) and (B.4), we know that for westward incoming flows in both layers (\( F_B > 0, A > 0 \)), solution (B.1) always exists no matter how strong the shear is or how big \( A \) is. But if we have an eastward flow in some layer, i.e. \( F_B < 0 \) or \( A < 0 \), we might have no IBC. In fact, the sufficient and necessary condition for the existence of IBC here is that we have at least two decaying eigenvalues. One sufficient condition is that \( \{(D_1 \pm D_2)/A\} > 0 \). For \( A > 0 \), this means \( D_1 \pm D_2 > 0 \). From (B.4), we always have \( D_2 > 0 \). So we need \( D_1 > D_2 > 0 \). Compared with (B.2), we must have \( AF_B > 0 \), implying \( F_B > 0 \). Otherwise, if \( AF_B < 0 \), we can prove that at most one decaying solution exists. This means that there is no IBC. Since now \( AF_B < 0 \) is equivalent to \( F_B < 0 \), or both layers have eastward incoming flows, there is no IBC in the case of eastward flow in both layers. For the cases of different directions of flow in different layers, or \( A < 0 \), it becomes more complicated (Spiegel and Robinson, 1968). For the CSBC, we can get

\[ \psi_{C1} = \left[ F_1e^{-s\xi} + Ge^{-\lambda_1\xi} + Ie^{-\lambda_2\xi} \right] \frac{A\eta_0}{J(s)}, \]  
\[ \psi_{C2} = \left[ (-s^2 + AF_1 + F_B)e^{-s\xi} + B_1Ge^{-\lambda_1} + B_2Ie^{-\lambda_2} \right] \frac{A\eta_0}{J(s)}, \]

where

\[ J(s) = A(s^2 - \lambda_1^2)(s^2 - \lambda_2^2) \]  
\[ G = -(F_B - s^2 + AF_1(1 - B_2))/(B_1 - B_2) \]  
\[ I = [F_B - s^2 + AF_1(1 - B_1)]/(B_1 - B_2), \]

\( B_n \) is given in (B.2) and \( J(s) \) is the same as (B.3). It is obvious that the existence condition is the same as the IBC.
2. The limiting case

To check the \( A \to 0 \) case, we manipulate the formulas further. (B.2) (B.3) and (B.5) yield

\[
B_1 - B_2 = (\lambda_2^2 - \lambda_1^2)/(F_1A) \\
E = (F_B - \lambda_2^2)/(\lambda_2^2 - \lambda_1^2), \quad F = (\lambda_1^2 - F_B)/(\lambda_1^2 - \lambda_2^2) \\
I = F_1A(\lambda_1^2 - s^2)/(\lambda_2^2 - \lambda_1^2), \quad G = F_1A(s^2 - \lambda_2^2)/(\lambda_2^2 - \lambda_1^2).
\]

Then (B.1a,b) and (B.5a,b) can be rewritten as

\[
\psi_{r1} = \left( 1 + \frac{1}{\lambda_2^2 - \lambda_1^2} \right) \left( (F_B - \lambda_2^2) e^{-\lambda_1 t} + (\lambda_1^2 - F_B) e^{-\lambda_2 t} \right) \\
\psi_{r2} = A\psi_{r1} = \left( 1 + \frac{1}{\lambda_2^2 - \lambda_1^2} \right) \left( (F_B - \lambda_2^2) B_1 e^{-\lambda_1 t} + (\lambda_1^2 - F_B) B_2 e^{-\lambda_2 t} \right) \\
\psi_{c1} = \frac{\eta_0 F_1}{(s^2 - \lambda_1^2)(s^2 - \lambda_2^2)} \left( e^{-s t} + \frac{1}{\lambda_2^2 - \lambda_1^2} \left( (s^2 - \lambda_2^2) e^{-\lambda_1 t} + (\lambda_1^2 - s^2) e^{-\lambda_2 t} \right) \right) \\
\psi_{c2} = \frac{\eta_0}{(s^2 - \lambda_1^2)(s^2 - \lambda_2^2)} \left( (AF_1 + F_B - s^2) e^{-s t} + \frac{AF_1}{\lambda_2^2 - \lambda_1^2} \left( (s^2 - \lambda_2^2) B_1 e^{-\lambda_1 t} + (\lambda_1^2 - s^2) B_2 e^{-\lambda_2 t} \right) \right).
\]

As \( A \to 0 \), we obtain \( D_1 \to F_2 + F_B + AF_B, D_2 \to F_2 + F_B - AF_B \). Thus

\[
\lambda_1^2 \to F_B, \quad \lambda_2^2 \to \frac{F_2 + F_B}{A}.
\]

Using the second part for \( B_n \) in (B.2) to \( B_1 \) gives

\[
B_1 = F_2/(F_2 + F_B - A\lambda_1^2) \to \frac{F_2}{F_2 + F_B},
\]

while the first part for \( B_n \) in (B.2) to both \( B_1 \) and \( B_2 \), noticing the \( B_1 \) limit above, yields

\[
\frac{F_B - \lambda_1^2}{F_1A} = B_1 - 1 \to \frac{F_2}{F_2 + F_B} - 1
\]

\[
B_2 = (AF_1 + F_B - \lambda_2^2)/AF_1 \to -\lambda_2^2/AF_1 \to -\frac{F_2 + F_B}{F_1A^2}.
\]

Therefore, noticing \( \lambda_1^2 \to 0(1), \lambda_2^2 \to 0(1/A), B_1 \to 0(1), B_2 \to 0(1/A^2) \), we may derive
the leading order solution of the IBC from (B.6)

\[
\psi_{I1} \rightarrow \varphi_0 \left[ 1 - e^{-\sqrt{F_B} \xi} \right]
\]

\[
\psi_{I2} \rightarrow A \varphi_0 \left[ 1 - e^{-\sqrt{F_2 + F_B} \sqrt{A} \xi} + \frac{F_2}{F_2 + F_B} \left( e^{-\sqrt{F_2 + F_B} \sqrt{A} \xi} - e^{-\sqrt{F_B} \xi} \right) \right].
\]

For the CSBC, there are two different cases. The first is \( s \sim 1 \), in which we have the leading order expression

\[
\psi_{C1} \rightarrow \frac{\eta_0 F_1 A}{(F_B - s^2)(F_2 + F_B)} \left[ e^{-\sqrt{A} \xi} - e^{-s \xi} \right]
\]

\[
\psi_{C2} \rightarrow \frac{A \eta_0}{F_2 + F_B} \left[ e^{-\sqrt{A} \xi} - e^{-\sqrt{F_2 + F_B} \sqrt{A} \xi} \right].
\]

In the second case where \( s \sim s_0 / A \sim 1 / A \), we obtain the leading order solutions

\[
\psi_{C1} \rightarrow 0(A^2)
\]

\[
\psi_{C2} \rightarrow \frac{\eta_0 A}{s_0^2 - (F_2 + F_B)} \left[ e^{-s_0 \xi / \sqrt{A}} - e^{-\sqrt{F_2 + F_B} \sqrt{A} \xi} \right].
\]

REFERENCES


Received: 2 May, 1989; revised: 7 March, 1990.