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An Ascending Auction for Interdependent Values: Uniqueness and Robustness to Strategic Uncertainty*

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Abstract

We consider an single object auction environment with interdependent valuations and a generalized Vickrey-Clark-Groves allocation mechanism that allocates the object almost efficiently in a strict ex post equilibrium. If there is a significant amount of interdependence, there are multiple rationalizable outcomes of this direct mechanism and any other mechanism that allocates the object almost efficiently. This is true whether the agents know about each others' payoff types or not.

We consider an ascending price dynamic version of the generalized VCG mechanism. When there is complete information among the agents of their payoff types, we show that the almost efficient allocation is the unique backward induction (i.e., extensive form rationalizable) outcome of the auction, even when there are multiple rationalizable outcomes in the static version. This example illustrates the role that open auctions may play in obtaining efficient allocations by reducing strategic uncertainty.

Keywords: Dynamic Auction, Rationalizability, Extensive Form, Uniqueness, Strategic Uncertainty.

Jel Classification: C79, D82

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1 Introduction

The important role of dynamic auctions, in particular ascending price auctions, for the revelation of private information has long been recognized. The advantage of sequential procedures is the ability to reveal and communicate private information in the course of the mechanism. The revelation of private information can decrease the uncertainty faced by the bidders and ultimately improve the final allocation offered by the mechanism. In auctions, the source of the uncertainty can either be payoff uncertainty (uncertainty about others' payoff relevant information) or strategic uncertainty (uncertainty about their bidding strategies).

The ability of dynamic auctions to reduce payoff uncertainty is well documented in the literature. In a setting with interdependent values, the seminal paper by Milgrom and Weber (1982) shows that the ascending price auction leads to larger expected revenues by weakening the winner's curse problem. The ascending price auction leads to sequential revelation of good news for the active bidder. As the bidding contest proceeds and the price for the object increases, each active bidder revises upwards his estimate of the private information of the remaining bidders. The continued presence of active bidders represents a flow of good news about the value of the object. In consequence, each active bidder becomes less concerned about exposure to the winner's curse.

The objective of this paper is to argue that dynamic auctions also offer benefits for the reduction of *strategic* uncertainty. We consider an environment with interdependent values. We show that - under ex post incentive compatible allocation rules - strategic uncertainty (i.e., multiple rationalizable outcomes) necessarily occurs in a static mechanism. But we study a dynamic auction format in the case of complete information among the bidders. The complete information assumption removes payoff uncertainty and focusses our analysis on the role of strategic uncertainty.

We introduce strategic uncertainty by analyzing the rationalizable outcomes of static and dynamic versions of a generalized Vickrey-Clark-Groves mechanism. The relationship between rationalizability and strategic uncertainty has been established in Brandenburger and Dekel (1987). In a complete information environment they show that the set of rationalizable outcomes is equivalent to the set of outcomes of Nash equilibria in some type space. We appeal to this epistemic result and analyze the outcomes of static and dynamic auction formats under rationalizability.

An important difference emerges as we compare the set of rationalizable outcomes in the static and dynamic auction format. In the static auction, the efficient outcome is the unique rationalizable outcome if and only if the interdependence in the valuation of the agents is moderate. In contrast, the efficient outcome will remain the unique rationalizable outcome in the dynamic auction as long as a much weaker single crossing condition prevails.

In the interdependent value environment, the reports of the bidders are strategic substitutes. If bidder i increases his bid for a given valuation, then bidder j has an incentive to lower his report. An increase in the report by bidder i makes the object more costly to obtain without changing its value. In consequence bidder j will lower his report to partially offset the increase in the payment for the object induced by bidder i. The element of strategic substitutes between the reports of bidder i and j is generated by the incentive compatible transfer scheme rather than by the signal of the agents directly.

The discrepancy between the static and the dynamic version of the auction is due to the ability of the dynamic mechanism to partially synchronize the beliefs of the agents. In the static auction a low bid by agent i can be justified by high bids of the remaining bidders. But in turn, a large bid by bidder j requires bidder j to believe in low bids by the remaining bidders. The beliefs of bidder i and j about the remaining bidders are thus widely divergent. In the dynamic auction, the current report of each bidder represent a lower bound on the beliefs of all the agents and hence imposes a synchronization on the belief. In addition, in the dynamic auction, the bidders look ahead and only consider rationalizable future outcomes in their consideration. This forces each bidder to have a belief about the future actions of the other bidders which are rationalizable.

2 Model

We consider an auction environment with interdependent values. There are I agents competing for a single object offered by a seller. The payoff type of agent i given by a realization $\theta_i \in [0,1]$. The type profile is given by $\theta = (\theta_i, \theta_{-i})$ and agent i's valuation of the object at the type profile θ is given by $v_i(\theta_i, \theta_{-i}) = \theta_i + \gamma \sum_{j \neq i} \theta_j$, with $\gamma \in \mathbb{R}_+$. The net utility of agent i depends on his probability q_i of receiving the object and the monetary transfer t_i:

$$u_i(\theta, q_i, y_i) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j\right) q_i - y_i. \tag{1}$$

The socially efficient allocation rule is given by:

$$\widehat{q}_i(\theta) = \begin{cases} \frac{1}{\#\{j:\theta_j \geq \theta_k \text{ for all } k\}}, & \text{if} \quad \theta_i \geq \theta_k \text{ for all } k, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Dasgupta and Maskin (2000) have shown that a generalized Vickrey-Clark-Groves (VCG) auction leads to truthful revelation of private information in ex post equilibrium. In the generalized VCG auction, the monetary transfer of the winning agent i is given by:

$$\widehat{y}_i(\theta) = \max_{j \neq i} \left\{\theta_j + \gamma \sum_{j \neq i} \theta_j\right\}, \tag{2}$$

and the losing bidders all have a zero monetary transfer. The generalized VCG mechanism only guarantees weak rather than strict ex post incentive compatibility conditions. We seek to analyze the strategic behavior in the auction in terms of rationalizable behavior. As rationalizability involves the iterative elimination of strictly dominated actions, we modify the generalized VCG mechanism to display strict ex post incentive constraints everywhere. We add to the VCG allocation rule $\widehat{q}_i$ an allocation rule which increases proportionally in the report of agent i:

$$q_i(\theta') = \frac{\theta_i'}{I} \text{ for all } i. \tag{3}$$

The modified VCG allocation rule is now defined for some $\varepsilon > 0$ by

$$q_i^*(\theta) = \varepsilon q_i(\theta) + (1-\varepsilon)\widehat{q}_i(\theta). \tag{4}$$

The modified allocation rule is supported by an associated set of transfers:

$$y_i^*(\theta) = \frac{\varepsilon}{2I}\theta_i^2 + \frac{\varepsilon\theta_i}{I}\left(\gamma \sum_{j \neq i} \theta_j\right) + (1-\varepsilon)\left(\max_{j \neq i}\left\{\theta_j + \gamma \sum_{j \neq i} \theta_j\right\}\right)\widehat{q}_i(\theta). \tag{5}$$

The transfer rule $y_i^*(\theta)$ leads to strict truthtelling incentives everywhere. The outcome function of the direct mechanism is denoted by $f^* = (q_i^*, y_i^*)_{i=1}^I$.

Truth-telling is a strict ex post equilibrium of the above mechanism. This means that whatever the agents' beliefs and higher order beliefs about other agents' types, there exists a strict equilibrium where every agent tells the truth. However, this does not guarantee that there do not exist other, non-truth-telling equilibria. In the remainder of this paper, we fix this mechanism - which is designed to deal with incentive compatibility problems under general *incomplete information* structures - and examine the performance of static and dynamic versions of the mechanism under *complete information.*

3 Static Auction

We first analyze the generalized VCG mechanism in a static environment. The purpose of this section is to provide a background for the analysis of the ascending auction. We then show that the ascending auction leads to a unique rationalizable outcome under very weak condition on the interaction parameter γ. More precisely, the set of rationalizable outcome consists of a singleton for each bidder if $\gamma < 1$. This condition is weak as $\gamma < 1$ is necessary and sufficient for the single crossing condition to hold. In contrast, the static version of generalized VCG auction leads to the unique rationalizable outcome if and only if the interdependence is moderate, or $\gamma < \frac{1}{I-1}$.

Proposition 1 is a special case of a general uniqueness result in environments with interdependent values and incomplete information in Bergemann and Morris (2005). The analysis in the present case is substantially simplified by the complete information assumption as well as the linear and symmetric valuation structure.

The net utility of agent i in the modified VCG mechanism depends on the true type profile θ and the reported profile θ':

$$u_i\left(f^*\left(\theta'\right),\theta\right) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j\right) q_i^*\left(\theta'\right) - y_i^*\left(\theta'\right).$$

We insert the outcome function f^* given by (4) and (5) to obtain the net utility of i:

$$\begin{aligned} u_i\left(f^*\left(\theta'\right),\theta\right) &= \left(\theta_i + \gamma \sum_{j \neq i} \theta_j\right)\left(\frac{\varepsilon}{I}\left(\theta'_i + \gamma \sum_{j \neq i} \theta'_j\right) + (1-\varepsilon)\,\widehat{q}_i\left(\theta'\right)\right) \\ &\quad -\frac{\varepsilon}{2I}\theta_i'^2 - \frac{\varepsilon\gamma\theta'_i}{I}\sum_{j \neq i}\theta'_j - (1-\varepsilon)\left(\max_{j \neq i}\left\{\theta'_j + \gamma \sum_{j \neq i}\theta'_j\right\}\right)\widehat{q}_i\left(\theta'\right). \end{aligned}$$

The net utility function is a linear combination of the efficient allocation rule and the proportional allocation rule. It is straightforward to compute the best response of each agent i given a point belief about the reports θ'_{-i} of the remaining agents. The best response is linear in the true valuation and the size of the downward or upward report of the other agents:

$$\theta'_i = \theta_i + \gamma \sum_{j \neq i} \left(\theta_j - \theta'_j \right).$$

From here, it follows that the report of agent i and agent j are strategic substitutes. If agent j increases his report, then in response agent i optimally chooses to lower his report. The linear best response structure facilitates the analysis. In order to establish the largest possible report of agent i, it suffices to look at the lowest possible reports by all other agents. The process of elimination can therefore proceed based on specific point beliefs about minimal and maximal reports by the agents.

Proposition 1 (Static Auction)
The rationalizable outcome is unique and coincides with truthtelling if and only if $\gamma < \frac{1}{I-1}$.

4 Ascending Auction

We now consider a dynamic version of the generalized Vickrey Clark Groves mechanism, namely an ascending auction in continuous time. The auction begins with the clock running and each bidder participating in the auction. Each bidder can choose at any point in time to exit the auction. The exit decision is irrevocable and presents a commitment. Similarly, a decision to stay in the game may be viewed as a partial commitment to bid at least as much as indicated by the current decision. The decision of each player is therefore to let the clock continue or to stop it. The time interval is the unit interval $t \in [0, 1]$ and the game ends at $T = 1$.

Agents will choose strategies that are rationalizable in every subgame. Given the perfect information, simultaneous move nature of the game, this will imply that we can characterize rationalizable outcomes essentially by backward induction in terms of recursive best response functions. Thus we do not need to appeal to the forward induction logic built into Pearce's (1984) notion of extensive form rationalizability.

In the ascending auction, the i-th bidder to exit the auction will choose a best response to the actions of the other bidders. However, conditional on being the i-th bidder to leave, his best response will *distinguish* between the actions of bidders who left before him and those who leave after him. As bidder i cannot influence the timing of bidders who already left the game, he will choose a best response to their actions. As for the actions of the bidders leaving the game after i, bidder i will have rational expectations as to how his timing will affect their future choices. Without loss of generality, we relabel the bidders so that we have ascending bidding times in the index i: $t_1 \leq t_2 \leq \cdots \leq t_I$. Given the stopping times of all other bidders, the stopping time t_I is simply the best response to the past stopping times. We denote the best response of bidder I by $\beta_I(t_1, t_2, ..., t_{I-1})$, and the best response of i to the stopping decisions of the preceding bidders is $\beta_i(t_1, ..., t_{i-1})$.

We can solve for the best response functions recursively. The stopping time of the last remaining bidder is his best response to the stopping times of the preceding bidders:

$$t_I = \theta_I + \gamma \sum_{i=1}^{I-1} (\theta_i - t_i) . \tag{6}$$

The best response of the penultimate bidder $I-1$:

$$t_{I-1} = \theta_{I-1} + \gamma \sum_{i \neq I-1} (\theta_i - t_i) = \theta_{I-1} + \gamma \sum_{i=1}^{I-2} (\theta_i - t_i) + \gamma (\theta_I - t_I) .$$

Now bidder $I-1$ anticipates the best response of bidder I to all previous stopping decisions. We can thus insert the best response of bidder I, (6), into the best response of bidder $I-1$. In consequence, we obtain the best response of bidder $I-1$ to all preceding bidders:

$$t_{I-1} = \theta_{I-1} + \frac{\gamma}{1+\gamma} \sum_{i=1}^{I-2} (\theta_i - t_i) .$$

We inductively obtain the best response for bidder $I-j$ for all $j = 0, 1, ..., I-1$ as:

$$\beta_i(t_1, ..., t_{i-1}) = \theta_i + \alpha_i \sum_{j=1}^{i-1} (\theta_j - t_j) , \tag{7}$$

where the slope α_i of the best response depends on the exit position of bidder i:

$$\alpha_i = \frac{\gamma}{1 + (I-i)\gamma}. \tag{8}$$

While bidder i with an early exit responds more moderately to preceding bidders, at the same time an early exit by i gives bidder i the possibility to influence the decision of all succeeding bidders. In this sense, an early exit gives bidder i more strategic influence than a late exit would give bidder i. In order to induce truthtelling in the dynamic bidding game, we therefore have to account for the strategic influence in the monetary transfers. The strategic weight of each exit position suggests that the transfer functions should account for the difference across exit decisions. We therefore modify the monetary transfer (5) to account for the strategic weight:

$$y_i\left(t\right) = \frac{1}{2I} w_i t_i^2 + \frac{t_i}{I} \gamma \sum_{j \neq i} t_j, \tag{9}$$

where w_i is the strategic weight of bidder i. In the static mechanism we implicitly assigned each agent the same strategic weight equal to one. In the dynamic game the weights are given by the direct and indirect effects that bidder i has on the stopping times of all successive bidders. The weight w_i is simply the marginal effect that an increase in the stopping time of agent i has on the behavior of successive bidders, or:

$$w_i = \frac{1 + \left(I - i - 1\right)\gamma + \left(I - i\right)\gamma^2}{1 + \left(I - i - 1\right)\gamma} = 1 + \frac{\left(I - i\right)\gamma^2}{1 + \left(I - i - 1\right)\gamma}. \tag{10}$$

The dynamic game is solved recursively by means of the best response functions (7).

Proposition 2 (Ascending Auction)
The rationalizable outcome is unique and coincides with truthtelling if $\gamma < 1$*.*

The dynamic game introduces the possibility that a player can strategically commit in order to affect the behavior of the other agents. The above analysis suggests that the strategic value of the commitment does not interfere with our analysis. The absence of a strategic value of commitment is due here to the careful design of the monetary transfers which neutralize the strategic value of commitment.

5 Discussion and Conclusion

Incomplete Information. The current analysis consider a game in which the bidding agents had complete information about their types. The focus on the complete information environment allowed us to interpret the remaining uncertainty about the actions of the players as pure strategic uncertainty. We could then appeal to the epistemic analysis of Brandenburger and Dekel (1987) to interpret the set of rationalizable strategies as Nash equilibria in some type space. Naturally, it is of interest to ask how the results presented here would be affected by the introduction of incomplete information among the bidders.

The nature of the argument in proposition 2 suggests that the result may partially survive in an incomplete information environment. The best response of each bidder was largely a function of his own payoff type and his exit position in the auction. This information will still be available to him in the incomplete information game. Moreover, in the inductive process the payoff type of agents exiting after i dropped out and hence the information about future bidders did not enter either.

Partial Commitment. An ascending auction requires players to make partial commitments during the play of the game: by not dropping out at the current price, I commit to bid something strictly higher than the current price, but I do not commit to what it will be. There is a literature looking at how gradual partial commitment can help resolve multiplicity of equilibrium outcomes (e.g., Caruana and Einav (2006)); in our setting, the partial commitment reduces multiple static rationalizable outcomes to a unique dynamic rationalizable outcome. However, there is an important difference. For us, partial commitment reduces outcomes only by reducing strategic uncertainty, and we carefully adjust the transfers to ensure that players do not have an incentive to use their commitment to alter others' reports. Our dynamic selection is analogous to the classic observation that in a simple coordination game with multiple equilibria, a Pareto efficient equilibrium is selected if players choose sequentially (Gale (1995)).

Implementation in Refinements of Nash Equilibrium. Moore and Repullo (1988), Abreu and Sen (1990) and others have examined abstract settings where sequential rationality refinements of Nash equilibrium in dynamic mechanisms can be used to strengthen implementation results. Our example in this note belongs to this tradition, although our results seem to have a more direct intuition than the canonical mechanisms in that literature.

Conclusion. We analyzed strategic bidding behavior in a static and dynamic auction in an environment with interdependent values. We analyzed the static and dynamic version of the auction under rationalizability. We interpreted the results from rationalizability with a well-known epistemic point of view. The rationalizable behavior in the ascending auction was determined to be unique in substantially larger class of environments than the static auction. The dynamic auction allowed the agents to update their beliefs about the behavior of their competitors. As the decision to stay or to exit is common knowledge among the bidders, the ascending auction makes the strategic decision of the agent public. In consequence, the ascending auction reduces the strategic uncertainty among the bidders and leads to tighter prediction of the behavior of the agents. This suggests a new and important advantage of ascending auctions over sealed bid auctions. By reducing the strategic uncertainty, the ascending auction severely limits the possibility of multiple equilibria to arise from the auction.

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Supplementary Material for

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The appendix contains the proofs of Proposition 1 and 2.

Proof of Proposition 1. The uniqueness result is an immediate consequence of Theorem 1 and Theorem 3 in general incomplete information environments in Bergemann and Morris (2006).■

The proof of Proposition 2 assumes that the payoff types of the bidders are distinct: $\theta_1 < \theta_2 < ... < \theta_I$. In addition we allow the exit time of each agent to be in the interval $[0, I-1]$ rather than $[0,1]$. With a larger set of feasible exit times we avoid the possibility of corner solutions in the best response functions. The resulting set of rationalizable exit times will nonetheless be in $[0,1]$. Proposition 2 continues to hold without distinct payoff types and with exit times restricted to the unit interval. The only consequence is a longer proof caused by the necessity of case distinctions due to corner solutions.

Proof of Proposition 2. The proof proceeds by induction on the number j of bidders still to leave the auction. We show that the best response of bidder $I-j$ to the reports of the bidders exiting before him is given by:

$$\beta_{I-j}(t_1, t_2, ..., t_{I-j-1}) = \theta_{I-j} + \frac{\gamma}{1+j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i). \tag{11}$$

We begin with the final bidder I and thus $j = 0$. We showed earlier in (6) that the best response function of bidder I is indeed given by:

$$\beta_I(t_1, t_2, ..., t_{I-1}) = \theta_I + \gamma \sum_{i=1}^{I-1} (\theta_i - t_i).$$

We now proof the general inductive step. With the outcome function defined by (4), (9) and (10), we can write the payoff of agent $I-j$ for $j > 0$ as follows:

$$\varepsilon\left(\left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i\right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j}\left(\sum_{i \neq I-j} t_i\right)\right). \tag{12}$$

We can rewrite the payoff function of bidder $I-j$ by separating the bidders who exited before and those who will exit after $I-j$:

$$\varepsilon\left(\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}-\frac{1}{2}w_{I-j}t_{I-j}^2-\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}t_i\right)\right). \tag{13}$$

By hypothesis, the inductive step holds for all $k<j$. For all $i>I-j$, we can therefore replace the report by the best response given by (11):

$$\varepsilon\left(\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}-\frac{1}{2}w_{I-j}t_{I-j}^2-\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\beta_i\left(t_1,...t_{i-1}\right)\right)\right). \tag{14}$$

In particular, as the inductive step holds for all $i>I-j$, we can rewrite the best response of every agent $i>I-j$ as follows:

$$\beta_i\left(t_1,t_2,...,t_{I-j}\right)=\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right). \tag{15}$$

We can now insert $\beta_i\left(t_1,t_2,...,t_{I-j}\right)$ for all $i>I-j$ into (14) to get:

$$\varepsilon\left(\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}-\frac{1}{2}w_{I-j}t_{I-j}^2-\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\left(\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right)\right)\right)\right).$$

We find the best response of bidder $I-j$ by determining the optimal exit time t_{I-j} in response to past exit times and differentiate the above payoff with respect to the exit time t_{I-j}:

$$\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)-w_{I-j}t_{I-j}-\gamma\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\left(\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right)\right)-\frac{j\gamma}{1+j\gamma}t_{I-j}\right)=0.$$

We can collect terms to obtain

$$\left(\theta_{I-j}+\left(\gamma-\frac{j\gamma^2}{1+\left(j-1\right)\gamma}\right)\sum_{i<I-j}\left(\theta_i-t_i\right)\right)-w_{I-j}t_{I-j}-\gamma\left(+\frac{j\gamma}{1+\left(j-1\right)\gamma}\theta_{I-j}-\frac{j\gamma}{1+\left(j-1\right)\gamma}2t_{I-j}\right)=0,$$

or

$$\left(\theta_{I-j}\left(1-\frac{j\gamma^2}{1+\left(j-1\right)\gamma}\right)+\left(\gamma-\frac{j\gamma^2}{1+\left(j-1\right)\gamma}\right)\sum_{i<I-j}\left(\theta_i-t_i\right)\right)=\left(w_{I-j}-\frac{2j\gamma^2}{1+\left(j-1\right)\gamma}\right)t_{I-j}. \tag{16}$$

We can then solve for t_{I-j} and find that

$$t_{I-j} = \theta_{I-j} + \frac{\gamma}{1 + j\gamma} \sum_{i=1}^{I-j-1} (\theta_i - t_i) \, ,$$

thus proving the inductive step.■