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A numerical model study of internal tides on the Australian Northwest Shelf

by Peter D. Craig

ABSTRACT
Models of internal tides have been difficult to apply to realistic ocean situations because of restrictive assumptions on the form of either the bottom topography or the vertical density structure. A numerical model, based on the analytic model of Craig (1987a), is applied to a section through the Australian Northwest Shelf, and the results compared with data collected at the North Rankin location. The model simulates accurately the amplitude and phase structure of the internal tide. The predicted temperature and current amplitudes are within one-third of those observed, but the phase relationship between the temperature and surface tide is different from that estimated from the data. The regions of strongest internal tidal generation are identified, by estimating numerically the horizontal energy flux, to be in water considerably deeper than the shelf break.

1. Introduction
Internal tides frequently dominate ocean dynamics over the upper continental slope. In thermally stratified tropical waters, the vertical and horizontal excursions associated with internal tides may be sufficient to cause significant nutrient fluxes onto the continental shelf (e.g. Holloway et al., 1985), while the currents can be large enough to be of engineering significance. Bottom intensification of the internal tidal currents can lead, for example, to scour around the base of an offshore structure.

The wavelength of internal tides in shelf and upper slope waters may be only a few tens of kilometers. Rarely will a mooring array be sufficiently intensive and extensive to resolve the structure of the internal tides. There is thus the potential for the use of numerical models, in conjunction with limited data collection, to determine details of the internal tidal generation and propagation that have both biological and engineering application.

Internal tides were first modelled by Rattray (1960) with a two-layered ocean over stepped topography. This model was subsequently extended to include continuous stratification (Rattray et al., 1969) and non-vertical, but strictly supercritical topography (Prinsenberg and Rattray, 1975).

Three further models, by Baines (1973, 1974, 1982), Sandstrom (1976) and Craig...
(1987a), represent a continuously stratified ocean over potentially arbitrary bottom

topography. Each model is essentially a different analytic technique for solving the

same equation. The model by Craig (1987a), in particular, is straightforward in its

application to topography that contains both sub- and supercritical segments. All three

models are, however, limited in their applicability because they are effectively valid

only for density profiles such that the buoyancy frequency is independent of depth,

although Baines (1973) showed that his technique could be applied to a density

gradient which varies as an inverse quartic in the depth.

The equation governing internal tidal dynamics is hyperbolic, and analytic solution

techniques are possible because it can be integrated analytically along characteristics.

If the buoyancy frequency is a general function of depth, the integration must,

however, be undertaken numerically. Chuang and Wang (1981) described a solution

procedure that uses a standard stepping technique on a rectangular grid, but it takes no

account of the properties of the characteristics, nor of the relationship between the

characteristic slope and the bottom slope. It is attractive for its simplicity, but its

accuracy and versatility have not been fully established.

An alternative numerical technique is to integrate the equation along the character-

istics. Craig (1987b) presented a preliminary description of such an approach that is,

in effect, an extension of his analytic model. This extension is described in Section 2 of

the present paper.

Because of the limitations of the models, they have seldom been applied to realistic

coastal situations. Baines (1974) presented a brief application to the Atlantic New

England continental shelf. His model was subsequently applied to the northwest

African coast (Huthnance and Baines, 1982), with disappointing results, which the

authors attributed to the influence of longshore currents and longshore variations in

topography, neither of which is included in the model.

Other applications of model predictions to data sets have been qualitative. Barbee et

al. (1975) and Torgrimson and Hickey (1979) attempted to identify high-energy

beams of the type predicted by the Rattray et al. (1969) model. Other authors (e.g.

Horn and Meincke, 1976; Gordon, 1979; Leaman, 1980 and Holloway, 1985) have

noted bottom intensification of internal tidal energy as predicted by the continuous

stratification, continuous topography models.

The Australian Northwest Shelf is an area of high internal tidal activity. Satellite

images reproduced in Baines (1981) show internal tides over a large area of the shelf.

During summer, the waters are strongly thermally stratified, with a buoyancy

frequency of the order of 0.015 s\(^{-1}\). The area also has a strong semi-diurnal tidal signal,

with the \(M_2 + S_2\) amplitude varying close to linearly from 0.7 m at Point Murat in the

south (approximately 22S latitude) to 2.8 m at Broome in the north (18S latitude)

(Holloway, 1983a).

The Northwest Shelf has been the subject of considerable petroleum industry

attention. Woodside Offshore Petroleum recently established a production platform at
North Rankin, 120 km offshore from Dampier (see Fig. 1). Oceanographic conditions at North Rankin have been monitored intensively by Woodside, who have subsequently provided their data to the scientific community.

Fortuitously for the study of internal tides, North Rankin is located only 20 km seaward of the shelf break. Holloway (1983b, 1984, 1988) analyzed in detail the internal tidal signal in the Woodside data. His observations, summarized in Section 3 of the present paper, are suitable for use in assessing the relevance and accuracy of the internal tidal model.

The results of a model simulation of the Northwest Shelf are presented in Section 4. The model simulates well many features of the internal tide as observed at North Rankin and, through energy flux calculations, provides insight into the nature of the internal tidal generation offshore from North Rankin. The phase relationship between the internal tide and the surface tide is not, however, well predicted. The reason for this and other discrepancies between the model and the data are explored in Section 5.
2. The model

Internal tides may be described by the stream function $\psi$ satisfying

$$\psi_{zz} - \frac{N^2(z)}{\omega^2 - f^2} \psi_{xx} = 0,$$  \hspace{1cm} (2.1)

(e.g. Baines, 1973; Craig, 1987a), in which $z$ and $x$ are the vertical and cross-shore coordinates, and alphabetic subscripts indicate differentiation. The tidal and Coriolis frequencies are $\omega$ and $f$ respectively, and $N$ is the buoyancy frequency, assumed to be a function only of $z$. The motion described by (2.1) is linear, inviscid, hydrostatic, Boussinesq, independent of the longshore coordinate, and has time-dependence $\exp(-i\omega t)$. For the semi-diurnal tide in tropical and mid-latitudes, $\omega$ is greater than $f$ so that (2.1) is a hyperbolic equation having real characteristics.

The solution domain has the form shown in Figure 2 in which the depth profile is a cross-section through the Australian Northwest Shelf (see Fig. 1). The $(x, z)$ origin is defined, as shown, at the deepest offshore point. Figure 2 also shows a typical summer profile of $N$ for the region, and a pair of the corresponding characteristics.

It is convenient, numerically, to work in a new coordinate system $(s, y)$ defined by

$$y = \frac{1}{\alpha} \int_0^z N(z) \, dz,$$

$$s = \frac{(\omega^2 - f^2)^{1/2}}{\alpha} x,$$ \hspace{1cm} (2.2)

where $\alpha$ is a scaling factor chosen, arbitrarily, so that $0 \leq y \leq 1$. In these coordinates, (2.1) becomes

$$\psi_{ss} - \psi_{yy} = D(y) \psi_y,$$ \hspace{1cm} (2.3)

where

$$D(y) = \frac{1}{N} \frac{dN}{dy}.$$ \hspace{1cm} (2.4)

In the transformed domain, the characteristics are straight, with slopes of $\pm 1$, and regions of large $N$ are expanded relative to those of low $N$. This may be seen in Figure 3, which shows the domain of Figure 2 in $(s, y)$ coordinates. Also shown in Figure 3 are a pair of characteristics and plot of the function $D$. The coastline is $s = S$, and the bottom boundary is defined as $y = h(s)$. The bottom slope is assumed to be non-negative, and is zero for $s < s_1$ and $s > s_2$, representing the deep ocean and continental shelf respectively.

The motion is forced by a specified barotropic wave, representing the surface tide. All internal wave energy in the domain is generated in the region of nonzero slope, $s_1 < s < s_2$. On $s < s_1$ the solution has the form

$$\psi = \beta(s, y) + \sum_{n \neq 1} a_n \exp(-ik_n s) \Psi_n(y).$$ \hspace{1cm} (2.5)
The function $\beta(s, y)$ is the barotropic forcing, representing the surface tide and given by

$$\beta(s, y) = -iw\xi_0 \frac{\alpha}{(\omega^2 - f^2)^{1/2}} (s - S) \frac{Z(y) - Z(h)}{Z(1) - Z(h)},$$

(2.6)
in which $z = Z(y)$ is the inverse of the function defined in (2.2), and $\xi_0$ is the surface tidal displacement at the coast. The function $\beta$ is the two-dimensional, inviscid solution given by Battisti and Clarke (1982). The functions $\Psi_n(y)$ in (2.5) are the baroclinic modes (e.g. Gill, 1982) satisfying

$$(N\Psi_{ny})_y - k^2_n N\Psi_n = 0,$$

(2.7)

with

$$\Psi_n(y) = 0 \quad \text{at} \quad y = 0, 1,$$

Figure 2. Bathymetry profile, and a typical summer buoyancy frequency profile and associated characteristics for the section through North Rankin shown in Figure 1.

Figure 3. The bathymetry and characteristics of Figure 2, transformed to $s-y$ coordinates, together with the function $D(y)$. 
and normalized (for later energy flux considerations) so that
\[
\int_0^1 N \Psi_n^2 \, dy = 1.
\]

For positive \( \omega \) and \( k_n \), the negative exponents in (2.5) indicate energy propagation in the negative \( s \) direction (e.g. Baines, 1973), and the \( a_n \)'s are the constant, but unknown, amplitudes of the internal waves propagating into the deep ocean.

For \( s > s_2 \), \( \psi \) has a form similar to (2.5):
\[
\psi = \beta(s, y) + \sum_{n=1} b_n \exp \left( i k_n s \right) \Psi_n(y),
\]
where \( \Psi_n^\prime \) and \( K_n^\prime \) are the shallow water eigensolutions, and the \( b_n \)'s are the amplitudes of the internal waves propagating onto the shelf.

Eqs. (2.5) and (2.8) express the lateral boundary conditions. The bottom boundary condition is the requirement of no normal flow, given by
\[
\psi_s = -h_s \psi_y, \quad \text{on } y = h(s).
\]

The linearized surface boundary condition, consistent with (2.6), is
\[
\psi_s = -\frac{i \omega s_0 \alpha}{(\omega^2 - f^2)^{1/2}}, \quad \text{on } y = 1
\]
(e.g. Craig, 1987a).

Eq. (2.3) is expressed in differential form along the characteristics as
\[
d(\psi_s \pm \psi_y) \pm D(y) \psi_y \, dy = 0, \quad \text{on } \frac{dy}{ds} = \pm 1,
\]
(e.g. Ames, 1977). These equations may be discretized into standard second-order difference form (Ames, 1977) to allow numerical integration on a difference grid defined by characteristics. Figure 4 shows such a grid for the domain of Figure 3.

The numerical integration across the grid, to determine the coefficients \( a_n \) and \( b_m \) is performed as follows. The initial characteristic AB (Fig. 4) is defined with positive slope and point \( A \) lying on \( s < s_1 \). The final characteristic CD has negative slope, with point \( C \) on \( s > s_2 \). The deep- and shallow-water modal representations are then valid on AB and CD respectively. For any mode, \( n \), the values of \( \psi_s \) and \( \psi_y \) are calculated at each discrete point on AB. Using the difference forms of (2.11), the solution is integrated across to the final characteristic CD, yielding values of \( \psi_s \) and \( \psi_y \) at the discrete points on CD. We define \( U_n \) to be the vector whose elements are the values of the normal derivative \( \psi_y - \psi_s \) at each of the interior points on CD. This integration is undertaken for the barotropic mode, yielding vector \( U_0 \), and \( M \) baroclinic modes, for some integer \( M \). The value of the right-hand side in the surface boundary condition (2.10) is kept
nonzero for the integration of the barotropic mode, but is set to zero for the baroclinic integrations.

Vectors $V_n$ are now formed by calculating the values of $\psi_y - \psi$, on CD, using the shelf modes. Continuity of the total stream function at CD then requires that

$$U_0 + \sum_{n=1}^{M} a_n U_n - V_0 + \sum_{n=1}^{N} b_n V_n. \quad (2.12)$$

$N$ is the integer such that $M + N$ is the total number of discrete interior points on AB. If the topography is strictly subcritical, as in Figure 4, then (2.12) constitutes $M + N$ linear algebraic equations in the $M + N$ amplitudes $a_n, n = 1, \ldots, M,$ $b_n, n = 1, \ldots, N$. These equations can be solved for the amplitudes, thus allowing the full solution to be determined.

If the topography is anywhere supercritical, then (2.12) defines less than $M + N$ equations. In this case, additional equations arise from the requirement that the bottom boundary condition (2.9) be satisfied at the supercritical boundary points. This situation is described in more detail in Craig (1987b).

The equation (2.11) and the boundary conditions (2.9) and (2.10) are expressed strictly in terms of the derivatives of $\psi$. There is thus no need to calculate $\psi$ during the
construction of the system (2.12); it is only calculated once the $a_n$ and $b_n$'s have been determined.

In practice, the accuracy of the solution obtained is very sensitive to specification of the characteristic grid. In Figure 4, the grid is constructed simply by initiating negatively sloped characteristics at equal intervals along the starting characteristic AB. It is clear from that figure that the shape of the bottom topography determines the subsequent configuration of the characteristics.

The accuracy of an analytic solution can be assessed by invoking energy conservation (Craig, 1987a). For the numerical solution, however, errors associated with estimation of the vertical energy fluxes at the surface are, in themselves, too large to allow such an assessment. This difficulty may be overcome, in a given case study, by considering first the scattering of a first-mode wave by the topography. The barotropic wave in (2.5) is replaced by an incoming first-mode wave of amplitude 1, to give, on $s < s_1$,

$$
\psi = \exp (ik_1 s) \Psi_1(y) + \sum_{n=1}^{\infty} a_n \exp (-ik_n s) \Psi_n(y).
$$

The barotropic mode in (2.8), and the right-hand side of the surface condition (2.10) are both set to zero, and the solution is calculated numerically. In this case, energy conservation may be expressed analytically as

$$
\sum_{n=1}^{\infty} k_n |a_n|^2 + \sum_{n=1}^{\infty} k'_n |b_n|^2 = k_1,
$$

where the terms on the left-hand side of (2.14) are the (scaled) reflected and transmitted energy fluxes, and the right-hand side is the incident energy flux.

The extent to which the numerically determined coefficients satisfy (2.14) is then an indication of the solution accuracy. This, in turn, provides an indication of the integrity of the solution to the full barotropic-baroclinic problem.

Analytically and numerically integrated solutions for both subcritical and supercritical topography are compared in Craig (1987b). Examples of errors introduced by minor variations in the grid configuration are also presented.

In the present application, the grid was specified in the following manner. If the number of deep water modes was $M$, then $M - 1$ shelf modes were calculated, and $2M$ characteristics were to cross AB. The first $I$ characteristics, where $I \leq 2M$, were initiated at equally spaced points (including the surface point B) on AB. The remaining $2M - I$ were generated at the midpoints of the maximum intervals on CD. The value of $I$ can be adjusted, by experimentation, to minimize the error in (2.14).

3. The observations

Since 1978, Woodside Offshore Petroleum has collected current and temperature data at North Rankin and on the section across the Northwest Shelf shown in Figure 1. In summer the current and, particularly, the temperature signals at North Rankin tend
to be dominated by internal tides. Holloway’s (1983b, 1984, 1985, 1988) analysis of
the internal tides on the Northwest Shelf, with emphasis on the signal at North
Rankin, is briefly summarized here; in the next section, his results are compared with
the model output.

The summer temperature profile on the Northwest Shelf in the vicinity of North
Rankin is consistently close to linear, resulting in a near-constant buoyancy frequency
over the top 120 m. The internal tidal signal is predominantly first modal in structure
at North Rankin, and the flow is predominantly cross-shore.

Holloway (1988) summarized the internal tidal data collected at North Rankin
from 1982 and 1984. In each of the three years, there is a consistent seasonal trend in
the internal tidal signal, but within each year, and from year to year, there is a high
degree of variability. The average January amplitudes provide a reasonable indication
of the summer signal. For this month, the average value of $N^2$ is $2.1 \times 10^{-4}$ s$^{-2}$ and the
average first mode vertical displacement ($M_2 + S_2$) is 17 m. The average for the sum
of the semi-major axes of the $M_2$ and $S_2$ baroclinic current ellipses is 0.17 ms$^{-1}$ at 23 m
depth, and 0.15 ms$^{-1}$ at 120 m. The direction of the semi-major axes is within 30° of
cross-shore.

The vertical displacements are consistently close to in-phase with the surface tide at
all depths in the water column. The phase of the currents, is much less consistent; it is
difficult, from the three-year analysis, to generalize about the phase relationship
between the currents and the surface tide.

However, Holloway (1984) presented details of amplitude and phase calculations
between 11 January and 8 April 1982. The analysis was undertaken for both North
Rankin and the ‘Station 5’ mooring 22 km away, just inshore of the shelf break (see
Fig. 1). Relative to the phase of the $M_2$ surface tide measured at Station 5, the
first-mode vertical displacement at North Rankin had a phase of 56°, while the phase
of the currents at 23 and 83 m was 209° and 20° respectively. Thus, the near-surface
currents are approximately 180° out-of-phase, and the near-bottom currents in-phase
with the vertical displacements, as is required for a shoreward propagating first-mode
wave.

Further, the motion at Station 5 is close to in-phase with that at North Rankin,
indicating that the two stations are one wavelength apart. The upper and lower water
column currents had phases of 203° and 16°. The first-mode vertical displacement
phase was 95°. The displacement phase may, however, be slightly misleading, since the
Station 5 signal is less strongly dominated by the first-mode. The first to second mode
amplitude ratio is 1.6, compared with 8.5 for North Rankin.

The amplitude of the baroclinic signal is dramatically reduced from North Rankin
to Station 5. The first-mode vertical displacement amplitude, for 11 January to 8 April
1982, was 10.2 m at North Rankin and 2.2 m at Station 5. Holloway (1984) estimated
a 98% reduction in onshore energy flux in the 22 km between the stations, which
indicates rapid energy dissipation.
Holloway's (1984) results are all for the \( M_2 \) tide. On the assumption of linearity, the behavior of the \( S_2 \) tide would be expected to be similar. Three tide gauges moored between the coast and the shelf break gave an \( S_2 \) to \( M_2 \) amplitude ratio, for the surface tide, of 0.65 (Holloway, 1983a). The \( S_2 \) to \( M_2 \) ratio for the vertical displacements at North Rankin is, however, highly variable (Holloway, 1988). This variability probably largely reflects the difficulty of harmonic analysis of baroclinic data.

4. Model simulation of Northwest Shelf

The model described in Section 2 was applied to the cross-section through the Northwest Shelf shown in Figure 2. For the purpose of the simulation, the topography has been artificially flattened at 68 m on the shelf, and at 1200 m in the deep ocean. The profile of \( N \) is also shown in Figure 2. Following the discussion in Section 3, the value of \( N \) is constant over the upper 120 m, with \( N^2 \) over this depth equal to \( 2 \times 10^4 \text{ s}^{-2} \). (The transformed domain, on which the baroclinic solution is calculated, is shown in Fig. 3.)

a. Barotropic solution. The amplitude and phase of the \( M_2 \) barotropic tide, calculated using (2.6), are compared in Figure 5 with measurements (from Holloway, 1983a) across the continental shelf. The coastal amplitude, \( \xi_0 \), was set to 1 m, and the velocity amplitude was calculated using the actual, rather than the flattened topography.

It can be seen from Figure 5 that the barotropic model gives a good representation of the phase of both the surface displacement and the horizontal currents. The inshore surface amplitudes and the velocities near the shelf break and upper slope are also well represented, but the nearshore velocities are underestimated and the offshore displacements overestimated.

Holloway (1983a) showed that the discrepancies apparent in Figure 5 between the model and the data could be reduced by introducing longshore pressure gradients. However, the agreement in Figure 5 is considered ample for the present study.

b. Baroclinic solution. The solution for the baroclinic stream function calculated over the Northwest Shelf, using the technique described in Section 2, is shown in Figure 6. The baroclinic stream function is defined as the total stream function minus the barotropic solution (2.6). The solution in Figure 6 was calculated using 14 modes in the deep ocean and 13 over the shelf.

The solution grid was formed as described in Section 2, with 16 equally spaced characteristics generated on the starting characteristic in the deep water, and the remaining 12 initiated over the shelf. With this grid configuration, the energy conservation test, Eq. (2.14), was satisfied to within 0.1%.

The modal coefficients for the solution in Figure 6 are listed in Table 1. Also listed are the cumulative energy fluxes \( C_n \), where \( C_n \) is defined as the energy flux due to all modes up to and including the nth. The cumulative energy fluxes give a good indication
Figure 5. Comparison between measured and model-predicted barotropic $M_2$ tidal surface displacements (a) and currents (b).

Figure 6. The model-predicted $M_2$ baroclinic stream function (a) in-phase and (b) out-of-phase with the surface tide. Arrowheads indicate the circulation direction. The streamline spacing is $1 \text{ m}^2 \text{s}^{-1}$. 
Table I. Modal amplitudes and cumulative energy fluxes $C_n$, in Wm$^{-1}$, for the baroclinic solution of Figure 7.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Deep-water modes</th>
<th>Shelf modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td>a_n</td>
</tr>
<tr>
<td>1</td>
<td>9.57</td>
<td>56.2</td>
</tr>
<tr>
<td>2</td>
<td>1.12</td>
<td>57.8</td>
</tr>
<tr>
<td>3</td>
<td>0.27</td>
<td>58.0</td>
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</tr>
<tr>
<td>5</td>
<td>0.07</td>
<td>58.1</td>
</tr>
<tr>
<td>6</td>
<td>0.14</td>
<td>58.2</td>
</tr>
<tr>
<td>7</td>
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</tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.08</td>
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</tr>
<tr>
<td>11</td>
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<tr>
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<tr>
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<td>58.8</td>
</tr>
<tr>
<td>14</td>
<td>0.03</td>
<td>58.8</td>
</tr>
</tbody>
</table>

of the solution convergence. It is clear from Table 1 that the 14 deep water modes and 13 shelf modes account accurately for the energy of the baroclinic motion.

The motion at the North Rankin location is clearly first modal in structure (see Fig. 6). This can also be seen in Figure 7, which shows the distribution of vertical displacements and horizontal currents over the upper slope region. The first mode-like behavior, with maximum displacements in the central water column, is limited to a region around North Rankin less than 10 km wide. Over the upper slope, the maximum displacements and currents occur along a beam defined by characteristics, as shown in Figure 7a. High velocity shears near the shelf break and over the shelf are clear in both Figures 6 and 7.

Profiles for the displacements and currents at North Rankin, for comparison with observations, are shown in Figure 8. These profiles are for the $M_2$ tide, so that the amplitudes must be multiplied by a factor of 1.65 to give the $M_2 + S_2$ amplitudes described in Section 3.

The predicted maximum $M_2 + S_2$ vertical displacement amplitude is 11 m, (Fig. 8), compared with an average of 17 m from the observations. The horizontal surface and bottom currents are predicted at 0.13 ms$^{-1}$; the observed average is between 0.15 and 0.17 ms$^{-1}$. Consistent with observation, the phase of the vertical displacement is approximately constant over the depth, while the bottom currents are in-phase and the near-surface currents are 180° out of phase with the displacements. Further, as indicated in Figure 6, North Rankin and Station 5 are almost exactly one wavelength apart, so that the motion is in-phase at both locations, as observed.

The phase relationship between the surface tide and the internal tides at North
Figure 7. Distribution of (a) vertical displacement amplitudes and (b) horizontal current amplitudes over the upper slope region calculated from the solution in Figure 6. Characteristics through the largest amplitudes are shown in (a).

Figure 8. Model-predicted internal tidal displacements and current profiles at North Rankin.
Rankin is not quite so well predicted. Analysis of the data from summer, 1982, indicated a phase delay of 56° between the surface and internal displacement. The model predicts a delay of 180°, approximately 120° in error. A phase of 120° corresponds to a distance of one-third of a wavelength, that is, a location error of approximately 8 km. This discrepancy will be discussed in the following section.

The model was run with surface $N^2$ values ranging from $1.6 \times 10^{-4}$ to $2.4 \times 10^{-4}$ s$^{-2}$. Over this range, the profiles at the North Rankin location exhibit only minor variations in magnitude and structure. It was also run with an $N^2$ profile which tended toward zero more rapidly below 400 m, again with no significant change in the results. Further, including topography down to 3000 m depth had little effect at North Rankin, because the stratification in the deep water is so weak.

The model contains no damping, so that all of the energy generated to propagate up the slope must, of course, propagate onto the shelf. This is contrary to observation. However, in Figure 6 it can be seen that, over the upper slope and on the shelf, the baroclinic solution exhibits high shears. The maximum vertical shear in the shoremost profile in Figure 7b is 0.02 s$^{-1}$. Based on this value, the local Richardson number, defined as $|N/u_z|^2$, is estimated at 0.5. Thus, strong damping of the internal tides is expected inshore of North Rankin.

Generation of the internal tides occurs over the whole of the nonzero slope region. It is of interest to use the model to identify the regions of strongest generation. The horizontal energy flux across any vertical section can be numerically estimated. At best, this estimate will be crude, being the numerical integral of a product of two numerically calculated quantities. Figure 9a shows the energy flux so calculated, plotted as a function of position across the topography. The gradient of the energy flux provides an indication of the internal energy production, which is plotted in Figure 9b.

The energy production curve of Figure 9b indicates three regions of strong baroclinic energy generation. Not surprisingly (e.g. Baines, 1982), in two of these regions the bottom slope is changing rapidly. The region of strongest generation is the deepest, at approximately 400 m, where the buoyancy frequency is of the order of $4 \times 10^{-3}$ s$^{-1}$.

The three regions of strong generation lie on a continuous (but reflecting) characteristic (Fig. 9c). It thus appears that the Northwest Shelf may experience its strong internal tidal signal because of a coincidence of stratification and topography. The separation between the two deep shelf-break-type locations in particular is almost exactly a baroclinic wavelength. This equality of topographic and baroclinic length scales is highly conducive to the generation of strong internal tides.

5. Discussion

The vertical structure of the vertical displacements and horizontal currents observed at North Rankin are reproduced well in the model. The model-predicted displacement amplitudes are approximately two-thirds of those observed, while the currents are predicted to within 25%. The phase relationship between the internal tidal currents and
displacements at North Rankin is also accurately predicted by the model, as is the phase relationship between the motion at North Rankin and Station 5. However, the modelled phase-difference between the surface tide and the internal tide differs from that observed by approximately 120°.

The phase discrepancy may, to some extent, be due to difficulties in the data analysis. Harmonic analysis requires that the amplitude and phase of the tidal constituents remain constant over the duration of the time series, which must be at least 15 days to allow separation of the $M_2$ and $S_2$ components. This condition will rarely be fulfilled for internal tides. It may be because of this that there are inconsistencies in the analyzed data: for example, the phase of the vertical displacements is always close to that for the surface tide, the phase of the currents is much more variable than that of the displacements, and the $M_2$ to $S_2$ amplitude ratio is highly variable (Holloway, 1988).

In addition, there are, of course, assumptions in the derivation of the governing
equation (2.1) that will also lead to discrepancies between the predictions and observations. It is informative to examine the influence of some of the physics that has been ignored in the formulation.

The most severe assumption implicit in (2.1) is probably that of two-dimensionality. It is clear from Figure 1 that the topography is not strictly two-dimensional near North Rankin. In particular, there is a very steep topographic feature approximately 60 km to the southwest that is apparent even in the 1000 m contour. There well may be waves propagating toward North Rankin from directions other than that of the chosen cross-section.

It is simple to test the validity of the two-dimensionality assumption. According to the longshore momentum equation, in the absence of longshore pressure gradients, the ratio of the amplitude of the longshore velocity to that of the cross-shore velocity should be $|f/\omega|$. At North Rankin, for the $M_2$ tide, $|f/\omega| = 0.35$. Holloway (1984) found the longshore to cross-shore velocity ratio for the $M_2$ tide to be approximately 0.88. There are obviously significant longshore pressure gradients on the scale of the baroclinic wavelength, which are presumably topographically generated.

Longshore pressure gradients can be included in the model in a heuristic way, as in the barotropic models of Battisti and Clarke (1982) and Holloway (1983a). The topography continues to be regarded as two-dimensional, but the dependent variables are given a wavelike longshore dependence. In this case, the equation for the pressure is hyperbolic and similar to (2.1), and can be solved using the technique described in Section 2. By choosing a wavelength similar to the longshore topographic length scales, the influence of the longshore topography could be assessed, at least conceptually.

Two further influences, not included in Eq. (2.1) but demonstrably present in the data, are nonlinearities and friction. A linear tidal record is sinusoidal in time. Time series of currents and temperature collected at North Rankin frequently exhibit, once per period, a sudden change in magnitude and, for the currents, direction (Holloway, 1987). These sudden changes introduce asymmetry into the time series. They appear similar to hydraulic jumps, but there is little or no local mixing associated with them. Frictional effects have already been mentioned in the context of the strong damping of the shoreward propagating internal tide between North Rankin and the shelf break (Holloway, 1984). Both of these phenomena are obviously beyond the scope of the model described in the present paper. The model will, it is hoped, explain the underlying sinusoidal behavior, and can be used, as was done in Section 4, to infer regions of strong damping results from high predicted shears.

The effects of three-dimensionality, nonlinearity and friction are easily identified in the data and are apparent every tidal cycle. The influence of other phenomena, such as horizontal and temporal variations in $N$, and large-scale longshore and cross-shore current shears, is more difficult to identify. To some extent, comparison with long-term "average" conditions eliminates the necessity to consider these shorter time-scale effects. The larger variations of the internal tidal amplitudes on shorter time scales
seldom seem to correlate with variations in the local buoyancy frequency. They are caused by influences not identifiable in measurements from a single location.

In summary, the solution technique described in the present paper provides a means by which the "traditional" equations of internal tidal generation may be applied to cases of realistic bathymetry and density profiles. The application to the Australian Northwest Shelf does not provide an exhaustive test of the model, because of the lack of horizontal coverage available from only two moorings. However, indications are that the essential physics of the internal tides is well simulated. There is thus some confidence in the model prediction that the strongest generation occurs in water 400 m deep. The next stage in the modelling of the Northwest Shelf, motivated by the Holloway et al. (1985) study, should probably be an examination of the influence, on the internal tidal amplitudes, of the seasonal cycle in the density structure and its implications for the nutrient supply to the shelf. The next stage in the model development should be the inclusion of some form of longshore structure.

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REFERENCES


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