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A least-squares fit of the advective-diffusive equations
to Levitus Atlas data

by Nelson G. Hogg

ABSTRACT

The distributions of temperature, oxygen and potential vorticity on two isopycnals and the
geostrophic shear between them are used to obtain the flow fields and diffusivities through a
least-squares inversion of the steady state, advective-diffusive equations. The source of the data
is the Levitus Atlas, and the two isopycnals are located in a 24° square centered on the
Mediterranean Water tongue in the eastern North Atlantic. One isopycnal is in the thermocline
and the other beneath so chosen to be near the level of maximum salinity and temperature
anomaly.

Although tracer distributions on the two levels are similar, the inversion suggests that they
result from quite different processes. In the thermocline a warm, salty tongue protrudes into the
subtropical gyre and results from downward diffusion of heat and salt in the north and lateral
exchange with Antarctic Intermediate Water in the south. At the true Mediterranean Water
level the flow is down the direction of the tongue and the advective flux is balanced by a lateral
diffusive flux.

The least-squares technique gives estimates for the streamfunction which are quite insensitive
to some of the subjective choices that must be made while the diffusion coefficients are relatively
more sensitive. In particular, in order to overdetermine the problem the cross-isopycnal variation
of the cross-isopycnal diffusivity must be constrained. A method based on the difference in
values between the two levels reduces the residual variance substantially more than one based on

1. Introduction

Property fields in the ocean are established through the combined influences of
advection and diffusion. Descriptive oceanographers have long used the measured
distributions and some intuitive notions of how water masses should “spread” to
develop circulation schemes (i.e. the “core method” introduced by Wüst, 1935). These
ideas are no doubt approximately correct in a large scale sense but are undoubtedly
wrong in detail. For example, tongue-like distributions of tracer are often taken to be
indications of flow in the direction of the tongue (down the gradient). However, the
ocean is thought to be primarily advective, and in the limit of no diffusion flow must be
along the isopleths of the tracer if it is to be conserved.

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One of the most pronounced and best studied tongue-like distributions in the ocean is that ascribed to the outflow of salty, warm water from the Mediterranean. Figure 1 gives the distribution of temperature on two isopycnals for the whole North Atlantic. One isopycinal is centered in the main thermocline ($\sigma_1 = 31.8$) and the other beneath, chosen to be near the level of maximum anomaly ($\sigma_1 = 32.3$). There is a similarity
between the two temperature distributions: both have a maximum to the east which protrudes westward with, perhaps, a slight southward shift of the upper level. On the basis of these distributions alone one might intuit that both resulted from similar advective-diffusive ("spreading") fields.

However, there are other water properties available from the Levitus Atlas which are believed to obey known conservation laws, and these are contoured in Figure 2 for the more limited region denoted "East Block" in Figure 1. The pressures of the two surfaces are given in Figures 2a and 2b, the former showing the bowl of the subtropical gyre, the latter deepening steadily to the south. Temperature is repeated in Figures 2c and 2d. Potential vorticity (Figs. 2e and 2f) can be inferred from the vertical gradient of potential density (multiplied by the Coriolis parameter) assuming that relative vorticity is negligible. Potential vorticity has a different distribution from either oxygen (Figs. 2g and 2h) or temperature and more resembles a blend of the two—a general decrease from north to south with an indication of an extremum protruding from the east in the upper level. Oxygen follows the same conservation law as temperature with the addition of a biological consumption term. The oxygen distribu-
tions in this region bear very little similarity to those of temperature. There is no discernible anomaly associated with the Mediterranean outflow. However, the patterns at the two levels are visually similar—a gradual decrease from north to south with the upper level skewed clockwise relative to the deeper one.

Finally, on the basis of hydrographic data alone the relative motion between the two isopycnals can be calculated from the Montgomery streamfunction (Montgomery, 1937). This distribution (Fig. 2i) closely resembles the pressure of the upper level (Fig. 2a) and indicates the relative anticyclonic flow of the upper level (i.e. the subtropical gyre) with respect to the deeper one of a few mm/sec. There is a strong gradient region in the northwest corner arising from the North Atlantic Current.
The object of this study is to rationalize these various distributions with each other and with the advective-diffusive equations of motion. The method of solution is similar in spirit to the recent inversion of numerical model tracer fields by Fiadeiro and Veronis (1984). Essentially the advective-diffusive equations are written in finite difference form and solved for the velocities and diffusion parameters using the property distributions. A surprise conclusion will be that the temperature distributions that look so similar in Figures 2c and 2d actually result from very different processes: in particular the thermocline temperature anomaly apparently does not originate from the Mediterranean.

2. Formulation

In the absence of diffusion parcels of water preserve their potential density. The advective-diffusive equations are, therefore, transformed using:

\[ x = x' \]
\[ y = y' \]
\[ \sigma = \sigma(x', y', z') \]

where \( x', y' \) and \( z' \) are the cartesian coordinates for east, north and the vertical (primes will be dropped henceforth) while \( \sigma \) is the potential density (here referenced to 1000 dbar). The gradient and divergence operators become:

\[
\text{grad} = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \sigma_z \frac{\partial}{\partial \sigma}
\]
\[
\text{div} = \sigma_z \left( i \frac{\partial}{\partial x} \sigma_z^{-1} + j \frac{\partial}{\partial y} \sigma_z^{-1} + k \frac{\partial}{\partial \sigma} \right)
\]

neglecting small factors with respect to unity in the metric terms for the lateral derivatives. In these coordinates Montgomery (1937) has shown that the geostrophic balance is:

\[ f u = \frac{\partial}{\partial y} \int_0^p \frac{dp}{\rho} - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad f v = - \frac{\partial}{\partial x} \int_0^p \frac{dp}{\rho} + \frac{1}{\rho} \frac{\partial p}{\partial x} \]

(3)

with \( p \) being pressure and \( u \) and \( v \) the east and north velocity components, respectively. With the further assumption that the \textit{in situ} density \( \rho \) is independent of \( x \) and \( y \), the Montgomery streamfunction \( \psi \) can be defined as:

\[ \psi = - \int_0^p \frac{dp}{\rho} + \frac{p}{\rho} = - \int_0^p \frac{1}{\rho} \frac{\partial p}{\partial \rho} d\rho + \frac{p}{\rho} = - \int_0^p \frac{p}{\rho^2} d\rho \]

(4)

so that the geostrophic balance is:

\[ f u = - \frac{\partial \psi}{\partial y}, \quad f v = \frac{\partial \psi}{\partial x} \]

(5)
Note that the *in situ* density will not be constant on a potential density surface, but its variations will be presumed unimportant. As is usual in the dynamic height method, there exists an undetermined constant of integration in the above formulation. Only the difference in streamfunctions between the two levels:

$$\psi_1 - \psi_2 = \int_{\rho_1}^{\rho_2} \frac{dp}{\rho} + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}$$

(6)
can be determined. This is the usual cartesian formulation with a correction for the pressure dependence of the reference surfaces.

The continuity equation is rewritten to isolate the contribution from the planetary divergence by defining nondivergent velocity components, $u_g$ and $v_g$:

$$f_o u_g = fu, \quad f_o v_g = fv$$

(7)
with $f_o$ being a local constant. This gives the planetary vorticity equation:

$$\nabla_h \cdot u_g - \frac{\beta v_g}{f_o} + w^*_\sigma = 0$$

(8)
where $\nabla_h$ is the horizontal part of the divergence operator in Eq. (2) and $w^*_\sigma$ is the cross-isopycnal velocity component which exists only through the action of diffusive processes. This equation resembles the cartesian counterpart, but the along-isopycnal divergence is nonzero (because of the metric term in the divergence operator) and $w^*_\sigma$ arises from diffusive processes. Eq. (8) can be rewritten as:

$$u \cdot \nabla_h (f \sigma_z) = f \sigma_z^2 w^*_\sigma$$

which expresses the manner in which potential vorticity $f \sigma_z$ varies along streamlines through the action of the cross-isopycnal stretching term.

The horizontal divergence term in Eq. (8) can be rewritten using (4) and (5) as follows:

$$\nabla_h \cdot u_g = \sigma_z \frac{\partial u_g}{\partial x} \sigma_z + \sigma_z \frac{\partial v_g}{\partial y} \sigma_z$$

$$= \sigma_z \left[ u_g \frac{\partial}{\partial x} \sigma_z^{-1} + v_g \frac{\partial}{\partial y} \sigma_z^{-1} \right]$$

$$= \sigma_z \left[ \frac{\partial}{\partial \sigma} (u_g \sigma_z + v_g \sigma_z) - u_g z_x + v_g z_y \right].$$

However, Eq. (4) differentiated by $\sigma$ gives the transformed thermal wind equations:

$$f_o u_{\sigma z} = -\psi_{\sigma y} = -\psi_{\sigma \rho}$$

$$= + \frac{p_{\sigma}}{\rho^2} \big|_{\sigma},$$

and $f_o v_{\sigma z} = +\psi_{\sigma x} = -\frac{p_{\sigma}}{\rho^2} \big|_{\sigma}.$
With $p$ and $z$ being approximately equal, the last two terms in the horizontal divergence cancel, and Eq. (8), after an integration with respect to density, becomes:

$$(w_I^* - u_{ig} \cdot \nabla p_1) - (w_2^* - u_{2g} \cdot \nabla p_2) = \frac{\beta}{f} \int_{p_1}^{p_2} v_g \, dp. \quad (9)$$

This relates the difference of the true vertical velocities at the two isopycnals (indicated by subscripts 1 and 2) to the integrated planetary divergence. Using the Montgomery streamfunction (Eqs. 4 and 5) the integral on the RHS takes the form:

$$f_o \int_{p_1}^{p_2} v_g \, dp = f_o \int_{p_1}^{p_2} v_g \frac{\partial p}{\partial \rho} \, dp$$

$$= f_o \left[ pv_g \bigg|_{p_1}^{p_2} - \int_{p_1}^{p_2} v_g \rho \, dp \right]$$

$$= p_2 \psi_{2x} - p_1 \psi_{1x} - \frac{1}{2} \frac{\partial}{\partial x} \int_{p_1}^{p_2} p^2 \, dp^{-1}$$

$$= (p_2 - p_1) \psi_{1x} + \bar{p} (\psi_{2x} - \psi_{1x}) - \frac{1}{2} \frac{\partial}{\partial x} \int_{p_1}^{p_2} (p - \bar{p}) \, dp^{-1}$$

$$= (p_2 - p_1) \frac{\partial}{\partial x} \bar{\psi} + \bar{p} \frac{\partial}{\partial x} \int_{p_1}^{p_2} (p - \bar{p}) \, dp^{-1}$$

$$- \frac{1}{2} \frac{\partial}{\partial x} \int_{p_1}^{p_2} (p^2 - \bar{p}^2) \, dp^{-1} \quad (10)$$

with $\bar{p} = (p_1 + p_2)/2$ and $\bar{\psi} = (\psi_1 + \psi_2)/2$. The dominant part of this expression is the first term on the RHS—a trapezoidal rule approximation to the integrated meridional velocity. The other terms are minor corrections for curvature.

The conservation equation for a water property with concentration, $C$ (i.e. temperature, oxygen or density), is:

$$\nabla_h \cdot (u_g C) - \frac{\beta v_g}{f} C + \sigma_z \frac{\partial}{\partial \sigma} (w^* C) = \nabla_h \cdot A \nabla_h C + \sigma_z \frac{\partial}{\partial \sigma} \left( K_{\sigma_z} \frac{\partial C}{\partial \sigma} \right) - \lambda C \quad (11)$$

where $A$, and $K$ are along- and cross-isopycnal diffusivities and $\lambda$ is the possible biological consumption coefficient (for oxygen only). For potential density this equation assumes the simplified form:

$$w^* = \frac{\partial}{\partial \sigma} (K_{\sigma_z}) \quad (12)$$

as the lateral property gradients vanish.

Some experiments were carried out using different cross-isopycnal diffusivities for salt and heat—no significant differences were found (although errors for these coefficients are substantial) and this approach was abandoned.

Tracers such as temperature and oxygen have a small range of variation around a larger average value. In order to prevent errors in the advection of this large offset from...
dominating the residuals, the planetary vorticity equation (8) is multiplied by the average, $\overline{C}$, and then subtracted from the conservation equation (11) to give an equation identical to (11) except for the tracer anomaly, $C - \overline{C}$.

Eqs. (5) through (12) represent the steady state balances of mass, momentum and property concentrations that are assumed to be satisfied by motions along the two isopycnals. In these equations it is the property fields and their variation that are known and the velocity components (i.e. streamfunction) and diffusion parameters that are to be determined. The streamfunction difference equation (6) and, to a lesser extent, the integrated planetary vorticity equation (9) are the two relations which make the problem inhomogeneous and give rise to the possibility of a unique solution.

The equations are written in centered difference form on the grid shown in Figure 3 by areal integration over the grid elements in the manner of Fiaideiro and Veronis (1984). An example is given in Appendix B. The problem can then be cast in matrix form:

$$ Gp = d + \epsilon. \quad (13) $$

The vector $p$ contains the unknowns and the matrix $G$ the terms multiplying them in the finite difference forms of the equations. The data vector $d$ contains the inhomogeneous terms from the streamfunction and integrated vorticity equations, and the vector $\epsilon$ is the error in the balance equations which is to be reduced to a minimum in a least squares sense through appropriate choice of $p$.

Choosing the domain to be a square $n \times n$ grid, the number of constraints and unknowns are those summarized in Table 1. In general there are more unknowns than equations, a situation which can be dealt with by using the minimum length formulation of inverse theory (e.g. Wunsch, 1978; see also Menke, 1984) where a
Table 1. Number of equations and constraints (grid is $n \times n$, polynomial is order $m$).

<table>
<thead>
<tr>
<th>Model</th>
<th>Constraints</th>
<th>Number of equations</th>
<th>Number of unknowns</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diffusion</td>
<td>$\psi_1(x_i, y_i) = 0$</td>
<td>$7(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$2(n - 1)^2$</td>
<td>overdetermined</td>
</tr>
<tr>
<td>Diffusion on a</td>
<td>$\psi_1(x_i, y_i) = 0$ only</td>
<td>$9(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$10(n - 2)^2 + 2(n - 1)^2 + 2(n^2 - 4)$</td>
<td>underdetermined</td>
</tr>
<tr>
<td>grid</td>
<td>+ Derivatives; $K_p, w_p$</td>
<td>$11(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$10(n - 2)^2 + 2(n - 1)^2 + 2(n^2 - 4)$</td>
<td>underdetermined</td>
</tr>
<tr>
<td></td>
<td>+ Gargett Law</td>
<td>$9(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$6(n - 2)^2 + 2(n - 1)^2 + 2(n^2 - 4) + 1$</td>
<td>underdetermined</td>
</tr>
<tr>
<td>Diffusion as a</td>
<td>$\psi_1(x_i, y_i) = 0$ only</td>
<td>$9(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$2(n - 1)^2 + 6(m + 1)(m + 2)$</td>
<td></td>
</tr>
<tr>
<td>degree $m$</td>
<td>+ Derivatives; $K_p, w_p$</td>
<td>$11(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$2(n - 1)^2 + 6(m + 1)(m + 2)$</td>
<td></td>
</tr>
<tr>
<td>polynomial</td>
<td>+ Gargett Law</td>
<td>$9(n - 2)^2 + (n - 1)^2 + 1$</td>
<td>$2(n - 1)^2 + 4(m + 1)(m + 2) + 1$</td>
<td></td>
</tr>
</tbody>
</table>
solution is determined which is consistent with the constraints and has the smallest (defined in some suitable way) departure from an initial guess. For this study another approach has been taken based on the belief that the ocean is primarily advective. If there is no diffusion at all there are more constraints than unknowns (first line of Table 1). Diffusive processes arise through the action of the oceanic eddy and wave field whose energies vary on the large scale except near the western boundary within the direct influence of the Gulf Stream (Dantzler, 1977; Schmitz et al., 1983). Therefore, the diffusive coefficients have been rewritten as polynomial functions of \( x \) and \( y \) whose order could be varied to determine the effects of inclusion of smaller scales. In particular, the discrete form of the Tchebychev polynomial was chosen as it forms an orthogonal set on a discrete grid. Table 1 gives the number of unknowns with a polynomial of degree \( m \) while Figure 4 shows the boundary between an overdetermined and underdetermined system. Rather arbitrarily an \( 8 \times 8 \) \((n = 8)\) grid has been used in this study and, provided \( m < 6 \), the system is in principle overdetermined.

3. Additional constraints, scaling and solution

Although the artifice of using low degree polynomials to express the diffusion effects can make the problem technically overdetermined, in reality it is not. Firstly, the
stream function has a physically arbitrary reference value. This is specified by setting the value at one of the grid points (generally at the center) and one of the levels (top) equal to zero and removing this as one of the unknowns. That is:

$$\psi_i(x_i, y_j) = 0$$

for some $i, j$.

There is a more subtle form of indeterminacy in the problem. Note that the cross-isopycnal diffusion terms can be expanded and combined with the cross-isopycnal velocity to give the term:

$$\sigma_z \left( w^* - \sigma_z \frac{\partial K}{\partial \sigma} \right) \frac{\partial C}{\partial \sigma}$$

and that the only other equation that contains $w^*$ directly is the integrated vorticity equation (9) in which it occurs as a difference between the two layers. If the two values of $K_s$ (at the two isopycnals) are offset by the same amount and the two values of $w^*$ are compensated a like amount, the equations will not know the difference (the variation in $\sigma_z$ on an isopycnal is small). An additional constraint is required.

Two resolutions of this difficulty have been used. The simplest and least prejudiced is to require that the difference in values of $w^*$ and $K_s$ between the two levels be approximately consistent with the calculated size of their cross-isopycnal derivatives. This will be termed the "Derivative Constrained" solution. Formally:

$$w_1^* - w_2^* \approx \frac{\partial w^*}{\partial \sigma} \cdot (\sigma_1 - \sigma_2)$$

$$K_1 - K_2 \approx \frac{\partial K}{\partial \sigma} (\sigma_1 - \sigma_2).$$

Clearly this is an over-specification, but if just one constraint were used it would be satisfied exactly even though there is some uncertainty in it.

A second approach is to use the conclusions of Gargett (1984) that the vertical diffusivity is inversely proportional to the Brunt-Väisälä frequency. All the other diffusion coefficients are expressed in the discrete polynomial form while $K$ and $K_s$ are constrained by:

$$K_i = \frac{C}{(\sigma_{iz})^{1/2}}, \quad \frac{\partial K_i}{\partial \sigma} = -K_i \frac{\sigma_{izz}}{\sigma_{iz}^2}, \quad i = 1, 2.$$  

The only unknown is the (constant) proportionality factor $C$. This will be called the "Gargett Law" solution.

Finally, it is presumed that all the eddy diffusion processes are dissipative and it is required that they all be positive:

$$A, K, \lambda > 0.$$
In order to reduce the number of constraints, (16) was applied at a subset of the grid points whose number was a function of the polynomial order.

There could be even subtler indeterminacies in the formulation combined with the nature of the data. For instance, the upper and lower level temperature distributions are quite similar and there are regions in the south where several of the fields are parallel suggesting that, locally at least, the problem may not be adequately determined. An objective means of assessing this is to perform a singular value decomposition of the matrix $G$ in Eq. (13) and investigate the distribution of eigenvalues (Fig. 5). Very small eigenvalues and associated eigenfunctions indicate poorly determined parts of the solution. For this problem there is a steady decrease in value with no indication of a transition to a region of vanishingly small values. The condition number of the matrix $G$, the logarithm (to base 10) of the ratio of the largest to smallest eigenvalue, is approximately 2. A usual rule for rejection of small eigenvalues is that the condition number be greater than the number of significant digits in the data—in this case about 3. The so-called Levenberg-Marquardt stabilization technique (see p. 188 in Lawson and Hanson, 1974) was also used to judge whether small eigenvalues contribute significantly to a reduction in residual variance without inordinate increases in parameter variance. For this problem and data set, all eigenvalues contribute in a significant way to the decrease in residual variance. The problem appears to be of full rank, at least in the overdetermined regime.

Equation scaling is another important issue. As they stand, the various equations
apply to different properties as well as different depths and can be expected to have different residual errors. In order that equations with large error do not unnecessarily weight the outcome, it is necessary to apply a scaling to the equations such that the scaled residual will have the same expected variance for all equations. The errors are difficult to quantify (see Olbers et al., 1985 for a discussion). For the tracer conservation equations the units of oxygen and temperature are such that they can be expected to have about the same absolute error at both depths (the values .02°C and .02 ml/l were assumed). The advective fluxes through the sides of the individual boxes are the product of the lateral velocity times the concentration divided by the metric term $\sigma_z$ which will weight the equations proportionately more in the lower layer. Therefore a weighting factor for these conservation equations (including vorticity) was chosen to be $\sigma_z$. Further discussion is given in Appendix B.

The problem now is to obtain a solution to the weighted set of equations (13) augmented by the constraints (14) or (15) and subject to the positivity constraints (16). The solution is determined by minimizing the residual (subject to the constraints) using the methods and programs given in Lawson and Hansen (1974). Because the problem is overdetermined, the extra constraints allow determination of error bounds on the parameters, expressed as the vector $s^2$:

$$s^2 I = \frac{1}{N_{df}} \langle e^2 \rangle (G^T G)^{-1} I,$$

where $I$ is the identity matrix and $\langle e^2 \rangle$ is the expected error determined from the residual imbalances in the conservation equations. In principle the number of degrees of freedom, $N_{df}$, is the difference between the number of constraints and the number of unknowns. In practice, however, the number will be somewhat smaller as not all constraints give useful information. Error distributions displayed here should be taken as lower bounds.

4. Solutions

a. Nondiffusive. This is the starting point to assess the possibility of obtaining balances with reasonable residuals and no need for diffusive terms and their inherent difficulties with parameterization. The least-squares solution for the streamfunctions at the two levels and their standard errors are given in Figure 6a. It is important to note that there has been no a priori prejudice toward forcing an anticyclonic gyre in the thermocline level, but the subtropical gyre is apparent: except for the constraints provided by the conservation equations, the streamfunction difference could have been applied just to the bottom layer, for example. Instead, the upper layer flow accounts for most of the shear including the strong North Atlantic Current in the northwest corner (a recent discussion is given by Krauss, 1986). In the lower layer, flow is toward the west virtually everywhere with an intensification in the west.

The upper layer flow crosses isopleths of both temperature and oxygen and the
residuals of the purely advective balances (right column in Fig. 6b) are above the noise level. Perhaps the best way of seeing this is to observe the large scale structure in the residuals: for example, on the upper level the water must be warmed in the subtropical gyre as it moves southeast but then cooled as it swings to the southwest. Oxygen, decreasing as it does from north to south, continually decreases along streamlines in the upper layer and increases along streamlines in the lower layer.

The expected error in the streamfunction (lower panels of Fig. 6a) is relatively small, amounting to about 10% of the signal.

b. The derivative constrained advective-diffusive solution. In principle, the inclusion of dissipative terms will permit the concentrations to change along streamlines to bring about a balance in the conservation equations. With these terms written as polynomial functions, there is an additional uncertainty with the question of order. The residual variance and the standard error decrease with increasing order (Fig. 7) when all the potential degrees of freedom are used. The difference between the number of constraints and the number of equations is probably an overestimate of the number of degrees of freedom: not all constraints are really independent and not all of them contain much signal. If the standard error is recalculated removing the equivalent of two constraints at two levels and all grid points, there is a minimum standard error for
Figure 6b. Residual imbalances in the various equations for the nondiffusive solution. Contour interval is .005 units. Dashed lines are negative values. The heavier solid line is the zero contour. (n.a. = not applicable to this model.)
Figure 7. Residual standard deviation versus polynomial order for the Derivative Constrained solution. Values for the Gargett Law and a solution in which $A$, $K$ and $\lambda$ were not constrained to be positive are also given for a third order polynomial. The dashed line is for an adjustment of the effective number of degrees of freedom (see text). The order "-1" designation is for the nondiffusive solution.

a polynomial order of 3 (dashed curve, Fig. 7). This minimum is about .005 units, somewhat lower than the crude estimate of ~.01 units given in Appendix B.

The streamfunction for this diffusive solution (Fig. 8a) is remarkably similar to that for the nondiffusive case (Figure 6a). For comparison the separation between streamlines given by the average of the four Armi and Stommel (1983) "Beta triangle" calculations is also shown (581 and 1231 m depths) on Figure 8a. Agreement is very close.

The diffusion coefficients (Fig. 8b) contain appreciable spatial structure from the third order polynomial (negative values arise because the positivity constraint was not applied at all grid points). Magnitudes are not unreasonable although the cross-isopycnal diffusivity in the thermocline seems high in light of the small values generally found there in microstructure studies (e.g. Gregg, 1977). The spatial variations contain some surprises. The lateral diffusivity in the upper layer is maximum in the south while the cross-isopycnal coefficient is a maximum in the northwest. The lower layer lateral diffusivity is distributed closer to one's preconceptions with a general increase toward the northwest. Values of the oxygen consumption on the lower level are not significantly different from zero: here flow is generally toward higher oxygen. Values in the thermocline are similar to those reported by Jenkins (1980) while those at the deeper level in his study would be less than the expected error shown in Figure 8c.
Figure 8a. Streamfunction value and its expected error on the two isopycnals for the Derivative Constrained solution. The 1 mm/sec indicator shows the required separation between streamlines. Two values from the Armi and Stommel (1983) $\beta$ triangle analysis are also shown and are the average of four cruises. Shaded region corresponds to that in Figure 10.

The remaining imbalances (right hand column of Fig. 8c) are all smaller than the contouring level (.005) with the exception of those for the upper layer temperature equation which are slightly higher. The implications of Figure 8c in terms of the property distributions will be discussed in the next section. The residual error in the streamfunction difference between the two layers (not shown) is about .01 m$^2$/sec$^2$—less than that suggested by the error maps in Figure 8a.

c. *The Gargett Law solution.* Forcing the cross-isopycnal diffusivity to be inversely proportional to the square root of the vertical potential density gradient on each surface reduces the number of unknowns for the cross-isopycnal diffusivity to just one (the constant of proportionality) but the problem remains underdetermined without resort to the low order polynomial expansion of the remaining coefficients. For comparison with the results of the previous subsection solutions for an order three polynomial are given (Fig. 9). Even though there are now more degrees of freedom, the standard error of the residuals is higher for this solution (Fig. 7).

The streamfunction (Fig. 9a) is very similar to that of the previous two solutions
although the circulation on both surfaces is somewhat more intense. This is especially noticeable on the lower level where flow speeds are about twice as high although they are still uniformly toward the west. For this case the lateral diffusion coefficients have a somewhat more pleasing distribution (Fig. 9b) with values intensifying toward the northwest. Because variations in $\sigma_z$ are slight, the cross-isopycnal diffusivity is fairly uniform and increases with depth. The value of about $10^{-4}$ m$^2$/s is closer to accepted values in the thermocline. There is little similarity between the patterns of Figures 8b and 9b although the overall amplitudes are similar.

Reflecting the somewhat higher standard error of this solution, the residuals are also higher (Fig. 9c), and the manner in which the advective imbalances are reduced is different.
Figure 8c. Residuals for the Derivative Constrained solution.
Figure 9a. Streamfunction value and its expected error on the two isopycnals for the Gargett Law solution.

5. Discussion

a. Which solution is best? The three solutions to the inverse problem have statistically indistinguishable streamfunctions, but the diffusion parameters are quite different. The nondiffusive model is unsatisfactory as there remains large-scale structure of significant amplitude in the residuals (Fig. 6b). This is reinforced by Figure 7 in which the addition of diffusive terms rapidly reduces the residual standard deviation. Of the two diffusive solutions the residual variance for the Gargett Law solution is higher. The ratio of variances is 3.59 and this is greater than 1.6, the Fisher $F$ test value for significance at the 99 percent level, taking the pessimistic view of the number of degrees of freedom. Discussion of the advective-diffusive balances will be restricted to the Derivative Constrained solution.

b. Balances. Although the property fields at the two levels (Figs. 1 and 2) are similar, the derived streamfunctions (Fig. 8a) are very different. The thermocline flow is dominated by the anticyclonic subtropical gyre while flow on the deeper isopycnal is toward the west and zonal, the rationalization of which is contained in the maps of the various terms in the advective-diffusive balances (Fig. 8c). On the upper surface, temperature must first increase and then decrease as water makes its way around the gyre and through the temperature maximum. This is accomplished by two processes.
Heating in the northwest occurs primarily through strong downward diffusion which is only partially offset through upwelling of colder water from beneath. Cooling in the south results from lateral diffusion with colder, fresher water (presumably Antarctic Intermediate Water) and, although there are large contributions from the cross-isopycnal terms, these cancel one another to give a small resultant cross-isopycnal flux. On the lower isopycnal, water cools everywhere as it moves out of the warm tongue mainly through lateral mixing.

The Gargett Law solution actually behaves in a similar fashion (Fig. 9c) except that the strong cross-isopycnal diffusive flux in the northwest is replaced by a cross-isopycnal advection—most likely a result of the ambiguity between \( w^* \) and \( \partial K / \partial \sigma \) noted in Section 3.

The temperature distributions at the two levels, therefore, result from very different advective-diffusive balances. The upper surface gains a temperature maximum...
Figure 9c. Residuals for the Gargett Law solution.
through downward flux of heat in the north and a lateral loss of heat in the south. This surface shoals greatly to the north, and it is likely that the vertical mixing results from convective processes driven by the large wintertime loss of heat in this region (e.g. see the heat loss charts of Bunker, 1980). This process has been suggested previously by Fuglister (personal communication) and Pollard and Pu (1985). The lower layer conserves heat in a fashion which in consistent with a Mediterranean source.

The oxygen budget for the upper level is similar to the heat budget: oxygen is first gained but then mostly lost as the water makes its way round the gyre, and these changes arise through first vertical and then lateral fluxes. However, consumption is also important particularly in the northwest and east. On the lower surface streamlines and oxygen isopleths are nearly aligned resulting in only a small advective increase along streamlines which is balanced primarily by lateral diffusion (here the contouring interval is too large to show this).

The density equation is straightforward as it is a two-term balance between cross-isopycnal advection and diffusion. This changes sign on the upper surface because the vertical curvature of density does.

There are three terms in the mass continuity equation: the advective divergence of the isopycnal separation field, planetary divergence, and the cross-isopycnal divergence. Over most of the upper surface, rather surprisingly, the planetary contribution can be ignored although it is important near the southern boundary. On the lower surface, terms in this equation are weak and the residual imbalance is as large.

The integrated continuity equation contains five terms, two resulting from the difference in true vertical velocities resulting from flow along sloping isopycnals, two more from the difference in cross-isopycnal velocities and finally the integrated advection of planetary vorticity. Each term is important somewhere. In the northern half of the region the most important terms are the slope of the upper surface, planetary advection and the upward flow across the lower surface. In the southern half downward flow across the upper surface replaces that across the lower one. Flow is mainly along the isobaths of the lower layer resulting in a relatively small contribution almost everywhere except in the eastern central area.

c. A region to the southwest. In order to gain further confidence in the technique, a similar analysis was done on a second region which overlapped the first by about 25%: that is, the upper right corner for the streamfunction of the new region (the "West Block" in Fig. 1) was at the center of the old. The Derivative Constrained solution was used with the same third order polynomial for the diffusion coefficients and the same scaling of the equations. The streamfunction (Fig. 10) once again reveals the subtropical gyre on the upper surface and more or less zonal flow on the lower one. A comparison of the streamfunctions and diffusion coefficients in the overlap region (Fig. 11) shows no significant differences. The streamfunction is quite flat on the upper surface in the southeast corner suggestive of the "shadow zone" predicted by Luyten et al. (1983).
Figure 10. Streamfunction value and its expected error on the two isopycnals for the Derivative Constrained solution in an area to the southwest of that in Figure 8a chosen to overlap it by 25% in area. Overlapping portions are indicated by the shading.

d. Sensitivity to constraints. The solution for the streamfunction seems remarkably insensitive to the parameterization of the diffusive fields, in both the order of the polynomial expansion and the choice of cross-isopycnal dependence. The question naturally arises as to which of the constraints are most important in determining the solution. Because the nondiffusive case is inherently overdetermined, a sensitivity test on this solution was done by down-weighting, in turn, each of the constraints (by a factor of 100). As the streamfunction difference contains the only important inhomogeneous term, this was left alone. The sensitivity to the various constraints was quantified in terms of the average and r.m.s. difference between the resulting lower level solution and the original nondiffusive solution with the results plotted in Figure 12. Neither removal of the temperature or oxygen constraints has an appreciable effect on the solution. However, the vorticity and more importantly the integrated vorticity equations have a substantial influence.

e. Relation to other work. Olbers et al. (1985) have recently performed a similar analysis on the Levitus data set, and inspection of their circulation schemes reveals similarities and differences. They also show this region to be dominated by the
subtropical gyre in the thermocline, but the flow beneath is variable and weak with no hint of a large-scale flow to the west.

There are some fundamental differences between the Olbers et al. approach and the one used here. Theirs was forced to be local in the sense that the property fields were inverted one grid point at a time. This required keeping the velocity components as the

Figure 11a. A comparison of the streamfunction solutions for the upper layer from the overlapping regions of Figures 8a and 10. Dashed lines are for the "East Block."

Figure 11b. A comparison of the diffusion coefficients from the centers of overlapping regions of Figures 8a and 10.
Figure 12. The average and r.m.s. differences between the full solution for the lower level (Fig. 6a) and those obtained by down-weighting the indicated constraints. Only nondiffusive solutions were used.

unknowns in the equations and dropping the constraint that mass be conserved which links grid points. The solutions here demand mass continuity and therefore couple the whole domain in the inversion. Another difference is that Olbers et al. include a term in the vorticity balance for the cross-isopycnal diffusion of relative vorticity. Comparing this with the cross-isopycnal stretching term:

\[
\frac{f \sigma_z w^*}{\sigma_z \frac{\partial}{\partial \sigma} K \sigma_z \frac{\partial}{\partial \sigma} (v_x - u_y)} \approx \frac{f w^* H L}{K U} \approx \frac{10^{-4} \times 10^{-7} \times 10^3 \times 10^6}{10^{-4} \times 10^{-2}} \approx 10^4
\]

shows that it should be small. \(H\) is a density scale height (taken to be \(10^3\) m), \(L\) a gyre scale (\(10^6\) m), \(U\) a velocity scale (\(10^{-2}\) m/sec), \(w^*\) the cross-isopycnal flux (\(10^{-7}\) m/sec) and \(K\) the cross-isopycnal diffusivity (\(10^{-4}\) m\(^2\)/s). Finally the Olbers et al. work deals only with heat, salt and vorticity and ignores possible constraints from the oxygen distribution.

The first quantitative attempt to fit an advective-diffusive model to the Mediterranean salt plume is that of Needler and Heath (1975). Assuming uniform zonal advection and eddy diffusion (different in the horizontal and vertical), they fit solutions to the observed rate of spread of the salinity anomaly. Being a homogeneous mathematical problem only the ratios of diffusion coefficients to lateral advection
could be determined. Judging by Figure 8a the uniform zonal advection is not a bad approximation. Taking this to be 1 mm/sec Needler and Heath would predict values for $A$ of $5 - 15 \times 10^2$ m$^2$/sec and $K$ of $1.5 - 6 \times 10^{-5}$ m$^2$/sec. From Figure 8b both are similar to the estimates obtained herein.

6. Conclusions

A least-squares inversion of tracer properties on two isopycnals in the eastern North Atlantic has been found to give streamfunction and diffusion fields similar to preconceived notions. Although no a priori bias was imposed, the thermocline circulation is a subtropical gyre and a "North Atlantic Current" while that at the Mediterranean level is weak, quasi-uniform, westward motion. Diffusion processes were found necessary to bring the residual balances down to acceptably small values. Interpretation of the diffusive fluxes supports the ideas of Fuglister and Pollard and Pu (1985) that the similar temperature tongues on the two levels arise through very different advective-diffusive processes.

The advective-diffusive problem is not fully constrained in the sense that cross-isopycnal advection ($w^*$) cannot be easily distinguished from the cross-isopycnal derivative of diffusion ($K_w$). An attempt to resolve this ambiguity using the Gargett (1984) prescription for the dependence of $K$ on the Brunt-Väisälä frequency was significantly less successful than that based on making the differences in values of $w^*$ and $K$ match the averages of their derivatives at the two levels times the density difference.

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APPENDIX A

Interpolation and smoothing

The Levitus Atlas data set is available on a $1^\circ$ grid although the effective lateral smoothing is on the order of 700–1000 km. For the purposes of this analysis, values of temperature, oxygen and pressure were required on potential density surfaces. These were obtained by using a cubic spline to interpolate each to the required density. The same spline when differentiated gave values of the first and second derivatives of temperature and oxygen with respect to potential density. In order to reduce interpolation noise and because the data is already effectively filtered on scales of 700–1000 km, the resulting values were then smoothed over six degrees using a two-dimensional Gaussian filter and then subsampled at three degree intervals. The horizontal Laplacians were computed as finite differences on this subsampled grid.
Finite differences, error estimation

The finite difference form of the steady state equations was determined by balancing the fluxes through the surfaces of the dashed box in Figure 3. For Eq. (11) for the tracer concentration \( C \) this gives:

\[
\frac{1}{f_j} \left\{ \left[ C''_{i,j+1} - C''_{i+1,j} \right] \psi_{i,j} + \left[ C''_{i+1,j} - C''_{i,j-1} \right] \psi_{i,j-1} + \left[ C''_{i,j-1} - C''_{i-1,j} \right] \psi_{i-1,j-1} \right. \\
+ \left[ C''_{i-1,j} - C''_{i,j+1} \right] \psi_{i-1,j+1} \right\} - \frac{1}{2} \beta \frac{L_y}{f_j^2} C''_{i,j} \cdot \left( \psi_{i,j} + \psi_{i,j-1} - \psi_{i-1,j} - \psi_{i-1,j-1} \right) \\
+ L_x L_y w^*_{i,j} C'_{i,j} = \frac{L_y}{2L_x} \left\{ \left[ A_{i+1,j} + A_{i,j} \right] \left[ C''_{i+1,j} - C''_{i,j} \right] - \left[ A_{i-1,j} + A_{i,j} \right] \left[ C''_{i,j} - C''_{i-1,j} \right] \right. \\
+ \left. \left[ A_{i,j+1} + A_{i,j} \right] \left[ C''_{i,j+1} - C''_{i,j} \right] - \left[ A_{i,j} + A_{i,j-1} \right] \left[ C''_{i,j} - C''_{i,j-1} \right] \right\} \\
+ L_x L_y K_{i,j} \frac{\partial}{\partial \sigma} \left( \sigma \frac{\partial C'_{i,j}}{\partial \sigma} \right) + L_x L_y K_{i,j} \sigma \frac{\partial C'_{i,j}}{\partial \sigma} - L_x L_y \lambda_{i,j} C''_{i,j}
\]

where \( C''_{i,j} = \frac{C'_{i,j}}{\sigma} \),

\( C'_{i,j} \) is the tracer concentration anomaly, and \( f_j \) is the Coriolis parameter evaluated at the center of the box. The other equations are treated similarly.

The errors in this equation are difficult to quantify because of the averaging scheme and variable number of samples per grid point associated with the Levitus data set. However, the metric term \( \sigma^{-1} \) introduces a scaling which emphasizes the equations from levels of weaker stability, probably the reverse of what should be true, and to remove this bias the concentration equations were scaled by \( \sigma_{i,j} \). The form of the density equation (12) has removed this scaling effect and no further adjustment was made.

Typical concentration differences are of order 0.2 units (°C or ml/l) while the streamfunction has an amplitude of order 0.1 m²/sec. If a 10% error in each of these quantities is assumed and neighboring grid points are uncorrelated, then the expected error from just the advective terms is of order 0.006 units m²/sec. The planetary advection and diffusive terms are likely to contribute an equal amount.

The vorticity equation (8) has fewer terms than the tracer conservation equation and the error could be expected to be lower. It was scaled by an additional factor of 2.

The integrated potential vorticity equation (9) contains advective terms representing the vertical velocity from flow along isopycnals. These are similar in amplitude to the scaled tracer advective terms. Consequently, this equation was not scaled.

The final equation is that for the streamfunction difference (Eq. 6). This is the only really inhomogeneous equation (the inhomogeneous part of (9) is very small) and an
attempt was made, through scaling, to keep the r.m.s. error at about 0.02 m$^2$/sec—a rough estimate of the error. For the order 3 polynomial, this meant a scaling factor of about 0.25.

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