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Abstract

Affective decision-making (ADM) is a refutable and predictive theory of individual choice under risk and uncertainty. It generalizes expected utility theory by positing the existence of two cognitive processes — the "rational" and the "emotional" process. Observed choice is the result of their simultaneous interaction. We present a model of affective choice in insurance markets, where risk perceptions are endogenous.

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1 Introduction

Behavioral theories of choice seek both to explain past choices as a function of changes in exogenous variables, and given data on past choice behavior, together with a probability model of the data generating process, to predict future choices. It is in this sense that affective decision-making, ADM, is a behavioral theory of choice. A property shared by consumer demand analysis — see part one in Deaton and Muellbauer (1980), but not evident in other strategic models of choice behavior such as Gul–Pessendorfer (2001), Bernheim–Rangel (2004) or Fudenberg–Levine (2006).

ADM is a game-theoretic model of individual decision-making under risk and uncertainty, which generalizes expected utility, and where the probability weights — perceived risk — are endogenous, as implied by optimism bias (Slovic 2000, Weinstein 1980). In our model of individual decision-making there are two distinct psychological processes that mutually determine choice. This approach is inspired in part by Kahneman (2003), who proposes two systems of reasoning that differ in several important aspects, such as emotion. We call these systems of reasoning the *rational process* and the *emotional process*. The rational process coincides with the expected utility model. That is, for a given risk perception, i.e., perceived probability distribution, it maximizes expected utility. The emotional process is where risk perception is formed. In particular, the agent selects an optimal risk perception to balance two contradictory impulses: (1) affective motivation and (2) a taste for accuracy. This is a definition of motivated reasoning, a psychological mechanism where emotional goals motivate agent's beliefs, e.g., Kunda (1990), and is a source of psychological biases, such as optimism bias. Affective motivation is the desire to hold a favorable personal risk perception — optimism — and is captured by the expected utility term. The desire for accuracy is the mental cost incurred by the agent for holding beliefs other than her *base rate,* given her desire for favorable risk beliefs. The base rate is the belief that minimizes the mental cost function of the emotional process. This is the agent's *correct* risk belief, if her risks are objective such as mortality tables.

As an application of affective decision-making, we present an example of the demand for insurance in a world with two states of nature: Bad and Good. The relevant probability distribution in insurance markets is personal risk, hence the demand for insurance may depend on optimism bias. Affective choice in insurance markets is defined as the insurance level and risk perception which constitute a pure strategy Nash Equilibrium of the ADM intrapersonal game.

The systematic departure of the ADM model from the expected utility model allows for both optimism and pessimism in choosing the level of insurance, and shows, consistent with consumer research (Keller and Block 1996), that campaigns intended to educate consumers on the loss size in the bad state can have the unintended consequence that consumers purchase less, rather than more, insurance. Hence, the ADM model suggests that the failure of the expected utility model to explain some data sets may be due to systematic affective biases.

To show that the ADM model explains past choice behavior in insurance markets, we introduce the ADM inequalities. These inequalities, are defined by the first order conditions for pure strategy Nash equilibria in the ADM intrapersonal game; the Afriat inequalities for the concave, Bernoulli utility function of the rational process; the Afriat inequalities for the convex, mental cost function of the emotional process; and the budget constraints. The ADM inequalities are a finite family of multivariate, polynomial inequalities, where the unknowns are the utility levels and marginal utilities for the rational process, and costs and marginal costs for the emotional process. Every solution of the ADM inequalities defines an ADM intrapersonal game.

An insurance market data set—a finite set of observations on state-contingent endowments of wealth, state-prices or insurance premia and the level of insurance purchased by the agent — is *rationalized* by the ADM model if the ADM inequalities are solvable for the parameter values derived from the data set. It follows from the Tarski-Seidenberg Theorem that the ADM model is *refutable* if the ADM inequalities are solvable for some but not all insurance market data sets. That is, the ADM model *explains* some, but not all past choice behavior in insurance markets. We show that this model is refutable if bounds on risk perception are known, consistent with Kahneman and Tversky (1979).

If the ADM inequalities are solvable, then the solution need not be unique. If the solution is unique, then it need not be stable, i.e., continuous under small, noisy perturbations of the data. Hence, the ADM inequalities are ill-posed in the sense of Hadamard (1932), a property they share with the Afriat inequalities. In this case, a unique, stable solution of the ADM inequalities is obtainable by Tikhonov regularization. Ridge regression is an example of Tikhovov regularization. Melkman and Micchelli (1979) show that regularization is the optimal algorithm for learning a function from noisy data.

The ADM intrapersonal game is a potential game, where the potential function is bi-concave. Potential games were introduced by Monderer and Shapley (1996). Neyman (1997) proved that potential games with a concave potential function have a unique pure strategy Nash equilibrium. The predictive ADM inequalities are defined by the ADM inequalities and the Afriat inequalities for a concave potential function. Every solution of the predictive ADM inequalities defines a predictive ADM intrapersonal game, with a unique pure strategy Nash equilibrium. In the insurance market example, every concave utility function, $u(W)$, and convex cost function, $c(\beta)$, where the associated potential function is concave, jointly define a predictive ADM intrapersonal game.

We identify the family of predictive, ADM intrapersonal games with the closed, convex cone of ordered pairs of concave utility functions and convex cost functions, denoted C_P. The optimal predictive ADM intrapersonal game is the argmin over $(u(W)), c(\beta)) \in C_P$ of the regularized, mean square deviation between the observed demand for insurance and the affective demand for insurance. As such, it is a unique, stable solution of the predictive ADM inequalities.

If we assume that the insurance market data sets are i.n.i.d., identical but not identically distributed, samples from a stratified population where the data generating process is random sampling within and across strata, then we can derive the optimal prediction of the agent's future affective demand for insurance. The choice of the rational process, in the optimal predictive ADM intrapersonal game, is the best mean square prediction of the agent's affective demand for insurance on future data sets. If insurance markets are actuarially fair then the predictive ADM model reduces to the expected utility model. In this case, our predictive analysis can be applied to the expected utility model.

This machine learning perspective is due to Poggio and Smale (2003) who propose a "predictive algorithm for fitting the best multivariate function to data." They discuss both a PAC learning algorithm and a regularized, learning algorithm. PAC learning was independently introduced in economic theory by Gil Kalai (2003) in the context of social choice theory.

In the next section of the paper, we present the ADM intrapersonal game. In the third section, we derive the ADM inequalities in the ADM state-preference model and discuss our results on the refutable implications of affective decision-making. In the fourth section of the paper, we follow Brown, Calsamiglia and Jones (2007), and extend the Poggio–Smale regularized, learning algorithm to fitting the best predictive, ADM intrapersonal game to insurance market data.

2 The ADM Intrapersonal Game

Affective decision-making (ADM) is a theory of choice, which generalizes expected utility theory by positing the existence of two cognitive processes — the *rational* and the *emotional process*. Observed choice is the result of their simultaneous interaction. This theory accommodates endogenity of beliefs, probability perceptions and tastes. In this paper, we present a model of affective choice in insurance markets, where probability perceptions are endogenous.

Consider an agent facing two possible future states of the world, Bad and Good with associated wealth levels ω_B and ω_G, where $\omega_B < \omega_G$. The agent has a strictly increasing, strictly concave, smooth utility function of wealth, $u(W)$, with $\lim_{w\to-\infty} Du(W) = \infty$, $\lim_{w\to\infty} Du(W) = 0$.[1] Risk perception is defined as the perceived probability $\beta \in [0,1]$ of the Bad state occurring. To avoid (perceived) risk, the agent can purchase or sell insurance $I \in (-\infty, \infty)$ to smooth her wealth across the two states of the world. The insurance premium rate, $\gamma \in (0,1)$ is fixed for all levels of insurance purchases.

The *rational process* chooses an optimal insurance (I^*) to maximize expected utility given a perceived risk β. Specifically, the rational process maximizes the

[1] All qualitative results remain the same for the case of $\lim_{W\to 0} Du(W) = \infty$, $\lim_{W\to\infty} Du(W) = 0$.

following objective function:

$$\max_{I}\left\{\beta u(\omega_B+(1-\gamma)I)+(1-\beta)u(\omega_G-\gamma I)\right\}.$$

The *emotional process* chooses an optimal risk perception (β^*) given an insurance level I, to balance affective motivation and taste for accuracy. Specifically, the emotional process maximizes the following objective function:

$$\max_{\beta}\left\{\beta u(\omega_B+(1-\gamma)I)+(1-\beta)u(\omega_G-\gamma I)-c(\beta;\beta_0)\right\}.$$

Affective motivation is captured with the expected utility term — the agent would like to assign the highest possible weight to her preferred state of the world. Taste for accuracy is modeled by introducing a mental cost function $c(\beta;\beta_0)$ that is a nonnegative, and smooth function of β. It is strictly convex in β, and reaches a minimum at $\beta=\beta_0$, where β_0 is the objective probability. The farther away β is from β_0, the greater are the psychological cost.[2] We will assume that $c(\beta;\beta_0)$ is a smooth function of β_0. It is well-known that agents attribute a special quality to situations corresponding to the extreme beliefs $\beta\in\{0,1\}$ (Kahneman and Tversky 1979). Hence we assume that there exist limits $\underline{\beta},\bar{\beta}\in(0,1)$ such that for $\beta\in(\underline{\beta},\bar{\beta})$, $c(\beta;\beta_0)$ is finite, and $\lim_{\beta\to\underline{\beta}}c(\beta;\beta_0)=\lim_{\beta\to\overline{\beta}}c(\beta;\beta_0)=+\infty$.

The interaction of the two processes in decision-making is modeled using an intrapersonal simultaneous-move game. Modeling the interaction of the two processes as a simultaneous move game reflects a recent view in cognitive neuroscience; namely, both processes mutually determine the performance of the task at hand (Damasio 1994).

Definition 1 *An intrapersonal game is a simultaneous move game of two players, namely, the rational and the emotional processes. The strategy of the rational process is an insurance level,* $I\in(-\infty,\infty)$, *and the strategy of the emotional process is a risk perception,* $\beta\in(\underline{\beta},\bar{\beta})$. *The payoff function for the rational process* $g:(\underline{\beta},\bar{\beta})\times(-\infty,\infty)\to R$ *is* $g(\beta,I)\equiv\beta u(\omega_B+(1-\gamma)I)+(1-\beta)u(\omega_G-\gamma I)$. *The payoff function for the emotional process* $\psi:(\underline{\beta},\bar{\beta})\times(-\infty,\infty)\to R$ *is* $\psi(\beta,I)\equiv g(\beta,I)-c(\beta;\beta_0)$, *where* $c(\cdot)$ *is the mental cost function of holding belief* β, *which reaches a minimum at* β_0.

The pure strategy Nash equilibria of this game, if they exist, are the natural candidates for the agent's choice, as they represent mutually determined choice and reflect consistency between the rational and emotional processes. The intrapersonal game defined above is a potential game, where the potential function can be interpreted as the utility function of the composite agent.

[2] To justify favorable beliefs agents may use strategies such as the availability heuristic, which can be unconsciously manipulated to arrive at the desired beliefs. Such mental strategies, or justification processes, are likely to be costly and are captured by the cost function. We assume that biased recall becomes increasingly more costly as the distance between desired beliefs β and the objective odds β_0 increases.

Proposition 2 *The intrapersonal game is a potential game, in which the emotional process's objective function is the potential function for the game. Because the potential function is strictly concave in each variable (risk perception and insurance), its critical points are the pure strategy Nash equilibria of the game.*

Proof. Denote the rational process's payoff function as (R) and the emotional process's payoff function as (E). A necessary and sufficient condition for the intrapersonal game to have a potential function (Monderer and Shapley 1996) is $\frac{\partial^2 R}{\partial\beta\partial I} = \frac{\partial^2 E}{\partial\beta\partial I}$. This condition clearly is satisfied in the ADM model. The potential function $P(\beta, I)$ is a function such that (Monderer and Shapley, 1996): $\frac{\partial P}{\partial\beta} = \frac{\partial E}{\partial\beta}$, $\frac{\partial P}{\partial I} = \frac{\partial R}{\partial I}$. Because $\frac{\partial E}{\partial I} = \frac{\partial R}{\partial I}$, (E) can serve as a potential function. The critical points of the potential function are $\frac{\partial P}{\partial\beta} = \frac{\partial E}{\partial\beta} = 0$, $\frac{\partial P}{\partial I} = \frac{\partial R}{\partial I} = 0$. The potential function is strictly concave in each variable, so at each critical point, each process is maximizing its objective function, given the strategy of the other process. Therefore, the critical points of the potential function are the pure strategy Nash equilibria of the intrapersonal game, and all pure strategy Nash equilibria are critical points of the potential function. ■

The potential function allows us to find sufficient condition for uniqueness, conduct welfare analysis, and make predictions about future behavior.

Excluding the case of tangency between the best responses of the two processes, we have the following existence theorem.

Proposition 3 *The ADM intrapersonal game has an odd number of pure strategy Nash equilibria. The set of Nash equilibria is a chain in R^2, under the standard partial order on points in the plane.*[3]

Proof. By the boundaries on risk perception, $0 < \underline{\beta} < \bar{\beta} < 1$, $\beta^* \in (\underline{\beta}, \bar{\beta})$, and insurance $I^* \in [I^*(\underline{\beta}), I^*(\bar{\beta})]$. Hence, all Nash equilibria will have perceived probabilities in the interval $[\beta^*(I^*(\underline{\beta})), \beta^*\left(I^*(\bar{\beta})\right)]$ where $0 < \underline{\beta} < \beta^*(I^*(\underline{\beta})) < \beta^*\left(I^*(\overline{\beta})\right) < \overline{\beta} < 1$. Define $\beta^*(I^*(\underline{\beta})) \equiv \underline{\beta}'$, $\beta^*\left(I^*(\bar{\beta})\right) \equiv \bar{\beta}'$; because all the Nash equilibria of the intrapersonal game for $\beta \in (\underline{\beta}, \bar{\beta})$ are $\in [\underline{\beta}', \bar{\beta}']$ the focus can remain on the latter probability space.

The existence and chain results can be shown by defining a restricted intrapersonal game in which the insurance pure strategy space is restricted to $[I^*(\underline{\beta}), I^*(\bar{\beta})]$ and the perceived probabilities are restricted to $\beta \in [\underline{\beta}', \bar{\beta}']$, such that the equilibria points of the intrapersonal game are not altered. The restricted game is a supermodular game, and thus, these results follow from the properties of this class of games (see Topkis 1998). To Show that the game admits odd number of equilibria, think of the geometry of the game. As $\beta \to \bar{\beta}$, the best response of the emotional process is above the best response of the rational process, while this relationship is reversed for

[3]The existence of a pure strategy Nash equilibrium also can be derived for the case of a logarithmic utility function, in which the agent's income in each state is not negative.

$\beta \to \underline{\beta}$. Since the best responses are monotonically increasing, it follows that there exists odd number of Nash equilibria. ■

To derive the predictive properties of the ADM model, we require the intrapersonal game to have a unique pure strategy Nash equilibrium. A sufficient condition for uniqueness follows:

Proposition 4 *A sufficient condition for a unique pure strategy Nash equilibrium of the intrapersonal game is:*

$$\frac{\partial^2 c(\beta;\beta_0)}{\partial \beta^2} > -\frac{\left[Du(\omega_B+(1-\gamma)I)(1-\gamma)+Du(\omega_G-\gamma I)\gamma\right]^2}{\left[\beta D^2u(\omega_B+(1-\gamma)I)(1-\gamma)^2+(1-\beta)D^2u(\omega_G-\gamma I)\gamma^2\right]},$$
$$\forall (I,\beta)\in[I^*((\underline{\beta}'),I^*(\bar{\beta}')]\times[\underline{\beta}',\bar{\beta}'],$$

where $\underline{\beta}' \equiv \beta^*(I^*(\underline{\beta}))$ *and, similarly,* $\bar{\beta}' \equiv \beta^*(I^*\left(\bar{\beta}\right))$.

Proof. The emotional process's objective function $\beta u(\omega_B + (1-\gamma)I) + (1-\beta)u(\omega_G - \gamma I) - c(\beta;\beta_0)$ is the potential function of the game. The maximization of (P) with respect to a pair (I,β) gives rise to a pure strategy Nash equilibria of the game.$\beta \in [\underline{\beta}', \bar{\beta}']$ and $I \in [I^*(\underline{\beta}'), I^*(\bar{\beta}')]$ (see proof of Proposition 3), hence only the restricted intrapersonal game in which both players' strategy spaces are compact need be considered. Neyman (1997), proved that a potential game with a strictly concave, smooth potential function, in which all players have compact, convex strategy sets, has a unique pure strategy Nash equilibrium. That is, the Hessian of the potential function is negative definite, as follows from the condition given above. ■

For large mental costs, the equilibrium is unique (think of $\lambda > 0$, $\hat{c}(\cdot) = \lambda c(\cdot)$). Moreover, for very large mental costs, the ADM model reduces to the expected utility model.[4]

However, considering the general case, where the mental costs are not very large, risk perceptions are endogenous and the ADM model systematically departs from the expected utility model. This suggests that the failure of the expected utility model to explain some data sets may be due to systematic affective biases. How exactly does affective choice in insurance markets differ from the demand for insurance in the expected utility model? Proposition 5 below shows that the expected utility outcome in the case of an actuarially fair insurance market (full insurance) falls within the choice set of the ADM agent. However, if the insurance market is not actuarially fair, then this is no longer the case.

Proposition 5 *If* $\gamma = \beta_0$*, there exists at least one Nash equilibrium* (β^*, I^*) *with* $\beta^* = \beta_0 = \gamma$*, and* $I^* =$ *full insurance.*

If $\gamma > \beta_0$*, there exists at least one Nash equilibrium* (β^*, I^*) *with* $\beta^* < \beta_0$ *and* $I^* < I^*(\beta_0)$,

[4] As $c \to \infty$, $\beta^* \to \beta_0$ for all values of I. As a result, the ADM model converges to the expected utility model.

If $\gamma < \beta_0$, there exists at least one Nash equilibrium (β^*, I^*) with $\beta_0 < \beta^*$ and $I^* > I^*(\beta_0)$.

Proof. Consider the case in which $\gamma = \beta_0$. At full insurance, there is no mental gain for holding beliefs $\beta \neq \beta_0$ but there exists mental cost. Therefore, at full insurance, the mental process's best response is $\beta = \beta_0$. Given that $\gamma = \beta_0 = \beta$, the rational process's best response is full insurance. Consequently, full insurance and $\beta = \beta_0$ is a Nash equilibrium of this case. Next, consider the case $\gamma > \beta_0$; because the insurance premium is higher than β_0, $I^*(\beta = \beta_0) < z$. Also, $\beta^* = \beta_0$ only at full insurance, where $I = z$. Therefore, at $\beta = \beta_0$ the mental process's best response falls above the rational process's best response. This relationship is reversed at the limit $\beta \to \underline{\beta}$, and both the mental and the rational best responses increase; therefore, there exists a Nash equilibrium with $\beta < \beta_0$ and less insurance than predicted by the expected utility model. A similar argument can be used to prove the result when $\gamma < \beta_0$. ■

To understand the intuition behind these results, consider a standard myopic adjustment process where the processes alternate moves. If $\gamma > \beta_0$, at β_0 the rational process, similar to the expected utility model, prescribes buying less than full insurance. The emotional process, in turn, leads the decision maker to believe "this is not going to happen to me" and determines that she is at a lower risk. This effect causes a further reduction in the insurance purchase, with a result of less than full insurance, even less than what the expected utility model would predict. Note that proposition 5 also implies that, from the viewpoint of an outside observer, both optimism and pessimism (relative to β_0) are possible. This is due to the characteristics of insurance: if an agent purchases more than full insurance, then the "bad" state becomes the "good" state, and vice versa. Consequently, if there is no effective action, i.e., one cannot change the bad state to a good state, we would observe optimism and less-than-optimal insurance.

Here is another example of the difference between affective choice and the demand for insurance in the expected utility model. In the expected utility model, if people realize that they face a higher potential loss, due to educational campaigns that make them aware of the possible catastrophe, then they purchase more insurance. In the ADM model, if an agent realizes she faces higher possible loss, then she might purchase less insurance. Because the increased loss size affects both the emotional and the rational processes in different directions; the rational process prescribes more insurance, the emotional process prescribes lower risk belief to every insurance level (due to greater incentives to live in denial). If the emotional effect is stronger the agent will buy less insurance than previously. That is, if the loss is great, agents might prefer to remain in denial and ignore the possible catastrophes altogether, which will lead them to take fewer precautions such as buying insurance. This is consistent with consumer research showing that high fear arousal in educating people on the health hazards of smoking leads to a discounting of the threat (Keller and Block 1996).

Proposition 6 below summarizes the conditions for educational campaigns to produce the counter-intuitive affective result.

Proposition 6 *An educational campaign result in less insurance if*

$$\frac{r(\omega_B - \gamma I)}{Du(\omega_B - \gamma I)} > \frac{r(\omega_G + (1-\gamma)I)}{Du(\omega_G + (1-\gamma)I)},$$

where $r(\cdot)$ is the absolute risk aversion property of the utility function $u(\cdot)$

Proof. Define $\tilde{I}(\beta;\beta_0)$ as the inverse function β^{*-1}. Define $\Pi(\beta;\beta_0) = I^*(\beta) - \tilde{I}(\beta;\beta_0)$, $\Pi : [\underline{\beta}', \bar{\beta}'] \to R$

Educational campaigns on impending catastrophes increase the loss size, z. Because $\Pi(\beta;\beta_0) = 0$ is a NE, $\frac{\partial \Pi}{\partial z} < 0$ represent the unintended consequence of such campaigns.

$$\frac{\partial \Pi}{\partial z} < 0 \Leftrightarrow \frac{\frac{\partial \tilde{I}}{\partial z}}{\frac{\partial I^*}{\partial z}} > 1$$

.

$$\frac{\partial I^*}{\partial z} = \frac{[u''(w_2 - z + (1-\gamma)I^*)][u'(w_2 - \gamma I^*)]^2}{[u'(w_2-\gamma I^*)][u''(w_2 - z + (1-\gamma)I^*)u'(w_2 - \gamma I^*)(1-\gamma) + u'(w_2 - z + (1-\gamma)I^*)u''(w_2 - \gamma I^*)\gamma]};$$

$$\begin{aligned} \frac{\partial \tilde{I}}{\partial z} &= \frac{[u'(w_2 - z + (1-\gamma)\tilde{I})]}{[u'(w_2 - z + (1-\gamma)\tilde{I})(1-\gamma) + u'(w_2 - \gamma\tilde{I})\gamma]} \Rightarrow \frac{\partial \Pi}{\partial z} < 0 \\ &\Leftrightarrow \frac{r(w_2 - \gamma I)}{u'(w_2 - \gamma I)} > \frac{r(w_1 + (1-\gamma)I)}{u'(w_1 + (1-\gamma)I)}, \text{ where } r(x) = -\frac{u''(x)}{u'(x)}. \end{aligned}$$

■

In Proposition 6, if the utility function $u(\cdot)$ exhibits constant or increasing absolute risk aversion, educational campaigns will lead to higher insurance purchase if and only if initially the agent buys more than full insurance. Insurees who initially buy less than full insurance will buy even less after the educational campaign. Hence, for such utility functions, educational campaigns divide the insurance market into a set of agents who purchase more insurance — the intended consequence — and a set of agents who purchase less insurance — the unintended consequence.

3 A Refutable Model of ADM

This section formulates the ADM model in the state preference framework, presents the ADM inequalities and derives the *Axiom of Revealed Affective Choice or ARAC*. The rational process's choice can be formulated in the state-preference model:

$$\begin{aligned} \max_{W_B, W_G \in R^2} \quad & \beta u(W_B) + (1-\beta)u(W_G) \\ \text{s.t.} \quad & \gamma W_B + (1-\gamma)W_G = \gamma\omega_B + (1-\gamma)\omega_G. \end{aligned}$$

Because the income level and the insurance premium are given, choosing the wealth levels W_B and W_G is equivalent to selecting an insurance level I.

The first-order conditions of the state preference model are:

$$\beta Du(W_B) = \lambda\gamma \tag{1}$$
$$(1-\beta)Du(W_G) = \lambda(1-\gamma) \tag{2}$$

These conditions can be written as

$$\frac{Du(W_G)}{Du(W_B)} = \frac{(1-\gamma)}{\gamma}\frac{\beta}{(1-\beta)} \tag{3}$$

Thus, every solution of the intrapersonal game can be translated into a point on the budget line in the (W_B, W_G) plane; this point is observable and the slope of the budget line is the ratio $\frac{(1-\gamma)}{\gamma}$. However, the inequality is transformed such that $\frac{Du(W_G)}{Du(W_B)}$ equals $\frac{(1-\gamma)}{\gamma}\frac{\beta}{(1-\beta)}$, which is labeled the perceived price ratio and is determined by the perceived probabilities. These perceived probabilities must satisfy the first-order condition of the emotional process:

$$u(W_B) - u(W_G) = Dc(\beta). \tag{4}$$

Figure 1 illustrates a possible choice.

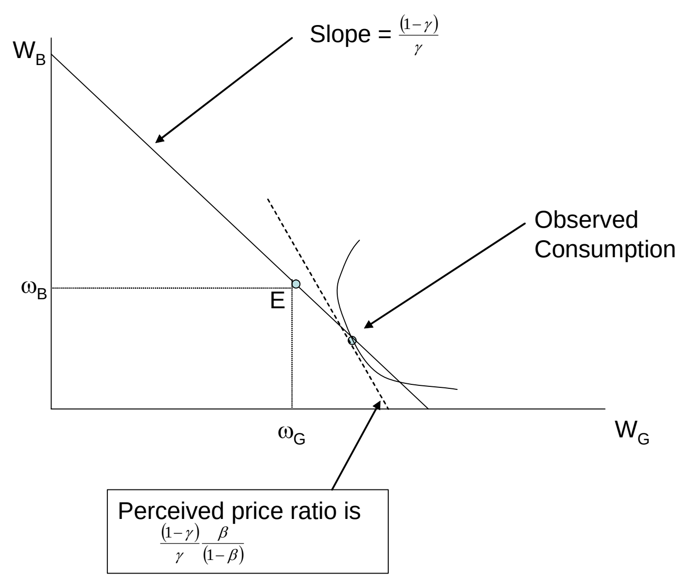

Figure 1 – Affective Choice in the State-Preference Framework.

Definition 7 *An affective choice is a wealth point (W_B, W_G) and price $0 < \gamma < 1$, such that the agent maximizes utility subject to her budget constraint and satisfies her emotional process's first-order condition.*

The distinction between objective and perceived probabilities enables the ADM model to support data sets that cannot be rationalized by expected utility theory, as Figure 2 shows.

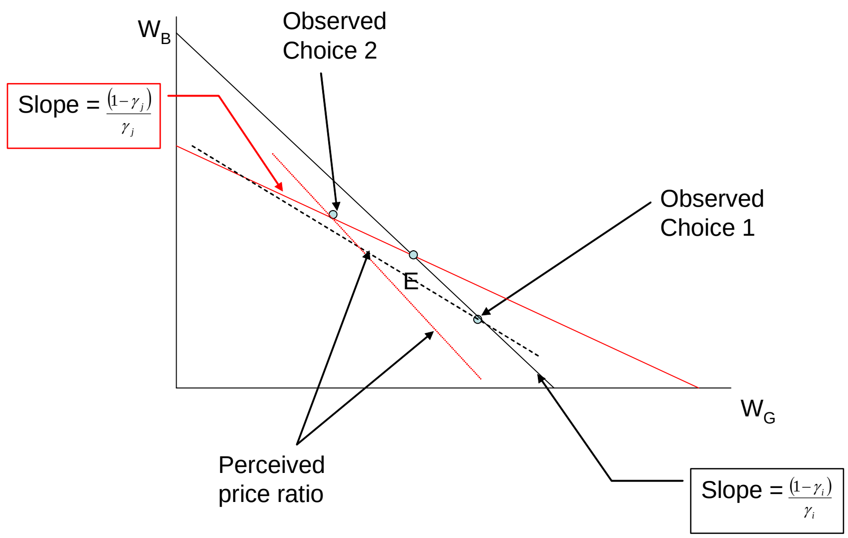

Figure 2 – Data Rationalized by ADM, But Not EU.

In Figure 2, with objective probabilities, the two observations violate the weak axiom of revealed preference (WARP). Therefore, this agent is not an expected utility maximizer; these observations refute the expected utility paradigm. However, if the agent is an affective agent, these observations need not violate WARP. Recall that with ADM, perceived probabilities generally differ from objective probabilities, so the agent's perceived price ratio is different from the one observed, and the data might not violate WARP; hence there exists a utility and a mental cost function such that the agent acts as if she is an affective expected utility maximizer. Thus the ADM model can rationalize the data.

To show that the ADM model is refutable, we introduce the ADM inequalities. Let the insurance market data set $D = \{(x^i; y^i)\}_{i=1,\ldots,N}$. The Bernoulli agent's utility of W dollars is $u(W)$, and the marginal utility is $Du(W)$. ω_B and ω_G are the agent's endowments of wealth in the bad and the good state, respectively; $\gamma \in (0,1)$ is the insurance premium, and I is the agent's insurance level. $c(\beta\ )$ is the mental cost of the perceived risk β and the marginal cost is $Dc(\beta)$. If $(x; y) \equiv (\omega_B, \omega_G, I; \gamma)$, then $W_B(x; y) \equiv (\omega_B + (1-\gamma)I)$, wealth in the bad state, and $W_G(x; y) \equiv (\omega_G - \gamma I)$, wealth in the good state.

The ADM inequalities consist of:

(a) *The first order conditions for a pure strategy Nash equilibrium in the ADM game*:

Rational Process: $\beta_i Du(W_B(x^i; y^i)) = \gamma^i; (1-\beta_i)Du(W_G(x^i; y^i)) = 1-\gamma^i$, *and*

Emotional Process: $u(W_B(x^i; y^i)) - u(W_G(x^i; y^i)) = Dc(\beta_i)$, *for* $i = 1, ..., N$;

(b) *The Afriat inequalities for the concave, Bernoulli utility function of the rational process*:

$$u(W_i) \leq u(W_j) + Du(W_j)(W_i - W_j), \text{ for all } W_i \text{ and } W_j;$$

(c) *The Afriat inequalities for the convex, mental cost function of the emotional process*:

$$c(\beta_i) \geq c(\beta_j) + Dc(\beta_j)(\beta_i - \beta_j), \text{ for all } \beta_i \text{ and } \beta_j;$$

(d) *The budget constraints*:

$$\gamma^i W_B(x^i; y^i) + (1 - \gamma^i) W_G(x^i; y^i) = \gamma^i \omega^i_B + (1 - \gamma^i)\omega^i_G, \text{ for } i = 1, ..., N.$$

This is a finite family of multivariate, polynomial inequalities, where the unknowns are the utility levels and the marginal utilities for the rational process together with the costs and the marginal costs for the emotional process. An insurance market data set is rationalized by the ADM model if the ADM inequalities are solvable for the parameter values derived from the data set. It follows from the Tarski–Seidenberg Theorem that the ADM model is refutable if the ADM inequalities are solvable for some but not all insurance market data sets — see chapter one in Brown and Kubler (2008).

That is, there exists a system of multivariate, polynomial inequalities where the unknowns are the agent's demands for insurance and prices of insurance. These inequalities constitute the Axiom of Revealed Affective Choice or *ARAC*. The ADM model rationalizes the data if and only if the data satisfies *ARAC*. This axiom exhausts all the refutable implications of the ADM model, just as GARP exhausts all the refutable implications of utility maximization subject to a budget constraint — see Varian (1983). In the Appendix we derive a necessary condition for rationalizing an insurance data set with the ADM model: The Weak Axiom of Revealed Affective Choice or WARAC, analogous to WARP. That is, we derive an explicit expression for ARAC in the case of two observations.

It is surprising that the ADM model is refutable, given that perceived probabilities are not observable. After all, if probabilities are allowed to vary, the observed insurance level could always be rationalized. However, this is not true if the perceived probabilities have known bounds in $(0, 1)$, as has been experimentally supported (Kahneman and Tversky 1979).

Recall the bounds $\underline{\beta} < \beta < \bar{\beta}$, such that $0 < \underline{\beta} < \bar{\beta} < 1$. Let (γ_i, I_i) for $i = 1, ..., T$ be a finite number of observations of insurance premia and levels, respectively. Using the state preference formulation of the ADM model we show that:

Proposition 8 *The ADM model is refutable.*

Proof. The proof has two parts: an existence proof or example to demonstrate that the inequalities have a solution, and an example in which the inequalities do not have a solution. Proposition 3 ensures that for a given insurance premium γ_i and endowments of wealth (ω_B, ω_G), a solution (I_i, β_i) exists. Given (ω_B, ω_G), γ_i and I_i the corresponding wealth levels are (W_B, W_G). With (W_B, W_G) and the endowment point, the state prices are determined, which establish existence. For refutability, one needs to show that some observations (I_i, γ_i) can not be rationalized by the model. If the state preference model represents the rational account choice, observing a consumption choice (equivalent to observing (I_i, γ_i)) can be illustrated as in Figure 3.

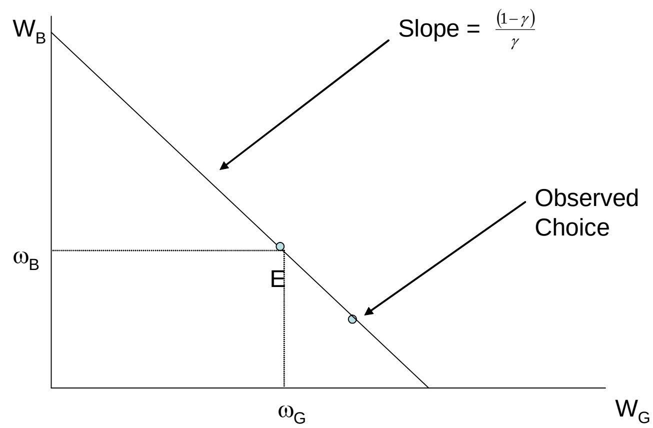

Figure 3 – Choice Representation.

Observing another choice pair (I_j, γ_j) can be illustrated as in Figure 4.

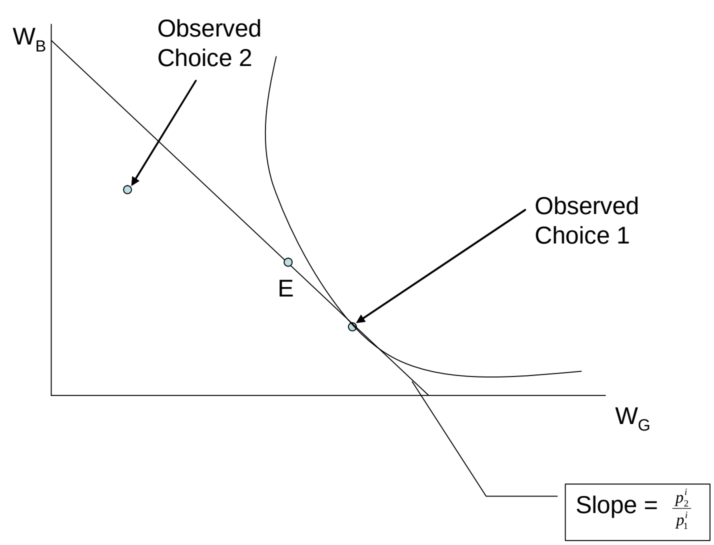

Figure 4 – Illustration of Two Choices.

Because the endowment point E must lie on the budget line, as must the consumption choice, the budget line may be determined with a single observation point, as is illustrated in Figure 5.

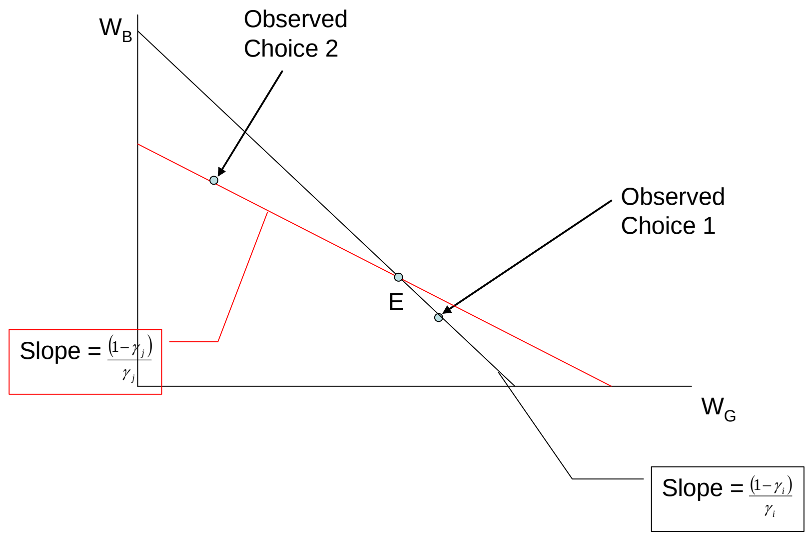

Figure 5 – Construction of the Budgent Line.

This case cannot be rationalized by an expected utility maximizer agent with constant probabilities, whether objective or subjective, nor can it be rationalized by the ADM model. The ADM model solves:

$$\frac{(1-\gamma)}{\gamma} = \frac{(1-\beta)}{\beta}\frac{Du(W_G)}{Du(W_B)} \tag{5}$$

Following Brown and Calsamiglia (2004), the rational account can be represented as a Bernoulli expected utility maximizer, such that

$$Du(W_B) = \frac{\lambda p}{\beta} \tag{6}$$

$$Du(W_G) = \frac{\lambda(1-p)}{(1-\beta)} \tag{7}$$

$$\frac{Du(W_G)}{Du(W_B)} = \frac{\lambda(1-p)\beta}{(1-\beta)\lambda p} \tag{8}$$

$$= \frac{\beta}{(1-\beta)}\frac{(1-p)}{p} \tag{9}$$

$$= \frac{\beta}{(1-\beta)}\frac{(1-\gamma)}{\gamma} \tag{10}$$

Thus, the perceived probability effect can be *transferred* to the price line, thereby illustrating the previous choice as the choice of a Bernoulli agent, as illustrated in

Figure 6:

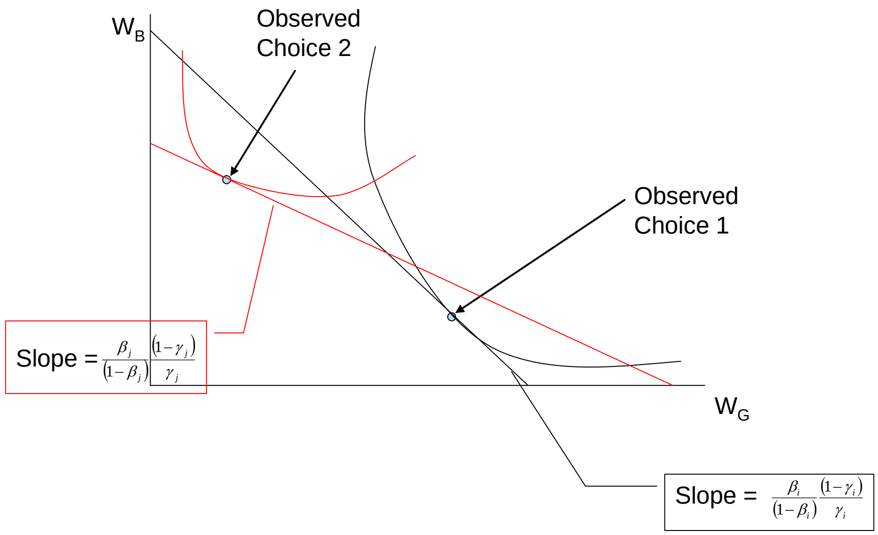

Figure 6 – The Bernoulli Agent Choice.

Recall that the marginal utilities of the Bernoulli agent are independent of the perceived probabilities,where the perceived probabilities are embodied in the (perceived) prices facing the agent. To refute the ADM model, it suffices that the weak axiom of revealed preference (WARP) is contradicted for any set of perceived prices. See the example in Figure 7.

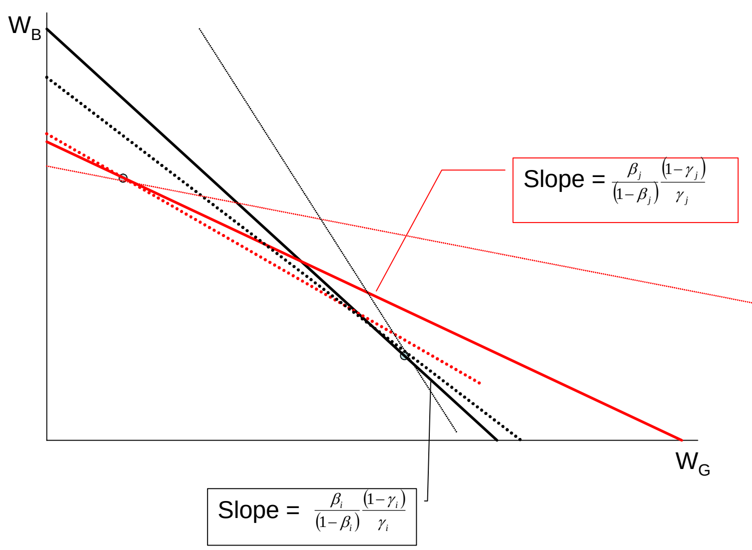

Figure 7 – Refuting the ADM model.

The model imposes bounds on the perceived probabilities which in turn bound the perceived prices; therefore, the model can be refuted. Note that only the upper bound on perceived probabilities for choice j and the lower bound on the perceived probabilities for choice i are binding. ■

The proof relies on the multiplicative separability of the marginal utility of a Bernoulli agent and the perceived probabilities, similar to the case of random utility functions, analyzed by Brown and Calsamiglia (2004). As a consequence, we can view the observed choice as a choice of a Bernoulli agent with a *perceived price ratio*, as illustrated in the previous section. The remaining difficulty pertains to the unobservability of the perceived price ratio. Fortunately, if the perceived probabilities have known bounds in $(0, 1)$ (as supported by experiments), the perceived price ratio also has known bounds, and the ADM model is refutable.[5]

In the next section, we will need the predictive ADM inequalities, defined as the ADM inequalities and the Afriat inequalities for the strictly concave potential function of the ADM game. That is, the objective function of the emotional process, denoted $P(\beta, I)$:

$$\begin{aligned} P(\beta^i, I^i) \;&<\; P(\beta^j, I^j) + \partial_\beta P(\beta^j, I^j)(\beta^i - \beta^j) + \partial_I P(\beta^j, I^j)(I^i - I^j) \\ &\quad \text{for } i, j = 1, ..., N \text{ and } i \neq j. \end{aligned}$$

4 A Predictive Model of ADM

This section shows that the ADM model is predictive. That is, if the data generating process is i.n.i.d. then the regularized, least squares solution of the predictive ADM inequalities is the *best prediction* of the agent's future affective demand for insurance. Our analysis utilizes the elements of the theory of Hilbert-function spaces with reproducing kernels, denoted RKHS, and some elementary material from the theory of Sobolev–Hilbert spaces. There are several excellent introductions to both theories and we recommend Schaback (2007) for RKHS and Lieb and Loss (2001) for a discussion of the Sobolev- Hilbert spaces considered in this paper.

For finite a and $b \in R^1$, let $H^{1,1}(a,b) \equiv$ the completion of $\{f \in C^1(a,b) : \|f\|_1 < \infty\}$ with respect to the norm $\|f\|_1 \equiv \|f\|^{1,1} + \|Df\|^{1,1} = \langle f^0, h^0 \rangle_{L_2[a,b]} + \langle f^1, h^1 \rangle_{L_2[a,b]}$, where $\langle f^j, h^j \rangle_{L_2[a,b]} = \int_a^b f^j(t)h^j(t)dt$ and r^j is the j^{th} derivative of $r \in H^{1,1}(a,b)$. $H^{1,1}(a,b)$ is a Sobolev–Hilbert space — see Lieb and Loss (2001). Berlinet and Thomas-Agnan (2004) prove that $H^{1,1}(-\infty, +\infty)$ is also a Sobolev–Hilbert space. The norm on $H^{1,1}(-\infty, +\infty)$ is $\|f\|^2_{H^{1,1}(-\infty,+\infty)} = \int_{-\infty}^{+\infty} f(s)^2 d\nu(s) + (1/\lambda^2) \int_{-\infty}^{+\infty} Df(s)^2 d\nu(s)$.

The predictive ADM inequalities are functions of the Bernoulli utility function and the marginal utility function of the rational process together with the mental cost function and the marginal cost function of the emotional process. We show that these functions are members of Hilbert-function spaces with reproducing kernels. This

[5] Brown and Calsamiglia (2004) show in their model that if probabilities are not bounded it is not refutable. This holds true for the ADM model as well.

property allows us to use the Representer Theorem of Micchelli and Pontil (2005) to reduce the infinite dimensional regularized, least squares regression for the ADM model to a finite dimensional regularized, regression model. The regularized, least squares regression proposed in this section is computed in the direct product of these Hilbert spaces.

Let $H_E \equiv (H^{1,1}(0,1))^2$ and C_E be the closed, convex cone of ordered pairs $(c(\beta), Dc(\beta)) \in H_E$, where $c(\beta)$ is a smooth, monotone, convex function on $(0,1)$, with derivative $Dc(\beta)$. $c(\beta)$ is the mental cost of the perceived risk β and $Dc(\beta)$ is the marginal mental cost of the perceived risk β. In a similar fashion, we define $H_R \equiv (H^{1,1}(-\infty,+\infty))^2$.

Let C_R be the closed, convex cone of ordered pairs $(u(W), Du(W)) \in H_R$, where $u(W)$ is a smooth, monotone, concave function on the real line, with derivative $Du(W)$. The agent's utility of W dollars is $u(W)$, and her marginal utility is $Du(W)$. ω_B and ω_G are the agent's endowments of wealth in the bad and the good state, respectively, $\gamma \in (0,1)$ is the insurance premium, and I is the agent's insurance level. If $(x;y) \equiv (\omega_B, \omega_G, I; \gamma)$, then $W_B(x;y) \equiv (\omega_B + (1-\gamma)I)$ is the wealth in the bad state, and $W_G(x;y) \equiv (\omega_G - \gamma I)$ is the wealth in the good state.

To formulate the regularized, least squares regression model for estimating the affective demand, we first recall the first order conditions for a pure strategy Nash equilibrium in the ADM intrapersonal game.

Rational Process: $\beta Du(W_B(x;y)) = \gamma$; $(1-\beta)Du(W_G(x;y)) = 1-\gamma$,

Emotional Process: $u(W_B(x;y)) - u(W_G(x;y)) = Dc(\beta)$.

To compute the affective demand for insurance, we use the indirect Afriat inequalities, introduced by Brown and Shannon (1996). If $V(\frac{p}{I},1)$ is the agent's indirect utility function, then it follows from Roy's identity that $\frac{\nabla V(\frac{p}{I},1)}{(\frac{p}{I})\times \nabla V(\frac{p}{I},1)} = x$, the agent's Marshallian demand at the normalized price vector $\frac{p}{I}$.

If $D = \{p^j, I^j\}_{j=1}^N$, then the indirect Afriat inequalities are of the form:

(a) $V^i - V^j \geq q^j \cdot \left(\frac{p^i}{I^i} - \frac{p^j}{I^j}\right)$ for $i,j = 1,...,N$

(b) $\lambda^j > 0$, $q^j \ll 0$, $j = 1,...,N$

(c) $\frac{q^j}{I^j} = -\lambda^j x^j$, $j = 1,...,N$,

where $q^j = \nabla V(\frac{p^j}{I^j},1) \in R^L$, $p^j \in R^L_{++}$, $I^j > 0$,$V^j > 0$, for $j = 1,...,N$.

An affective agent is characterized by the convex, indirect Bernoulli utility function of her rational process, $u(\cdot)$, and the convex, mental cost function of her emotional process, $c(\cdot)$.

The two states of the world are Bad and Good and the market data can be expressed as

$$(x;y) \equiv \left(\omega_B, \omega_G, \frac{\gamma}{\gamma\omega_B + (1-\gamma)\omega_G}, \frac{(1-\gamma)}{\gamma\omega_B + (1-\gamma)\omega_G}; I\right).$$

If $(u(W), Du(W)) \in H_R$, and $(c(\beta), Dc(\beta)) \in H_E$, then there exists a unique β, where $\beta = \arg\min\langle \|u(W_B(x;y)) - u(W_G(x;y)) - Dc(\beta)\|^2 \mid \beta \in [0,1]\rangle$.

$V(W_B(x;y), W_G(x;y)) \equiv \beta u(W_B(x;y)) + (1-\beta)u(W_G(x;y))$ is the expected indirect utility function of the rational process and $c(\beta)$ is the mental cost function of the emotional process. The state-contingent demands for wealth in the bad state, $X_B(x;y)$, and wealth in the good state, $X_G(x;y)$, are:

$$
\begin{aligned}
X_B(x;y) &= \frac{\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}{\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)+\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}, \\
X_G(x;y) &= \frac{\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}{\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)+\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)} &= \frac{\beta\times\partial u(W_B(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)} \text{ and} \\
\frac{\partial V(W_B(x;y),W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)} &= \frac{(1-\beta)\times\partial u(W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
X_B(x;y) &= \frac{\left(\frac{\beta\times\partial u(W_B(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}{\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\beta\partial u(W_B(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)+\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{(1-\beta)\partial u(W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}, \\
X_G(x;y) &= \frac{\left(\frac{(1-\beta)\partial u(W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)}{\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{\beta\partial u(W_B(x;y))}{\partial\left(\frac{\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)+\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)\left(\frac{(1-\beta)\partial u(W_G(x;y))}{\partial\left(\frac{1-\gamma}{\gamma\omega_B+(1-\gamma)\omega_G}\right)}\right)},
\end{aligned}
$$

and $I(x;y) = (X_B(x;y) - \omega_B) - (X_G(x;y) - \omega_G)$ is the demand for insurance.

Let $C_{R\times E} \equiv C_R \times C_E$, then $C_{R\times E} \subset H_R \times H_E \equiv H_K$ and $C_P \subset C_{R\times E}$ is also a closed, convex cone. Then, for each observation $(x;y)$ and $(u, Du, c, Dc) \in C_P$, the *theoretical* regularized, mean square deviation between observed demand $\widehat{I}(x;y)$ and the affective demand $I(x;y)$ is $Q(u, Du, c, Dc) = \int ||\hat{I}(x;y) - I(x;y)||^2 d\nu(x,y) + \|f\|^2_{H_K}$, where $d\nu(x,y)$ is the fixed, but unknown bivariate distribution over $(x;y)$. The theoretical, regularized, least squares problem is $\min\{Q(u, Du, c, Dc) \,|\, (u, Du, c, Dc) \in C_P\}$. For each observation $(x^i; y^i)$ and $(u, Du, c, Dc) \in C_P$, the *empirical* regularized, mean square deviation between the observed demand $\hat{I}(x^i; y^i)$, and the affective demand $I(x^i; y^i)$ is $Q_N(u, Du, c, Dc) = \frac{1}{N}\sum_{i=1,..,N} ||\hat{I}(x^i; y^i) - I(x^i; y^i)||^2 + \|f\|^2_{H_K}$. The empirical, regularized, least squares problem is $\min\{Q_N(u, Du, c, Dc) \,|\, (u, Du, c, Dc) \in C_P\}$.

We convert this infinite dimensional, constrained optimization problem, to a finite dimensional, unconstrained optimization problem by formulating the predictive ADM intrapersonal game as a model in vector-valued learning, and employ the analysis developed by Micchelli and Pontil (2005). That is, we use the ADM inequalities to convert the empirical, regularized, least squares problem over C_P to a regularized, least squares problem over H_K, using Lagrange multipliers. Next we show that H_K inherits the Reproducing Property, defined by Micchelli and Pontil, from the classical Reproducing Property of the factors of H_K, defined below as $H_{r,s}$. The final step is to verify the assumptions of Micchelli and Pontil's Representer Theorem – also originally due to Kimeldorf and Wahba (1970). The result is a finite dimensional, unconstrained optimization problem.

The domain of functions in H_K is $X = (0, 1) \times (-\infty, +\infty)$ and the range is R^4. If $f \in H_K$, then $f(x) = f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, where $f_1(x_1) = (f_{1,1}(x_1), f_{1,2}(x_1)) = (c(x_1), Dc(x_1))$ and $f_2(x_2) = (f_{2,1}(x_2), f_{2,2}(x_2)) = (u(x_2), Du(x_2))$.

That is, $f(x) = (f_{1,1}(x_1), f_{1,2}(x_1), f_{2,1}(x_2), f_{2,2}(x_2)) = (c(x_1), Dc(x_1), u(x_2), Du(x_2))$. Micchelli and Pontil (2005) define a vector-valued, reproducing kernel Hilbert space H_K as a Hilbert function space taking values in a Hilbert space H_V, where for all x and v the map $(v, f(x))_E$ is a continuous linear functional on H_K. That is, there exists a $g_{x,v} \in H_k$ such that for all $f \in H_K : (v, f(x))_E = (f, g_{x,v})_{H_k}$. We call this condition the Reproducing Property. If $Y = R^1$, then this definition reduces to the classical, Reproducing Property in Hilbert-function spaces with reproducing kernels — see Schaback (2007).

Proposition 9 *H_K has the Reproducing Property.*

Proof. Schaback (2007) shows that $H^{1,1}(0, 1)$ is a RKHS with reproducing kernel $K(x, t)$. The reproducing kernel $K_{r,s}(x, t)$ for $H_{r,s} = H^{1,1}(0, 1)$ for $r = 1$ and $s = 1, 2$ is given by the formulas: $K_{r,s}(x, t) = K(x, t) = \cosh(x - b)\cosh(t - a)/\sinh(b - a)$ for $a \leq t \leq x \leq b$ and $K(x, t) = \cosh(x-a)\cosh(t-b)/\sinh(b-a)$ for $a \leq x \leq t \leq b$.

Berlinet and Thomas-Agnan (2004) show that $H^{1,1}(-\infty, +\infty)$ is a RKHS with reproducing kernel $K(x, t)$, the reproducing kernel $K(x, t)$ for $H_{r,s} = H^{1,1}(-\infty, +\infty)$ for $r = 2$ and $s = 1, 2$ is given by the formula: $K_{r,s}(x, t) = K(x, t) = (\lambda/2)\exp(-\lambda|x-t|)$.

Let $H_V = R^4$ with the Euclidean inner product $(\cdot, \cdot)_E$. Recall that the inner product on the direct product of Hilbert spaces is the sum of the inner products on the factor spaces.

If $f(x) \in H_K$ then $f(x) = (f_{1,1}(x_1), f_{1,2}(x_1), f_{2,1}(x_2), f_{2,2}(x_2))$.

If $v = (v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}) \in H_V$ then $(v, f(x))_E = v_{1,1}f_{1,1}(x_1) + v_{1,2}f_{1,2}(x_1) + v_{2,1}f_{2,1}(x_2) + v_{2,2}f_{2,2}(x_2)$. Denote the factors of H_K as $H_{r,s}$, then $f_{r,s}(x) = (f_{r,s}(t), K_{r,s}(x,t))_{H_{r,s}}$ for $r,s = 1$ or 2, where $K_{r,s}(x,z)$ is the reproducing kernel for $H_{r,s}$, since each $H_{r,s}$ is a $RKHS$.

Hence $v_{r,s}f_{r,s}(x) = (f_{r,s}(t), v_{r,s}K_{r,s}(x,t))_{H_{r,s}}$ for $r,s = 1$ or 2.

For fixed $x = (x_1, x_2)$ and $v = (\nu_{1,1}, \nu_{1,2}, \nu_{2,1}, \nu_{2,2})$ let $g_{x,v} = (v_{1,1}K_{1,1}(x_1,t_1), v_{1,2}K_{1,2}(x_1,t_1), v_{2,1}K_{2,1}(x_2,t_2), v_{2,2}K_{2,2}(x_2,t_2))$, then $(v, f(x))_E = (f, g_{x,v})_{H_K}$. ■

Definition 10 *The kernel $K(x,t)$ in Micchelli and Pontil is the operator-valued map $K(x,t) : X^2 \to H_V^{H_V}$, where $K(x,t)(v) \equiv g_{t,v}(x)$ and $X \subseteq R^2$.*

Proposition 11 *If $H_V = R^4$ then for all (x,t) in X^2, $K(x,t)$ has the 4×4 matrix representation $(g_{x,e_i}, g_{t,e_j})_{H_K}$ for $i,j = 1,..,4$ and $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, $e_3 = (0,0,1,0,0)$, $e_4 = (0,0,0,1,0)$.*

Proof. See Theorem 1 in Micchelli and Pontil (2005). ■

Definition 12 *Suppose $\tilde{D} = \{x^i\}_{i=1,..,N}$, where for all i, $(x^i) = (x_1^i, x_2^i) \in R^2$*

Definition 13 *For any $(c^1, ..., c^N)$, where $c^i \in R^4$:*

$$\tilde{f}_{(c^1,..,c^N)}(\cdot) \equiv \sum_{i=1,...,N} g_{x^i,c^i}(\cdot) \in H_K.$$

Following Micchelli and Pontil, let $E : (H_V)^N \times R_+ \to R$ be given and consider minimizing $E(f(x), \|f\|)$ over all $f \in H_K$. We now state their Representer Theorem.

Proposition 14 *If for every $v \in (H_V)^N$, the function $h : R_+ \to R_+$ defined by $h(t) = E(v,t)$ is strictly increasing in t and $f_0 \in H_K$ minimizes $E(f(x), \|f\|)$ then $f_0 = \tilde{f}_{(c^1,...,c^N)}$ for some $(c^1, ..., c^N) \in (H_V)^N$.*

Proof. See Theorem 5 in Micchelli and Pontil (2005). ■

Definition 15

$$\begin{aligned} W(x^i, y^i, u, Du, c, Dc, \lambda_j) &= \frac{1}{N} \sum_{i=1,...,N} \left\| \hat{I}^i(x,y) - I^i(x,y) \right\| \\ &\quad + \sum_{j=i,...,M} \lambda_i \phi_j(x^i, y^i, u, Du, c, Dc). \end{aligned}$$

Definition 16

$$\mathcal{L}_R(x^i, y^i, u, Du, c, Dc, \lambda_j, \mu) = W(x^i, y^i, u, Du, c, Dc, \lambda_j) + \mu \|f\|_{H_K}^2$$

is the regularized Lagrangian, where $\phi_j(\cdot)$, are the ADM (state-preference) inequalities — see Section 3.

Proposition 17 *If* $(\overline{u, Du, c, Dc})$ *minimized* $\mathcal{L}_R(x^i, y^i, u, Du, c, Dc, \lambda_j, \mu)$ *over all* $(u, Du, c, Dc) \in H_K$ *then* $(\overline{u, Du, c, Dc}) = \tilde{f}_{(c_1,...,c_N)}(\cdot)$ *for some* $(c^1, ..., c^N) \in (H_V)^N$.

Proof. By the Representer Theorem we can substitute $\tilde{f}_{(c_1,...,c_N)}(x^i)$ for the (u, Du, c, Dc) in the Lagrangian. The Lagrangian $\mathcal{L}_R(x^i, y^i, c^i, \lambda_j, \mu)$ is now a function of the c_i. We compute (u, Du, c, Dc) by minimizing $\mathcal{L}_R(x^i, y^i, c^i, \lambda_j, \mu)$ over all $(c^1, ..., c^N) \in (H_V)$. If there are K agents in the market for mutual insurance, then the best (ADM) prediction of the future aggregate for insurance is simply the sum of the individual affective demands for insurance on future data sets. ■

Appendix: The Weak Axiom of Affective Choice

It is convenient to change the notation in this section of the paper: $x_1 \equiv W_B$, $x_2 \equiv W_G$ and $u' \equiv Du$.

Consider two choices $(\gamma, x_1, x_2, \beta)$,and $(\hat{\gamma}, \hat{x}_1, \hat{x}_2, \hat{\beta})$ of some affective agent, where $\gamma, \hat{\gamma}$ are prices, (x_1, x_2) , $(\hat{x}_1, \hat{x}_2)$ are consumption bundles, and $\beta, \hat{\beta}$ are the associated risk beliefs. However, for an outside observer, the data consist of the consumption choices and prices only — (γ, x_1, x_2) and $(\hat{\gamma}, \hat{x}_1, \hat{x}_2)$. On the basis of the preceding discussion, if the ADM rationalizes the data (i.e., if these are the choices of an affective agent), the observables must satisfy the rational and emotional processes' first-order conditions, as well as the Afriat's inequalities. For the case of two observations, these conditions total 18 inequalities that can be reduced to the following:

$$[\beta - \hat{\beta}][\hat{u}_1 - u_2 - \hat{u}_1 + \hat{u}_2] \geq 0 \tag{11}$$

and

$$x_i > x_j \Rightarrow u'_i < u'_j \tag{12}$$

where

$$\beta = \frac{u'_2 \gamma}{u'_1(1-\gamma) + u'_2 \gamma}; \; \hat{\beta} = \frac{\hat{u}'_2 \hat{\gamma}}{u'_1(1-\hat{\gamma}) + \hat{u}'_2 \hat{\gamma}}.$$

In other words, if there exists numbers $u_1, u_2, \hat{u}_1, \hat{u}_2,\ u'_1 \geq 0,\ u'_2 \geq 0, \hat{u}'_1 \geq 0,\ \hat{u}'_2 \geq 0$ that satisfy the inequalities in (11(and (12), then there exist a concave utility function and a convex mental cost function such that the observed choices constitute a pure strategy Nash equilibrium in the intrapersonal game of the affective agent. In the following proposition we give neccesary and sufficient conditions on the data such that the inequalities in (11) and (12) are satisfied. These conditions constitute the Weak Axiom of Revealed Affective Choice or WARAC. This axiom is presented as a finite family of multivariate, polynomial inequalities in the data and the assumed known bounds on risk preferences.

The possible rankings on $(x_1, x_2, \hat{x}_1, \hat{x}_2)$ can be divided into three exhaustive and mutually exclusive groups of eight (8) rankings, denoted A, B and C.

Proposition 18 *Consider two observations (γ, x_1, x_2), $(\hat{\gamma}, \hat{x}_1, \hat{x}_2)$, where γ, $\hat{\gamma}$ are insurance prices and (x_1, x_2), $(\hat{x}_1, \hat{x}_2)$ are state-contingent choices of wealth. For known bounds on risk beliefs $\underline{\beta}, \bar{\beta}$, such that $0 < \underline{\beta} < \bar{\beta} < 1$ the data is rationalized by the ADM model if and only if:*

(i) The rankings lie in Group A and $\frac{\gamma}{(1-\gamma)} \frac{1-\bar{\beta}}{\bar{\beta}} < \frac{\hat{\gamma}}{(1-\hat{\gamma})} \frac{1-\underline{\beta}}{\underline{\beta}}$, *or*

(ii) The rankings lie in Group B and $\frac{\hat{\gamma}}{(1-\hat{\gamma})} \frac{1-\bar{\beta}}{\bar{\beta}} < \frac{\gamma}{(1-\gamma)} \frac{1-\underline{\beta}}{\underline{\beta}}$, *or*

(iii) The rankings lie in Group C.

Proof. Consider two observations (γ, x_1, x_2), $(\hat{\gamma}, \hat{x}_1, \hat{x}_2)$ with associated risk belief β, $\hat{\beta}$, respectively; if these observations are choices of some affective agent, then we must find numbers $u_1, u_2, \hat{u}_1, \hat{u}_2,\ u_1' \geq 0, u_2' \geq 0,\ \hat{u}_1' \geq 0,\ \hat{u}_2' \geq 0$ that satisfy the rational and the emotional processes' first-order conditions, as follows:

$$\beta u_1'(1-\gamma) - (1-\beta)u_2'\gamma = 0 \tag{13}$$
$$\hat{\beta}\hat{u}_1'(1-\hat{\gamma}) - (1-\hat{\beta})\hat{u}_2'\hat{\gamma} = 0 \tag{14}$$
$$u_1 - u_2 - c'(\beta) = 0 \tag{15}$$
$$\hat{u}_1 - \hat{u}_2 - c'(\hat{\beta}) = 0 \tag{16}$$

where $u_i = u(x_i)$, and $c(\cdot)$ is the mental cost function. The observed behavior should also satisfy the Afriat's (1967) inequalities:

$$u_1 \leq u_2 + u_2'(x_1 - x_2) \tag{17}$$
$$u_2 \leq u_1 + u_1'(x_2 - x_1) \tag{18}$$
$$\hat{u}_1 \leq \hat{u}_2 + \hat{u}_2'(\hat{x}_1 - \hat{x}_2) \tag{19}$$
$$\hat{u}_2 \leq \hat{u}_1 + \hat{u}_1'(\hat{x}_2 - \hat{x}_1) \tag{20}$$
$$u_1 \leq \hat{u}_1 + \hat{u}_1'(x_1 - \hat{x}_1) \tag{21}$$
$$u_1 \leq \hat{u}_2 + \hat{u}_2'(x_1 - \hat{x}_2) \tag{22}$$
$$u_2 \leq \hat{u}_1 + \hat{u}_1'(x_2 - \hat{x}_1) \tag{23}$$
$$u_2 \leq \hat{u}_2 + \hat{u}_2'(x_2 - \hat{x}_2) \tag{24}$$
$$\hat{u}_1 \leq u_1 + u_1'(\hat{x}_1 - x_1) \tag{25}$$
$$\hat{u}_1 \leq u_2 + u_2'(\hat{x}_1 - x_2) \tag{26}$$
$$\hat{u}_2 \leq u_1 + u_1'(\hat{x}_2 - x_1) \tag{27}$$
$$\hat{u}_2 \leq u_2 + u_2'(\hat{x}_2 - x_2) \tag{28}$$

$$c(\beta) \geq c(\hat{\beta}) + c'(\hat{\beta})(\beta - \hat{\beta}) \tag{29}$$
$$c(\hat{\beta}) \geq c(\beta) + c'(\beta)(\hat{\beta} - \beta) \tag{30}$$

From Equations (15), (16), (29), and (30),

$$(\beta - \hat{\beta})[u_1 - u_2 - \hat{u}_1 + \hat{u}_2] \geq 0 \tag{31}$$

From Equations (13) and (14),

$$\beta = \frac{u_2'\gamma}{u_2'\gamma + u_1'(1-\gamma)};\ \hat{\beta} = \frac{\hat{u}_2'\hat{\gamma}}{\hat{u}_2'\hat{\gamma} + \hat{u}_1'(1-\hat{\gamma})} \tag{32}$$

Conditions (17)–(28) can be reduced to

$$\begin{aligned}
(x_1 - x_2)(u_2' - u_1') &\geq 0 \\
(\hat{x}_1 - \hat{x}_2)(\hat{u}_2' - \hat{u}_1') &\geq 0 \\
(\hat{x}_1 - x_1)(u_1' - \hat{u}_1') &\geq 0 \\
(x_1 - \hat{x}_2)(\hat{u}_2' - u_1') &\geq 0 \\
(\hat{x}_1 - x_2)(u_2' - \hat{u}_1') &\geq 0 \\
(\hat{x}_2 - x_2)(u_2' - \hat{u}_2') &\geq 0
\end{aligned}$$

which can be summarized as:

$$x_i > x_j \Rightarrow u_i' < u_j' \tag{33}$$

Therefore, we restrict attention to conditions (31), (32), and (33). Splitting the observations into three groups of eight rankings, where the rankings in Group A are:

$$\begin{aligned}
x_1 &> \hat{x}_1 > \hat{x}_2 > x_2 \\
x_1 &> x_2 > \hat{x}_2 > \hat{x}_1 \\
x_1 &> \hat{x}_2 > \hat{x}_1 > x_2 \\
x_1 &> \hat{x}_2 > x_2 > \hat{x}_1 \\
\hat{x}_2 &> \hat{x}_1 > x_1 > x_2 \\
\hat{x}_2 &> x_1 > \hat{x}_1 > x_2 \\
\hat{x}_2 &> x_1 > x_2 > \hat{x}_1 \\
\hat{x}_2 &> x_2 > x_1 > \hat{x}_1
\end{aligned}$$

If numbers $u_1' \geq 0$, $u_2' \geq 0$, $\hat{u}_1' \geq 0$, $\hat{u}_2' \geq 0$ satisfy condition (33), all members of group A must satisfy:

$$\frac{u_1'}{u_2'} < \frac{\hat{u}_1'}{\hat{u}_2'}$$

We assume each risk belief β falls between known bounds $\{\underline{\beta}, \bar{\beta}\}$, where $0 < \underline{\beta} < \bar{\beta} < 1$. According to condition (32),

$$\begin{aligned}
\underline{\beta} &< \beta = \frac{u_2'\gamma}{u_2'\gamma + u_1'(1-\gamma)} < \bar{\beta} \\
&\Rightarrow \\
\frac{\gamma}{1-\gamma}\frac{1-\bar{\beta}}{\bar{\beta}} &< \frac{u_1'}{u_2'} < \frac{\gamma}{1-\gamma}\frac{1-\underline{\beta}}{\underline{\beta}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{\beta} &< \hat{\beta} = \frac{\hat{u}_2'\hat{\gamma}}{\hat{u}_2'\bar{\gamma} + \hat{u}_1'(1-\hat{\gamma})} < \bar{\beta} \\
&\Rightarrow \\
\frac{\hat{\gamma}}{1-\hat{\gamma}}\frac{1-\bar{\beta}}{\bar{\beta}} &< \frac{\hat{u}_1'}{\hat{u}_2'} < \frac{\hat{\gamma}}{1-\hat{\gamma}}\frac{1-\underline{\beta}}{\underline{\beta}}
\end{aligned}$$

if

$$\frac{\hat{\gamma}}{1-\hat{\gamma}}\frac{1-\underline{\beta}}{\underline{\beta}} < \frac{\gamma}{1-\gamma}\frac{1-\bar{\beta}}{\bar{\beta}}$$

Therefore, it can not be that

$$\frac{u'_1}{u'_2} < \frac{\hat{u}'_1}{\hat{u}'_2},$$

a contradiction.

If

$$\frac{\hat{\gamma}}{1-\hat{\gamma}}\frac{1-\underline{\beta}}{\underline{\beta}} > \frac{\gamma}{1-\gamma}\frac{1-\bar{\beta}}{\bar{\beta}}$$

then there exists numbers u'_1, u'_2 and $\hat{u}'_1, \hat{u}'_2$, such that

$$\frac{u'_1}{u'_2} < \frac{\hat{u}'_1}{\hat{u}'_2}$$

This guarantees that $u_1 - u_2 - \hat{u}_1 + \hat{u}_2 > 0$. Therefore, for the ADM model to be consistent with the data, condition (31) must be satisfied. That is, $(\beta - \hat{\beta}) \geq 0$

$$\beta = \frac{u'_2\gamma}{u'_2\gamma + u'_1(1-\gamma)} \geq \hat{\beta} = \frac{\hat{u}'_2\hat{\gamma}}{\hat{u}'_2\bar{\gamma} + \hat{u}'_1(1-\hat{\gamma})}$$

$$\begin{aligned}
\frac{u'_2\gamma}{u'_2\gamma + u'_1(1-\gamma)} &\geq \frac{\hat{u}'_2\hat{\gamma}}{\hat{u}'_2\bar{\gamma} + \hat{u}'_1(1-\hat{\gamma})} \\
u'_2\gamma[\hat{u}'_2\hat{\gamma} + \hat{u}'_1(1-\hat{\gamma})] &\geq \hat{u}'_2\hat{\gamma}[u'_2\gamma + u'_1(1-\gamma)] \\
u'_2\gamma\hat{u}'_2\hat{\gamma} + u'_2\gamma\hat{u}'_1(1-\hat{\gamma}) &\geq \hat{u}'_2\hat{\gamma}u'_2\gamma + \hat{u}'_2\hat{\gamma}u'_1(1-\gamma) \\
u'_2\gamma\hat{u}'_1(1-\hat{\gamma}) &\geq \hat{u}'_2\hat{\gamma}u'_1(1-\gamma) \\
\frac{u'_1}{u'_2}\frac{\hat{\gamma}}{(1-\hat{\gamma})} &\leq \frac{\gamma}{(1-\gamma)}\frac{\hat{u}'_1}{\hat{u}'_2}
\end{aligned}$$

Therefore numbers u'_1, u'_2 and $\hat{u}'_1, \hat{u}'_2$ must satisfy

$$\frac{u'_1}{u'_2}\frac{\hat{\gamma}}{(1-\hat{\gamma})} \leq \frac{\gamma}{(1-\gamma)}\frac{\hat{u}'_1}{\hat{u}'_2}$$

If $\hat{\gamma} < \gamma$ this result is obvious, but if $\hat{\gamma} > \gamma$, then there exists numbers u'_1, u'_2 and $\hat{u}'_1, \hat{u}'_2$ such that$\frac{u'_1}{u'_2} \leq \frac{\gamma(1-\hat{\gamma})}{\hat{\gamma}(1-\gamma)}\frac{\hat{u}'_1}{\hat{u}'_2} = k\frac{\hat{u}'_1}{\hat{u}'_2}$.

The rankings in Group B are:

$$\begin{aligned}
x_2 &> x_1 > \hat{x}_1 > \hat{x}_2 \\
x_2 &> \hat{x}_1 > x_1 > \hat{x}_2 \\
x_2 &> \hat{x}_1 > \hat{x}_2 > x_1 \\
x_2 &> \hat{x}_2 > \hat{x}_1 > x_1 \\
\hat{x}_1 &> \hat{x}_2 > x_2 > x_1 \\
\hat{x}_1 &> x_2 > x_1 > \hat{x}_2 \\
\hat{x}_1 &> x_2 > \hat{x}_2 > x_1 \\
\hat{x}_1 &> x_1 > x_2 > \hat{x}_2
\end{aligned}$$

If condition (33) holds, all observations that belong to this group satisfy:

$$\frac{u_1'}{u_2'} > \frac{\hat{u}_1'}{\hat{u}_2'}$$

In turn, $u_1 - u_2 - \hat{u}_1 + \hat{u}_2 < 0$. Therefore, arguments similar to arguments used for group A, prove that this group can be rationalized iff $\frac{\hat{\gamma}}{(1-\hat{\gamma})}\frac{\underline{\beta}}{1-\underline{\beta}} < \frac{\gamma}{(1-\gamma)}\frac{\bar{\beta}}{1-\bar{\beta}}$.

Finally,the rankings in Group C are:

$$\begin{aligned}
x_1 &> x_2 > \hat{x}_1 > \hat{x}_2 \\
x_1 &> \hat{x}_1 > x_2 > \hat{x}_2 \\
x_2 &> x_1 > \hat{x}_2 > \hat{x}_1 \\
x_2 &> \hat{x}_2 > x_1 > \hat{x}_1 \\
\hat{x}_1 &> \hat{x}_2 > x_1 > x_2 \\
\hat{x}_1 &> x_1 > \hat{x}_2 > x_2 \\
\hat{x}_2 &> \hat{x}_1 > x_2 > x_1 \\
\hat{x}_2 &> x_2 > \hat{x}_1 > x_1
\end{aligned}$$

Suppose the Afriat (1967) inequalities are satisfied for members of group C; then the sign of $[u_1 - u_2 - \hat{u}_1 + \hat{u}_2]$ is undetermined. Hence there always exists a set of numbers $u_1, u_2, \hat{u}_1, \hat{u}_2$, $u_1' \geq 0$, $u_2' \geq 0$, $\hat{u}_1' \geq 0$, $\hat{u}_2' \geq 0$ that satisfy conditions (31), (32), and (33). That is, the data can always be rationalized by the ADM model. ■

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