A Note on a ‘Square-Root Rule’ for Reinsurance

Michael R. Powers
Martin Shubik

Follow this and additional works at: https://elischolar.library.yale.edu/cowles-discussion-paper-series

Recommended Citation
https://elischolar.library.yale.edu/cowles-discussion-paper-series/1805

This Discussion Paper is brought to you for free and open access by the Cowles Foundation at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Cowles Foundation Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.
A NOTE ON A “SQUARE-ROOT RULE” FOR REINSURANCE

By

Michael R. Powers and Martin Shubik

June 2005

COWLES FOUNDATION DISCUSSION PAPER NO. 1521

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/
A Note on a

“Square-Root Rule” for Reinsurance

Michael R. Powers,¹ Martin Shubik²

Abstract

In previous work, the current authors derived a mathematical expression for the optimal (or “saturation”) number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). In the current paper, we show analytically that, for large numbers of primary insurers, this mathematical expression provides a “square-root rule”; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers. We note further that an analogous “fourth-root rule” applies to markets for retrocession (the reinsurance of reinsurance).

Keywords: Primary insurance, reinsurance, market size, square-root rule.

JEL Classification:

¹ Professor and Director, Advanta Center for Financial Services Studies, Temple University.
² Seymour H. Knox Professor of Mathematical Institutional Economics, Yale University.
1. Introduction

In previous work, the current authors derived a mathematical expression for the optimal (or “saturation”) number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). Specifically, we identified the optimal number of reinsurers as the maximum value of $n_1 \in \{2, 3, \ldots, n_0\}$ such that the price per unit of primary insurance in a market with $n_0$ primary insurers and $n_1$ reinsurers remains less than the price of insurance in a market with $n_0 + 1$ primary insurers and $n_1 - 1$ reinsurers. This marginal analysis yields

$$n_1^* = \text{Max} \left\{ n_1 : \left( \frac{n_1}{n_1 - 1} \right) \left( \frac{n_0}{n_0 - 1} \right) \left( \frac{n_0}{n_0 - n_1} \right) < \left( \frac{n_0 - 1}{n_0 - 2} \right) \left( \frac{n_0 + 1}{n_0} \right) \left( \frac{n_0 + 1}{n_0 - n_1 + 2} \right) \right\}$$

and the existence of a unique solution $n_1^* \in \{2, 3, \ldots, n_0 - 1\}$ is guaranteed by the fact that the inequality in (1) is satisfied for $n_1 = 2$, but not for $n_1 = n_0$.

Having computed $n_1^*$ for values of $n_0$ in the interval $[10, 5000]$, we presented a graph of these results, which is reproduced in Figure 1.

Figure 1. Insurance/Reinsurance Market
From a tabular display of the same results (see Table 1), it is easy to see that the solution to (1), \( n_1^* \), is approximately equal to the square root of \( n_0 \) (although this observation was not made in Powers and Shubik, 2001).

<table>
<thead>
<tr>
<th>( n_0 ) (Number of Primary Insurers)</th>
<th>( n_1^* ) (Optimal Number of Reinsurers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>40</td>
<td>6</td>
</tr>
<tr>
<td>50</td>
<td>7</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
</tr>
<tr>
<td>200</td>
<td>14</td>
</tr>
<tr>
<td>300</td>
<td>17</td>
</tr>
<tr>
<td>400</td>
<td>20</td>
</tr>
<tr>
<td>500</td>
<td>22</td>
</tr>
<tr>
<td>1000</td>
<td>31</td>
</tr>
<tr>
<td>2000</td>
<td>44</td>
</tr>
<tr>
<td>3000</td>
<td>54</td>
</tr>
<tr>
<td>4000</td>
<td>63</td>
</tr>
<tr>
<td>5000</td>
<td>70</td>
</tr>
<tr>
<td>10,000</td>
<td>101</td>
</tr>
</tbody>
</table>

Table 1. Optimal Numbers of Reinsurers for Selected Numbers of Primary Insurers

2. A Square-Root Rule

In the current research, we show analytically that, for large numbers of primary insurers, the solution to problem (1) is indeed a “square-root rule”; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers, as stated formally in the following result.
Theorem 1: For sufficiently large \( n_0 \), there exists a unique solution 
\[
n_1^* = n_1^*(n_0) \in \{2, 3, \ldots, n_0 - 1\}
\]
to problem (1), where 
\[
n_1^*(n_0) \sim \sqrt{n_0}
\]
as \( n_0 \to \infty \).

Proof: First, we extend problem (1) from the two-dimensional integer grid 
\[
\{(n_0, n_1) : 2 \leq n_1 \leq n_0\}
\]
to the corresponding two-dimensional real space \( \{(a, x) : 2 \leq x \leq a\} \) by considering the inequality 
\[
\left( \frac{x}{x-1} \right) \left( \frac{a}{a-1} \right) < \left( \frac{x-1}{x-2} \right) \left( \frac{a+1}{a} \right) \left( \frac{a+1}{a-x+2} \right). \tag{2}
\]
Apart from its points of unboundedness, (2) is equivalent to the cubic polynomial inequality
\[
f(x) = (3a^2 - 3a + 1)x^3 - (5a^3 + 3a^2 - 5a + 2)x^2
\]
\[
+ (9a^3 - 3a^2 - a + 1)x + (a^3 - 3a^2 + 3a^2 - a) > 0. \tag{3}
\]
Thus, the unique solution specified by the theorem—if it exists—is given by 
\[
n_1^* = \lfloor x^* \rfloor
\]
where \( x^* = x^*(a) \in (2, a) \) is a positive real root of \( f(x) \) such that \( f'(x) < 0 \).

For large \( a \), one can rewrite (2) as
\[
\left( \frac{x}{x-1} \right) \left( \frac{x-2}{x-1} \right) \left( \frac{a-x+2}{a-2} \right) \left( \frac{a}{a-1} \right) \left( \frac{a+1}{a} \right) < \left( \frac{a-1}{a} \right) \left( \frac{a+1}{a} \right) = 1 + O(a^2),
\]
from which it follows that we seek the roots \( x(a) \) of
\[
f(x) = \left[ 1 + (x-1)^2 O(a^2) \right] x^2 - \left[ x^3 - x - 1 + (x-1)^3 O(a^2) \right] x
\]
\[
- \left[ x^3 - 2x^2 + x + x(x-1)^2 O(a^2) \right] = 0. \tag{4}
\]
Solving (4) for \( a \) as a function of \( x \) yields
\[
a = \frac{\left[ x^2 - x - 1 + (x-1)^3 O(a^{-2}) \right]}{2\left[ 1 + (x-1)^2 O(a^{-2}) \right]}
\]

\[
\pm \sqrt{\left[ x^2 - x - 1 + (x-1)^3 O(a^{-2}) \right] + 4\left[ 1 + (x-1)^2 O(a^{-2}) \right] x^3 - 2x^2 + x(x-1)^2 O(a^{-2})} \over 2\left[ 1 + (x-1)^2 O(a^{-2}) \right].
\]  (5)

Anticipating that there exists at least one positive root \( x(a) = o(a^{2/3}) \) to (4), we find

exactly one solution to (5); namely,

\[
a = \frac{\hat{x}^2 + O(\hat{x})}{2[1 + o(1)]} + \frac{\sqrt{\hat{x}^4 + O(\hat{x}^3)}}{2[1 + o(1)]} \approx \hat{x}^2,
\]

which implies

\[
\hat{x}(a) \approx \sqrt{a}.
\]

For sufficiently large \( a \), it is clear that \( \hat{x}(a) \in (2, a) \).

To confirm that \( \hat{x}(a) \) is the desired root of \( f(x) \), we consider the local extrema of this polynomial, given by

\[
f'(x) = 3(3a^2 - 3a + 1)x^2 - 2(5a^3 + 3a^2 - 5a + 2)x + (9a^3 - 3a^2 - a + 1) = 0,
\]

or equivalently,

\[
x = \frac{(5a^3 + 3a^2 - 5a + 2) \pm \sqrt{(5a^3 + 3a^2 - 5a + 2)^2 - 3(3a^2 - 3a + 1)(9a^3 - 3a^2 - a + 1)}}{3(3a^2 - 3a + 1)}. \]  (6)

As \( a \to \infty \), the two solutions to (6) are

\[
x_L(a) = \frac{(5a^3 + 3a^2 - 5a + 2) - \sqrt{(5a^3 + 3a^2 - 5a + 2)^2 - 3(3a^2 - 3a + 1)(9a^3 - 3a^2 - a + 1)}}{9a^2 + O(a)}
\]
and

\[ x_U(a) = \frac{5a^3 + O(a^4) + \sqrt{25a^5 + O(a^6)}}{9a^2 + O(a)} \approx \frac{10}{9} a, \]

respectively. For sufficiently large \( a \), it follows that \( \bar{x}(a) \) lies between \( x_L(a) \) and \( x_U(a) \), implying \( f'(\bar{x}) < 0 \). Therefore, \( x^*(a) = \bar{x}(a) \).

3. Extension to Retrocession

Beyond the world of ordinary reinsurance lie the misty realms of retrocession (second-order reinsurance), second-order retrocession (third-order reinsurance), and so on. Although the model in our 2001 paper was extended to an arbitrary number of reinsurance levels, we acknowledged that such clearly defined levels are not reflective of the real world. While a few specialized purveyors of retrocession do exist, higher-order reinsurance is typically provided by ordinary reinsurers through the packaging and repackaging of risk through various types of pooling arrangements.

It may be argued that the absence of distinct higher-order reinsurance markets is consistent with the analytical results of our game-theoretic model. Assuming that there exist at least two second-order reinsurers in the market, and again using the minimization of price in the primary insurance market as the optimality criterion, our expression for the optimal number of second-order reinsurers is given by:
\[
\begin{align*}
n_2^* &= \text{Max} \left\{ n_2 : \left( \frac{n_1}{n_1 - 1}, \frac{n_1}{n_1 - n_2}, \frac{n_0}{n_0 - n_1} \right) \right\} \\
&< \left( \frac{n_2 - 1}{n_2 - 2} \right) \left( \frac{n_1 + 1}{n_1} \right) \left( \frac{n_1 + 1}{n_1 - n_2 + 2} \right) \left( \frac{n_0}{n_0 - n_1 - 1} \right) \right\}. 
\end{align*}
\] (7)

By methods analogous to those employed in the proof of Theorem 1, it is quite straightforward to show the following result.

**Theorem 2:** For sufficiently large \( n_0 \), if \( n_1 \sim \sqrt[4]{n_0} \), then there exists a unique solution

\[
n_2^* = n_2^* (n_0) \in \{2, 3, \ldots, n_1 - 1\}
\]

problem (7), where

\[
n_2^* (n_0) \sim \sqrt[4]{n_0}
\]

as \( n_0 \to \infty \).

In short, the number of retrocessionaires in a market should be approximately equal to the fourth-root of the number of primary insurers, which for most national insurance markets (other than that of the U.S.) is rather small. Thus, according to the model, one should not expect to see distinct significant retrocession markets, except perhaps in the U.S. This result agrees with empirical observation.

**4. A Comment on Paper on Paper**

More generally, for all financial instruments involving risk and transaction costs, the principles dictating how many levels of paper are economically optimal need to be considered. We suspect that four or five is an extreme upper bound, and our result here conforms to this.
Reference