9-1-2005

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September 2005

COWLES FOUNDATION DISCUSSION PAPER NO. 1535

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CONTINUOUS VERSUS DISCRETE MARKET GAMES

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Abstract. De Meyer and Moussa Saley [4] provide an endogenous justification for the appearance of Brownian Motion in Finance by modeling the strategic interaction between two asymmetrically informed market makers with a zero-sum repeated game with one-sided information. The crucial point of this justification is the appearance of the normal distribution in the asymptotic behavior of $\frac{V_n(P)}{\sqrt{n}}$. In De Meyer and Moussa Saley's model [4], agents can fix a price in a continuous space. In the real world however, the market compels the agents to post prices in a discrete set. The previous remark raises the following question: Does the normal density still appear in the asymptotic of $\frac{V_n}{\sqrt{n}}$ for the discrete market game? The main topic of this paper is to prove that for all discretization of the price set, $\frac{V_n(P)}{\sqrt{n}}$ converges uniformly to 0. Despite of this fact, we do not reject De Meyer, Moussa analysis: when the size of the discretization step is small as compared to $n^{-\frac{1}{2}}$, the continuous market game is a good approximation of the discrete one.

1991 *Mathematics Subject Classification.* 91A20, 91B26.

Key words and phrases. Insider trading, game of incomplete information, Brownian Motion.

1. Introduction

Financial models of the price dynamic on the stock market often incorporate a Brownian term (see for instance Black and Scholes [3]). This Brownian term is often explained exogenously in the literature: the price of an asset depends on a very long list of parameters which are subject to infinitesimal random variations with time (as for instance the demographic parameters). Due to an aggregation result in the spirit of the Central Limit theorem, these variations are responsible for the Brownian term in the price dynamic. However, this kind of explanation does not apply to discontinuous parameters that are quite frequent in the real world. For instance, the technological index of a firm will typically jump whenever a new production process is discovered. With the above exogenous explanation, such a discontinuity of the parameter process (a shock) would automatically generate a discontinuity of the price process. In [4], De Meyer and Moussa Saley provide an endogenous justification for the appearance of the Brownian term even in case of discontinuous parameters. They also explain how the market will preserve the continuity of the price process. Their explanation is based on the informational asymmetries on the market. When such a shock happens, some agent are informed and others are not. At each transaction, the optimal behavior of the informed agents will be a compromise between an intensive use of his information at that period and a constant concern of preserving his informational advantage for the next periods. To obtain this compromise, the insiders will slightly noise their actions day after day and asymptotically these noises will aggregate in a Brownian Motion.
To support this thesis, De Meyer and Moussa Saley analyze the interaction between two asymmetrically informed market makers : Two market makers, player 1 and 2, are trading two commodities N and R. Commodity N is used as numéraire and has a final value of 1. Commodities R(R for risky asset) has a final value depending on the state k of nature $k \in K := \{L, H\}$. The final value of commodity R is 0 in state L and 1 in state H. By final value of an asset, we mean its liquidation price at a fixed horizon T, when the state of nature will be publicly known.
The state of nature k is initially chosen at random once for all, the probability of H and L being respectively P and $1 - P$. Both players are aware of this probability. Player 1 is informed of the resulting state k while player 2 is not.

The transactions between the players, up to date T, take place during n consecutive rounds. At round q ($q = 1, \ldots, n$), player 1 and 2 propose simultaneously a price $p_{1,q} \in D$ and $p_{2,q} \in D$ for 1 unit of commodity R ($D \subset \mathbb{R}$). The maximal bid wins and one unit of commodity R is transacted at this price. If both bids are equal, no transaction happens.

In other words, if $y_q = (y_q^R, y_q^N)$ denotes player 1's portfolio after round q, we have $y_q = y_{q-1} + t(p_{1,q}, p_{2,q})$, with

$$t(p_{1,q}, p_{2,q}) := 1\!\!1_{p_{1,q} > p_{2,q}}(1, -p_{1,q}) + 1\!\!1_{p_{1,q} < p_{2,q}}(-1, p_{2,q}).$$

The function $1\!\!1_{p_{1,q}>p_{2,q}}$ takes the value 1 if $p_{1,q} > p_{2,q}$ and 0 otherwise.

At each round the players are supposed to remind the previous bids including those of their opponent. The final value of player 1's portfolio y_n is then $1\!\!1_{k=H} y_R + y_N$. We consider the players are risk neutral, so that the utility of the players is the expectation of the final value of their own final portfolio. There is no loss of generality to assume that initial portfolios are $(0,0)$ for both players. With that assumption, the game $G_n^D(P)$ thus described is a zero-sum repeated game with one-sided information as introduced by Aumann and Maschler [1].

As indicated above, the informed player will introduce a noise on his actions. Therefore, the notion of strategy we have in mind here is that of behavior strategy. More precisely, a strategy σ of player 1 in $G_n^D(P)$ is a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$, where σ_q is the lottery on D used by player 1 at stage q to selects his price $p_{1,q}$. This lottery will depend on player 1's information at that stage which includes the state as well as both player's past moves. Therefore σ_q is a (measurable) mapping from $\{H, L\} \times D^{q-1}$ to the set $\Delta(D)$ of probabilities on D. In the same way, a strategy τ of player 2 is a sequence $(\tau_1, \ldots, \tau_n)$ such that $\tau_q : D^{q-1} \to \Delta(D)$.

A pair of strategies (σ, τ) joint to P induces a unique probability $\Pi_{P,\sigma,\tau}$ on the histories $k \in \{H, L\}, p_{1,1}, p_{2,1}, \ldots, p_{1,n}, p_{2,n}$. The payoff $g(P, \sigma, \tau)$ in $G_n^D(P)$ corresponding to the pair of strategy (σ, τ) is then $E_{\Pi_{P,\sigma,\tau}}[1\!\!1_{k=H} y_R + y_N]$.
The maximal amount player 1 can guarantee in $G_n^D(P)$ is

$$\underline{V}_n^D(P) := \sup_\sigma \inf_\tau g(P, \sigma, \tau)$$

and the minimal amount player 2 can guarantee not to pay more is $\overline{V}_n^D(P) := \inf_\tau \sup_\sigma g(P, \sigma, \tau)$. If both quantities coincide the game is said to have a value. A strategy σ (resp. τ) such that $\underline{V}_n^D(P) = \inf_\tau g(P, \sigma, \tau)$ (resp. $\overline{V}_n^D(P) := \sup_\sigma g(P, \sigma, \tau)$ is said to be optimal.

Before dealing with the main topic of this paper, let us discuss the economical interpretation of this model. A first observation concerns the fact that the model is a zero sum game with positive value: This means in particular that the uninformed market maker will lose money in this game, so, why should he take part to this game? To answer this objection, we argue that, once an institutional has agreed to be a market maker, he is committed to do so. The only possibility for him not to participate to the market would be by posting prices with a huge bid-ask spread. However, there are rules on the market that limit drastically the allowed spreads. In this model the spread is considered as null since the unique price posted by a player is both a bid and an ask price. The above model has to be considered as the game between two agents that already have signed as Market Makers, one of which receives after this some private information.

The second remark we would like to do here is on the transaction rule: The price posted by a Market Maker commits him only for a limited amount: when a bigger number of shares is traded, the transaction happens at a negotiated price which is

not the publicly posted price. We suppose in this model that the price posted by a Market Maker only commits him for one share.
Now, if two market makers post a prices that are different, say $p_1 > p_2$, there will clearly be a trader that will take advantage of the situation: The trader will buy the maximal amount (one share) at the lowest price (p_2) and sell it to the other market maker at price p_1. So, if $p_1 > p_2$, one share of the risky asset goes from market maker 1 to market maker 2, and this is indeed what happens in the above model.
The above remark also entails that each market maker trades the share at his own price in numéraire. This is not taken into account in De Meyer Moussa Saley model, since the transaction happens there for both market makers at the maximal price. Introducing this in the model would make the analysis much more difficult: the game would not be zero sum any more, and all the duality techniques used in [4] would not apply. The analysis of a model with non zero sum transaction rules goes beyond the scope of this paper, but will hopefully be the subject of a forthcoming publication.

De Meyer- Moussa Saley were dealing with the particular case $D = [0, 1]$ and the corresponding game will be denoted here $G_n^c(P)$ (c for continuous) and their main results, including the appearance of the Brownian motion, are reminded in the next section.

It is assumed in G_n^c that the prices posted by the market makers are any real numbers in [0,1]. In the real world however, market makers are committed to use only a limited numbers of digits, typically four. In this paper, we are concerned with the same model but under the additional requirement that the prices belong to some discrete set: we will also consider the discretized game $G_n^l(P) := G_n^{D_l}(P)$ where $D_l := \{\frac{i}{l-1}, i = 0, \ldots, l-1\}$. The main topic of this paper is the analysis of the effects of this discretization.

As we will see, the discretized game is quite different from the continuous one: It is much more costly to noise his prices for the informed agent in G_n^l than in G_n^c: he must use lotteries on prices that differ at least by the tick $\delta := \frac{1}{l-1}$ while in G_n^c, the optimal strategies are lotteries whose support is asymptotically very small (and thus smaller than δ).

The question we address in this paper is the following: As $n \to \infty$, does the Brownian motion appear in the asymptotic dynamics of the price process for the discretized game?
As we will see in section 3, the answer is negative. At first sight, this result questions the validity of De Meyer, Moussa's analysis.
We compare therefore in section 5 the discrete game with the continuous one. In particular, we show that the continuous model remains a good approximation of the discrete one, as far as $\sqrt{n}\delta$ is small, where δ is the discretization step and n is the number of transactions. When this is the case, we prove that discretizing the optimal strategies of the continuous game provides good strategies for G_n^l. The fact that $\sqrt{n}\delta$ is small in general explains why the analysis made in [4] remains valid.

2. Reminder on the continuous game G_n^c

De Meyer, Moussa Saley prove in [4] that the game $G_n^c(P)$ has a value $V_n^c(P)$. Furthermore, they provide explicit optimal strategies for both players.

The keystone of their analysis is the recursive structure of the game, and a new parametrization of the first stage strategy spaces. Namely, at the first stage, player 1 selects a lottery σ_1 on the first price p_1 he will post, lottery depending on his information $k \in \{H, L\}$. In fact, his strategy may be viewed as a probability distributions π on (k, p_1) satisfying: $\pi[k = H] = P$.
In turn, such a probability π may be represented as a pair of functions (f, Q) ($[0,1] \to [0,1]$) satisfying :

$$\begin{array}{ll} (1) & f \text{ is increasing} \\ (2) & \int_0^1 Q(u)du = P \\ (3) & \forall x, y \in [0,1] : f(x) = f(y) \Rightarrow Q(x) = Q(y) \end{array} \tag{2.1}$$

The set of these pairs will be denoted by $\Gamma_1^c(P)$ in the sequel.
Given such a pair (f, Q), player 1 generates the probability π as follows: he first selects a random number u uniformly distributed on $[0,1]$, he plays then $p_1 := f(u)$ and he then chooses $k \in K$ at random with a lottery such that $p[k = H] = Q(u)$.

In the same way, the first stage strategy of player 2 is a probability distribution for $p_2 \in [0,1]$. To pick p_2 at random, player 2 may proceed as follows: given a increasing function h :$[0,1] \to [0,1]$, he selects a random number u uniformly distributed on $[0,1]$ and he plays $p_2 = h(u)$. Any distribution can be generated in this way and therefore we may identify the strategy space of player 2 with set Γ_2^c of these functions h.

Based on that representation of player 1 first stage strategies, the recursive formula for V_n^c becomes:

Theorem 2.1. *[The primal recursive formula]*

$$V_{n+1}^c = T^c(V_n^c),$$

where

$$T^c(g)(P) = sup_{(f,Q)\in\Gamma_1^c(P)} inf_{p_2\in[0,1]} F((f,Q), p_2, g),$$

with

$$F((f,Q), p_2, g) := \int_0^1 \{1\!\!1_{f(u)>p_2}(Q(u) - f(u)) + 1\!\!1_{f(u)<p_2}(p_2 - Q(u))\}du + \int_0^1 g(Q(u))du$$

A first move optimal strategy σ_1 in $G_{n+1}^c(P)$ for player 1 corresponds to a pair (f°, Q°) which verifies:

$$V_{n+1}^c(P) = inf_{p_2\in[0,1]} F((f^\circ, Q^\circ), p_2, V_n^c).$$

After the first stage, player 1 plays optimally in $G_n^c(Q(u))$.

Another useful tool in De Meyer, Moussa Saley analysis is Fenchel duality: it is quite natural to use it in this framework since V_n^c is proved to be concave.

Definition 2.2. the Fenchel conjugate f^* (or simply conjugate) of f is defined as follows: $f^* : \mathbb{R} \rightarrow [-\infty, +\infty)$ such that :

$$f^*(x) = inf_{P\in[0,1]} xP - f(P)$$

From this definition, it is obvious that:

$$\text{If } f \leq g \text{ then } g^* \leq f^* \tag{2.2}$$

The Fenchel conjugate $W_n^c := (V_n^c)^*$ of V_n^c may be interpreted as the value of a dual game. The recursive structure of this dual game is particularly well suited to analyze the optimal strategies of player 2.

Theorem 2.3. *[The dual recursive formula] For all $x \in \mathbb{R}$:*

$$W_{n+1}^c(x) = \Lambda^c(W_n^c)(x),$$

where $\Lambda^c(g)(x) = sup_{h\in\Gamma_2^c} inf_{p_1\in[0,1]} R[x](p_1, h, g)$,
with

$$R[x](p_1, h, g) := g(x - \int_0^1 1\!\!1_{h(u)<p_1} - 1\!\!1_{h(u)>p_1} du) - \int_0^1 1\!\!1_{h(u)<p_1}(-p_1) + 1\!\!1_{h(u)>p_1} h(u) du$$

An optimal strategy for player 2 is a function h° which verifies :

$$W_{n+1}^c(x) = inf_{p_1\in[0,1]} R[x](p_1, h^\circ, W_n^c)$$

The following Formulas, corresponding to the formula (6) and (8) in [4], provide explicit optimal strategies for player 1 in $G_n^c(P)$. For all $u \in [0, 1]$

$$\begin{array}{rcl} u^2 f(u) & = & \int_0^u 2sQ(s)ds \\ Q(u) & = & (W_n^c)'(\lambda + 1 - 2u) \end{array}$$

where $(W_n^c)'$ is the derivative of the function W_n^c and λ is such that the expectation of Q is equal to P. The following explicit expression for optimal h^* is given in formula (20) in [4]: for all $u \in [0, 1]$

$$h(u) = u^{-2} \int_0^u 2s(W_n^c)'(x - 2s + 1)ds$$

The main result of [4] is the appearance of Brownian Motion in the asymptotic dynamic of the price process in $G_n^c(P)$ as n goes to infinity: Since optimal strategy of players are explicitly known, we may compute the distribution of the proposed price process of player 1 $(p_{1,1}^n, \dots, p_{1,n}^n)$ in $G_n^c(P)$. This process p_1^n may be viewed as a continuous time process Π^n on $[0, 1]$ with $\Pi_t^n = p_{1,q}^n \; if \; \frac{q}{n} \leq t < \frac{q+1}{n}$.
With the previous notation, De Meyer and Moussa Saley (see [4]) prove the following asymptotic result:

Theorem 2.4. *As n goes to ∞ , the process Π_t^n converges in law, in the sense of finite distributions, to the following process Π:*

$$\Pi_t = F(\frac{z_p + B_t}{\sqrt{1-t}})$$

Where $F(x) = \int_x^\infty f(z)dz$, z_p *is such that* $F(z_p) = p$ *and* B_t *is a Brownian Motion. The process* Π *is a* $[0,1]$*-valued continuous martingale starting at* P *at time* 0*. Furthermore* Π_1 *belongs almost surely to* $\{0,1\}$*.*

This result is in fact related to the following one:

Theorem 2.5. *Let* f *the normal density:* $f(z) := exp(-\frac{z^2}{2})/\sqrt{2\pi}$*.*
As n *goes to* ∞*,* $\Psi^c_n(P) := \frac{V^c_n}{\sqrt{n}}(P)$ *converges to* $\frac{1}{\sqrt{3}} f(z_P)$*, where* z_P *is such that* $P = \int_{z_P}^\infty f(s)ds$*.*

In the next section, we prove that the value $V^l_n(P)$ of the discretized game doesn't have the same asymptotic as $V^c_n(P)$. There is therefore no hope for the appearance of a Brownian Motion in the dynamic of the discretized price process. This phenomena could heuristically be explained as follows.

From theorem 15 and lemma 9 in [4], there exists a constant C such that for all n, m, with $m < n$:

$$|p^n_{1,m} - p^n_{2,m}| \leq C/\sqrt{n-m},$$

where $p^n_{i,m}$ is the price posted by player i in the m'th stage of $G^c_n(P)$.

Therefore, once $C/\sqrt{n-m}$ is less than the discretization step $\frac{1}{l-1}$ the players should post the same price. Due to the transaction rules, this means a zero payoff for both players in the beginning of the game. This will be true as far as $m \leq n - ((l-1)C)^2$, so only $((l-1)C)^2$ transactions could give a positive payoff (smaller than 1) to player 1: the value of the discrete market game would be bounded by $((l-1)C)^2$. This is the content of theorem 3.1.

3. The discretized game G^l_n

In this section we are concerned with the game $G^l_n := G^{D_l}_n$
where $D_l := \{\frac{i}{l-1}, i = 0, \ldots, l-1\}$.

This game is in fact a standard repeated game as introduced in Aumann Mashler with D_l as action set and with A^H, A^L as payoff matrices:

$$A^H = \begin{pmatrix} 0 & \delta-1 & ... & i\delta-1 & (i+1)\delta-1 & ... & 0 \\ 1-\delta & 0 & ... & ... & ... & ... & 0 \\ ... & ... & 0 & i\delta-1 & ... & ... & ... \\ 1-i\delta & ... & 1-i\delta & 0 & (i+1)\delta-1 & ... & ... \\ 1-(i+1)\delta & ... & ... & 1-(i+1)\delta & 0 & ... & ... \\ ... & ... & ... & ... & ... & 0 & 0 \\ 0 & ... & ... & ... & ... & 0 & 0 \end{pmatrix}$$

and :

$$A^L = \begin{pmatrix} 0 & \delta & ... & i\delta & (i+1)\delta & ... & 1 \\ -\delta & 0 & ... & ... & ... & ... & 1 \\ ... & ... & 0 & i\delta & ... & ... & ... \\ -i\delta & ... & -i\delta & 0 & (i+1)\delta & ... & ... \\ -(i+1)\delta & ... & ... & -(i+1)\delta & 0 & ... & ... \\ ... & ... & ... & ... & ... & 0 & 1 \\ -1 & ... & ... & ... & ... & -1 & 0 \end{pmatrix}$$

(Line i corresponds to price $p_1 = i\delta$ with $\delta = \frac{1}{l-1}$, and similarly for column j.)

From Aumann and Maschler's paper, the game $G_n^l(P)$ has a value hereafter denoted by $V_n^l(P)$ and both players have optimal strategies.

The next section is devoted to the proof of the next theorem:

Theorem 3.1. *For $n = 0, 1, \ldots$, for all $P \in [0,1]$, $V_n^l(P)$ is an increasing sequence in n with limit $g^l(P)$, where $g^l(P)$ is linear for P in $[\frac{k}{l-1}, \frac{k+1}{l-1}]$ for each k in $\{0, \ldots, l-2\}$ and such that for P in D_l, $g^l(P) := P(1-P)\frac{1}{2\delta}$.*

The proof of this theorem is based on the well known recursive structure of the Aumann and Maschler repeated games that expresses V_{n+1}^l as $T(V_n^l)$ where T is the following recursive operator:

$$T(g)(P) = \max_{(\sigma_H,\sigma_L)} \min_{\tau} [\sum_{k\in\{H,L\}} P^k \sigma_k A^k \tau + \sum_{i=1}^{l} \sigma(i) g(P(i))] \tag{3.1}$$

with $\sigma = P\sigma_H + (1-P)\sigma_L$ and, if $\sigma(i) > 0$, $P(i) = \frac{P\sigma_H(i)}{\sigma(i)}$.

The pair (σ_H, σ_L) joint to P induces a probability distribution on $K \times D_l$ which in turn can be represented by its marginal distribution σ on D_l and by $P(.)$, where $P(i)$ is as above the conditional probability of H given i. In particular we have $E_\sigma[P(i)] = P$. In this framework, T may be written as:

(3.2)

$$T(g)(P) = \max_{\{(\sigma(i),P(i))\ st\ E_\sigma[P(i)]=P\}} [\min_j (\sum_{i=1}^{l} \sigma(i)[1\!\!1_{i>j}(P(i)-i\delta) + 1\!\!1_{i<j}(j\delta - P(i)) + g(P(i))])]$$

To play optimally in $G_n^l(P)$, player 1 proceeds as follows: At the first stage, he plays σ_H and σ_L optimal in $T(V_{n-1}^l)(P)$ and he then computes the a posteriori $P^1(i_1) := P(i_1)$. From there on, he plays optimally in $G_{n-1}^l(P^1(i_1))$. In particular, he plays at the second stage an optimal move in $T(V_{n-2}^l)(P^1(i_1))$. He then computes the a posteriori probability $P^2(i_1, i_2)$ of H and plays for the remaining stages an optimal strategy in $G_{n-2}^l(P^2(i_1, i_2))$. So that the a posteriori martingale $P^1, \ldots, P^n$ may be viewed as a stage variable for player 1: at stage q, he just has to remind P^q to play optimally in $G_n^l(P)$.

The fact that V_n^l is increasing in n just results from the fact that for all concave continuous function V, $V \leq T(V)$ (see lemma 4.2).

We then have to prove that V_n^l is bounded from above by g^l. Since T is an increasing operator (if $h \leq g$ then $T(h) \leq T(g)$), a positive fixed point g for operator T will be an upper bound for V_n^l (see lemma 4.3). We have then to find such a fixed point, but the operator T is a bit complicated to analyze directly so we introduce an operator T^* that dominates T (for all V, $T(V) \leq T^*(V)$) for which we prove that g^l is a fixed point and therefore also a fixed point for T (see lemma 4.4).
It then remains to prove the convergence of V_n^l to g^l and this is obtained as follows: Since we suspect that for high n, V_n^l should be close to g^l, the optimal strategy in $T(V_n^l)$ should be close to an optimal strategy in $T(g^l)$. We then consider a strategy $\sigma^{n,l}$ of player 1 in $G_n^l(P)$ that consists at stage q in playing the optimal strategy in $T(g^l)(P^q)$, where P^q is the a-posteriori after stage q. The amount $C_n^l(P)$ guaranteed by that strategy in $G_n^l(P)$ is clearly a lower bound of $V_n^l(P)$.

We next prove that C_n^l converges to g^l as follows:
When P belongs to $D_l\backslash\{0,1\}$, we prove in theorem 4.11 that the following strategy (σ_H, σ_L) is optimal in $T(g^l)(P)$: let $P^+ := P+\delta$ and $P^- := P-\delta$. Both σ_H and σ_L are lotteries on the prices P and P^- with $\sigma_H(P) = \frac{P^+}{2P}$ and $\sigma_L(P) = \frac{1-P^+}{2(1-P)}$. With such a strategy, player 1 plays P with probability $P\sigma_H(P) + (1-P)\sigma_L(P) = \frac{1}{2}$ and therefore $P^1(P)$ is equal to $2P\sigma_H(P) = P^+$. Similarly player 1 plays P^- with probability $P\sigma_H(P^-) + (1-P)\sigma_L(P^-) = \frac{1}{2}$ and therefore $P^1(P^-)$ is equal to $2P\sigma_H(P^-) = P^-$. Therefore, with that strategy the a posteriori P^1 and the price posted by player 1 differ at most by δ. Furthermore, the a posteriori belongs clearly to D_l.

The price process induced by the strategy $\sigma^{n,l}$ remains at most at a distance δ of the a posteriori martingale $(P^q)_{q=1,\ldots,n}$. If P^q is in $]0,1[$, then P^{q+1} is equal to $P^{q,+}$ or $P^{q,-}$, each with probability $\frac{1}{2}$. Furthermore, if P^q is equal to 0 or 1 then $P^{q+1} = P^q$ and the prices fixed by player 1 are respectively 0 and 1. So, the process $(P^q)_{q=1,\ldots,n}$ is a D_l-valued symmetric random walk stopped at the time τ when it reaches 0 or 1.

As proved in theorem 4.11, the best reply of player 2 against $\sigma^{n,l}$ is to post at stage q a price equal to P^{q-1}. So, this allows us to compute explicitly C_n^l. At stage q, player 1 get exactly

$$E[\mathbb{1}_{p_1>P^{q-1}}(P^q - p_1) + \mathbb{1}_{p_1<P^{q-1}}(P^{q-1} - P^q)]$$

The price posted by player 1 is either P^{q-1} or $P^{q-1}-\delta$, so the first term is always equal to 0. The second term takes only the value δ when the price posted by player 1 is $P^{q-1}-\delta$ which happens with probability $\frac{1}{2}$. Hence, the expectation is just $\frac{\delta}{2}$, if P^{q-1} is not equal to 0 or 1. In case $P^{q-1}=0$ or 1, the previous expectation is equal to 0. As a consequence, C_n^l is just equal to:

$$C_n^l = E[\sum_{q=1}^{n} \mathbb{1}_{q\leq \tau}\frac{\delta}{2}] = \frac{\delta}{2}E[\tau \wedge n].$$

Let us observe that for a symmetric D_l-valued random walk with jumps of size δ, $((P^q)^2 - q\delta^2)_{q=0,1,\ldots}$ is a martingale. Therefore, due to the stopping theorem for uniformly integrable martingales, if P is in D_l then $\delta^2 E[\tau]$ is equal to $E[(P^\tau)^2 - P^2]$. Since P^τ belongs almost surely to $\{0,1\}$ and $E[P^\tau] = P$, we get $E[(P^\tau)^2 - P^2] = P(1-P)$. Due to the monotone convergence theorem, we get

$$\lim_{n\to+\infty} C_n^l = \frac{\delta}{2}E[\lim_{n\to+\infty} \tau \wedge n] = \frac{\delta}{2}E[\tau] = P(1-P)\frac{1}{2\delta} = g^l(P)$$

The convergence of $V_n^l(P)$ to $g^l(P)$ is thus proved for $P \in D_l$. Due to the concavity of V_n^l the convergence will hold clearly for all point in $[0,1]$, and the theorem is proved.

Let us observe that the above described strategy $\sigma^{n,l}$ is in fact not an optimal strategy in the game $G_n^l(P)$. The amount $C_n^l(P)$ it guarantees is symmetrical around $\frac{1}{2}$, $C_n^l(P) = C_n^l(1-P)$ while $V_n^l(P)$ is not (see graphs 1 and 2). We have no explicit expression of the optimal strategies in $G_n^l(P)$, but heuristically, these strategies should be close to $\sigma^{n,l}$, at least for large n.

As a corollary of theorem 3.1, we have the uniform convergence of $\frac{V_n^l}{\sqrt{n}}$ to 0. This indicates that the continuous and the discrete models are quite different. In particular, we do not expect to have the appearance of a Brownian motion as n goes to infinity for a fixed l in the asymptotic dynamics of the price process in the discretized games. More precisely, let us consider player 1's price process in $G_n^l(P)$ when using $\sigma^{n,l}$. Up to an error δ, this process is equal to the a posteriori martingale. As in [4] (see theorem 2.4 in this paper), this a posteriori martingale may be represented by the continuous time process Π^n, with $\Pi_t^n := P^q$ if $t \in [\frac{q}{n}, \frac{q+1}{n}[$. Now, if $q \geq \tau$, then $P^q \in \{0,1\}$. Therefore $\Pi_t^n \in \{0,1\}$ whenever $t \geq \tau/n$. We get therefore:

Theorem 3.2. *As n increases to ∞, the process Π^n converges in law to a splitting martingale Π that jumps at time 0 to 0 or 1 and then remains constant.*

However, we prove in the last section of the paper that, in some sense, for moderate n, the continuous model remains a good approximation of the discrete one: more precisely, we discretize the optimal strategies in the continuous game, and we show that these discretized strategies guarantee $V_n^l(P) - \epsilon$ in $G_n^l(P)$, with ϵ proportional to $n\delta$. As a consequence, if l depends on n, we get that $\frac{V_n^{l(n)}(P)}{\sqrt{n}}$ converge to the same limit as $\frac{V_n^c(P)}{\sqrt{n}}$ whenever $\sqrt{n}/l(n) \to 0$

The next section is devoted to the lemmas used in the proof of theorem 3.1: we analyze the properties of the recursive operator of the game and we find out its positive fixed point g^l.

4. A positive fixed point for T

4.1. Some properties of T.

We start this section by proving some easy properties of T.

Let us first observe that the value $u(P)$ of the average game with antisymmetric payoff matrix $A(P) := PA^H + (1-P)A^L$ is equal to 0. The optimal strategy for both players is the pure strategy $\lfloor P \rfloor$ defined as follows:

Definition 4.1. For all P in $[0,1]$:
let $\lfloor P \rfloor = [\frac{P}{\delta}]\delta$ and $\lceil P \rceil = \lfloor P \rfloor + \delta$ ($[x]$ is the highest integer less or equal to x).

If player 1 uses the pure strategy $\lfloor P \rfloor$, independently of H, L in the definition (3.1) of $T(g)(P)$, he plays a non revealing strategy ($P^1 = P$). The first stage payoff in $T(g)(P)$ is just the payoff in the average game which is clearly positive. This leads to the following lemma:

Lemma 4.2. *T is increasing and, for all g: $g \leq T(g)$.*

As a consequence, we have:

Lemma 4.3. *A positive fixed point of T is an upper bound for V_n^l.*

Let indeed g be a positive fixed point of T then we have for $n = 0$: $V_0^l = 0 \leq g$. By induction we get next that, if $V_n^l \leq g$, then $V_{n+1}^l = T(V_n^l) \leq T(g) = g$. □

Unfortunately, the fixed points of T are not easy to find, we will therefore bound T from above by an operator T^* and we will apply the next lemma.

Lemma 4.4. *Let T^* such that $T \leq T^*$.*
Then a fixed point of T^* is a fixed point of T.

Indeed, $g \leq T(g) \leq T^*(g) = g$.

We will next introduce the operator T^*.

The definition (3.2) of $T(g)(P)$ contains a minimization over player 2's action $j\delta$. If instead of minimizing, Player 2 plays in that formula $j\delta = \lfloor P \rfloor$, we obtain an operator T^0 such that $T(g) \leq T^0(g)$, where

$$T^0(g)(P) := \max_{\{(\sigma(i),P(i))\ st\ E_\sigma[P(i)]=P\}} [\sum_{i=1}^{l} \sigma(i)[\mathbb{1}_{i\delta > \lfloor P \rfloor}(P(i) - i\delta) + \mathbb{1}_{i\delta < \lfloor P \rfloor}(\lfloor P \rfloor - P(i)) + g(P(i))]$$

In turn, whenever $i\delta > \lfloor P \rfloor$ then $P(i) - i\delta \leq P(i) - \lceil P \rceil$. Therefore

$$T(g) \leq T^0(g) \leq T^1(g)$$

where:

$$T^1(g)(P) = \max_{\{(\sigma(i),P(i))\ st\ E_\sigma[P(i)]=P\}} [\sum_{i=1}^{l} \sigma(i)[\mathbb{1}_{i\delta > \lfloor P \rfloor}(P(i) - \lceil P \rceil) + \mathbb{1}_{i\delta < \lfloor P \rfloor}(\lfloor P \rfloor - P(i)) + g(P(i))]$$

Finally, $(\sigma, P(.))$ generates a probability distribution on $K \times D_l$. As mentioned above, the max in the definition of $T^1(g)$ is in fact a max over all probability distribution Π on $K \times D_l$ such that $\Pi[k = H] = P$. A general procedure to generate such probabilities is as follows: given $\sigma, P(.)$ and a one to one mapping i from L to L where $L = \{0, \ldots, l-1\}$, the lottery σ is used to select a virtual action $\tilde{\imath}$, player 1 plays in fact $i(\tilde{\imath})$. The state of nature is chosen according to the lottery $P(\tilde{\imath})$. Therefore we infer that

$$T^1(g) = \max_{\{(\sigma_{\tilde{\imath}}, P_{\tilde{\imath}}) \text{ st } E_\sigma[P_{\tilde{\imath}}]=p\}} \max_{\{i: permutation\ L\to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[\mathbb{1}_{i(\tilde{\imath})\delta > \lfloor P\rfloor}(P_{\tilde{\imath}} - \lceil P\rceil) + \mathbb{1}_{i(\tilde{\imath})\delta < \lfloor P\rfloor}(\lfloor P\rfloor - P_{\tilde{\imath}}) + g(P_{\tilde{\imath}})]$$

Simply by relaxing hypothesis that i is a permutation, we get a new inequality:

$$T^1(g) \le T^*(g)$$

where,

$$T^*(g) = \max_{\{(\sigma_{\tilde{\imath}}, P_{\tilde{\imath}}) \text{ st } E_\sigma[P_{\tilde{\imath}}]=p\}} \max_{\{i: L\to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[\mathbb{1}_{i(\tilde{\imath})\delta > \lfloor P\rfloor}(P_{\tilde{\imath}} - \lceil P\rceil) + \mathbb{1}_{i(\tilde{\imath})\delta < \lfloor P\rfloor}(\lfloor P\rfloor - P_{\tilde{\imath}}) + g(P_{\tilde{\imath}})]$$

The max over $i : L \to L$ in the last formula can be solved explicitly:
Whenever $P_{\tilde{\imath}} \ge (\frac{\lfloor P\rfloor + \lceil P\rceil}{2})$ (or equivalently $P_{\tilde{\imath}} - \lceil P\rceil \ge \lfloor P\rfloor - P_{\tilde{\imath}}$), $i(\tilde{\imath})$ must be chosen above $\frac{\lfloor P\rfloor}{\delta}$.
Similarly, if $P_{\tilde{\imath}} < (\frac{\lfloor P\rfloor + \lceil P\rceil}{2})$, then $i(\tilde{\imath}) < \frac{\lfloor P\rfloor}{\delta}$.
We obtain in this way that:

Lemma 4.5.

$$\max_{\{i: L\to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[\mathbb{1}_{i(\tilde{\imath})\delta > \lfloor P\rfloor}(P_{\tilde{\imath}} - \lceil P\rceil) + \mathbb{1}_{i(\tilde{\imath})\delta < \lfloor P\rfloor}(\lfloor P\rfloor - P_{\tilde{\imath}})] = E_\sigma[F_p(P_{\tilde{\imath}})]$$

with:

$$F_P(P_{\tilde{\imath}}) = \mathbb{1}_{P_{\tilde{\imath}} \ge (\frac{\lfloor P\rfloor + \lceil P\rceil}{2})}(P_{\tilde{\imath}} - \lceil P\rceil) + \mathbb{1}_{P_{\tilde{\imath}} < (\frac{\lfloor P\rfloor + \lceil P\rceil}{2})}(\lfloor P\rfloor - P_{\tilde{\imath}}).$$

Note that for all P in $[0,1]$, $F_P = F_{\lfloor P\rfloor}$.
The above result leads us to a new expression of T^*: For all P in $[0,1]$:

$$T^*(g)(P) = \max_{\{(\sigma_{\tilde{\imath}}, P_{\tilde{\imath}}) \text{ st } E_\sigma[P_{\tilde{\imath}}]=p\}} E_\sigma[F_{\lfloor P\rfloor}(P_{\tilde{\imath}}) + g(P_{\tilde{\imath}})]$$

Definition 4.6. The concavification $cav(f)$ of a function f is the smallest concave function higher than f which is concave.

With that definition, we obtain that:

$$T^*(g)(P) = cav_{P'}(F_{\lfloor P\rfloor}(P') + g(P'))(p)$$

In particular, the fixed point of T^* are concave.

4.2. A fixed point of T^*.

In this section, we seek for a fixed point of T^*.
T^* is increasing $(g \leq T(g) \leq T^*(g))$. As a consequence:

Proposition 4.7. *g is a fixed point of T^* if and only if*
$\forall P \in [0,1]$, $cav_{P'}(F_{\lfloor P \rfloor}(P') + g(P'))(P) \leq g(P)$.

We will seek for a fixed point g with the particularity that $g = \min_{d \in D_l} g_d$, where for all d, g_d linear on $\mathbb{R}$ and for all $P \in [0,1]$ $g(P) = g_{\lfloor P \rfloor}(P)$. (this means that g is linear between two successive points of D_l)

To prove that g is a fixed point T^*, it is sufficient to verify the condition: for all P in $[0,1]$

$$cav_{P'}(F_{\lfloor P \rfloor}(P') + g(P'))(P) \leq g_{\lfloor P \rfloor}(P)$$

Since g_d is linear for all d in D_l, and since the concavification of a negative function is negative, we are led to the following lemma:

Lemma 4.8. *If for all P and P' in $[0,1]$,*

$$F_{\lfloor P \rfloor}(P') + g(P') - g_{\lfloor P \rfloor}(P') \leq 0 \tag{4.1}$$

then g is a fixed point of T^*.

We use the equality $g = \min_d(g_d)$ to simplify (4.1). The following lemma leads to an explicit expression of a fixed point of T^*.

Lemma 4.9. *If for all P and P' in $[0,1]$,*

$$1\!\!1_{P' \leq \lfloor P \rfloor}(\lfloor P \rfloor - P' + g_{\lfloor P \rfloor - \delta}(P') - g_{\lfloor P \rfloor}(P')) + 1\!\!1_{P' \geq \lceil P \rceil}(P' - \lceil P \rceil + g_{\lceil P \rceil}(P') - g_{\lfloor P \rfloor}(P')) \leq 0 \tag{4.2}$$

then $g = \min_{d \in D_l}(g_d)$ is a fixed point of T^*. With the convention $g_{-\delta} := g_0$.

Indeed, since $g = \min_{d \in D_l}(g_d)$, we get for all P and P' in $[0,1]$:
$g(P') \leq 1\!\!1_{P' \leq \lfloor P \rfloor} g_{\lfloor P \rfloor - \delta}(P') + 1\!\!1_{\lfloor P \rfloor < P' < \lceil P \rceil} g_{\lfloor P \rfloor}(P') + 1\!\!1_{P' \geq \lceil P \rceil} g_{\lceil P \rceil}(P')$,
therefore for all P and P' in $[0,1]$,

$$\begin{aligned} F_{\lfloor P \rfloor}(P') + g(P') - g_{\lfloor P \rfloor}(P') \quad \leq \quad & 1\!\!1_{P' \geq (\frac{\lfloor P \rfloor + \lceil P \rceil}{2})}(P' - \lceil P \rceil) + 1\!\!1_{P' < (\frac{\lfloor P \rfloor + \lceil P \rceil}{2})}(\lfloor P \rfloor - P') + \cdots \\ & \cdots + 1\!\!1_{P' \leq \lfloor P \rfloor}(g_{\lfloor P \rfloor - \delta}(P') - g_{\lfloor P \rfloor}(P')) + \cdots \\ & \cdots + 1\!\!1_{P' \geq \lceil P \rceil}(g_{\lceil P \rceil}(P') - g_{\lfloor P \rfloor}(P')) \end{aligned}$$

Since $1\!\!1_{\lceil P \rceil > P' > (\frac{\lfloor P \rfloor + \lceil P \rceil}{2})}(P' - \lceil P \rceil) \leq 0$ and $1\!\!1_{\lfloor P \rfloor < P' < (\frac{\lfloor P \rfloor + \lceil P \rceil}{2})}(\lfloor P \rfloor - P') \leq 0$, we infer that g is a fixed point of T^* whenever for all P and P' in $[0,1]$,

$$1\!\!1_{P' \leq \lfloor P \rfloor}(\lfloor P \rfloor - P' + g_{\lfloor P \rfloor - \delta}(P') - g_{\lfloor P \rfloor}(P')) + 1\!\!1_{P' \geq \lceil P \rceil}(P' - \lceil P \rceil + g_{\lceil P \rceil}(P') - g_{\lfloor P \rfloor}(P')) \leq 0$$

□

In particular, if the linear functions g_d satisfy to $g_{\lceil P \rceil}(P') = g_{\lfloor P \rfloor}(P') - P' + \lceil P \rceil$ for all P and P' in $[0,1]$ then the function $g^l = \min_d g_d$ verifies the condition of the previous lemma.

The following set of g_d has all those properties:

$\forall i \in \{0, l-1\}$, $\forall P \in [0,1]$, $g_{i\delta}(P) := (\frac{l}{2} - 1 - i)P + i(i+1)\frac{\delta}{2}$

The resulting function g^l may be computed explicitly: $g_{i\delta}(P)$ is a quadratic convex expression of $i\delta$. It is symmetric around $i\delta = P - \delta/2$. The minimum on $i\delta \in D_l$ is thus reached at the point of D_l that is closest to $P - \delta/2$. This point is clearly $i\delta = \lfloor P \rfloor$, and thus

$$g^l(P) = g_{\lfloor P \rfloor}(P) = (l/2 - 1 - \lfloor P \rfloor/\delta)(P - \lfloor P \rfloor) + \lfloor P \rfloor(1 - \lfloor P \rfloor)\frac{1}{2\delta}$$

This is exactly the function g^l introduced in theorem 3.1. It is symmetric around $\frac{1}{2}$ on $[0,1]$).

As a consequence of the previous discussion, we get the following theorem

Theorem 4.10. *g^l is a positive fixed point of T^* and thus of T.*

We next compute optimal strategies of player 1 in $T(g^l)(P)$ as well as best replies of player 2:

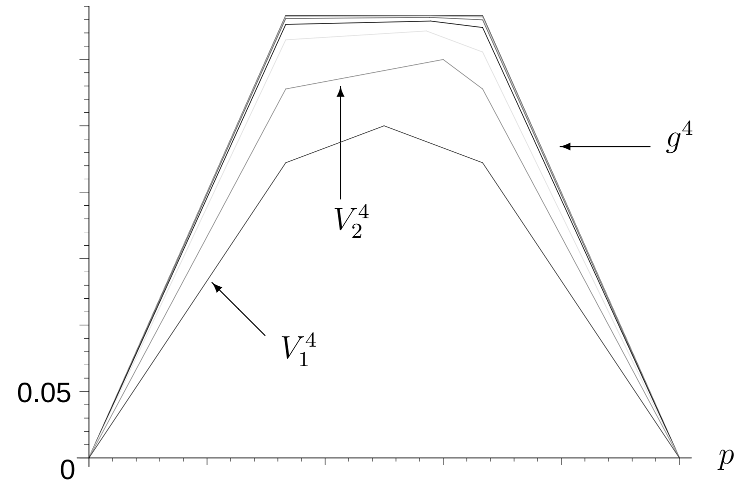
Theorem 4.11. *If P belongs to $D_l \backslash \{0,1\}$ then the following strategy (σ_H, σ_L) is optimal in $T(g^l)(P)$: σ_H and σ_L are lotteries on the prices P and P^- with $\sigma_H(P) = \frac{P^+}{2P}$ and $\sigma_L(P) = \frac{1-P^+}{2(1-P)}$ where $P^+ := P + \delta$ and $P^- := P - \delta$.*
The best reply of player 2 in $T(g^l)(P)$ against that strategy is to post a price equal to P.

Proof :
With that strategy, player 1 plays P with probability $P\sigma_H(P) + (1-P)\sigma_L(P) = \frac{1}{2}$ and therefore $P^1(P)$ is equal to $2P\sigma_H(P) = P^+$. Similarly player 1 plays P^- with probability $P\sigma_H(P^-) + (1-P)\sigma_L(P^-) = \frac{1}{2}$ and therefore $P^1(P^-)$ is equal to $2P\sigma_H(P^-) = P^-$. So, when player 1 uses that strategy, the first stage payoff in $T(g^l)(P)$ is equal to

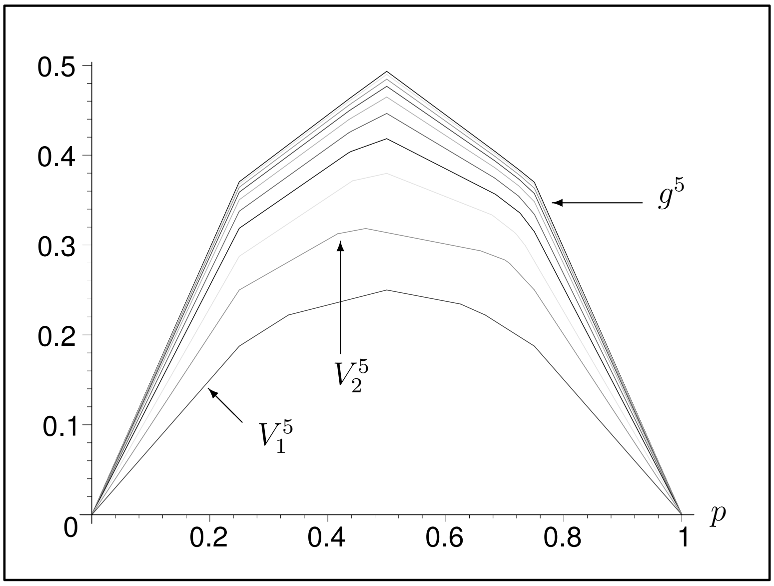
$$\frac{1}{2}[\mathbb{1}_{P>j\delta}(P^+ - P) + \mathbb{1}_{P<j\delta}(j\delta - P^+)] + \frac{1}{2}[\mathbb{1}_{P^->j\delta}(P^- - P^-) + \mathbb{1}_{P^-<j\delta}(j\delta - P^-)]$$

In case $j\delta \leq P^-$, only the first term is not equal to 0 and so the payoff is equal to $\frac{\delta}{2}$. In case $j\delta = P$, only the last term remains and the expectation is also $\frac{\delta}{2}$. The last case to consider is $j\delta \geq P^+$, then we obtain $j\delta - \frac{1}{2}(P^+ + P^-) = j\delta - P \geq \delta$. From this, we obtain that the price $j\delta = P$ is a best reply against that strategy and the first stage payoff is $\frac{\delta}{2}$. The second term payoff is then $\frac{1}{2}g^l(P^+) + \frac{1}{2}g^l(P^-) = g^l(P) - \frac{\delta}{2}$, so as announced the above strategy guarantees $g^l(P) = T(g^l)(P)$ and it is thus optimal in $T(g^l)(P)$.$\square$

Remark 4.12. The following graphs are drawn from numerical computation of V_n^l. It indicates in particular that V_n^l is not symmetric around $\frac{1}{2}$ and thus V_n^l does not coincide with C_n^l.



l=4



l=5

5. Continuous versus discrete market game

As indicated in the previous section, the continuous and the discrete games are quite different. However, we prove in this section that, in some sense, for moderate

n, the continuous model remains a good approximation of the discrete one: more precisely, we discretize the optimal strategies in the continuous game, and we show that these discretized strategies guarantee $V_n^l(P) - \epsilon$ in $G_n^l(P)$, with ϵ proportional to $n\delta$. As a consequence, if l depends on n, we get that $\frac{V_n^{l(n)}(P)}{\sqrt{n}}$ converge to the same limit as $\frac{V_n^c(P)}{\sqrt{n}}$ whenever $\sqrt{n}/l(n) \to 0$. This is the content of theorem 5.2.
Let us remark that the expression of V_n^c involves the sum of n independent random variables. For n too small ($n < 20$), even in the continuous model, there is not enough independent random variables in these sums for the central limit theorem to be applied. However, as it results from the next theorem, if l is large enough, for middle values of n ($20 < n \ll l$), the continuous game is a good approximation of the discrete game. The discretized optimal strategies of the continuous game are close to be optimal in the discrete game, and the resulting price process will be the discretization of the price process in the continuous game: For n high enough, it involves a Brownian motion.

As reminded in section 2, player 1's strategies in the first stage of G_n^l are represented by a pair (f_l, Q_l) satisfying (1), (2) and (3) of (2.1) with the additional requirement on f_l to be D_l valued. We denote $\Gamma_1^l(P)$ the space of these strategies. Similarly player 2 strategy space Γ_2^l will be the set of increasing functions $h_l : [0,1] \to D_l$.
In this section we will compare the payoff guaranteed in $G_n^l(P)$ by the discretization (f_l°, Q_l°) (resp h_l°) of the optimal strategy (f°, Q°) (resp h°) in $G_n^c(P)$ to get the next theorem.

Definition 5.1. If $\lceil x \rceil$ denotes the smallest $d \in D_l$ that dominates x, the discretization $\Pi^l(f,Q) := (f_l, Q_l)$ of the strategy (f,Q) is defined as: $f_l := \lceil f \rceil$ and $Q_l(\alpha)$ is the expectation of $Q(u)$ given that $f_l(u) = f_l(\alpha)$ where u is a uniform random variable on $[0,1]$. (Similarly $\Pi^l(h) := \lceil h \rceil$)

Theorem 5.2. *The discretized optimal strategies of $G_n^c(P)$ are $n\delta$-optimal strategies in $G_n^l(P)$. Therefore:*

$$\forall l, \forall n \geq 1 : \|V_n^c - V_n^l\|_\infty \leq n\delta$$

where $\delta = \frac{1}{l-1}$.

With the previous strategy spaces, the recurrence operator T for V_n^l, defined in (3.2), can be written as:
For all $P \in [0,1]$:

$$T(g)(P) := sup_{(f,Q)\in\Gamma_1^l(P)} inf_{p_2 \in D_l} F((f,Q), p_2, g),$$

with F as in theorem 2.1.

Lemma 5.3. *For all n in $I\!N$, if (f°, Q°) are optimal strategies in the first stage of $G_n^c(P)$, for all $p_2 \in D_l$:*

$$F((f^\circ, Q^\circ), p_2, V_n^c) \leq F(\Pi^l(f^\circ, Q^*), p_2, V_n^c) + \delta$$

In particular $T^c(V_n^c) \leq T(V_n^c) + \delta$

Indeed, if $p_2 \in D_l$ and $(f_l^\circ, Q_l^\circ) := \Pi^l(f^\circ, Q^\circ)$:

$$\begin{array}{rcl} F((f^\circ,Q^\circ),p_2,V_n^c) & = & E[1\!\!1_{f^\circ>p_2}(Q^\circ - f^\circ) + 1\!\!1_{p_2>f^\circ}(p_2 - Q^\circ) + V_n^c(Q^\circ)] \\ & = & E[1\!\!1_{f_l^\circ>p_2}(Q^\circ - f^\circ) + 1\!\!1_{p_2>f_l^\circ}(p_2 - Q^\circ) + V_n^c(Q^\circ)] \\ & & + E[1\!\!1_{\{f_l^\circ=p_2 \& f^\circ<p_2\}}(p_2 - Q^\circ)] \\ & = & E[1\!\!1_{f_l^\circ>p_2}(Q^\circ - f_l^\circ) + 1\!\!1_{p_2>f_l^\circ}(p_2 - Q^\circ) + V_n^c(Q^\circ)] \\ & & + E[1\!\!1_{f_l^\circ>p_2}(f_l^\circ - f^\circ) + 1\!\!1_{\{f_l^\circ=p_2 \& f^\circ<p_2\}}(f_l^\circ - f^\circ)] \\ & & + E[1\!\!1_{\{f_l^\circ=p_2 \& f^\circ<p_2\}}(f^\circ - Q^\circ)] \end{array}$$

Since we have $0 \le f_l^\circ - f^\circ \le \delta$ and as proved on page 298 in [4], $f^\circ - Q^\circ \le 0$, the second expectation in last equation is clearly bounded by δ and thus:
$F((f^\circ,Q^\circ),p_2,V_n^c) \le E[1\!\!1_{f_l^\circ>p_2}(Q^\circ - f_l^\circ) + 1\!\!1_{p_2>f_l^\circ}(p_2 - Q^\circ) + V_n^c(Q^\circ)] + \delta$
Since $Q_l^\circ = E[Q^\circ|f_l^\circ]$ and both $1\!\!1_{f_l^\circ>p_2}$ and $1\!\!1_{f_l^\circ<p_2}$ are f_l° measurable, we may replace Q° by Q_l° in the two first terms of the last inequality. Furthermore, due to Jensen inequality and the concavity of V_n^c, we get $E[V_n^c(Q^\circ)] \le E[V_n^c(E[Q^\circ|f_l^\circ])] = E[V_n^c(Q_l^\circ)]$.

The inequality $F((f^\circ,Q^\circ),p_2,V_n^c) \le F((f_l^\circ,Q_l^\circ),p_2,V_n^c)+\delta$ follows then immediately. Finally, since (f°,Q°) is optimal, we have

$$\begin{array}{rcl} T^c(V_n^c) & = & \min_{p_2\in[0,1]} F((f^\circ,Q^\circ),p_2,V_n^c) \\ & \le & \min_{p_2\in D_l} F((f^\circ,Q^\circ),p_2,V_n^c) \\ & \le & \min_{p_2\in D_l} F((f_l^\circ,Q_l^\circ),p_2,V_n^c) + \delta \\ & \le & \max_{(f,Q)\in\Gamma_1^l(P)} \min_{p_2\in D_l} F((f,Q),p_2,V_n^c) + \delta \\ & \le & T(V_n^c) + \delta \end{array}$$

□

Proposition 5.4. *$\forall l$, $\forall n \ge 1$: $V_n^c - V_n^l \le n\delta$*

The proof is by induction:
The result is clearly true for $n = 0$ ($V_n^c = V_n^l = 0$). Next, if the result is true for n then it holds also for $n+1$:
Indeed,

$$\begin{array}{rcl} V_{n+1}^c(P) & = & T^c(V_n^c)(P) \\ & \le & T(V_n^c)(P) + \delta \\ & \le & T(V_n^l + n\delta)(P) + \delta \\ & = & T(V_n^l)(P) + (n+1)\delta \\ & = & V_{n+1}^l(P) + (n+1)\delta \end{array}$$

□

To deal with the reverse inequality $\forall l$, $\forall n \ge 1 : V_n^l - V_n^c \le n\delta$, we will work on the dual model:
Let us consider the concave functions W_n^c and W_n^l respectively defined as the Fenchel conjugate of V_n^c and V_n^l. Due to (2.2), we just have to prove that

$$\forall l, \forall n \ge 1 : W_n^c - W_n^l \le n\delta$$

These functions are the value of dual games characterized by a recursive structure. The recursive formula for W_n^c was proved in theorem 4.5 in [4], and reminded in theorem 2.3. The same argument as in lemma 4.4 in [4], but with D_l valued strategies, gives us a similar recursive formula for W^l_n.

$$W^l_{n+1}(x) \geq \Lambda(W^l_n)(x) := sup_{h\in\Gamma^l_2} inf_{p_1\in D_l} R[x](p_1,h,W^l_n),$$

with R as in theorem 2.3.
The inequality in the last formula could be replaced by an equality, and this would lead to the dual recursive formula for the finite games as define in [5].

Lemma 5.5. *For all x in $I\!R$, for all n in $I\!N$, if h° is optimal strategy in the first stage of the dual game, for all $p_1 \in D_l$:*

$$R[x](p_1,h^\circ,W^c_n) - R[x](p_1,\Pi^l(h^\circ),W^c_n) \leq \delta$$

In particular $\Lambda^c(W^c_n) \leq \Lambda(W^c_n)+\delta$.

Indeed, with the notation $h^\circ_l := \Pi^l(h^\circ)$ and if $p_1 \in D_l$:

$$R[x](p_1,h^\circ,W^c_n) = W^c_n(x-\int_0^1 1\!\!1_{h^\circ(u)<p_1} - 1\!\!1_{h^\circ(u)>p_1} du) - \int_0^1 1\!\!1_{h^\circ(u)<p_1}(-p_1) + 1\!\!1_{h^\circ(u)>p_1} h^\circ(u)du$$

To simplify the notations, let us consider $h^\circ(u)$ (with u uniformly distributed) as a random variable h° then $\int_0^1 1\!\!1_{h^\circ(u)<p_1} - 1\!\!1_{h^\circ(u)>p_1} du$ is just equal to $A(h^\circ) := -1 + 2Prob(h^\circ < p_1) + Prob(h^\circ = p_1)$.
Next:

$$\begin{aligned} A(h^\circ) &= -1 + 2Prob(h^\circ_l < p_1) + 2Prob(h^\circ_l = p_1 \& h^\circ < p_1) + \cdots \\ &\quad \cdots + Prob(h^\circ_l = p_1) - Prob(h^\circ_l = p_1 \& h^\circ < p_1) \\ &= -1 + 2Prob(h^\circ_l < p_1) + Prob(h^\circ_l = p_1) + Prob(h^\circ_l = p_1 \& h^\circ < p_1) \\ &= A(h^\circ_l) + Prob(h^\circ_l = p_1 \& h^\circ < p_1) \end{aligned}$$

Therefore, due to the concavity of W^c_n:

$$\begin{aligned} W^c_n(x - A(h^\circ)) &= W^c_n(x - A(h^\circ_l) - Prob(h^\circ_l = p_1 \& h^\circ < p_1)) \\ &\leq W^c_n(x - A(h^\circ_l)) - Prob(h^\circ_l = p_1 \& h^\circ < p_1)(W^c_n)'(x - A(h^\circ_l)), \end{aligned}$$

where $(W^c_n)'$ stands for the derivative of W^c_n. Next, $(W^c_n)'(x - A(h^\circ_l)) = (W^c_n)'(x + 1 - 2\zeta)$, with $\zeta := Prob(h^\circ_l < p_1) + \frac{Prob(h^\circ_l = p_1)}{2}$. As proved in formula (18) in [4], $h^\circ(u) = \int_0^u 2s(W^c_n)'(x+1-2s)ds/u^2$.
Due to the concavity of W^c_n, $(W^c_n)'$ is a decreasing decreasing function, therefore, if $s \leq u$, then $(W^c_n)'(x+1-2s) \leq (W^c_n)'(x+1-2u)$. We get in this way:

$$h^\circ(u) \leq \int_0^u 2s(W^c_n)'(x+1-2u)ds/u^2 = (W^c_n)'(x+1-2u)$$

and so: $-(W^c_n)'(x - A(h^\circ_l)) \leq -h^\circ(\zeta) \leq -h^\circ_l(\zeta) + \delta$.

We claim next that $Prob(h^\circ_l = p_1 \& h^\circ < p_1)h^\circ_l(\zeta) = Prob(h^\circ_l = p_1 \& h^\circ < p_1)p_1$. Indeed, we just analyze the case $Prob(h^\circ_l = p_1 \& h^\circ < p_1) > 0$: let us define $x_0 := Prob(h^\circ_l \leq p_1 - \delta)$ and $x_1 := Prob(h^\circ_l \leq p_1)$. Since h° is continuous and increasing and since $x \to \lceil x \rceil$ is left continuous, increasing, h°_l is left continuous, increasing. Therefore $\{u | h^\circ_l(u) \leq p_1 - \delta\}$ is the closed interval $[0,\alpha]$ whose length is precisely $Prob(h^\circ_l \leq p_1 - \delta)$. Therefore $\alpha = x_0$ and thus $h^\circ_l(x_0) \leq p_1 - \delta$. We find similarly $h^\circ_l(x_1) \leq p_1$. Now, since $0 < Prob(h^\circ_l = p_1) = x_1 - x_0$, we infer that on $]x_0, x_1]$,

h_l° assumes the constant value p_1. Observing that, by definition, $\zeta \in]x_0, x_1]$, we conclude that $h_l^\circ(\zeta) = p_1$.
Thus,

$$W_n^c(x - A(h^\circ)) \leq W_n^c(x - A(h_l^\circ)) - Prob(h_l^\circ = p_1 \& h^\circ < p_1)(p_1 - \delta)$$

We next deal with the term $-\int_0^1 1\!\!1_{h^\circ(u)<p_1}(-p_1) + 1\!\!1_{h^\circ(u)>p_1} h^\circ(u)du$ in $R(p_1, h^\circ, W_n^c)$. It is just equal to:

$$\begin{array}{l} -\int_0^1 1\!\!1_{h_l^\circ(u)<p_1}(-p_1) + 1\!\!1_{h_l^\circ(u)>p_1} h_l^\circ(u)du + Prob(h_l^\circ = p_1 \& h^\circ < p_1)p_1 + \cdots \\ \cdots + \int_0^1 1\!\!1_{h_l^\circ(u)>p_1}(h_l^\circ(u) - h^\circ(u))du \end{array}$$

Therefore:

$$\begin{array}{rcl} R[x](p_1, h^\circ, W_n^c) & \leq & R[x](p_1, h_l^\circ, W_n^c) + Prob(h_l^\circ = p_1 \& h^\circ < p_1)\delta + \cdots \\ & & \cdots + \int_0^1 1\!\!1_{h_l^\circ(u)>p_1}(h_l^\circ(u) - h^\circ(u))du \end{array}$$

Since $h_l^\circ - h^\circ \leq \delta$, the inequality $R[x](p_1, h^\circ, W_n^c) \leq R[x](p_1, h_l^\circ, W_n^c) + \delta$ follows then immediately.
Finally, since h° is optimal, we have

$$\begin{array}{rcl} \Lambda^c(W_n^c)(x) & = & \min_{p_1 \in [0,1]} R[x](p_1, h^\circ, W_n^c) \\ & \leq & \min_{p_1 \in D_l} R[x](p_1, h^\circ, W_n^c) \\ & \leq & \min_{p_1 \in D_l} R[x](p_1, h_l^\circ, W_n^c) + \delta \\ & \leq & \max_{h \in \Gamma_2^l} \min_{p_1 \in D_l} R[x](p_1, h, W_n^c) + \delta \\ & \leq & \Lambda(W_n^c)(x) + \delta \end{array}$$

□

Proposition 5.6. *$\forall l$, $\forall n \geq 1$: $W_n^c - W_n^l \leq n\delta$*

The proof is by induction:
The result is clearly true for $n = 0$ ($W_0^c = W_0^l$). If the result is true for n then it holds also for $n + 1$:
Indeed,

$$\begin{array}{rcl} W_{n+1}^c & = & \Lambda^c(W_n^c) \\ & \leq & \Lambda(W_n^c) + \delta \\ & \leq & \Lambda(W_n^l + n\delta) + \delta \\ & = & \Lambda(W_n^l) + (n+1)\delta \\ & = & W_{n+1}^l + (n+1)\delta \end{array}$$

The result holds thus for all n. □

6. Conclusion

The results of section 3 indicate that the normal density does not appear in the asymptotic behavior of Ψ_n^l, as n goes to infinity for a fixed l. In particular, we have seen in that case (see theorem 3.2) that the limit price process Π is a splitting martingale that jumps at time 0 to 0 or 1 and then remains constant. The effect of the discretization is to force the informed player to reveal is information much sooner than in the continuous model. The discretization improves the efficiency of the prices.

Theorem 5.2 in terms of Ψ_n reads:

Corollary 6.1. *$\forall l$, $\forall n \geq 0$, $\|\Psi_n^c - \Psi_n^l\|_\infty \leq \frac{\sqrt{n}}{l-1}$*

This implies in particular that if the size $l(n)$ of the discretization set increases with the number n of transaction stages in such a way that $\lim_{n\to+\infty} \frac{l(n)}{\sqrt{n}} = +\infty$, then $\Psi_n^{l(n)}$ converges to the same limit as Ψ_n^c, and in that case, the normal distribution does appear. The discretized optimal strategies of the continuous games are then close to be optimal in the discrete game, and the brownian motion will appear in the asymptotic of the price process. Therefore, the continuous game remains a good model for the real world discretized game as far as $\frac{\sqrt{n}}{l-1}$ is small.

References

1. Aumann, R.J. and M. Maschler. 1995. Repeated Games with Incomplete Information, MIT Press.
2. Bachelier, L. 1900, Théorie de la spéculation. Ann. Sci. Ecole Norm. Sup., 17, 21-86.
3. Black, F. and M. Scholes (1973), The pricing of options and corporate liabilities, Jounal of Political Economy, 81, 637-659.
4. De Meyer, B. and H. Moussa Saley. 2002. On the origin of Brownian motion in finance. Int J Game Theory, 31, 285-319.
5. De Meyer, B. 1995. Repeated games, duality and the Central Limit Theorem. Mathematics of Operations Research, 21, 235-251.