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INFORMATION LOSS IN VOLATILITY MEASUREMENT WITH FLAT PRICE TRADING

By

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Information Loss in Volatility Measurement with Flat Price Trading[1]

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Abstract

A model of price determination is proposed that incorporates flat trading features into an efficient price process. The model involves the superposition of a Brownian semimartingale process for the efficient price and a Bernoulli process that determines the extent of flat price trading. A limit theory for the conventional realized volatility (RV) measure of integrated volatility is developed. The results show that RV is still consistent but has an inflated asymptotic variance that depends on the probability of flat trading. Estimated quarticity is similarly affected, so that both the feasible central limit theorem and the inferential framework suggested in Barndorff-Nielson and Shephard (2002) remain valid under flat price trading.

Keywords: Bernoulli process, Brownian semimartingale, Flat trading, Quarticity function, Realized volatility.

JEL classification: C15, G12

1. Introduction

The expression 'flat trading' refers to situations in market trading where consecutively sampled prices take on the same value. The phenomenon of flat pricing is extremely common in stock market trading, affecting almost all traded stocks, especially (but not exclusively) over small time intervals. An immediate implication of the phenomenon is that both returns and volatility are zero over the flat price subinterval, an outcome that has null probability of occurrence in any model where price behaves like a continuous Brownian semimartingale. This characteristic of the realized data inevitably has implications for the econometric measurement of volatility.

The present paper seeks to explore some of these implications in the context of the use of realized volatility (RV) estimates of integrated variance (IV). Part of the task is to develop a model that compounds the presumed semimartingale behavior of underlying efficient market prices with a mechanism that produces periods of flat prices in practical trading. Flat trading is a regular feature of many financial markets, especially for stock price data that is sampled at modest to high frequencies, where it may be regarded as a market microstructure phenomenon arising from discrete trading practices, information arrival in discrete packets, and trading volume effects. Without developing a full microstructure theory, we posit a stochastic mechanism that accords a constant probability of the occurrence of a trading flat over each given subinterval. The formulation leads to the compounding of the efficient price Brownian semimartingale with a Bernoulli process that determines the timing and length of the flat trading periods.

Under this new model, we develop a limit theory for standard econometric estimates of volatility by nonparametric RV measures. It turns out that when we allow for flat trading RV is still consistent, converges to IV, and follows a mixed Gaussian limit theory under standard regularity conditions corresponding to those used in the original work of Barndorff-Nielson and Shephard (2002, BNS hereafter). These new results generalize the standard theory on empirical quadratic variation estimates. Notably, however, there is some information loss when using RV to do inference about IV due to the presence of flat price effects. This loss takes the form of an increase in the asymptotic variance. The effects are of a magnitude to be very significant in practical applications. For example, if the RV estimate is constructed from 5-minute returns for Alcoa (AA) stock prices on April 5, 1995, the proportion of flat pricing on this day amounts to some 60% of the sample and our results imply that the correct variance quadruples that of the variance

obtained from a semimartingale process without flat pricing.

As with much other recent research on volatility, our interest in the use of RV measures is motivated by the availability of ultra-high frequency data which has made it feasible to measure volatility accurately in a direct nonparametric way. The idea is well explained in earlier work and simply involves the calculation of the sum of squared intra-day returns obtained from observed intra-day prices. The theoretical justification for measuring volatility in this way relies on standard properties of the empirical quadratic variation process for semimartingales (e.g., Protter, 2004), a set up which is commonly assumed for financial asset prices in the literature (see, for example, Andersen, Bollerslev, Diebold and Labys (ABDL, hereafter) (2001)). The main object of interest in this research is the value of IV over a specific time period such as a day. This approach to measuring volatility has attracted a great deal of attention in the last 5 years and has led to numerous successful applications – see, for example, ABDL (2001, 2003), Andersen, Bollerslev, Diebold and Ebens (ABDE, hereafter 2001), Andersen, Bollerslev, Diebold and Wu (2005), Andersen, Bollerslev and Meddahi (2005), Bandi and Russell (2006) and Fleming, Kirby and Ostdiek (2003). For overviews of the literature, see Andersen, Bollerslev, Diebold (2005) and BNS (2007).

Direct application of empirical quadratic variation limit theory requires that efficient or equilibrium prices be observed. This requirement appears too strong at ultra-high frequencies, such as the tick-by-tick frequency, because of the presence of various market microstructure effects. These market microstructure effects may be regarded as contaminating the efficient price process and may be, albeit somewhat crudely, modeled as noise. Ignoring these effects produces bias and inconsistency in realized volatility estimates.

While maintaining the assumption of martingale-like behavior for efficient prices, the literature has produced three different strands of research on how to deal with microstructure noise in realized volatility calculations with intra-day data. One strand of research is to use all available tick-by-tick data and seek to explicitly model microstructure noise in this fine-grain sampling context. Assumptions about the properties of the microstructure noise are typically made for analytic convenience and include both iid and stationarity conditions. Important contributions to this literature include Zhang, Mykland and Aït-Sahalia (2005), Aït-Sahalia, Mykland and Zhang (2005), and Barndorff-Nielson, Hansen, Lunde and Shephard (2006).

A second strand of research in the literature is to sample sparsely relative to the available sampling frequency, usually at modest frequencies, of 5 or 10 minute intervals.

This approach is motivated by the fact that many sources of microstructure noise (such as bid/ask bounce), which occur in ultra-high frequency data, are mitigated when prices are sampled at these modest frequencies. Correspondingly, it has been argued that these more sparsely sampled prices better approximate the efficient price process, and therefore standard semimartingale theory can be invoked. Under such semimartingale conditions, the consistency of RV was used in ABDL (2001) and the asymptotic distribution of RV was developed in Jacod (1994) and BNS (2002).

In the third strand of the literature, researchers have focused on the finite sample properties of the RV estimates. Here it is argued that the choice of sampling frequency effectively trades off estimation variance against bias. When microstructure noise is explicitly modelled, an "optimal" sampling frequency, which minimizes the mean squared error of the RV estimate, may be calculated. Studies following this approach include Zhou (1996), Hansen and Lunde (2006) and Bandi and Russell (2005).

None of the above analyses explicitly models or allows for flat trading in observed prices even though flat trading is a salient feature in actual stock data at most of the frequencies that have been used in this literature, from tick-by-tick data through to 15-minute trading data. Flat trading is a characteristic of both actively traded and inactively traded stocks. To illustrate the former, Fig. 1 plots transaction prices for the stock AA from the Dow Jones Industrial Average (DJIA) on the New York Stock Exchange (NYSE) at three different frequencies: tick-by-tick, 1-minute, and 15-minute frequencies on April 5, 2000. Flat trading is obvious at all three frequencies and it becomes a dominant feature in the tick-by-tick data. Table 1 reports the proportions of flat transaction prices for AA when sampling is performed at five different frequencies (1-, 2-, 3-, 4-, 5-minute intervals) on the first Wednesday in April from 1993 to 2004. Although flat pricing effects are less pronounced after the decimalization of trading in January 2001, they remain a non-negligible feature of these data. Flat pricing also takes place in tick sampling and in quote data; see, for example, Table 1 in Hansen and Lunde (2006) for the percentages of flat quote prices at the tick-by-tick level for 30 DJIA stocks. Note that AA is a DJIA stock and DJIA stocks are among the most actively traded equities. Flat trading is naturally even more of an issue for less liquid stocks. This feature of trading data deserves attention both in financial modeling and econometric volatility estimation with high frequency data.

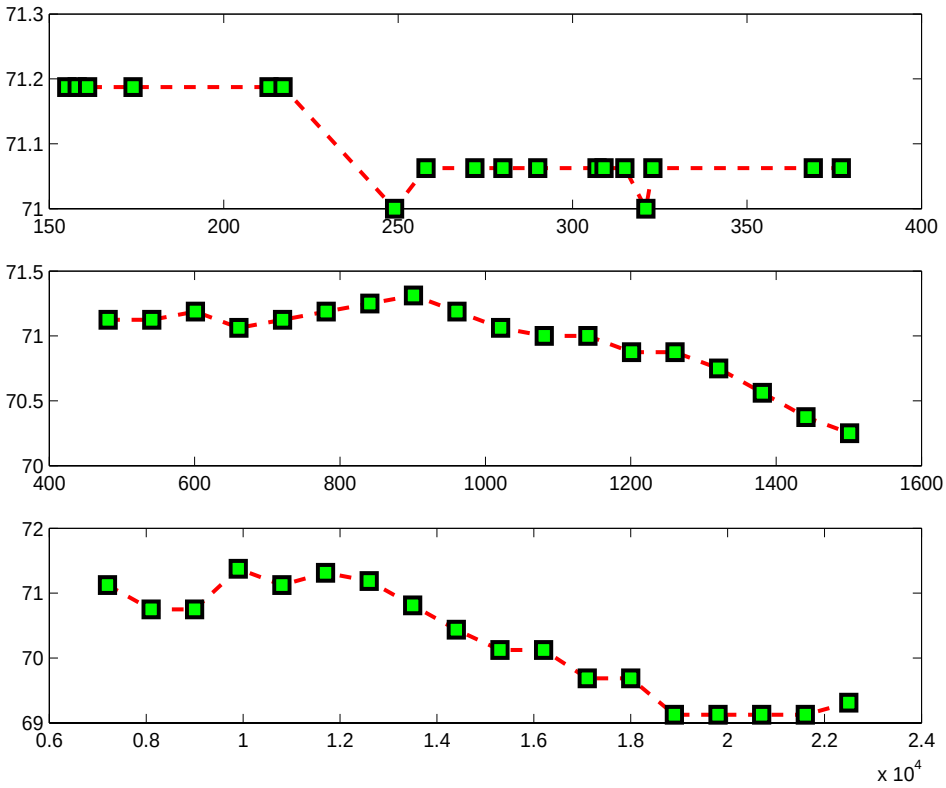


Figure 1: Time series plots of transaction prices for AA on April 5, 2000 at three different frequencies. The horizontal axis is the time stamp (in seconds) since the market opening at 9:30am. The first panel is based on tick-by-tick observations. The second panel is based on data that are sampled every 1 minute. The third panel is based on data that are sampled every 15 minutes. The prices at the 1- and 15-minute frequencies are obtained using the previous tick method. See Hansen and Lunde (2006) for a detailed discussion of the different sampling schemes.

Table 1: Proportion of flat trading in AA stock prices

Date	# of ticks	Proportion of flat trading				
		1-min	2-min	3-min	4-min	5-min
April 7, 1993	213	.8793	.8000	.7615	.7041	.6282
April 6, 1994	174	.8897	.8051	.7308	.6429	.6154
April 5, 1995	171	.8872	.7949	.7154	.6633	.6026
April 3, 1996	246	.8795	.7897	.6846	.6327	.5897
April 2, 1997	243	.8179	.6923	.5923	.5612	.4872
April 1, 1998	394	.7846	.6359	.5538	.4898	.4103
April 7, 1999	794	.5513	.4154	.3538	.2245	.2692
April 5, 2000	1007	.4436	.2564	.2231	.2041	.1154
April 4, 2001	1788	.1872	.0872	.0692	.0510	.0128
April 3, 2002	1281	.3872	.2615	.1385	.1224	.0128
April 2, 2003	1986	.3795	.2923	.1769	.1939	.0897
April 7, 2004	3845	.2641	.1590	.1692	.0306	.0513

The contribution of the present note to these issues relates to the second strand of the literature on modest frequency sampling and to studies on market microstructure noise. First, the model introduced here extends the models used in ABDL (2001) and BNS (2002) by gaining some additional realism in its allowance for flat trading sample paths. Second, we extend the limit theory of RV to the new model, showing that while RV still consistently estimates IV and asymptotically follows a mixed Gaussian law in the presence of flat trading, the asymptotic variance of the RV estimate is inflated, thereby revealing the loss of a substantial amount of information about underlying efficient price volatility in flat trading. Third, we show that the estimated variation of RV based on empirical quarticity is similarly affected by the occurence of trading flats. In consequence, and importantly for empirical research, both the feasible central limit theorem and the inferential framework developed in BNS (2002) remain valid under flat price trading.

We proceed as follows. After briefly reviewing the literature, we introduce the new model and develop the corresponding limit theory in Section 2. Section 3 reports the results of some Monte Carlo experiments to assess the accuracy of the theory in finite

samples. Section 4 concludes. Proofs are given in the Appendix.

2. A Flat Trading Model and Limit Theory

Let $p^*(t)$ be the logarithm of the efficient price and assume $p^*(t)$ evolves according to a Brownian semimartingale process on a filtered probability space $(\Omega, \mathfrak{F}, P)$. This assumption is justified by Back (1991) in a frictionless, arbitrage-free economy. As it is typical in the high frequency volatility literature, we further assume that $p^*(t)$ follows the (driftless) diffusion

$$dp^*(t) = \sigma(t)dB(t), \tag{1}$$

where $B(t)$ is a standard Brownian motion and $\sigma(t)$ is a càdlàg volatility process. The quantity of interest is $IV = \int_0^1 \sigma^2(t)dt$, the IV of $p^*(t)$ over a certain unit time period, say a day. The integral may be defined as the limit of the empirical quadratic variation

$$IV = \text{plim}_{h\to 0} \sum_{i=1}^{m} [p^*_{i,m} - p^*_{i-1,m}]^2, \tag{2}$$

where $p^*_{i,m} = p^*(t_{i,m})$, $0 = t_{0,m} < t_{1,m} < \cdots < t_{m,m} = 1$ is a sequence of deterministic partitions of $[0,1]$, and $h = \sup_i |t_{i,m} - t_{i-1,m}|$ is the grid size. Sometimes, it is convenient to assume that the partition involves a simple grid of equi-spaced points $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$ with $h = \frac{1}{m}$.

The limiting value IV in (2) is a (unit time period) segment of the quadratic variation process of p^*. The sample counterpart is the empirical quadratic variation

$$\sum_{i=1}^{m} [p^*_{i,m} - p^*_{i-1,m}]^2 := RV^{(m)}(p^*),$$

which is now commonly referred to as RV in financial economics.

Since $RV^{(m)}(p^*) \xrightarrow{p} IV$ (e.g., Protter, 2004), RV is a natural candidate for estimating IV, motivating the recent interest in this approach to volatility measurement. To quantify the statistical difference between RV and IV, Jacod (1994) and BNS (2002) used the limit theory

$$\sqrt{m}\left[RV^{(m)}(p^*) - IV\right] | \sigma^2(t) \xrightarrow{d} MN\left(0, 2\int_0^1 \sigma^4(t)dt\right), \tag{3}$$

where MN signifies mixed normality. A feasible version of this limit involves the estimation of the quarticity functional $\int_0^1 \sigma^4(t)dt$ using empirical quarticity. BNS obtained the following result

$$\frac{RV^{(m)}(p^*) - IV}{\sqrt{\frac{2}{3}\sum_{i=1}^{m}[p^*_{i,m} - p^*_{i-1,m}]^4}} \xrightarrow{d} N(0,1), \tag{4}$$

which is convenient for use in inference.

These asymptotic results all require knowledge of the log-efficient price, $p^*_{i,m}$. At ultra high frequencies market microstructure effects challenge this requirement, contaminating observations with microstructure noise so that the actual price data $p_{i,m} = p(t_{i,m})$ differs from $p^*_{i,m}$ and $RV^{(m)}(p) \neq RV^{(m)}(p^*)$. To mitigate such market microstructure effects, ABDL (2001), ABDE (2001) and BNS (2002) suggested sampling sparsely, say at five minute intervals, so that the accumulative effects of noise are less important and $p_{i,m}$ is treated the same as $p^*_{i,m}$. ABDL justified the choice of five minute intervals using the signature plot, a graphical device used to assess the degree of bias caused by market microstructure effects at different sampling frequencies. Signature plots typically suggest that RV is more severely biased when the sampling frequency increases but stabilizes at modest frequencies. This observation has prompted researchers to view the observed price as a good approximation to the efficient price and has the same semimartingale characteristics at these modest frequencies.

The impact of market microstructure noise has also been examined in the more specific analytic framework

$$p(t) = p^*(t) + u(t), \tag{5}$$

where $u(t)$ is microstructure noise. Most studies assume that the noise process $u(t)$ and price process $p^*(t)$ are independent. However, there are many different proposals in the literature about how to model the noise process and how to treat the presence of noise in the estimation of IV. Some studies (e.g., Zhou, 1996, Bandi and Russell, 2005, Zhang, Mykland and Aït-Sahalia, 2005, Sun, 2006) assume a pure noise structure for $u(t)$. Some other studies (e.g. Hansen and Lunde, 2006 and Aït-Sahalia, Mykland and Zhang, 2005) assume $u(t)$ is covariance stationary.

Neither pure noise nor covariance stationary microstructure effects explain flat trading. In fact, when the efficient price follows a Brownian semimartingale as in (1), then during periods of flat trading prices the microstructure noise effect completely offsets

the efficient price fluctuations to produce a sustained flat transactions price. The noise process therefore inherits the same local martingale-like behavior of the efficient price process over this subinterval. Inspection of trading data such as that shown in Fig. 1 shows that while sampling at modest frequencies reduces the effects of flat trading it too does not completely resolve the problem. Accordingly, we propose to build a model that directly incorporates flat trading features, so that the effects of flat pricing on RV asymptotics can be assessed.

We follow the existing literature and assume that the efficient price process $p^*(t)$ follows (1). This specification implies that, for any $t_i \in [0,1]$, $p^*_{i,m}$ has the martingale structure $p^*_{i,m} = \sum_{j=1}^{i} \varepsilon_{j,m}$, where $\varepsilon_{j,m} = \int_{t_{j-1,m}}^{t_{j,m}} \sigma(s) dB(s)$. Also, conditional on the volatility path $\sigma^2(s)$ over $s \in [t_{i-1,m}, t_{i,m}]$,

$$\varepsilon_{i,m} = p^*_{i,m} - p^*_{i-1,m} =_d N\left(0, \int_{t_{i-1,m}}^{t_{i,m}} \sigma^2(s) ds\right) \sim N(0, \sigma^2(t_{i-1,m})(t_{i,m} - t_{i-1,m})) \qquad (6)$$

for small grid size h.

Working within this framework, the new model adds a simple Bernoulli process to determine the trading price

$$p_{i,m} = \begin{cases} p^*_{i,m} & \text{if } \xi_i = 1 \\ p_{i-1,m} & \text{if } \xi_i = 0 \end{cases}, \qquad (7)$$

where ξ_i is a Bernoulli sequence with $E(\xi_i = 1) = \pi$, and $p_{0,m} = p^*_{0,m} = O_p(1)$. Thus, while $p^*_{i,m}$ follows an underlying martingale in the background, the observed price compounds this efficient process with a Bernoulli sequence that determines whether flat trading occurs in price realization . Whenever $p_{i,m} \neq p_{i-1,m}$, the realization follows the efficient price and we observe $p^*_{i,m}$. Otherwise, flat trading occurs. In that event, the microstructure noise effect completely offsets the efficient price movement over the subinterval in which flat trading occurs.

This model allows for flat trading with a constant probability of $1-\pi$, so that there is a positive probability of flat trading at each point on the temporal grid when $\pi \in [0,1)$. When $\pi = 1$, $p_{i,m} = p^*_{i,m}$ almost surely and the model reduces to the earlier model of ABDL (2001), ABDE (2001) and BNS (2002). If $p_{i-1,m} = p^*_{i-1,m}$ and $p_{i,m} = p_{i-1,m}$, then $p_{i,m} - p^*_{i,m} = p^*_{i-1,m} - p^*_{i,m} = -\varepsilon_{i,m}$. So the new model allows for noise in the observed price and the noise depends on the efficient price. The noise can be interpreted as a discrete price effect, according to which the realized price changes only when the information

content is strong enough. Eventually, of course, the observed price will change and follow the efficient price provided $\pi > 0$. One consequence of the specification is that when noise occurs in the model it takes the form $p_{i,m} - p^*_{i,m} = -\varepsilon_{i,m} = -\int_{t_{i-1,m}}^{t_{i,m}} \sigma(s) dB(s)$ and is therefore negatively correlated with the efficient price process. Negative correlation between microstructure noise and the efficient price has been empirically documented in Hansen and Lunde (2006, page 132). However, since $p_{i,m} = p^*_{i,m}$ when $p_{i,m} \neq p_{i-1,m}$, the present model eliminates noise effects when the price changes. Thus, the model may be more appropriate at modest frequencies rather than at ultra-high frequencies.

Our first result shows that the compound model preserves the martingale property for trading prices.

THEOREM 2.1 (Martingale Property): *If $p^*(t)$ follows (1) and the trading price $p(t)$ follows (7) with $\pi \in (0, 1]$, then $\{p_{i,m}\}$ is a martingale with $E(p_{i,m}|\mathfrak{F}_{i-1,m}) = p_{i-1,m}$ and the natural filtration $\mathfrak{F}_{i,m} = \sigma(p_{i,m}, p_{i-1,m}, \cdots)$.*

We now present the main results of the paper. Theorem 2.2 shows that RV still consistently estimates IV, extending the standard theory of empirical quadratic variation (ABDL, 2001) to the case of flat trading. Theorem 2.3 derives the corresponding central limit theorem (CLT) for RV and Theorem 2.4 provides a feasible version of the CLT for inference about IV using an empirical quarticity estimate. For the CLT results it is convenient to assume that the discrete sampling grid involves equi-spaced observations, so that $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$. This requirement might be dispensed with at the cost of some additional complexity, but fits in with earlier conditions used in BNS on RV limit theory without flat trading.

THEOREM 2.2 (Consistency): *If $\pi \in (0, 1]$, then as $m \to \infty$,*

$$RV^{(m)}(p) \xrightarrow{p} IV. \tag{8}$$

THEOREM 2.3 (Infeasible CLT): *Assume the observation grid is equi-spaced with $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$. If $\pi \in (0, 1]$, then*

$$\sqrt{m}\left[RV^{(m)}(p) - IV\right] \xrightarrow{d} MN\left(0, \frac{4 - 2\pi}{\pi} \int_0^1 \sigma^4(t) dt\right), \tag{9}$$

stably as $m \to \infty$, where MN signifies mixed normal.

Remark A: Stable convergence in law means here, as in Barndorff-Nielson, Graversen, Jacod and Shephard (2006), that there is joint convergence of the pair $\left(\int_0^1 \sigma^4(t) dt, \sqrt{m}\left[RV^{(m)}(p) - IV\right]\right)$,

as $m \to \infty$,

$$\frac{\sqrt{m}\left[RV^{(m)}(p) - IV\right]}{\left\{\frac{4-2\pi}{\pi}\int_0^1 \sigma^4(t)dt\right\}^{1/2}} \xrightarrow{d} N(0,1).$$

This type of convergence in law is useful because it ensures that normings can be interchanged in the statistic when there is a mixed normal limit (see Hall and Heyde, 1980, pp. 56-59), thereby facilitating inference as in Theorem 2.4 below.

Remark B: When $\pi = 0.5$, some 50% of the data involves flat trading and the asymptotic variance in (9) is three times as large as when $\pi = 1$. This magnitude seems to be in line with what has been documented empirically in Hansen and Lunde (2006, page 137). Table 2 shows the ratio of the asymptotic variance to the case where there is no flat trading for various values of π and Fig. 2 plots this nonlinear relationship. As π becomes small, the ratio blows up rapidly.

Remark C: Interestingly, result (9) holds even when flats are removed from the sample. This is because the empirical quadratic variation is unaffected by the presence of flat trading periods. Hence removing flat prices from data does not reduce the asymptotic variance or change the limit theory. In effect, the limit result shows that, when trading which does not reflect the true efficient price occurs, the asymptotic variance of the RV estimate increases proportionately. That is, when there is flat price trading there is less information about the efficient price $p^*(t)$, and the asymptotic theory reflects this reduction in information by an inflation of the variance.

Table 2: Ratio of asymptotic variance with flats to that without flat

π	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\frac{2-\pi}{\pi}$	19.00	9.00	5.67	4.00	3.00	2.33	1.86	1.50	1.22	1

To use the CLT (9) in practice the asymptotic variance must be estimated, which involves estimating the integrated quarticity functional $\int_0^1 \sigma^4(t)dt$. Following BNS, integrated quarticity can be estimated consistently and used in a feasible CLT that is suitable for inference about IV.

Lemma 2.4: *Under the conditions of Theorem 2.3, as* $m \to \infty$

$$\frac{\pi}{6-3\pi} m \sum_{i=1}^{m} \left[p_{i,m} - p_{i-1,m}\right]^4 \xrightarrow{p} \int_0^1 \sigma^4(t)dt. \tag{10}$$

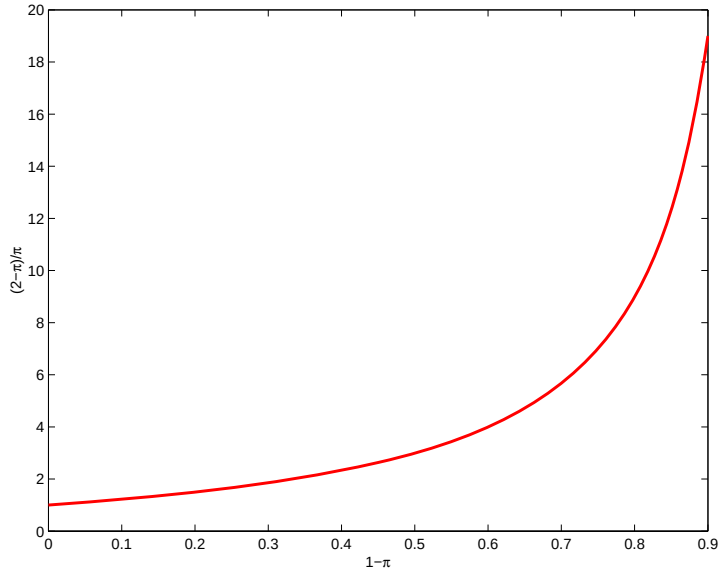


Figure 2: Ratio of the asymptotic variance under flat trading to that with no flat trading ($\pi = 1$).

Remark D: When $\pi = 1$, $\frac{\pi}{6-3\pi} = \frac{1}{3}$ and result (10) is identical to that of BNS.

THEOREM 2.5 (Feasible CLT): *Under the conditions of Theorem 2.3, as* $m \to \infty$

$$\sqrt{\frac{3}{2}} \frac{\left(RV^{(m)}(p) - IV\right)}{\sqrt{\sum_{i=1}^{m} \left[p(t_{i,m}) - p(t_{i-1,m})\right]^4}} \xrightarrow{d} N(0,1). \tag{11}$$

Remark E: Interestingly, the standardization in the feasible CLT (11) does not depend on π and the feasible CLT result is therefore the same as that given in BNS for the case where there is no flat trading ($\pi = 1$). In effect, the quantity involving π appears as a factor $\frac{2-\pi}{\pi}$ in the asymptotic variance and the estimated quarticity functional and is therefore scaled out in the feasible CLT. Nonetheless, the effects of flat trading are implicitly embodied in the feasible CLT since they are carried in the empirical measure $\sum_{i=1}^{m} \left[p(t_{i,m}) - p(t_{i-1,m})\right]^4$, which is correspondingly reduced by periods of flat pricing. Thus, the asymptotic inferential apparatus of BNS continues to hold under the present model where flat trading is manifest.

3. Monte Carlo Study

Data are simulated over a day so that $t_{0,m} = 0$ and $t_{m,m} = 1$. A day is assumed to have 6.5 hours and 23400 seconds. In our Monte Carlo design we chose m =39, 78, 130, 195, and 390. These values correspond to frequencies of 10, 5, 3, 2, 1 minutes, respectively.

3.1 The Brownian motion model

This subsection reports simulations from a simple Brownian motion model where volatility is a known constant ($\sigma^2(t) = 1$) so that

$$dp^*(t) = dB(t). \tag{12}$$

This formulation allows us to assess the accuracy of CLT (9) as $\int_0^1 \sigma^4(t)dt = 1$ and then the asymptotic variance in (9) is simply $\frac{4-2\pi}{\pi}$.

Table 3 shows both the asymptotic and finite sample simulated variances of the statistic $\sqrt{m}\left[RV^{(m)}(p) - IV\right]$ based on 5000 replications for various combinations of π and m. The asymptotic formula is clearly very accurate except for very small values of π. The effect of flat trading on the asymptotic variance is dramatic, producing a three fold increase in variance when $\pi = 0.5$.

Table 3: Asymptotic and finite sample variances

π	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Asymp. variance $\frac{4-2\pi}{\pi}$	38	18	11.33	8	6	4.67	3.71	3	2.44	2
Asymp variance $\pi = 1$	2	2	2	2	2	2	2	2	2	2
Variance ($m = 39$)	24.31	14.84	10.18	7.61	5.82	4.56	3.74	3.06	2.51	2.04
Variance ($m = 78$)	30.38	16.56	10.88	7.71	5.79	4.61	3.70	3.02	2.49	2.02
Variance ($m = 130$)	33.47	17.15	11.16	7.85	5.88	4.66	3.71	3.00	2.45	2.02
Variance ($m = 195$)	34.39	17.06	10.87	7.74	5.95	4.69	3.74	3.04	2.45	2.02
Variance ($m = 390$)	36.32	17.41	11.03	7.92	6.03	4.74	3.76	3.04	2.47	2.03

3.2 A stochastic volatility model

In this subsection, price data is simulated from Heston's stochastic volatility model with volatility following a square root model (Heston, 1993):

$$\begin{cases} dp^*(t) &= \sigma(t)dB_1(t), \\ d\sigma^2(t) &= \kappa(\mu - \sigma^2(t))dt + \eta\sigma(t)dB_2(t). \end{cases} \tag{13}$$

Feller (1951) showed that the density of $\sigma^2(t+h)$ conditional on $\sigma^2(t)$ is $ce^{-u-v}(v/u)^{q/2}I_q(2(uv)^{1/2})$ and the marginal density of $\sigma^2(t)$ is $w_1^{w_2}\sigma^{2(w_2-1)}e^{-w_1\sigma^2}/\Gamma(w_2)$, where $c = 2\kappa/(\eta^2(1 - e^{-\kappa h}))$, $u = c\sigma^2(t)e^{-\kappa h}$, $v = c\sigma^2(t+h)$, $q = 2\kappa\mu/\eta^2 - 1$, $w_1 = 2\kappa/\eta^2$, $w_2 = 2\kappa\mu/\eta^2$, and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The conditional density together with the marginal density are used for data simulation. The parameters in the model are set at $\kappa = 0.01, \mu = 1$ and $\eta = 0.05$.

Table 4: **Asymptotic and finite sample variances**

π	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Our asymp variance	1	1	1	1	1	1	1	1	1
Variance ($m = 39$)	15.06	5.76	4.61	2.15	1.90	1.62	1.45	1.21	1.16
Variance ($m = 78$)	5.85	2.71	2.05	1.54	1.50	1.35	1.26	1.16	1.11
Variance ($m = 130$)	3.17	1.95	1.54	1.31	1.29	1.28	1.21	1.17	1.11
Variance ($m = 195$)	1.89	1.52	1.27	1.20	1.20	1.20	1.14	1.12	1.08
Variance ($m = 390$)	1.37	1.35	1.21	1.20	1.15	1.16	1.14	1.12	1.08

The aim of the experiment is to assess the accuracy of the empirical quarticity formula in the feasible CLT (11). Table 4 gives the Monte Carlo results. In particular, we report the variance of the standardized statistic $\sqrt{\frac{3}{2}}\frac{(RV^{(m)}(p)-IV)}{\sqrt{\sum_{i=1}^{m}[p(t_{i,m})-p(t_{i-1,m})]^4}}$ from 1000 replications, for various combinations of π and m, shown against the asymptotic variance of unity.

Some conclusions can be drawn from the table. First, the asymptotic theory clearly works better for large π and large m. This is unsurprising because larger values of π imply fewer flat price trading periods and therefore larger effective sample sizes. Second, the asymptotic theory does eventually work well even for small π, but needs larger values of m to provide a good approximation. The main reason for these effects is that it is

more difficult to estimate the integrated quarticity than the integrated volatility. This corroborates existing findings in the literature on realized volatility without flat pricing. Table 4 shows that these effects are exacerbated when there is flat trading, especially when π is small, because of the smaller effective sample size.

4. Conclusion

When trading does not reflect the efficient price because of flat trading effects, the variance of the RV estimate of integrated volatility increases because we have correspondingly less information about the efficient price than the number of observations might indicate. Of course, the same conclusion holds when the flats in the trading price are simply ignored and the previous tick method of constructing the RV estimate is employed. Furthermore, since fitted quarticity is similarly affected by flat trading, the framework suggested in BNS for inference about RV remains valid. These conclusions are intuitively obvious in that the operationally useful data for IV estimation are simply those observations that reflect the underlying efficient price.

The flat trading model used here may be interpreted as embodying some noise effects as flat trades imply that past values such as $p_{i-K_i,m}$ may be observed rather than the current efficient price $p^*_{i,m}$, in which case the offset from the efficient price is a form of noise. But the present model is clearly limited by the fact that other sources of microstructure noise are neglected, especially those that occur at very high frequency. So the present results may be regarded as being most relevant at modest frequencies.

Appendix

Proof of Theorem 2.1: The specification of $p(t)$ implies that

$$p_{i,m} = p^*_{i,m}\xi_{i,m} + p_{i-1,m}(1-\xi_{i,m}), \tag{14}$$

$$p_{i,m} - p_{i-1,m} = (p^*_{i,m} - p_{i-1,m})\xi_{i,m}, \tag{15}$$

and

$$p^*_{i,m} - p_{i,m} = (p^*_{i,m} - p_{i-1,m})(1-\xi_{i,m}). \tag{16}$$

Taking conditional expectations of both sides of equation (14), we have

$$\begin{aligned} E(p_{i,m}|\mathfrak{F}_{i-1,m}) &= E(p^*_{i,m}|\mathfrak{F}_{i-1,m})\pi + p_{i-1,m}(1-\pi) \\ &= E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m})\pi + p_{i-1,m}(1-\pi) \end{aligned} \tag{17}$$

To compute $E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m})$, note that if $p_{i-1,m} \neq p_{i-2,m}$, then $p_{i-1,m} = p^*_{i-1,m}$ and hence $E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m}) = p_{i-1,m}$. If $p_{i-1,m} = p_{i-2,m}$ but $p_{i-2,m} \neq p_{i-3,m}$, then $p_{i-2,m} = p^*_{i-2,m}$, $p^*_{i-1,m} = p^*_{i-2,m} + \varepsilon_{i-1,m}$, and $\mathfrak{F}_{i-1,m} = \mathfrak{F}_{i-2,m}$. Hence $E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m}) = E(p^*_{i-2,m} + \varepsilon_{i-1,m}|\mathfrak{F}_{i-2,m}) = p_{i-2,m} = p_{i-1,m}$. Similarly, if $p_{i-1,m} = \cdots = p_{i-K,m}$ but $p_{i-K,m} \neq p_{i-K=1,m}$, then $p_{i-K,m} = p^*_{i-K,m}$, $p^*_{i-1,m} = p^*_{i-K,m} + \varepsilon_{i-K+1,m} + \cdots + \varepsilon_{i-1,m}$ and $\mathfrak{F}_{i-1,m} = \cdots = \mathfrak{F}_{i-K,m}$. Hence $E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m}) = E(p^*_{i-K,m} + \varepsilon_{i-K+1,m} + \cdots + \varepsilon_{i-1,m}|\mathfrak{F}_{i-K,m}) = p_{i-K,m} = p_{i-1,m}$. In general, we have $E(p^*_{i-1,m}|\mathfrak{F}_{i-1,m}) = p_{i-1,m}$,and so $E(p_{i,m}|\mathfrak{F}_{i-1,m}) = p_{i-1,m}$, as required.

Proof of Theorem 2.2: Unless specified, the analysis below is conditioned on the volatility path, $\{\sigma^2(t)\}$. To prove the theorem, we first need to recall the following result on the maximum run time of a Bernoulli process. Let K_i be the maximum time period of flat trading prior to $t_{i,m.}$ It is known (e.g. Schilling, 1990) that the maximum time, K_i, for a sequence of identical Bernoulli draws in a sample of size m has mean $E(K_i) = O\left(\log_{1/\pi}\{m(1-\pi)\}\right) = O\left(\frac{\log\{m(1-\pi)\}}{\log\frac{1}{\pi}}\right)$ and variance $Var(K_i) = \frac{\pi^2}{6\log^2\left(\frac{1}{\pi}\right)}$. It follows that

$$K_i = O_p(\log m). \tag{18}$$

From equation (15), we have

$$\begin{aligned} \sum_{i=1}^{m}[p_{i,m} - p_{i-1,m}]^2 &= \sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2\xi^2_{i,m} \\ &= \sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2 E[\xi^2_{i,m}] + \sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2\left(\xi^2_{i,m} - E[\xi^2_{i,m}]\right) \\ &= \pi\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2 + \sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2\left(\xi^2_{i,m} - E[\xi^2_{i,m}]\right). \end{aligned} \tag{19}$$

Write the sum $\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2$ in the first term above as follows

$$\begin{aligned}
\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^2 &= \sum(p^*_{i,m} - p^*_{i-1,m} + p^*_{i-1,m} - p_{i-1,m})^2 \\
&= RV^{(m)}(p^*) + 2\sum(p^*_{i,m} - p^*_{i-1,m})\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum(p^*_{i-1,m} - p_{i-1,m})^2 \\
&= RV^{(m)}(p^*) + 2\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum(p^*_{i-1,m} - p_{i-2,m})^2(1-\xi_{i-1,m})^2 \\
&= RV^{(m)}(p^*) + 2\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + \sum(p^*_{i-1,m} - p_{i-2,m})^2(1-\pi) \quad (20) \\
&\quad + \sum(p^*_{i-1,m} - p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2 - (1-\pi)\right\}.
\end{aligned}$$

Set $A_m = \sum(p^*_{i,m} - p_{i-1,m})^2$. So $A_{m-1} = A_m - (p^*_{m,m} - p_{m-1,m})^2$. Substituting out A_{m-1} in equation (20) gives

$$\begin{aligned}
A_m &= RV^{(m)}(p^*) + 2\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) + A_m(1-\pi) \\
&\quad -(1-\pi)(p^*_{m,m} - p_{m-1,m})^2 + \sum(p^*_{i-1,m} - p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2 - (1-\pi)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
A_m &= \frac{1}{\pi}RV^{(m)}(p^*) + \frac{2}{\pi}\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) \quad (21) \\
&\quad -\frac{1-\pi}{\pi}(p^*_{m,m} - p_{m-1,m})^2 + \frac{1}{\pi}\sum(p^*_{i-1,m} - p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2 - (1-\pi)\right\}.
\end{aligned}$$

Substituting (21) into (19) we have

$$\begin{aligned}
\sum[p_{i,m} - p_{i-1,m}]^2 &= RV^{(m)}(p^*) + 2\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right) - (1-\pi)(p^*_{m,m} - p_{m-1,m})^2 \\
&\quad + \sum(p^*_{i-1,m} - p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2 - (1-\pi)\right\} \\
&\quad + \sum(p^*_{i,m} - p_{i-1,m})^2\left(\xi^2_{i,m} - \pi\right) \\
&= RV^{(m)}(p^*) + \underbrace{2\sum \varepsilon_{i,m}\left(p^*_{i-1,m} - p_{i-1,m}\right)}_{A}\underbrace{-(1-\pi)(p^*_{m,m} - p_{m-1,m})^2}_{B} \\
&\quad \underbrace{-2\sum(p^*_{i,m} - p_{i-1,m})^2\xi_{i,m}(1-\xi_{i,m})}_{C} \\
&= RV^{(m)}(p^*) + A + B + C. \quad (22)
\end{aligned}$$

Standard quadratic variation theory implies $RV^{(m)}(p^*) \xrightarrow{p} IV$. We now consider the limit behavior of A, B and C.

First, for term C, since $\xi_{i,m}$ is a Bernoulli variable, $\xi_{i,m}\left(1-\xi_{i,m}\right)=0$ $a.s.$, and so $C=0$. Next consider term B. Note that

$$\begin{aligned} p^*_{m,m}-p_{m-1,m} &= p^*_{m,m}-p^*_{m-K_m,m} \text{ for some } K_m=O_p(\log m) \\ &= \int_{t_{m-K_m,m}}^{1}\sigma(s)dB(s) \\ &= O_p\left(\frac{\sqrt{\log m}}{\sqrt{m}}\right), \end{aligned}$$

since

$$E\left\{\int_{t_{m-K_m,m}}^{1}\sigma(s)dB(s)\right\}^2=\int_{t_{m-K_m,m}}^{1}E\left\{\sigma(s)^2\right\}ds=O_p\left(\frac{K_m}{m}\right)=O_p\left(\frac{\log m}{m}\right).$$

Hence

$$B=-(1-\pi)(p^*_{m,m}-p_{m-1,m})^2=-(1-\pi)O_p\left(\frac{\log m}{m}\right)=o_p(1), \tag{23}$$

Finally, for term A, note that

$$\varepsilon_{i,m}=\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)=MN\left(0,\int_{t_{i-1,m}}^{t_{i,m}}\sigma^2(s)ds\right), \tag{24}$$

where MN signifies mixed normal. Next,

$$\begin{aligned} p^*_{i-1,m}-p_{i-1,m} &= (p^*_{i-1,m}-p_{i-2,m})(1-\xi_{i-1,m}) \qquad (25) \\ &= (p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}), \text{ for some } K_{i-1}=O_p(\log m), \end{aligned}$$

and then

$$\sum\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right)=\sum\varepsilon_{i,m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m}).$$

Now $\varepsilon_{i,m}$ is independent of $\xi_{i-1,m}$, and $E(\varepsilon_{i,m})=0$ and $Var(\varepsilon_{i,m})\to 0$, as $m\to\infty$, while $\sum(p^*_{i-1,m}-p^*_{i-K_{i-1},m})^2$ is bounded as $m\to\infty$. It follows that $A=o_p(1)$.

Thus,

$$\begin{aligned} \sum[p_{i,m}-p_{i-1,m}]^2 &= RV^{(m)}(p^*)+o_p(1) \\ &= IV+o_p(1), \end{aligned}$$

giving consistency as stated.

Proof Theorem 2.3: From (22) we have

$$\begin{aligned}\sqrt{m}\left\{\sum[p_{i,m}-p_{i-1,m}]^2-IV\right\} &= \sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}+\sqrt{m}2\sum\varepsilon_{i,m}\left(p^*_{i-1,m}-p_{i-1,m}\right)\\ &\quad -\sqrt{m}(1-\pi)(p^*_{m,m}-p_{m-1,m})^2\\ &\quad +\sqrt{m}\sum(p^*_{i-1,m}-p_{i-2,m})^2\left\{(1-\xi_{i-1,m})^2-(1-\pi)\right\}\\ &\quad +\sqrt{m}\sum_{i=1}^{m}(p^*_{i,m}-p_{i-1,m})^2\left(\xi^2_{i,m}-\pi\right)\\ &= \sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}+\sqrt{m}A+\sqrt{m}B+\sqrt{m}C\\ &= \sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}+\sqrt{m}A+\sqrt{m}B, \qquad (26)\end{aligned}$$

since $C=0$, *a.s.* . From BNS (2002) and Barndorff-Nielson, Graversen, Jacod and Shephard (2006), we have the CLT

$$\sqrt{m}\left\{RV^{(m)}(p^*)-IV\right\}\xrightarrow{d}N\left(0,2\int_0^1\sigma^4(t)dt\right), \qquad (27)$$

stably as $m\to\infty$. We now study the asymptotic behavior of $\sqrt{m}A$, and $\sqrt{m}B$.

For term $\sqrt{m}A$, from (24) and (25) we get

$$\begin{aligned}\sqrt{m}A &= \sqrt{m}2\sum\varepsilon_{i,m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m})\\ &= 2\sqrt{m}\sum\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)(p^*_{i-1,m}-p^*_{i-K_{i-1},m})(1-\xi_{i-1,m})\\ &= 2\sqrt{m}\sum\nu_{i,m}(p^*_{i-1,m}-p^*_{i-K_{i-1},m}),\end{aligned}$$

where $\nu_{i,m}=\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)dB(s)(1-\xi_{i-1,m})$ is uncorrelated with $(p^*_{i-1,m}-p^*_{i-K_{i-1},m})$, because of the martingale property, and has mean 0 and conditional variance $(1-\pi)\,m\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)^2ds$. So $\sqrt{m}A$ is a martingale with conditional variance

$$m\,(1-\pi)\sum(p^*_{i-1,m}-p^*_{i-K_{i-1},m})^2\int_{t_{i-1,m}}^{t_{i,m}}\sigma(s)^2ds.$$

By stochastic Taylor series expansion we have

$$\begin{aligned}p^*_{i-1,m}-p^*_{i-K_{i-1},m} &= \int_{t_{i-K_{i-1},m}}^{t_{i-1,m}}\sigma(s)dB(s)\\ &= \left\{\sigma\left(t_{i-K_{i-1},m}\right)+O_p\left(\sqrt{K_{i-1}h}\right)\right\}\left(B\left(t_{i-1,m}\right)-B\left(t_{i-K_{i-1},m}\right)\right)\\ &= \sigma\left(t_{i-K_{i-1},m}\right)\left(B\left(t_{i-1,m}\right)-B\left(t_{i-K_{i-1},m}\right)\right)+O_p\left(K_{i-1}h\right)\\ &= \sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right)-B\left(\frac{i-K_{i-1}}{m}\right)\right)+O_p\left(\frac{K_{i-1}}{m}\right) \qquad (28)\end{aligned}$$

on the equi-spaced grid $\{t_{i,m} = \frac{i}{m} : i = 0, ..., m\}$ with $h = \frac{1}{m}$. Then

$$\begin{aligned}
& m\sum (p^*_{i-1,m} - p^*_{i-K_{i-1},m})^2 \int_{t_{i-1,m}}^{t_{i,m}} \sigma(s)^2 ds \\
= \; & m\sum \left\{ \sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right) - B\left(\frac{i-K_{i-1}}{m}\right)\right) + O_p\left(\frac{K_{i-1}}{m}\right)\right\}^2 \\
& \times \left\{ \sigma\left(\frac{i-1}{m}\right)^2 + O_p\left(\frac{1}{\sqrt{m}}\right)\right\}\frac{1}{m} \\
= \; & \sum \left\{ \sigma\left(\frac{i-1}{m}\right)^4 \left(B\left(\frac{i-1}{m}\right) - B\left(\frac{i-K_{i-1}}{m}\right)\right)^2 \left[1 + O_p\left(\frac{K_{i-1}}{m}\right)\right]\right\} \\
= \; & \sum \left\{ \sigma\left(\frac{i-1}{m}\right)^4 \frac{K_{i-1}-1}{m} \left[1 + O_p\left(\frac{K_{i-1}}{m}\right)\right]\right\} \\
\to \; & {}_p\left(\int_0^1 \sigma^4(t)dt\right) E(K_{i-1} - 1).
\end{aligned}$$

It follows by the martingale central limit theorem (e.g., theorem 3.2 of Hall and Heyde, 1980) that

$$\sqrt{m}A \xrightarrow{d} 2 \times MN\left(0, (1-\pi)\left(\int_0^1 \sigma^4(t)dt\right) E(K_{i-1} - 1)\right),$$

and the convergence is stable.

Observe that

$$K_{i-1} - 2 = \begin{cases} 0 & \text{with probability } \pi \\ 1 & \text{with probability } \pi(1-\pi) \\ 2 & \text{with probability } \pi(1-\pi)^2 \\ \vdots & \end{cases},$$

so that $E(K_{i-1}-2) = \pi(1-\pi) + 2\pi(1-\pi)^2 + \cdots = \frac{1-\pi}{\pi}$, which implies that $E(K_{i-1}-1) = \frac{1}{\pi}$. Thus

$$\sqrt{m}A \xrightarrow{d} MN\left(0, 4\frac{1-\pi}{\pi}\int_0^1 \sigma^4(t)dt\right), \tag{29}$$

stably.

Next consider term $\sqrt{m}B$. From (23) we have

$$\sqrt{m}B = -\sqrt{m}O_p\left(\frac{\log m}{m}\right)(1-\pi) = o_p(1). \tag{30}$$

Finally, note that the components of the term $\sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\}$ are quadratic in the increments $\varepsilon_{i,m} = p^*_{i,m} - p^*_{i-1,m}$, whereas the components of $\sqrt{m}A$ involve the product $\varepsilon_{i,m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m})(1 - \xi_{i-1,m})$, and

$$E\left\{\varepsilon^3_{i,m}(p^*_{i-1,m} - p^*_{i-K_{i-1},m})(1 - \xi_{i-1,m})\right\} = 0,$$

so the components are uncorrelated. It follows that $\sqrt{m}\left\{RV^{(m)}(p^*) - IV\right\}$ and $\sqrt{m}A$ are asymptotically independent, conditional on $\int_0^1 \sigma^4(t)dt$. Therefore

$$\sqrt{m}\left\{RV^{(m)}(p) - IV\right\} \xrightarrow{d} MN\left(0, \frac{4-2\pi}{\pi}\int_0^1 \sigma^4(t)dt\right),$$

stably, giving the required result.

Proof of Lemma 2.4: From equation (15), we have

$$m\sum_{i=1}^{m}[p_{i,m} - p_{i-1,m}]^4 = \pi m\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^4 + m\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^4\left(\xi^4_{i,m} - E[\xi^4_{i,m}]\right). \tag{31}$$

Consider $\sum_{i=1}^m (p^*_{i,m} - p_{i-1,m})^4$ in the first term,

$$\begin{aligned}
\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^4 &= \sum(p^*_{i,m} - p^*_{i-1,m} + p^*_{i-1,m} - p_{i-1,m})^4 \\
&= \sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + 4\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right) \\
&\quad +6\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2 \\
&\quad +4\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3 + \sum_{i=1}^{m}\left(p^*_{i-1,m} - p_{i-1,m}\right)^4 \\
&= \sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + 4\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right) \\
&\quad +6\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2 \\
&\quad +4\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3 + m\sum_{i=1}^{m}\left(p^*_{i-1,m} - p_{i-2,m}\right)^4(1-\pi) \\
&\quad +\sum_{i=1}^{m}\left(p^*_{i-1,m} - p_{i-2,m}\right)^4\left((1-\xi_{i-1,m})^4 - (1-\pi)\right).
\end{aligned} \tag{32}$$

Set $B_m = \sum (p^*_{i,m} - p_{i-1,m})^4$. So $B_{m-1} = B_m - (p^*_{m,m} - p_{m-1,m})^4$. Substituting out B_{m-1} in equation (32) and solving for B_m, we get

$$\begin{aligned} B_m &= \frac{1}{\pi}\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + \frac{4}{\pi}\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right) \qquad (33)\\ &+\frac{6}{\pi}\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2\\ &+\frac{4}{\pi}\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3\\ &-\frac{1-\pi}{\pi}(p^*_{m,m} - p_{m-1,m})^2 + \frac{1}{\pi}\sum (p^*_{i-1,m} - p_{i-2,m})^4\left\{(1-\xi_{i-1,m})^4 - (1-\pi)\right\}. \end{aligned}$$

Substituting (33) into (31) we have

$$\begin{aligned} m\sum[p_{i,m} - p_{i-1,m}]^4 &= m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + 4m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right)\\ &+6m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2\\ &+4m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3 - (1-\pi)(p^*_{m,m} - p_{m-1,m})^2\\ &+m\sum (p^*_{i-1,m} - p_{i-2,m})^4\left\{(1-\xi_{i-1,m})^4 - (1-\pi)\right\}\\ &+m\sum_{i=1}^{m}(p^*_{i,m} - p_{i-1,m})^4\left(\xi^4_{i,m} - \pi\right)\\ &= m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + \underbrace{4m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right)}_{A}\\ &+\underbrace{6m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2}_{B}\\ &+\underbrace{4m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3}_{C}\underbrace{-(1-\pi)m(p^*_{m,m} - p_{m-1,m})^4}_{D}\\ &+\underbrace{m\sum (p^*_{i,m} - p_{i-1,m})^4(1-\xi_{i,m})^2\xi^2_{i,m}}_{E}\\ &= m\sum_{i=1}^{m}\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 + A + B + C + D + E. \qquad (34) \end{aligned}$$

As in (6) and (28) we have

$$\begin{aligned} p^*_{i,m} - p^*_{i-1,m} &= \sigma\left(t_{i-1,m}\right)\left(B\left(t_{i,m}\right) - B\left(t_{i,m}\right)\right) + O_p\left(\frac{1}{m}\right) \\ &= \sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}} + O_p\left(\frac{1}{m}\right), \end{aligned} \tag{35}$$

where $\epsilon_{i,m}$ is $iid\ N\left(0,1\right)$. Hence

$$\begin{aligned} m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 &= \sum\sigma\left(t_{i-1,m}\right)^4\frac{\epsilon^4_{i,m}}{m} + O_p\left(\frac{1}{m^{1/2}}\right) \\ &= \sum\sigma\left(t_{i-1,m}\right)^4\frac{E\left(\epsilon^4_{i,m}\right)}{m} + O_p\left(\frac{1}{m^{1/2}}\right) \\ &\to_p 3\int_0^1\sigma^4(t)dt. \end{aligned} \tag{36}$$

Hence

$$\frac{2}{3}m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^4 \to_p 2\int_0^1\sigma^4(t)dt. \tag{37}$$

This corresponds with the result obtained in BNS (2002).

We now consider the limit behavior of terms A, B, C, D, and E. First, for term E, since $\left(1-\xi_{i,m}\right)^2\xi^2_{i,m} = 0$ almost surely, $E = 0$. Second, for term D, note that

$$D = -(1-\pi)m(p^*_{m,m} - p_{m-1,m})^4 = -(1-\pi)m \times O_p\left(\frac{\log^2 m}{m^2}\right) = o_p(1).$$

Next, consider term A, viz.,

$$\begin{aligned} & m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-1,m}\right) \\ = \;& m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^3\left(p^*_{i-1,m} - p_{i-2,m}\right)\left(1-\xi_{i-1,m}\right) \\ = \;& m\sum\left\{\sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}} + O_p\left(\frac{1}{m}\right)\right\}^3\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)\left(1-\xi_{i-1,m}\right) \\ & \text{for some } K_{i-1} = O_p(\log m) \\ = \;& \frac{1}{\sqrt{m}}\sum\sigma^3\left(t_{i-1,m}\right)\epsilon^3_{i,m}\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)\left(1-\xi_{i-1,m}\right) + o_p\left(1\right). \end{aligned} \tag{38}$$

The component $\sigma^3\left(t_{i-1,m}\right)\epsilon^3_{i,m}\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)\left(1-\xi_{i-1,m}\right)$ in the sum (38) has mean zero and conditional variance $E\left[\epsilon^6_{i,m}\right]\left(1-\pi\right)\sigma^6\left(t_{i-1,m}\right)\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)^2 = O_p\left(\frac{\log m}{m}\right)$

since from (28)

$$\begin{aligned} p^*_{i-1,m} - p^*_{i-K_{i-1},m} &= \sigma\left(\frac{i-1}{m}\right)\left(B\left(\frac{i-1}{m}\right) - B\left(\frac{i-K_{i-1}}{m}\right)\right) + O_p\left(\frac{K_{i-1}}{m}\right) \\ &= \sigma\left(\frac{i-1}{m}\right)\frac{\eta_{K_{i-1}}}{\sqrt{m}} + O_p\left(\frac{K_{i-1}}{m}\right) = O_p\left(\sqrt{\frac{\log m}{m}}\right), \end{aligned} \tag{39}$$

where

$$\eta_{K_{i-1}} := B\left(\frac{i-1}{m}\right) - B\left(\frac{i-K_{i-1}}{m}\right) = MN\left(0, K_{i-1}-1\right) = O_p\left(\sqrt{\log m}\right).$$

It follows that

$$E\left\{\frac{1}{\sqrt{m}}\sum \sigma^3\left(t_{i-1,m}\right)\epsilon^3_{i,m}\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)\left(1-\xi_{i-1,m}\right)\right\}^2 = O_p\left(\frac{\log m}{m}\right),$$

and so $A = o_p(1)$. Similarly, for term C, $4m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)\left(p^*_{i-1,m} - p_{i-1,m}\right)^3$ is $o_p(1)$.

Next, consider term B. Using (15) and (39), we have

$$\begin{aligned} & m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-1,m}\right)^2 \\ = \;& m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p_{i-2,m}\right)^2\left(1-\xi_{i-1,m}\right)^2 \\ = \;& m\sum\left(p^*_{i,m} - p^*_{i-1,m}\right)^2\left(p^*_{i-1,m} - p^*_{i-K_{i-1},m}\right)^2\left(1-\xi_{i-1,m}\right)^2 \text{ for some } K_{i-1} = O_p(\log m) \\ = \;& m\sum\left\{\sigma\left(t_{i-1,m}\right)\frac{\epsilon_{i,m}}{\sqrt{m}} + O_p\left(\frac{1}{m}\right)\right\}^2\left(\sigma\left(t_{i-1,m}\right)\frac{\eta_{K_{i-1}}}{\sqrt{m}} + O_p\left(\frac{\log m}{m}\right)\right)^2\left(1-\xi_{i,m}\right)^2 \\ = \;& \frac{1}{m}\sum\sigma^2(t_{i,m})\epsilon^2_{i,m}\sigma\left(t_{i-1,m}\right)^2\eta^2_{K_{i-1}}(1-\xi_{i-1,m})^2 + O_p\left(\sqrt{\frac{\log m}{m}}\right) \\ = \;& \frac{1}{m}\sum\sigma^4(t_{i,m})E\left[\epsilon^2_{i,m}\eta^2_{K_{i-1}}(1-\xi_{i-1,m})^2\right] \\ & + \frac{1}{m}\sum\sigma^4(t_{i,m})\left\{\epsilon^2_{i,m}\eta^2_{K_{i-1}}(1-\xi_{i-1,m})^2 - E\left[\epsilon^2_{i,m}\eta^2_{K_{i-1}}(1-\xi_{i-1,m})^2\right]\right\} \\ & + O_p\left(\sqrt{\frac{\log m}{m}}\right) \\ = \;& \frac{1}{m}\sum\sigma^4\left(t_{i,m}\right)(1-\pi)E(K_{i-1}-1) + O_p\left(\frac{\log m}{\sqrt{m}}\right) \\ \to \;& {}_p\left(\int_0^1\sigma^4(t)dt\right)(1-\pi)E(K_{i-1}-1) = \frac{(1-\pi)}{\pi}\int_0^1\sigma^4(t)dt, \end{aligned} \tag{40}$$

since $\epsilon_{i,m}$, $\eta_{K_{i-1}}$, and $\xi_{i-1,m}$ are independent. Therefore,

$$\begin{aligned} m\sum\left[p_{i,m}-p_{i-1,m}\right]^4 &= 3\int_0^1\sigma^4(t)dt+6\frac{(1-\pi)}{\pi}\int_0^1\sigma^4(t)dt+o_p(1) \\ &= \frac{6-3\pi}{\pi}\int_0^1\sigma^4(t)dt+o_p(1), \end{aligned}$$

leading to the required result.

Proof Theorem 2.5: The proof follows directly from Lemma 2.4 and Theorem 2.3.

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