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Implications of conservation equations for the determination of absolute velocities

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ABSTRACT

The consequences of assuming that density is conserved in the problem of determining absolute velocities are investigated. Two questions are considered: (i) the constraints that the density must satisfy to be compatible with the assumed geostrophic and hydrostatic dynamics and (ii) whether and to what extent the indeterminacy in this dynamics is removed by this additional assumption.

1. Introduction

With the dawn of ocean forecasting approaching, the need to know velocity fields for use as initial conditions in a prediction scheme is keenly felt. But direct velocity measurements are still difficult to make, and so we are forced to rely on the classical procedure which consists of inferring the velocity fields from temperature and salinity measurements. In this procedure, we are given the density field $\rho(x, y, z)$, and we must find velocity fields $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ and a pressure field $p(x, y, z)$ such that

$$fv = \rho_0^{-1} p_x \tag{1.1a}$$

$$fu = -\rho_0^{-1} p_y \tag{1.1b}$$

$$0 = -\rho^{-1} p_z - g \tag{1.1c}$$

$$u_x + v_y + w_z = 0 \tag{1.1d}$$

The notations in the above equations are standard: x, y, z are cartesian coordinates; f is the Coriolis parameter and g is the gravitational acceleration; ρ_0 is an average value of the density.

As is very well known, the integration of the hydrostatic equation from the ocean's bottom at $z = -h$ to a depth z, viz.

$$p = -g \int_{-h}^{z} \rho(x, y, z')\, dz' + \pi(x, y) \tag{1.2}$$

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introduces an unknown function of x, y denoted here by π. This 'barotropic' component of the pressure cannot be determined from the dynamics (1.1a, b, c, d).

The indeterminacy in the pressure entails a similar indeterminacy in the velocity fields. Indeed, if we define

$$P(x, y, z) = -g \int_{-h}^{z} \rho(x, y, z')\, dz' \tag{1.3}$$

then

$$\begin{aligned} u(x, y, z) &= u'(x, y, z) + \overline{u}(x, y) \\ v(x, y, z) &= v'(x, y, z) + \overline{v}(x, y) \end{aligned} \tag{1.4}$$

where

$$u' = -(\rho_0 f)^{-1} P_y \tag{1.5a}$$

$$v' = (\rho_0 f)^{-1} P_x \tag{1.5b}$$

and

$$\overline{u} = -(\rho_0 f)^{-1} \pi_y \tag{1.6a}$$

$$\overline{v} = (\rho_0 f)^{-1} \pi_x \tag{1.6b}$$

The vertical velocity, which is obtained by integrating the continuity equation in the vertical,

$$w = w(x, y, -h) - \int_{-h}^{z} (u_x + v_y)\, dz' \tag{1.7}$$

also contains the same indeterminacy. The search for the famous 'level of no motion' is intimately related to the determination of $\pi(x, y)$.

Following in the footsteps of Worthington (1976), Wunsch (1978), Fiadeiro and Veronis (1982) and others, we want to examine whether the addition of extra information in the form of the conservation of the density field, viz.

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{1.8}$$

can determine $\pi(x, y)$.

Mathematically, it is not clear whether Eq. (1.8) is compatible with the dynamics (1.1). We shall examine this question first on the f-plane and then on the β-plane. We shall show that in both cases the density field must satisfy certain constraints. If these constraints are satisfied, then for the β-plane case, but not for the f-plane, π can be determined. Our analysis is carried out in the simplest case of a flat bottom ocean. However, it can be extended to the case where bottom topography exists as well as to cases where quantities other than density are conserved.

2. A mathematical result

An essential step in our analysis consists in finding a *z-independent* solution of an equation of the form

$$A(x, y, z)\pi_x + B(x, y, z)\pi_y = 1 \tag{2.1}$$

which holds in a cylindrical basin $D = \{(x, y, z) : (x, y) \in D, -h < z < 0\}$, where D is the horizontal cross-section. If A and B themselves are independent of z, then the existence of such solutions is not too surprising. Of greater interest is the case in which such solutions exist even though A and B are z-dependent, i.e. when

$$A_z^2 + B_z^2 \not\equiv 0 \text{ in } D. \tag{2.2}$$

Relabeling coordinates if need be, we interpret (2.2) to mean that

$$B_z \not\equiv 0 \text{ in } D. \tag{2.2'}$$

With the above conditions we have the following result.

a. Theorem. There exists a z-independent solution $\pi(x, y)$ of (2.1) if and only if for all $(x, y, z) \in D$

$$\text{(i) } A(x, y, z) = k_0(x, y)B(x, y, z) + l_0(x, y) \tag{2.3a}$$

$$\text{(ii) } l_0(x, y) \neq 0 \tag{2.3b}$$

$$\text{(iii) } \partial_y(l_0^{-1}) - \partial_x(k_0/l_0) = 0. \tag{2.3c}$$

Furthermore, this solution $\pi(x, y)$ satisfies

$$\pi_x = l_0^{-1}(x, y) \tag{2.4a}$$

$$\pi_y = -k_0(x, y)/l_0(x, y). \tag{2.4b}$$

b. Proof. Suppose (2.3a, b, c) hold in D. We must show that a z-independent solution $\pi(x, y)$ of (2.1) exists, and that it satisfies (2.4a, b).

In view of (2.3b), we can write (2.3a) thus:

$$Al_0^{-1} - Bk_0l_0^{-1} = 1. \tag{2.5}$$

Now, (2.3c) implies the existence of a function, say $\pi(x, y)$, such that (2.4a, b) hold. Substituting l_0^{-1} and $-k_0l_0^{-1}$ by their expressions in terms of π in (2.4), we conclude that this function π is a solution of (2.1).

Conversely, suppose that π is a z-independent solution of (2.1) and that (2.2′) holds. Then we can differentiate (2.1) with respect to z and write

$$A_z\pi_x + B_z\pi_y = 0. \tag{2.6}$$

We note at this stage that $A_zB - B_zA \not\equiv 0$ in D. Indeed, if this were the case, then

$$B_z(A\pi_x + B\pi_y) = B_z,$$

which is obtained by multiplying (2.1) by B_z, could be written as

$$B(A_z\pi_x + B_z\pi_y) = B_z.$$

But as a result of (2.6), we reach a contradiction. Therefore

$$A_zB - B_zA \not\equiv 0 \tag{2.7}$$

and as a result, we can solve (2.1) and (2.6) for π_x and π_y to find

$$\pi_x = \frac{B_z}{AB_z - BA_z} \tag{2.8a}$$

$$\pi_y = -\frac{A_z}{AB_z - BA_z}. \tag{2.8b}$$

This step is not unlike that taken by Stommel and Schott (1977) and Needler (1985) in their study of this problem.

We must next insure that these expressions for π are indeed z-independent. By considering their ratio, we immediately see that

$$A_z = k_0(x, y)B_z$$

and after integration we arrive at the condition (2.3a) between A and B. Substituting this condition in (2.8a, b), we arrive at (2.4a, b). Condition (2.3c) arises from forming π_{xy} in two different ways. Finally, the condition (2.7) translates into (2.3b). We can easily verify that further z differentiations do not introduce other conditions. This completes the proof.

In closing this section, we give an example of a partial differential equation of the form (2.1) in which the coefficients A and B satisfy all the conditions of the theorem. The example is:

$$\left[B(x, y, z) + \frac{1}{f'(x - y)}\right]\pi_x + B(x, y, z)\pi_y = 1$$

where B is an analytic function of z. In this case,

$$k_0 = 1$$

$$l_0 = \frac{1}{f'(x - y)}$$

and

$$\pi = f(x - y).$$

3. The f-plane

We consider in this section the case of the f-plane, simply as a foil for the β-plane which is discussed next. In this case, the flow is horizontally nondivergent. This, together with the fact that the bottom is flat, implies that

$$w = 0. \tag{3.1}$$

Thus, the added conservation equation reads

$$u\rho_x + v\rho_y = 0. \tag{3.2}$$

Making use of the hydrostatic and geostrophic balances, this equation can be written as follows:

$$-p_y p_{xz} + p_x p_{yz} = 0 \tag{3.3}$$

and since $p_x^2 + p_y^2$ is not identically zero in D lest the ocean would be at rest, this implies that

$$\partial_z\left(\frac{p_x}{p_y}\right) = 0$$

i.e.

$$p_x = -p_y \tan \alpha_0(x, y) \tag{3.4}$$

or equivalently

$$v(x, y, z) = u(x, y, z) \tan \alpha_0(x, y). \tag{3.5}$$

Thus, at a given latitude and longitude, the velocity, which is purely horizontal, has the same direction for all depths! This result, in turn, places some constraint on the data. Indeed, substituting (3.5) in (3.2) shows that

$$\rho_x + \rho_y \tan \alpha_0 = 0. \tag{3.6}$$

This constraint is typical of the constraints the conserved quantity must satisfy for the data to be compatible with the assumed dynamics. What this means in practice is that at each station, z-independent α_0 are generated by using the best fitting line passing through points with abscissas ρ_y and ρ_x (or whatever the corresponding quantities are for the case at hand).

As we shall see, a similar constraint arises on the β-plane. Having satisfied this constraint, are we now able to determine π? The answer is unfortunately no. Indeed, from the definition (1.3) of P, it follows that:

$$P_x + P_y \tan \alpha_0 = 0. \tag{3.7}$$

This means that the baroclinic components of the flow field satisfy the conservation equation by themselves. Therefore, the need for a barotropic correction is not imperative. From (3.4) and (3.7) it follows that

$$\pi_x + \pi_y \tan \alpha_0 = 0. \tag{3.8}$$

If we were forced to proceed further, then we would need to know the distribution of sources and sinks at the edge of our basin in order to solve this equation for π.

4. The β-plane

Because of the β-effect, the horizontal mass divergence is $-\beta v/f_0$ and therefore the vertical velocity as given by (1.7) is

$$w(x, y, z) = \beta f_0^{-1} \int_{-h}^{z} v(x, y, z')\, dz' \tag{4.1}$$

or in terms of P and π

$$w = (\beta/f_0^2\rho_0)\left\{\int_{-h}^{z} P_x\, dz' + [z + h]\pi_x\right\}. \tag{4.2}$$

We shall find it convenient at times to write

$$W = (\beta/f_0^2\rho_0) \int_{-h}^{z} P_x\, dz'. \tag{4.3}$$

The conservation equation for ρ now becomes

$$\{\rho_y + (\beta[z + h]/f_0)\rho_z\}\pi_x - \rho_x\pi_y = -\rho_y P_x + \rho_x P_y - (\rho_0 f_0)\rho_z W. \tag{4.4}$$

The question once again is whether the right-hand side of (4.4) vanishes, i.e. whether the baroclinic fields identically satisfy the conservation equation. We recall that for $\beta = 0$ the answer to this question was always yes. If each side of (4.4) is separately zero, then the density field which makes up the data is constrained to satisfy two distinct equations. The equation stemming from the baroclinic part is

$$-\rho_y P_x + \rho_x P_y - (\rho_0 f_0)\rho_z W = 0 \tag{4.5}$$

whereas that from the barotropic part yields

$$\rho_x + \{\rho_y + (\beta[z + h]/f_0)\rho_z\} \tan \alpha_0 = 0 \tag{4.6}$$

where α_0 is, as previously, solely a function of x and y. Are there isopycnal surfaces which satisfy both of the above equations?

We shall show that this question must be answered negatively. Integrating (4.6) over z and using the definition (1.3) of P, we deduce that

$$P_x + \tan \alpha_0 \left\{P_y + \frac{\beta(z + h)}{f_0}\rho g + \frac{\beta}{f_0}P\right\} = 0. \tag{4.7}$$

Next, we eliminate ρ_y and P_y from (4.5) using (4.6) and (4.7). The result is:

$$\beta f_0^{-1}\rho_x\left\{P + \rho g\int_{-h}^{z} dz'\right\} = \beta f_0^{-1}\rho_z\left\{-P_x\int_{-h}^{z} dz' + \int_{-h}^{z} P_x\, dz'\right\}. \qquad (4.8)$$

If we differentiate this equation with respect to z, we see that

$$\beta f_0^{-1}\rho_{xz}\left\{P + \rho g\int_{-h}^{z} dz'\right\} = \beta f_0^{-1}\rho_{zz}\left\{-P_x\int_{-h}^{z} dz' + \int_{-h}^{z} P_x\, dz'\right\}. \qquad (4.9)$$

The above two equations imply that

$$\rho_z\rho_{xz} = \rho_x\rho_{zz}$$

i.e.

$$\rho_z = \rho_x \tan\gamma_0(x, y). \qquad (4.10)$$

We have thus reduced the problem to finding whether there exists a function $\rho(x, y, z)$ which satisfies the system of equations

$$E_1[\rho] \equiv \tan\gamma_0\rho_x - \rho_z = 0$$
$$E_1[\rho] \equiv \rho_x + \tan\alpha_0\rho_y + (\beta[z+h]/f_0)\tan\alpha_0\rho_z = 0.$$

At this stage we appeal to the theory of systems of first order partial differential equations (see e.g. Smirnov, 1964, p. 34.). The first step in the theory consists in attempting to derive further independent equations for ρ. Such additional equations are obtained by forming Poisson brackets. Thus, denoting temporarily x, y, z by x_1, x_2, x_3 and ρ_x, ρ_y, ρ_z by p_1, p_2, p_3, the theory leads us to consider

$$E_3[\rho] \equiv \sum_{i-1}^{3}\{(\partial E_1/\partial x_i)(\partial E_2/\partial p_i) - (\partial E_2/\partial x_i)(\partial E_1/\partial p_i)\} = 0$$

i.e.

$$-[(\partial\gamma_0/\partial x) + (\partial\gamma_0/\partial y)\tan\alpha_0(\cos\gamma_0)^{-2}]\rho_x + [(\partial\alpha_0/\partial y)\tan\gamma_0(\cos\alpha_0)^{-2}]\rho_y$$
$$+ [\beta(z+h)f^{-1}(\partial\alpha_0/\partial y)\tan\gamma_0(\cos\alpha_0)^{-2} - \beta f^{-1}\tan\alpha_0]\rho_z = 0 \qquad (4.11)$$

This equation is independent of the other two. Having obtained *three* independent, homogeneous, first order partial differential equation for ρ which is a function of three independent variables, the theory tells us that ρ = constant is the only possible solution.

Thus, on the β-plane the baroclinic velocity fields cannot satisfy the density conservation. Therefore, we can write (4.4) as

$$A\pi_x + B\pi_y = 1 \qquad (4.12)$$

where

$$A = \frac{\rho_y + \rho_z[\beta(z+h)/f_0]}{\rho_x P_y - \rho_y P_z - (\rho_0 f_0)\rho_z W}, \tag{4.13a}$$

and

$$B = -\frac{\rho_x}{\rho_x P_y - \rho_y P_z - (\rho_0 f_0)\rho_z W}. \tag{4.13b}$$

If we assume that A and B are functions of depth, then the theorem yields the following constraint on data

$$A = k_0(x,y)B + l_0(x,y)$$

and the usual expressions for the barotropic components, namely

$$\pi_x = l_0^{-1}(x,y)$$
$$\pi_y = -k_0(x,y)/l_0(x,y).$$

On the β-plane, the density conservation is stringent enough to determine the barotropic field completely.

Finally, we note that the same constraints which beta spiral calculations place on density and potential vorticity are present in our analysis: potential vorticity cannot be a function of density alone, i.e. material surfaces of density and potential vorticity cannot coincide. From (1.1) and (1.8) it follows that potential vorticity is conserved (Ertel's Theorem):

$$uq_x + vq_y + wq_z = 0 \tag{4.14}$$

where $q = f\rho_z$. If (4.14) is used to eliminate w from the density conservation equation one obtains

$$\pi_x(\rho_{zy}\rho_z + (\beta/f)\rho_z^2 - \rho_{zz}\rho_y) - \pi_y(\rho_{zx}\rho_z - \rho_{zz}\rho_x) =$$
$$- P_x(\rho_{zy}\rho_z + (\beta/f)\rho_z^2 - \rho_{zz}\rho_y) + P_y(\rho_{zx}\rho_z - \rho_{zz}\rho_x). \tag{4.15}$$

When $q = G(\rho)$ for some function G both sides of (4.15) are zero and the barotropic field cannot be determined. The same degeneracy is also present in (4.4): if $q = G(\rho)$ the right-hand side of (4.4) is zero and the barotropic field cannot be completely determined from density conservation alone. One way to see this is to note that the relationship

$$f\rho_z = G(\rho) \tag{4.16}$$

implies that

$$\rho(x,y,z) = K(z/f + Q(x,y)) \tag{4.17}$$

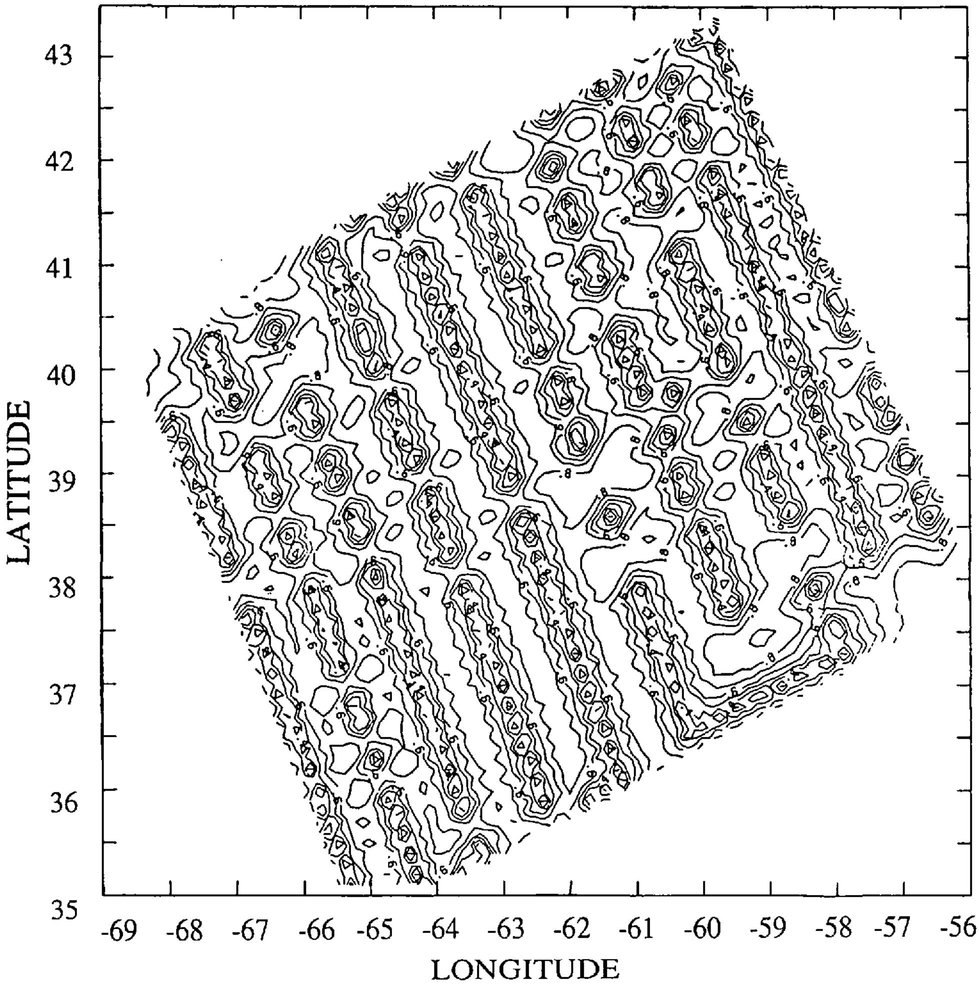

Figure 1. Normalized error, $\Gamma^2 = (E/S)^2$, in REX area. E is the error of the estimated field from true field. S is the signal variance.

where $K'(x) = G(K(x))$ and Q is some arbitrary function determined by boundary conditions. Substituting (4.17) into the right-hand side of (4.4) and integrating the resulting expression for W by parts then results in cancellation of all terms present there. Consequently the barotropic field cannot be completely determined by density conservation alone when material surfaces of density and potential vorticity coincide. We note that perhaps the discrepancies in the examples which follow, one including upper ocean data and the other not, suggest that this "$q = G(\rho)$" degeneracy occurs in the upper ocean at the site of the example. For further information about the occurrence of this possibility, we mention the work of Rhines and Young (1982).

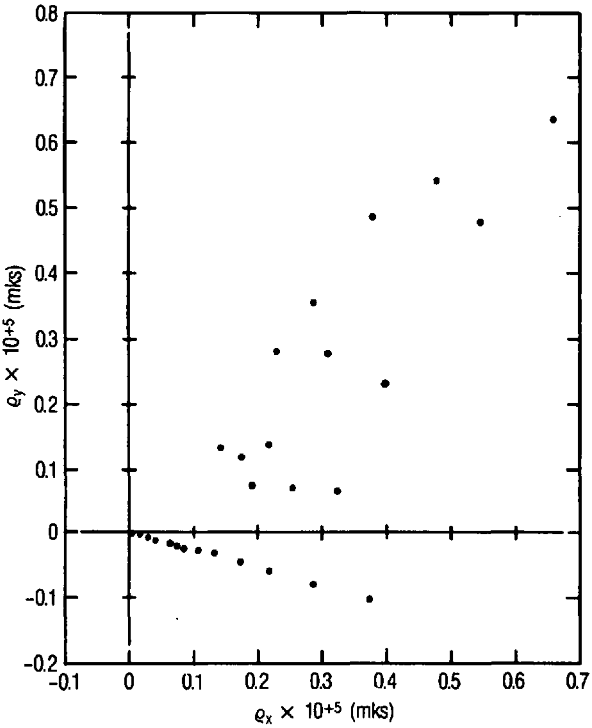

Figure 2. N-S density gradient vs. E-W density gradient at 37N and 64E for different depths. The points in the first quadrant represent measurements taken on August 13–15, 1985 as part of the REX experiment in the upper 700 m. Points in the fourth quadrant are from climatological data for depths greater than 700 m.

5. Concluding remarks

We conclude by illustrating the procedure with the analysis of a particular data set. The set selected was obtained in the REX (Regional Energetic Experiment) experiment. AXBT profiles were taken on August 13, 14, and 15, 1985. The drops were at 20 km intervals along altimeter tracks spaced about 90 km apart.

The AXBT profiles were smoothed, in order to remove the finescale variability, and combined with GDEM (Generalized Digital Environmental Model) below 700 m climatology depths.

The data was objectively analyzed for the three-day period using optimal estimation theory (Gandin, 1963). This technique interpolates mixed observations taken at random points in space and time to a uniform grid. It uses the statistical properties of the field and minimizes the difference between analysis and observations.

The *e*-folding correlation scales were assumed to be half a degree and the noise to signal ratio was .01. The analyzed temperature field was three-dimensional and contained all the characteristic features of the area. The details of the optimal estimation analysis for the REX area are described in Bennett and May (1988).

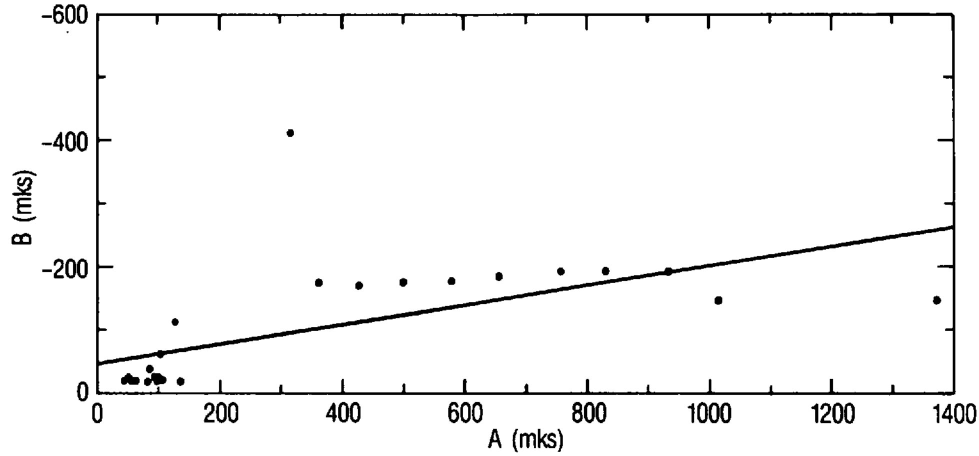

Figure 3. A and B fields as computed from (4.13a, b). The line passing through these points determines the parameters l_0 and k_0 at that location.

The normalized error obtained from the optimal estimation analysis is shown in Figure 1. It ranges from .4 at AXBT location, indicated by small triangles to 1.0 in regions of no data. Signal variance varied from .4°C at surface to .05°C below 700 m.

We chose a region centered at 37N and 64E for our calculations. This was an area that contained no surface signature of the Gulf Stream fronts and eddies. Below the surface the effects of the Gulf Stream were still present because of the tilt versus depth of the front (tilts away from shore). We used the optimally estimated fields for our calculations with a horizontal differentiation interval of one degree in latitude and longitude.

Figure 2 shows a scatter diagram of ρ_x versus ρ_y. The errors in ρ_x and ρ_y range from $.23 \times 10^{-5}$ mks at surface to $.05 \times 10^{-5}$ mks at 700 m and below. The points laying in the first quadrant are from the REX data and represent values from the upper 700 meters; whereas those in the fourth quadrant are climatological data for the layers below 700 meters. Figure 3 shows the corresponding variables A and B as evaluated from (4.13a, b) for all the data points. The points located in the region of $A = 200$ mks and $B = 200$ mks correspond to the upper 700 m region.

Figure 3 shows the corresponding values of the variables A and B as evaluated from (4.13a, b) for all the data points. The least square fit through these points yielded:

$$l_0 = -312.19 \text{ mks}$$
$$l_0/k_0 = 49.62 \text{ mks}$$

which implies barotropic velocity components of

$$u = 43 \text{ cm/s}$$
$$v = -6.8 \text{ cm/s}.$$

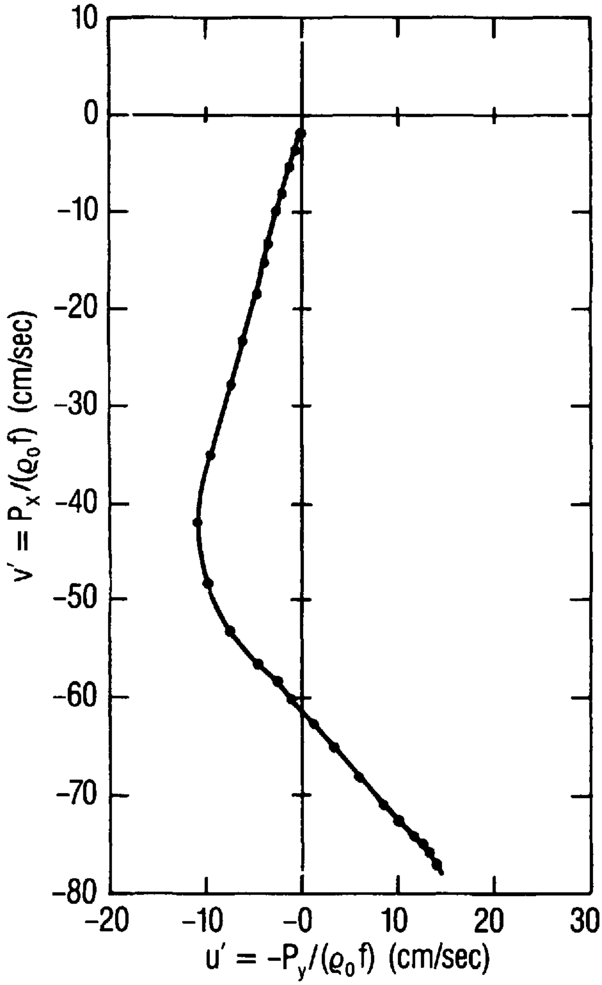

Figure 4. The baroclinic components of the velocity. To these must be added barotropic components of 43 cm/s East, 6.8 cm/s South.

The "baroclinic" velocity is shown in Figure 4, (obtained by integrating Eq. (1.3) from $-h$ to z and using (1.5e) and (1.5b)). Incidentally, if the climatological data alone were used, then

$$l_0 = -3219.5 \text{ mks}$$
$$l_0/k_0 = 152.4 \text{ mks}$$

and the corresponding barotropic velocity components would be

$$u = 14 \text{ cm/s}$$
$$v = -.7 \text{ cm/s}.$$

Measurements during REX, with an inverted echo sounder and a "pressure" gauge yields ratios of barotropic to baroclinic velocities of the order of 30% (Hallock, pers. comm.). According to these measurements our calculated barotropic velocities (u = 43 cm/sec, $v = -6.8$ cm/sec) are in the expected range.

All of our model equations are assumed to hold during our calculations. The upper ocean, however, is characterized by a mixed layer region. This region is approximately 50 m deep during the REX experiment and is controlled by atmospheric forcing. Its representation would require the introduction of a mixed layer model that would represent the time variation of density and vertical diffusion of momentum and heat flux. Since this region is only 50 m deep, we do expect our model equation to hold over most of the vertical column. Another deviation from the model equations can occur due to a degeneracy in the density conservation equation when the potential vorticity is a function of density alone. In this case the barotropic field cannot be completely determined by density conservation since material surfaces of density and vorticity coincide. The numerical analysis suggests such a possibility due to differences in behavior between the upper and lower regions of the ocean.

In summary, we have attempted to elucidate the questions of how a depth-dependent conserved field can remove the depth independent indeterminacy of the velocity fields. We have seen that this attempt leads naturally to a characterization of the class of acceptable density fields. In turn, this suggests a very natural variational procedure, namely that the "distance" between the data and this set of acceptable densities be minimized. For the β-plane, there is a unique velocity field associated with this projection.

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