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ABSTRACT

We examine the influence of mesoscale turbulence and random growth rate fields upon phytoplankton patchiness, on length scales from 1 km to 100 km and time scales from 1 day to 100 days. We consider phytoplankton concentrations with quite general nonlinear growth rate functions, such that the concentration is bounded for all time. We use, and justify the use of, particle separation statistics to deduce variance spectra of nonlinearly transformed concentration. Two growth rate models are examined: an advected field, and a locally specified field. Both lead to initial patchiness in the concentration, correlated with the growth rate field. The advected growth rate field leads to a temporal peak in the patchiness before the mesoscale turbulence causes the concentration variance to cascade to a "noisy" spatial distribution that retains no correlation with either the motion field, or the growth rate field. We outline numerical experiments to test these results, in particular the occurrence of the peak in patchiness and the time scales associated with its formation and decay.

1. Introduction

The analysis of spatial pattern in planktonic ecosystems differs fundamentally from that in benthic or terrestrial ecosystems because, by definition, planktonic organisms are passively transported around by the fluid medium they reside in. The first theoretical attempts to explain horizontal patterns or patches in the distributions of planktonic organisms (Skellam, 1951; Kierstead and Slobodkin, 1953) considered the coupled effects of biological growth factors and physical factors of the flow field. Phytoplankton, the microscopic plants that undergo photosynthesis in the surface layer of the ocean, especially can be considered as passive but nonconservative tracers of the fluid flow. Some species of phytoplankton can adjust their buoyancy to move vertically meters per day, but they have no known capability for directed horizontal movement. The crustaceous zooplankton that feed on phytoplankton often migrate vertically tens of meters in an hour. While they can move horizontally fractions of a meter to feed on prey organisms, they are unlikely to make directed movements of more than a few meters in the horizontal direction. In this paper, we will concentrate on the interaction of passive but growing phytoplankton with the horizontal flow field in the ocean. The phytoplankton growth rate will be random both in space and time. Zooplankton will be
considered only indirectly in the sense that their grazing or feeding on phytoplankton affects the net growth rate of the phytoplankton themselves. Kierstead and Slobodkin (1953) and Skellam (1951), in the commonly-called KISS models, considered a cylinder of fertile water containing phytoplankton with a constant growth rate surrounded by infertile water. They calculated the radius of the smallest cylindrical patch of growing phytoplankton that could maintain itself against erosion by small scale turbulent diffusion represented as Fickian diffusion. Later, Denman and Platt (1976) and Denman et al. (1977) considered models of the competing effects of constant growth and turbulent diffusion to obtain plausible shapes for the variance spectrum of phytoplankton concentration. Their models considered neither variable growth nor advection by varying flow. Wilson et al. (1979) extended a time dependent equation for a tracer in three-dimensional isotropic turbulence (Hinze, 1975) to include a growth term. For a typical observed spectrum as an initial condition, their simulated spectrum proceeded toward a flatter spectrum (more variance at higher wavenumbers) with increasing time up to several days. All these theories assumed three-dimensional isotropic turbulence, which probably is valid only on scales of less than 10 m.

Fasham (1978a, b), extending the work of Whittle (1962), added a nonspecific random forcing to the KISS-type models. The random forcing could be random temporal variations in growth rate or in grazing, for example. For a negative net growth rate, Fasham obtained a large time spectral shape similar to that of Denman and Platt (1976)—dropping off steeply with increasing wavenumber beyond some characteristic wavenumber, but flattening off at smaller wavenumbers. For a positive net growth rate, his spectrum diverged with decreasing wavenumbers below the characteristic value.

Segel and Jackson (1972) and Levin and Segel (1976) represented the zooplankton-phytoplankton coupling by a pair of nonlinear equations of the reaction-diffusion type. They found that diffusive instabilities could arise resulting in the generation of spatial patterns if two conditions were satisfied. First, a term must be present representing an autocatalytic effect in phytoplankton concentration whereby the relative impact of zooplankton grazing is reduced with increasing phytoplankton concentration. Second, the coefficient of diffusion must be greater for the zooplankton than the phytoplankton, as might be expected if the zooplankton have some random-walk type motility, or if they move vertically in a sheared flow. These models suggest an ecological mechanism for the onset of horizontal pattern even if ultimately, the control is by turbulent water motions.

All these previous models consider only the small scale mixing aspect of horizontal turbulence; they ignore the stirring aspect (Eckart, 1948) whereby the turbulence acts by advective distortion of the scalar tracer patterns until local gradients are so large as to be amenable to smoothing by small scale turbulent diffusion. In current analogs of horizontal turbulence, that is, either Lagrangian particle statistics as applied to flow...
tracers such as drifters, or pseudo-spectral numerical stimulations of Eulerian flow characteristics, the small scale turbulent diffusion (if it appears at all) is limited to representing a subgrid scale sink for enstrophy or scalar variance. One important exception is the case of large scale, long term variability (ocean basin and seasonal to intra-annual scales) where a large scale turbulent diffusion can be used to represent stirring over many individual eddies.

In this paper, we consider mesoscale ocean turbulence, which is dominated by motions that vary on scales of 100 km or less. It is quasi two dimensional on scales as short as about 1 km (Young et al., 1982), and we assume that the mesoscale turbulence on scales from 100 to 1 km is an inertial subrange in dynamic equilibrium. That is, the Reynolds' number is so large that dissipation of momentum on such scales is negligible, but there is a uniform cascade of enstrophy (mean squared vorticity) through the subrange, from larger to smaller scales (smaller to larger wavenumbers). Many of the properties of this subrange have been inferred using dimensional analysis and similarity assumptions. We will use these methods, and others to infer the stirring of a passive but nonconservative tracer representing neutrally buoyant growing phytoplankton organisms. In particular, we will consider the effects, if any, of random temporal and spatial variations in net growth rate on the generation and maintenance of spatial structures in phytoplankton concentration. Such growth rate variations might arise from variable light or nutrient availability or from nonuniform grazing by zooplankton.

Denman (1983) pointed out that for phytoplankton in the exponential growth phase being advected by mesoscale ocean turbulence, the growth rate can be factored out of the conservation equation unless it varies spatially. In fact this is true at all stages of concentration increasing toward the carrying capacity, so long as the growth rate is independent of position. The phytoplankton would thus behave as a passive conserved scalar and initial spatial structures could not be maintained or enhanced unless variance was somehow being added continually. The hypothesis that we wish to test is that a spatially variable growth rate can generate and maintain patchiness in a manner not possible for a conserved scalar, or a scalar with a spatially uniform growth rate. By "patchy" distributions we mean ones where the variance initially concentrated near a small wavenumber \( k_0 \) (large scale) will remain concentrated near that wavenumber with increasing time. By "noisy" distributions, we mean ones where the variance approaches white noise with increasing time, and \( k_0 \) no longer scales the spectrum. In the same manner as the velocity field, the growth rate field will be assumed also to be concentrated initially near the low wavenumber region characterized by \( k_0 \). For several prescribed growth rate fields, we will investigate the time evolution of a phytoplankton concentration field which is initially the same everywhere. Can it develop "patchiness," and can any "patchiness" persist or does it becomes "noisy" with increasing time?

In Section 2, we will state the conservation equation for a passive but growing scalar
concentration with nonlinear growth functions, and invoke properties of mesoscale oceanic turbulence to identify the relevant space and time scales. We shall use dimensional analysis to deduce the statistical equilibrium shape of the variance spectrum for the nonlinearly transformed concentration, in the enstrophy inertial subrange of wavenumber space. We shall also deduce the relaxation time scale for approach to this statistical equilibrium. From this relaxation time scale we shall infer that many turbulent eddy turnover times must pass before the transformed phytoplankton concentration variance, concentrated near $k_0$ for small times, cascades to high enough wavenumbers that small scale turbulent dissipation becomes important. Until such times we may neglect dissipation, and formulate the problem in terms of Lagrangian particle statistics. In Section 3 we derive formal expressions for the spatial covariance of nonlinearly transformed concentration, and its variance spectrum. Evaluation of these expressions requires explicit models for growth rate statistics, and statistics for particle-pair separations (the latter are described in Section 3). In Section 4 we introduce two conceptual models of random growth rate: a growth rate advected by the turbulence (e.g. caused by a spatially variable advected nutrient field), and a local growth rate specified in space and time independently of the turbulence (e.g. caused by bottom topography, local climatic variations, etc.). We then obtain the shapes of the variance spectra as they evolve in time. They approach the shape of the statistical equilibrium spectrum characteristic of "noisy" fields, as the relaxation time is approached.

Finally, in Section 5 we recapitulate the important points of Sections 2–6 without the mathematical detail, and suggest some numerical experiments that might test our conclusions.

2. Equilibrium states

In this section we describe the properties of a horizontally two-dimensional turbulent velocity field, and state the conservation equation for a growing, passive scalar field $C$ with a general nonlinear growth function. We then use dimensional arguments to derive the well-known $k^{-1}$ variance spectrum for a nonlinear transform of $C$ at statistical equilibrium, and we estimate the time scale for relaxation toward this equilibrium spectrum. The length and time scales at which this theory applies to the ocean are also identified.

a. Two-dimensional turbulence. Kraichnan (1967) has demonstrated theoretically the existence of an incompressible two-dimensional turbulent velocity field $u = u(x, t)$ with Cartesian components $(u, v)$ which depends on planar position $x = (x, y)$ and time $t$, and which has the following properties.

(i) It is in a statistical equilibrium state, in which driving forces and viscous dissipation are balanced, at least in the mean square sense.
(ii) The equilibrium is homogeneous and stationary. That is, the velocity moments such as mean and variance are independent of position and time, while correlations across space or time depend only on the spatial separation or temporal lag. The equilibrium is also isotropic, that is, correlations do not depend on the direction of the separation or the sign of the lag, but only on their magnitudes.

(iii) There is a wavenumber range $k_0 < k < k_1$ (where $k$ is the magnitude of the planar wavenumber vector $k$) in which viscous dissipation is negligible; however there is a steady net transfer or cascade of enstrophy (the variance of the vertical component of vorticity) from the lowest wavenumber $k_0$ to the highest wavenumber $k_1$. Dissipation is a fortiori negligible for $0 < k < k_0$ but is significant if $k_1 < k$. The net transfer of enstrophy from $k_0$ to $k_1$ takes place at the rate $\eta$, which has the dimensions of $(\text{time})^{-3}$.

(iv) Dimensional arguments show that, in the “inertial subrange” $k_0 < k < k_1$, the variance spectrum of $u$ must have the following form:

$$E_u(k) = a_1 \eta^{2/3} k^{-3}, \quad (k_0 < k < k_1)$$

(2.1)

where $a_1$ is a dimensionless constant of order unity. See Figure 1. Note that $E_u$
depends only on \( k = |k| \), as a consequence of the isotropy of the turbulent state. The total variance of \( u \) is given by

\[
U^2 = \langle |u|^2 \rangle = \int_0^\infty E_u(k)dk.
\]

The angle brackets in (2.2) denote an average with respect to the ensemble of possible realizations of \( u \). We shall use \( U \) as a velocity scale.

(v) Kraichnan (1971) has shown that a multiplicative correction, which is logarithmic in \( k \), should be included in (2.1). This will not concern us since we are interested only in the cascade rate \( \eta \), and the general steepness of (2.1).

(vi) The steepness or "redness" of \( E_u(k) \) indicates that the variance of \( u \) is concentrated in the "energy range" \( 0 < k \leq k_0 \). In fact the integral of (2.1) from \( k_0 \) to \( k_1 \) diverges as \( k_0 \to 0 \). We suppose \( E_u \) is peaked at some wavenumber \( k \sim k_0 \), that is, the dominant velocity eddies have a spatial scale \( S_u \sim k^{-1} \sim k_0^{-1} \). We shall refer to both \( S_u \) and \( k_0 \) as "scales." It may be assumed that \( \eta \) is related to \( S_u \) and \( U \) by

\[
\eta \sim T_u^{-3},
\]

where the time scale \( T_u \) is

\[
T_u = S_u/U.
\]

(vii) The statistical isotropy and stationarity of \( u \) implies that its mean value \( \langle u \rangle \) is independent of space and time. Such uniform translation has no effect on the spatial structure of a passive scalar, so we may assume without loss of generality that \( \langle u \rangle = 0 \). Otherwise a Galilean transformation of coordinates may be made, yielding the same result. Further properties of \( u \) will be introduced in Section 3. The subject of two-dimensional turbulence is reviewed by Kraichnan and Montgomery (1980).

We assume that mesoscale ocean turbulence in the wavenumber range \( k_0 \sim 10^{-3} \text{ m}^{-1} < k < k_1 \sim 10^{-3} \text{ m}^{-1} \) is approximately described by the two-dimensional enstrophy-cascading inertial subrange. Values for \( U \) of about \( 10^{-1} \text{ ms}^{-1} \) are observed, and \( S_u \sim k_0^{-1} \sim 10^5 \text{ m} \), so \( T_u \sim 10^6 \text{ s} \) or about 10 days. Robinson (1983) provides a comprehensive review of knowledge of oceanic mesoscale eddies.

b. Growing, passive scalar. Let \( C = C(x, t) \) be the phytoplankton concentration field which, like \( u \), varies with \( x \) and \( t \). If \( r = r(x, t) \) is the net specific growth rate of \( C \), then the nondissipating conservation equation for \( C \) becomes

\[
\frac{\partial C}{\partial t} + u \cdot \nabla C = rF(C)
\]
where \( \nabla = (\partial/\partial x, \partial/\partial y) \). The "growth function" \( F = F(C) \) in (2.5) is a general, nonlinear function of \( C \) which has the dimensions of \( C \), and which must vanish for at least two, distinct non-negative values \( C_1 \) and \( C_2 \). We consider nonlinear forms for \( C \), since the simple linear form \( (F(C) = C) \) with constant \( r \) leads to unbounded exponential growth of \( C \) in time, which is hardly realistic. On the other hand it is clear that if \( F \) has two distinct non-negative zeroes \( C_1 \) and \( C_2 \), then the solution of (2.5) cannot escape the interval \( C_1 < C < C_2 \) (provided the initial value of \( C \) lies in the interval). For example, the logistic model is given by \( F(C) = C(1 - C/C_p) \) where \( C_p \) is a constant carrying capacity. The corresponding solution of (2.5) lies in the interval \( 0 < C < C_p \).

The nonlinearity of (2.5) precludes the calculation of the variance spectrum of \( C \). However, if we introduce the concentration variable \( L \) defined by the nonlinear transformation

\[
L = L(C) = \int_C^C \frac{dC}{F(C)},
\]

then \( L \) satisfies the linear equation

\[
\frac{\partial L}{\partial t} + \mathbf{u} \cdot \nabla L = r.
\]

The solution of (2.7) will in general have unbounded linear growth in time for constant \( r \), but this simply implies that \( C \) is approaching either \( C_1 \) or \( C_2 \), so the integral in (2.6) diverges. In the logistic model (Levins, 1969):

\[
L = \ln C - \ln (1 - C/C_p),
\]

with \( C_1 = 0, C_2 = C_p \).

The linear model \( (F(C) = C, L = \ln C) \) may be recovered from (2.8) in the limit of large carrying capacity: \( C_p \rightarrow \infty \).

The incompressibility assumption for \( \mathbf{u} \) may be expressed as

\[
\nabla \cdot \mathbf{u} = 0.
\]

We assume that \( r \) is a random field, and allow the possibility that some realizations of \( r \) have negative values at some positions or times. Thus, grazing by zooplankton is crudely incorporated in this random model. We believe this approach to be, by virtue of simplicity, more constructive than, for example, introducing a random conservation equation for zooplankton, followed by a similar equation for their predators, and so on. In particular we assume in this section that \( r \) is statistically isotropic and stationary. We also assume that the variance spectrum of \( r \) in wavenumber space is strongly red; in fact, the variance of \( r \) is assumed to be highly concentrated around a wavenumber \( k \sim k_0 \). This corresponds to assuming that \( r \) is predominantly patchy, the patches having
Table 1. Table of scales of velocity and growth rate fields. $S_u$, $T_u$, $S$, and $T$, are correlation scales. $U$ and $R$ are amplitude scales. Other derived space and time scales are listed.

<table>
<thead>
<tr>
<th>Space Scale</th>
<th>Time Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_u \sim k_0^{-1} \sim 10^3 m$</td>
<td>$T_u \sim S_u/U \sim 10^6 s$</td>
</tr>
<tr>
<td>$S_r \sim S_u$</td>
<td>$10^3 s &lt; T_r &lt; 10^7 s$</td>
</tr>
</tbody>
</table>

Other Space and Time Scales

- $k_1^{-1} \sim 10^3 m$: largest scale at which small scale turbulent diffusion is significant; high wavenumber limit of enstrophy subrange.
- $\eta^{-1/3} \sim T_u$: time scale based on enstrophy cascade rate $\eta$.
- $T_r \sim T_u \ln (k_1/k_0) \sim 10^6 s$: equilibrium relaxation time scale.
- $t_0 \sim 3\eta^{-1/3}$: time scale of maximum patchiness due to advected growth rate.

The assumptions of statistical isotropy for $u$ and $r$ imply, via (2.7), that $L$ is also statistically isotropic. The stationarity of $u$ and $r$ do not lead to $L$ also being stationary. For example it is easily shown using (2.7) that the mean of $L$ (averaged over all realizations of $u$ and $r$) is constant in space, but a linear function of time. Nevertheless we believe that the second-order statistics of $L$ should approach an equilibrium state. The reason is that (2.7) is not valid on all spatial scales. At wavenumbers $k > k_1$, there is diffusion of $L$ variations due to mixing by three-dimensional turbulence. It is this small scale dissipation of the variance of $L$ which balances, on average, the input from the source $r$ at wavenumbers $k \sim k_0$. Thus, at equilibrium there is a net transfer, or cascade of variance of $L$ from $k_0$ to $k_1$ at a rate $\theta$, through a wavenumber range in which dissipation of $L$, and input from $r$, are negligible. In this subrange, variations of $L$ are effectively conserved, that is, they are nongrowing. They are influenced only by the turbulent velocity field and the cascade from smaller wavenumbers. Dimensional
analysis then leads to the conclusion that in this subrange, the variance spectrum of $L$ must have the form (Batchelor, 1959; Lesieur et al., 1981)

$$E_L(k) = a_2 \theta \eta^{-2/3} k^{-1}$$

(2.10)

where $a_2$ is a dimensionless constant of order unity (and so on for subsequent $a_3$, $a_4$, etc. . . ). See Figure 1. This spectrum does have a negative slope, but the variance is in effect distributed uniformly over all wavenumbers, in the following sense. The total variance in the subrange is

$$G_L = \int_{k_0}^{k_1} E_L(k) dk = a_2 \theta \eta^{-2/3} \ln \left( \frac{k_1}{k_0} \right).$$

(2.11)

The $k^{-1}$ spectrum marks the borderline between a red and a blue spectrum. Had the exponent of $k$ been $-1 - \epsilon$, for some $\epsilon > 0$, then we would have $G_L \propto k_0^{-1}$ as $k_1 \to \infty$, while an exponent of $-1 + \epsilon$ would lead to $G_L \propto k_1^\epsilon$ as $k_1 \to \infty$. The point is that at statistical equilibrium, the variance of $L$ is not predominantly in wavenumbers $k \sim k_0$ or smaller. Thus $L$ does not have the "eddy" or "patch" structure of $u$ and $r$ with length scales of $S_u \sim S_r \sim k_0^{-1}$. Instead the equilibrium field of $L$ may be described as having a very "noisy" appearance.

Some insight into the appearance of this "noisy" equilibrium field for $L$ has been obtained by theoretical considerations (Batchelor, 1959; Kraichnan, 1974; Garrett, 1983; Bennett, 1984) and numerical experiments (Holloway and Kristmannsson, 1984). The $L$ field is pervaded with long, thin streaks of width $\sim k_1$ rather than little blobs of diameter $k_1^{-1}$. The streaks have meanders with length scales $\sim S_u$. The contours of the field of the original concentration variable $C$ are identical in shape to those of the nonlinearly transformed variable $L = L(C)$, but the levels on the contours are of course different.

Finally we can estimate the time scale $T_e$ for relaxation to the equilibrium spectrum (2.10). Our estimate of $T_e$ is the time taken for the net variance of $L$ input at wavenumber $k_0$ to equal the total variance at equilibrium:

$$T_e \sim G_L/\theta = a_2 \eta^{-2/3} \ln \left( \frac{k_1}{k_0} \right) \sim a_2 T_u \ln \left( \frac{k_1}{k_0} \right).$$

(2.12)

This is of course only an order-of-magnitude estimate for $T_e$, since it neglects the variance of $L$ removed by dissipation during the relaxation process. Both (2.10) and (2.12) may be verified by detailed calculation of initial value problems for the evolution of $L$, with dissipation included in (2.7) (Batchelor, 1959; Kraichnan, 1974; and Bennett, 1985). The crucial aspect of (2.12) is that $T_e \gg T_u$ for $(k_1/k_0)$ large, that is, many "eddy turnovers" take place in the turbulent velocity field $u$ which is stirring $L$ before the variance of $L$ reaches statistical equilibrium. Another consequence of (2.12) is that, if initially (or shortly thereafter) the variance of $L$ is confined to wavenumbers $\leq k_0$ at which dissipation is negligible, then the $L$ field will have been stirred many times by $u$ before dissipation becomes significant.
The eddy turnover time $T_u$ (see Table 1) for mesoscale ocean turbulence is about 10 days. For our wavenumber range, $\ln (k_1/k_0) \sim \ln 100 \sim 5$ so $T_u \sim 50$–100 days. Evidence of the equilibrium state (2.10) should appear before seasonal influences overtake the growth rate $r$.

3. Formal solution for $L$.

In this section we shall give a formal solution for the transformed concentration $L$, corresponding to a general growth rate field $r$ which varies in space and time. We shall use the Lagrangian or particle path representation. A formal expression for the spatial covariance function for $L$ will be derived, along with an expression for the variance spectrum. It will be evident from these expressions that a knowledge of the statistics of particle pair separations is required. The statistics will be given at the end of this section. In Section 4, the formal expression for the variance spectrum will be evaluated using some idealized models for the growth rate field.

a. Particle path representation. We assume that $L = L(x, t)$ satisfies the nondissipating conservation law (2.7), subject to an initial condition for $L(x, 0)$. As discussed in the preceding section, if variance of $L$ appears first at the scale $S \sim k_0^{-1}$, then dissipation of $L$ variance is negligible initially, and remains insignificant until $t \sim T_u$ as defined by (2.12).

A formal solution of (2.7) may be obtained in terms of the particle paths $X = X(x, t, s)$ which satisfy the kinematic equation

$$\frac{\partial X}{\partial s} = u(X(x, t, s), s)$$

subject to the condition

$$X(x, t, t) = x.$$  \hspace{1cm} (3.1)

That is, $X(x, t, s)$ is the position at time $s$ of a particle which passes through position $x$ at some earlier or later time $t$. It is elementary to show that (2.7) is satisfied exactly by

$$L(x, t) = L(X(x, t, 0), 0) + \int_0^t r(X(x, t, s), s)ds.$$  \hspace{1cm} (3.3)

Eq. (3.3) expresses the change in $L$ from time zero to time $t$ as the integral of the growth rate along particle paths. A solution for $L(x, t)$ is determined by specification of an initial field $L(x, 0)$. We choose an initial concentration field $C$ which is uniform in space: $C(x, 0) = C_0$, a positive constant. Without loss of generality we may choose the lower limit of integration in (2.6) to be $C_0$, hence the initial field for $L$ is particularly simple:

$$L(x, 0) = 0.$$  \hspace{1cm} (3.4)
It is clear from (3.3) and (3.4) that $L$ will be independent of position $x$ for all time $t$ if $r$ is independent of $x$: the turbulent velocity field can only create spatial variability in $L$ (and hence $C$) if the stirring action causes different phytoplankton to experience different growth rates along their respective trajectories.

**b. The spatial covariance of $L$.** In general $L(x, t)$ will be a random field, due to the random nature of the velocity field $u$ and growth rate field $r$. Note that the particle paths are random, due to randomness in $u$. We may easily derive the first moment of $L$, using (3.3) and (3.4):

$$
(L),_u = \int_0^t \langle r(X(x, t, s)) \rangle, u ds = t \langle r \rangle
$$

(3.5)

The symbol $\langle \rangle, u$ denotes an average over all possible realizations of the independent random fields $r$ and $u$. We are assuming $r$ is statistically isotropic (and stationary), so the average of $r(X, s)$ with respect to realizations of $r$ alone is independent of $X$ (and $s$), hence a subsequent average over $u$ leads to no change. In particular it follows that $\langle L \rangle, u$ is independent of $x$. The redundant subscripts on $\langle r \rangle, u$ have, therefore, been deleted from (3.5) and from subsequent expressions when there is no ambiguity.

The simplest statistical description of the spatial structure of $L$ is provided by its spatial covariance function:

$$
V_L(D, t) = \langle L(x + D, t) L(x, t) \rangle, u - \langle L \rangle^2, u
$$

(3.6)

where $D = |D|$ is the initial separation distance. It is implied in (3.6) that $L$ is statistically isotropic, that is, the covariance is dependent only upon $D$. This will be seen to be a consequence of assuming isotropy for $u$ and $r$. However the statistics of $L$ do depend on the elapsed time $t$. For example, $\langle L \rangle, u$ is a linear function of time, as shown in (3.5).

Combining (3.3), (3.4), (3.5) and (3.6) leads to

$$
V_L(D, t) = \int_0^t ds_1 \int_0^t ds_2 \{ \langle r(X(x + D, t, s_1), s_1) r(X(x, t, s_2), s_2) \rangle, u - \langle r \rangle^2 \}.
$$

(3.7)

The two particle paths in (3.7), passing through $x$ and $x + D$ at time $t$, are sketched in Figure 2. Care is required in the choice of models for $r$ and $u$, in the evaluation of (3.7).

We require spatial continuity of $L$, at least in the mean square sense:

$$
\langle (L(x + D, t) - L(x, t))^2 \rangle, u \to 0 \text{ as } D \to 0,
$$

(3.8)

which is readily shown to be equivalent to the requirement that the spatial covariance function $V_L$ be continuous at $D = 0$:

$$
V_L(D, t) \to V_L(0, t) \text{ as } D \to 0.
$$

(3.9)
Figure 2. Positions $X(x, t, s)$ and $X(x + D, t, s)$ at times $s_1$ and $s_2$ respectively, of a pair of particles located at $x$ and $x + D$ when $s_1 = s_2 = t$. The horizontal axis is contained in the $x - y$ plane.

The condition (3.9) will be satisfied if $r$ is mean-square continuous, and

$$\xi = X(x + D, t, s) - X(x, t, s) \rightarrow 0 \text{ as } D \rightarrow 0,$$

(3.10)

for each realization of the velocity field $u$ and particle paths $X$. Note that $\xi$ is the vector separation, at the common time $s$, of two particles which are at $X + D$ and $x$ at time $t$. It may be shown that (3.10) is satisfied if each realization of $u$ is differentiable with respect to $x$.

Evaluation of (3.7) requires the choice of probability distribution functions for the particle paths. If the chosen model distributions do not imply (3.10) then there is a danger that (3.9) and hence (3.8) do not hold. In an analogous calculation Durbin (1980) introduced a model distribution for particle separations in an energy-cascading inertial subrange. His model did not imply (3.10), and in fact (3.9) did not hold. Durbin attributed this effect to the neglect of dissipation. However the model distribution used below does imply (3.10), and (3.9) does hold even though dissipation has been neglected here.

We now return to manipulating (3.7). We are assuming $r$ is statistically isotropic and stationary. Hence the space-time covariance of $r$, given by

$$V_r(\alpha; \tau) = \langle r(x + \alpha, t + \tau) r(x, t) \rangle_r - \langle r \rangle^2$$

(3.11)

is dependent only on $|\alpha|$ and $|\tau|$, where $\alpha$ is a spatial vector lag and $\tau$ is a time lag. Note the use of a semicolon in $V_r(\alpha; \tau)$ to denote that $|\tau|$ is a time lag and not a (total) elapsed time, as in the purely spatial covariance (3.6). Combining (3.7) and
(3.11) leads to

\[ V_L(D, t) = \int_0^t ds_1 \int_0^t ds_2 \langle V_r(\mathbf{X}(\mathbf{x} + \mathbf{D}, t, s_1) - \mathbf{X}(\mathbf{x}, t, s_2) |; | s_1 - s_2 |) \rangle_u. \]  

(3.12)

It remains to average over \( u \), or equivalently, over the particle paths in (3.12). This averaging may be expressed as

\[ V_L(D, t) = \int_0^t ds_1 \int_0^t ds_2 \int_0^{\infty} d\xi_{12} V_r(\xi_{12}; | s_1 - s_2 |) P_{12}(\xi_{12}; D, t, s_1, s_2) \]  

(3.13)

where \( P_{12} \) is the probability distribution function (pdf) for the two-time, two-particle separation \( \xi_{12} = | \xi_{12} | \) where

\[ \xi_{12} = \mathbf{X}(\mathbf{x} + \mathbf{D}, t, s_1) - \mathbf{X}(\mathbf{x}, t, s_2), \]  

(3.14)

given that \( \xi_{12} = \mathbf{D} \) when \( s_1 = s_2 = t \). Eq. (3.13) is our formal expression for the spatial covariance of \( L \), at elapsed time \( t \). Clearly its evaluation requires models for \( V_r \) and \( P_{12} \).

c. The variance spectrum of \( L \). The variance spectrum of \( L \) in wavenumber space, at time \( t \), is defined by the Fourier-Bessel transform

\[ E_L(k, t) = k \int_0^{\infty} J_0(kD)DV_L(D, t) dD, \]  

(3.15)

where \( J_0 \) is the zero-order Bessel function of the first kind, while \( k \) is the magnitude of the two-dimensional wavenumber vector \( \mathbf{k} \). The relation reciprocal to (3.15) is

\[ V_L(D, t) = \int_0^{\infty} J_0(kD)E_L(k, t) dk. \]  

(3.16)

In particular (3.16) yields

\[ V_L(0, t) = \int_0^{\infty} E_L(k, t) dk \]  

(3.17)

since \( J_0(0) = 1 \). That is, the variance of \( L \) is the area under the variance spectrum.

d. Separation statistics. Evaluation of (3.13), and hence (3.15), requires a knowledge of the separation pdf \( P_{12} \). We conclude Section 3 with a brief description of \( P_{12} \) in two limits.

(i) \( s_1 \approx s_2 \)

In Section 4, we will have greatest need of \( P_{12} = P_{12}(\xi_{12}; D, t, s_1, s_2) \) for values of \( s_1 \) and \( s_2 \) very nearly or exactly equal: \( s_1 \approx s_2 \), since small values of \( \xi_{12} \) (and hence large values of \( V_r(\xi_{12}; | s_1 - s_2 |) \)) are most probable when \( s_1 \approx s_2 \). In this case we shall
assume that $P_{12}$ is given approximately by $P_{11}$, that is, by its value when $s_1 = s_2 = s$, say. For simplicity we denote $P_{11}$ by $P = P(\xi, D, t, s)$ where again $\xi = |\xi|$ as defined by (3.10). If $D \ll S_u$, then the two particles at $x + D$ and $x$ at time $t$ are in covarying parts of the turbulent velocity field, and will remain so for considerable values of $|t - s|$. A bound on such values will be given below. Estimation of the statistics of $\xi$ in this case is the 
relative dispersion problem.

Kraichnan (1966) used his Lagrangian-History Direct Interaction approximation for turbulence, to derive a diffusion equation for $P$. For two-dimensional space it has the form

$$\frac{\partial P}{\partial s} = \text{sgn}(s - t) \frac{\partial}{\partial \xi} \left\{ \xi K(\xi, |s - t|) \frac{\partial}{\partial \xi} (\xi^{-1} P) \right\}$$

(3.18)

where the relative diffusivity $K = K(\xi, s)$ is given by

$$K(\xi, s) = \frac{1}{2} \xi^{-2} \int_0^\xi \xi' \left( \frac{d}{ds} \langle \xi'^2 \rangle \right) (\xi', s) d\xi'$$

(3.19)

and $\text{sgn}(t) = t/|t|$ is the algebraic sign of $t$. The appropriate initial condition for (3.18) is

$$P(\xi, D, t, t) = \delta(\xi - D)$$

(3.20)

where $\delta$ is the Dirac delta function (Lighthill, 1964). That is, $\xi$ is statistically sharp (deterministic, nonrandom) at time $s = t$, and it has the value $D$. Note that (3.18) and (3.20) ensure that $P$ is a suitably normalized pdf for all $s$:

$$\int_0^\infty P(\xi, D, t, s) d\xi = 1.$$  

(3.21)

Lundgren (1981) has an alternative, simpler derivation of (3.18) and (3.19) which assumes that the velocity field is a white noise process in time. Richardson (1926) guessed that $P$ was the solution of (3.18), with a form for $K$ to be deduced from dimensional considerations.

We require an estimate for $K$ with $\xi$ in the subrange $k_1^{-1} < \xi < k_0^{-1}$. This belongs to the enstrophy inertial subrange of $u$. Accordingly, Lin (1972) proposed on dimensional grounds that

$$\frac{d}{ds} \langle \xi^2 \rangle = a_s \eta^{1/3} \xi^2,$$

(3.22)

and hence from (3.19),

$$K = (a_s/8) \eta^{1/3} \xi^2.$$  

(3.23)

It may be shown (Bennett, 1984) that the dimensional argument is an oversimplification, but (3.22) is approximately correct for $|t - s| \geq T_u$ so neglect of dissipation in
and adoption of the relative diffusivity (3.22), confines us to the time interval $T_u \leq |t - s| \ll T_e$. From (3.19) and (3.22) we infer that

$$\langle \xi^2 \rangle = D^2 \exp \left( a_2 \eta^{1/3} |t - s| \right). \quad (3.24)$$

For $|t - s| \ll T_e$, we have from (3.24) and (2.12) the very rough estimate

$$\langle \xi^2 \rangle \approx D^2 (k_1/k_0)^2 \sim k_0^{-2} \sim S_u^2. \quad (3.25)$$

provided $D$ is not much greater than $k^{-1}_1$. We have, therefore, the criterion that if $k^{-1}_1 \lesssim D \ll k_0^{-1}$ then the pair of particle velocities remain covarying in the time interval of interest: $T_u \lesssim |t - s| \ll T_e$.

If we do adopt the relative diffusivity (3.23), then the Richardson-Kraichnan equation (3.18) and initial condition (3.20) may be solved (Lundgren, 1981), yielding a log-normal distribution:

$$P(\xi, D, t, s) = (2\pi f)^{-1/2} \xi^{-1} \exp \left[ - \left[ \ln (\xi/D) - f \right]^2 (2f)^{-1} \right] \quad (3.26)$$

where

$$f = (a_2/8)\eta^{1/3} |t - s|. \quad (3.27)$$

Again, (3.26) is strictly valid only for $D \ll k_0^{-1}$, so we may strictly use it only to estimate the covariance $V_\xi(D, t)$ for small lags, or equivalently to estimate the variance spectrum $E_\xi(k, t)$ for wavenumbers $k$ approaching $k_1$. However, this will prove sufficient to demonstrate the appearance of a spectral shape consistent with the equilibrium shape (2.10) deduced by dimensional arguments. A final remark is that we expect the single-time separation $pdf P$ to be a good approximation to the two-time separation $pdf P_{12}$ for $|s_1 - s_2| \leq \epsilon T_u$, where $\epsilon$ is perhaps as large as $1/2$, with $s$ in (3.26) taking the mean value $(s_1 + s_2)/2$.

\[ (ii) \ |s_1 - s_2| \text{ large.} \]

Suppose $|s_1 - s_2| \gg T_u$. Then $\xi_{12}$ in (3.14) is a pair separation at widely differing times. The particle velocities will be approximately uncorrelated at such times. This situation can arise only if $|t - s_1| \gg T_u$, or $|t - s_2| \gg T_u$. We shall describe the situation when both inequalities are satisfied; the analysis when only one is satisfied is similar. When both inequalities are satisfied, not only are the velocities uncorrelated but the displacements of the particles from their positions at time $t$ are uncorrelated, and the two Cartesian components of each displacement are uncorrelated, and are, asymptotically, normally distributed (Cocke; 1969, 1971, 1972). Hence the two Cartesian components of the separation vector $\xi_{12}$ are, asymptotically, independent normal random variables with means given by the respective components of $D$ and with the common variance

$$\sigma^2_{12} = \tilde{A} \left[ |t - s_1| + |t - s_2| \right]. \quad (3.28)$$
In (3.28), \( A \) is the constant absolute diffusivity (e.g. Monin and Yaglom, 1971; Okubo, 1980). It is \( O(U^2T_x) \).

This completes the statement of particle statistics required for the evaluation of (3.13), to be carried out in Section 4 for several models of the growth rate field \( r \).

4. Evolution of the variance spectrum for \( L \)

In this section we derive detailed estimates of the shape of the evolving variance spectrum of \( L \), using the formalism developed in the preceding section together with some choices for growth rate models.

\( a. \) Small time. For elapsed times \( t \) much smaller than \( T_u \) (or \( T_r \)) we may obtain simple results using either (2.7) and (3.4), or (3.13). Using the former we readily obtain an approximation for \( L \) with the first few terms of the power series expansion about \( t = 0 \):

\[
L(x, t) \approx tr(x, 0) + \frac{1}{2} t^2 \left\{ \frac{\partial r}{\partial t} (x, 0) - u(x, 0) \cdot \nabla r(x, 0) \right\} + O(t^3).
\]

(4.1)

Note that only \( r \) affects \( L \) to leading order in \( t \). Taking just the leading term in (4.1) (or approximating (3.13)) leads directly to the following simple relation between the variance spectra of \( L(x, t) \) and \( r(x, 0) \):

\[
E_L(k, t) \approx t^2 E_r(k, 0).
\]

(4.2)

Thus the spatial structure of \( L \) is, for small \( t > 0 \), the same as that of \( r(x, 0) \). In particular the variance of \( L \) is concentrated at \( k \sim k_0 \sim S_r^{-1} \). Of course \( L \) and \( r \) have not only the same structure statistically; the first term of (4.1) implies that for each realization, \( t^{-1}L \) and \( r \) are identical functions of position \( x \).

\( b. \) Intermediate times. We shall obtain estimates of \( E_L(k, t) \) for \( T_u \leq t \ll T_r \) using (3.13), (3.15) and the separation pdfs described in Section 3. To proceed, however, we must first characterize the growth rate field \( r(x, t) \). This field is the net sum of several competing processes such as growing by photosynthesis, losses due to zooplankton grazing, and possibly sinking. Accordingly \( r \) is not readily described with any precision. It seems to us that there are, qualitatively, two distinct types of processes contributing to \( r \). They are first, an advected growth rate which is conserved on particle paths (such as an inexhaustible supply of nutrients swept along with the phytoplankton concentration); and second, a local growth rate which is specified at each position and time independently of the horizontal velocity field (such as sunlight, or possibly a nonuniform vertical transport of nutrients induced by variable bottom topography). We shall examine the two types in isolation.
c. **Advected growth rate.** This type of growth rate satisfies the conservation law

\[
\frac{\partial r}{\partial t} + u \cdot \nabla r = 0, \tag{4.3}
\]

and as such is conserved along particle paths:

\[
r(x, t) = r(X(x, t, 0), 0). \tag{4.4}
\]

Eq. (4.4) expresses \( r(x, t) \) in terms of the initial field \( r(x, 0) \) and the initial particle position \( X(x, t, 0) \). If we note that (4.4) implies

\[
r(X(x, t, s), s) - r(X(x, t, 0), 0)
\]

then (3.7) may be integrated trivially, and (3.12) reduces to

\[
V_L(D, t) = \frac{1}{2} \langle V_x(|X(x + D, t, 0) - X(x, t, 0)|) \rangle \tag{4.6}
\]

where \( V_x(\xi) \) is the spatial covariance of the initial growth rate \( r(x, 0) \). We assume the latter is a statistically isotropic, random field. We take as our model of \( V_x \) the following:

\[
V_x(\xi) = R^2 \delta(k - k_0) \tag{4.7}
\]

where \( R^2 \), the total variance of \( r(x, 0) \), is independent of \( x \). The variance spectrum corresponding to (4.7) is

\[
E_x(k) = R^2 \delta(k - k_0), \tag{4.8}
\]

showing that in this model, the variance of growth rate is concentrated at the low wavenumber \( k_0 \). In particular the spectrum of \( r \) is "red."

Combining (4.6), (3.26), and (4.7) or (4.8), leads without further approximation to the variance spectrum of \( L \):

\[
E_L(k, t) = t^2 R^2 (2 \pi f_0)^{-1/2} k^{-1} \exp \left\{ - \left[ \ln \left( \frac{k}{k_0} \right) - f_0 \right]^2 (2 f_0)^{-1} \right\} \tag{4.9}
\]

where \( f_0 = a_d \eta^{1/3} \). The spectrum (4.9) has the following properties.

1. Strictly speaking, it is valid only for \( k_0 < k \leq k_1 \).
2. \( E_L \to 0 \) as \( (k/k_0) \to 0 \) or \( \infty \).
3. \( E_L \) has a solitary maximum at \( k = k_0 \).
4. The maximum value is

\[
E_L(k_0, t) = t^2 R^2 (2 \pi f_0)^{-1/2} k_0^{-1} \exp \left\{ -f_0/2 \right\}. \tag{4.10}
\]
(5) As a function of time, $E_L(k_0, t)$ has a solitary maximum at $t = t_0 = 3a_t^{-1} \eta^{-1/3}$ with the value

$$E_L(k_0, t_0) = 9t_0^2 R^2 (6\pi)^{-1/2} k_0^{-1} \exp\{-3/2\}. \tag{4.11}$$

For $t > t_0$, the new $L$ variance created by $r$ at wavenumber $k_0$ cascades to higher wavenumbers at a rate sufficient to erode the maximum (4.11).

(6) The total variance of $L$ is

$$V_L(0, t) = \int_0^\infty E_L(k, t) \, dk = t^2 R^2 \tag{4.12}$$

which grows (without bound) quadratically in time.

(7) The first moment of $E_L$ about $k = 0$ is

$$\bar{k}_1 - \bar{k}_1(t) = t^{-2} R^{-2} \int_0^\infty k E_L(k, t) \, dk - k_0 \exp\{+3f_0/2\} \tag{4.13}$$

which grows exponentially in time.

(8) There is a range of $k$ given by

$$f_0 - (2f_0)^{-1/2} \ll \ln (k/k_0) \ll f_0 + (2f_0)^{1/2} \tag{4.14}$$

in which (4.9) has an approximate power law behavior:

$$E_L(k, t) \approx t^2 R^2 (2\pi f_0)^{-1/2} k^{-1}. \tag{4.15}$$

The center of the range is at $k = k_0 \exp\{f_0\}$, which $\to \infty$ as $t \to \infty$; the bounds of the range are $\propto \exp\{(2f_0)^{1/2}\}$, which $\to \infty$ as $t \to \infty$.

See Figure 3 for $E_L(k, t)$ as a function of $(k/k_0)$, for three values of $f_0$: $f_0 = 3/2, 3, 6$. The middle value yields the maximum for the spectral peak at $k_0$. These results, especially (4.15), are consistent with the asymptotic equilibrium spectrum (2.10), on the assumption that dissipation eventually checks the rapidly growing variance which develops in the nondissipating model (2.7). The center of the range (4.14), in which (4.15) holds, indicates that the time taken for the $k^{-1}$ spectrum to become established at wavenumber $k_1$ is indeed given by the crude estimate (2.12).

The point of interest to us is that the initial patchiness in $L$ at the scale $k_0 \sim S_r^{-1}$, imposed by the growth rate field, reaches a maximum at $t = t_0 \approx \eta^{-1/3} \sim T_u$ and then becomes lost as significant variance accumulates at ever-increasing wavenumbers. That is, $L$ develops a "noisy" appearance for $t \gg T_u$.

We note that results (4.9) and (4.15) have been derived semi-quantitatively by Batchelor (1959) and especially Kraichnan (1974) for a passive scalar, such as $L$, advected by a velocity field with a random rate of strain which is spatially uniform but has a white-noise time dependence. Such a simple velocity field leads to a log-normal pdf for the component of separation along the principal axis of dilatation. Those
authors were able to include dissipation of $L$ in their simple models, leading to equilibration of the $L$ variance. The random-strain model provides a useful representation of advection in the enstrophy subrange, since the mean square strain rate and hence the particle-pair separation statistics are dominated by the energy-containing eddies (i.e. eddies with scales $S_u \sim k_0^{-1}$) even if the separation is $\sim k^{-1}$ where $k_0 \ll k$. Our more general analysis combines the formal result (3.13) derived from the passive scalar equation (2.7), the Richardson-Kraichnan equation (3.18), the relative diffusivity (3.23) for the inertial subrange, and the precisely specified, advected growth rate model (4.4), (4.7).

d. Local growth rate. A local growth rate is a field $r = r(x, t)$ specified independently of the velocity field $u$. We assume $r$ is statistically isotropic and stationary, with a
Note again that the semicolon in \( V_r(\alpha; |\tau|) \) indicates that \(|\tau|\) is a time lag. The model (4.16) has the same spatial structure as the initial field for the advected growth rate (see (4.7), (4.8)). That is, the local \( r \) field also has a “red” variance spectrum concentrated at \( k = k_0 \). The particularly simple structure of our model for \( V_r \), in terms of the time lag, is not entirely satisfactory. It leads to frequency spectra for \( r \) with some negative values. However we will not be estimating frequency spectra for \( r \); we need only simple integrals over time and these are facilitated by the elementary time structure of (4.16). Its essential feature is the (abrupt) decorrelation time \( T_r \).

It remains to combine (4.16), with the separation pdfs of Section 3, in the general expression (3.13). The result is clarified by a change of the time variables in (3.13), so that the integrations are over \(|s_1 - s_2|\) and \((s_1 + s_2)\):

\[
\int_0^t \int_0^t I(s_1, s_2) ds_1 ds_2 = \int_0^t d |s_1 - s_2| \int_{|s_1 - s_2|}^{2t-|s_1 - s_2|} d(s_1 + s_2) I(s_1, s_2)
\]  

(4.17)

where \( I \) denotes the integral over \( \xi_{12} \) in (3.13). Then, by virtue of (4.16), the repeated integral in (4.17) reduces to

\[
\int_0^{T_r} d |s_1 - s_2| \int_{|s_1 - s_2|}^{2t-|s_1 - s_2|} d(s_1 + s_2) I(s_1, s_2),
\]  

(4.18)

since we will only consider \( t > T_r \). These ranges of integration are shown in Figure 4. We have obtained approximations to (4.18) for two cases: \( T_r \ll T_u \) and \( T_r \gg T_u \). These will be discussed in turn.

(i) \( T_r \ll T_u \).

In this case, the growth rate decorrelates well before particles have been “stirred once” by the turbulent eddies. Eq. (4.18) may be approximated by

\[
2T_r \int_0^{T_r} ds I(s, s)
\]  

(4.19)

where \( s_1 = s_2 = s \), hence the average over \( u \) in \( I \) involves the single-time, two-particle, log-normal separation pdf (3.25). The integral in (4.19) may be expressed in terms of special functions, leading without further approximation to the following variance spectrum for \( L \):

\[
E_L(k, t) = a \eta^{-1/3} T R^2 k_0^{-1} G(\ln(\kappa/k_0), f_0)
\]  

(4.20)
where \( f_0 = a_4 \eta^{1/3} t \) again, and
\[
G(a, b) = -e^a \text{erfc}[(b/2)^{1/2} + a(2b)^{-1/2}] \\
+ e^{-a}[2 - \text{erfc}[(b/2)^{1/2} - a(2b)^{-1/2}]]
\]
for \( a, b > 0 \)  

(4.21)

where \( \text{erfc} \) is the complementary error function. (See Abramowitz and Stegun (1964), especially formula 7.4.33.) The spectrum (4.20) has the following properties.

1. Strictly speaking, it is valid only for \( k_0 < k \leq k_1 \).
2. As \( t \to \infty \), for fixed \( (k/k_0) \),
\[
E_L(k, t) \sim 2a_4^{-1} \eta^{-1/3} T_r R^2 k_0^{-1} \left\{
\begin{array}{l}
(k/k_0), k < k_0 \\
(k_0/k), k_0 < k
\end{array}
\right.
\]
(4.22)
3. As \( (k/k_0) \to 0 \) or \( \infty \), for fixed \( t \),
\[
E_L(k, t) \to 0.
\]
(4.23)
(4) $E_L(k, t)$ has a solitary maximum at $k = k_0$.

(5) The maximum value is

$$E_L(k_0, t) = 2a_4^{-1} \eta^{-1/3} T, R^2 k_0^{-1} \text{erfc} \left( \frac{f_0/2}{2} \right).$$

(4.24)

(6) As a function of time, the spectral peak at $k_0$ increases monotonically from zero toward the asymptotic value

$$E_L(k_0, \infty) = 2a_4^{-1} \eta^{-1/3} T, R^2 k_0^{-1}$$

(4.25)
as $t$ increases from 0 to $\infty$.

(7) The total variance of $L$ is given by

$$V_L(0, t) = \int_0^\infty E_L(k, t) dk = 2 T, t R^2$$

(4.26)

which grows (without bound) linearly in time.

(8) The first moment of $E_L$ about $k = 0$ is

$$\bar{k}_1 = \bar{k}_1(t) = (2 T, t R^2)^{-1} \int_0^\infty k E_L(k, t) \, dk$$

$$= \frac{2}{3} a_4^{-1} \eta^{-1/3} t \left( \exp \left( \frac{3 f_0}{2} \right) - 1 \right)$$

(4.27)

which $\to \infty$ as $t \to \infty$.

Inspection of (4.20), (4.21) shows that the $k^{-1}$ subrange in (4.22) is attained at $k \sim k_1$, provided $t \sim T$, as anticipated in Section 2. The spectrum (4.20), (4.21) is plotted in Figure 5 for several values of $t$.

The essential differences between (4.9), which is the variance spectrum of $L$ for an advected growth rate, and (4.20), (4.21) which is the variance spectrum of $L$ for a local growth rate which decorrelates rapidly in time, are as follows.

(1) The total variance in the former is $O(t^2)$; it is only $O(T, t)$ for the latter due to decorrelation of $r$ in time.

(2) The spectral peak at $k_0$ in the former decays as $t \to \infty$, due to the increasing probability that particles with a separation $\sim k_0^{-1}$ at time $t$ were much further apart at time zero, and hence had only weakly correlated growth rates for all times $s$ from 0 to $t$. (Note that the statistical stationarity of $u$ implies that particles tend to separate both in “forward” dispersion and “backward” dispersion.) The spectral peak at $k_0$ in the latter approaches a maximum asymptotically as $t \to \infty$, due to the continued imposition of growth rate at wavenumber $k_0$. 
Figure 5. Scaled variance spectrum $a \eta^{1/3} T_r^{-1} R^{-2} k_0 E_L(k, t) = F(|\ln (k/k_0)|, f_0)$ for local growth rate ($T_r/T_u \ll 1$), versus $(k/k_0)$ for $f_0 = a \eta^{1/3} t = 3/2, 3, 6$ (solid lines). The function $F$ is given by (4.21). The spectral peak at $(k/k_0) = 1$ approaches the maximum value (4.24) asymptotically and monotonically as $f_0 \to \infty$. Also shown (broken line) is $2 \exp \left(- |\ln (k/k_0)| \right)$ as a function of $(k/k_0)$, which is the asymptotic scaled variance spectrum in the limit $f_0 \to \infty$. The sharp peaks in the $L$ spectra at $(k/k_0) - 1$ are vestiges of the 'spike' in the $r$ spectrum at $(k/k_0) - 1$. See (4.16), (4.7) and (4.8).

(ii) $T_r \gg T_u$.

In this case, the growth rate stays correlated while the particles are “stirred many times” by the turbulent eddies. The integral over $|s_1 - s_2|$ in (4.18) may be divided, somewhat imprecisely, into two parts:

$$
\int_0^{T_r} \cdots d s_1 - s_2 = \int_0^{T_r} \cdots d s_1 - s_2 + \int_{\epsilon T_u}^{T_r} \cdots d s_1 - s_2, \quad (4.28)
$$

where $\epsilon$ is sufficiently small that the two-time separation $pdf P_{12}$ may be approximated by the single-time separation $pdf P$. We suggest $\epsilon$ might be as large as $\frac{1}{4}$ or $\frac{1}{2}$. The first part of (4.28) leads to the same results as in subsection (i) above, except that $T_r$ is replaced by $\epsilon T_u$. In particular, $P$ is the log-normal distribution, and the contribution of
the first part of (4.28) to the total variance \( V_L(0, t) \) is \( O(T_u t) \). The second part of (4.28) includes \( |s_1 - s_2| \gg T_u \), so we assume that the vector separation pdf is the normal distribution described at the very end of Section 3. A more refined treatment would recognize the existence of an intermediate region between \( t \ll T_u \) and \( T_u \ll t \), with an intermediate pdf which matches the log-normal and normal distributions at the appropriate ends of the region. However such refined treatment is not needed in what follows, where we draw conclusions only about order-of-magnitude estimates. Proceeding with the normal distribution in the second integral in (4.28), a tedious calculation shows that the resulting contribution to \( V_L(0, t) \) is only \( O(T_u^2) \), which is negligible in comparison with \( \epsilon T_u t \) if \( t \gg T_u \). We conclude that the first part of (4.28) dominates for \( t \gg T_u \), leading to the somewhat surprising conclusion that the shape of the variance spectrum is independent of \( (T_r / T_u) \). The reason is that the separations with the greatest weight (i.e. the smallest separations) are log-normally distributed irrespective of \( (T_r / T_u) \). The total variance of \( L \), however, is dependent on \( (T_r / T_u) \):

\[
V_L(0, t) = \begin{cases} 
0(T_r t), & T_r \ll T_u \\
0(\epsilon T_u t), & T_u \ll T_r .
\end{cases}
\] (4.29)

If we express \( \epsilon T_u \) as \( T_r(\epsilon T_u / T_r) \) then we see that the total variance of \( L \) has a reduction in amplitude scale as \( (T_u / T_r) \to 0 \). This is a consequence of the apparent reduction of the decorrelation time for \( r \) as sampled by a particle, during turbulent stirring through the random spatial variations of \( r \). We note that Levins (1969), who examined phytoplankton concentration as a random function of time alone, was in effect considering our case \( T_r \ll T_u \), in which turbulent spatial stirring has little effect.

In summary, our analysis of the evolving spatial structure of transformed phytoplankton concentration \( L \) shows that the variance of \( L \) cascades to ever higher wavenumbers with increasing time, leading to \( k^{-1} \) spectra. This cascade produces "noisy" \( L \) fields showing little evidence of the "patchiness" present, for small times, at the same scale as the growth rate patchiness. However in the case of advected growth rate there is a time \( t_0 \), comparable with the decorrelation time \( \eta^{-1/3} \) for the turbulence, at which the variance spectrum of \( L \) has a maximum amplitude in time for its peak at wavenumber \( k_0 \). After this time \( t_0 \), the peak erodes away to higher and lower \( k \).

5. Discussion

In Section 2 we have shown that, in general, phytoplankton patchiness distinct from that of a conserved scalar quantity (dye) in a turbulent flow field cannot be generated by a growth rate \( r - r(t) \), which is either constant or varying in time (see also Denman, 1983). In Sections 2–4, we have investigated mathematically whether a spatially variable growth rate is a sufficient condition to create and maintain patchiness in phytoplankton concentration, initially distributed horizontally either uniformly or as white noise. For the interpretation of the resulting spectra, a mathematical definition
of patchiness was necessary. We have defined a patchy distribution to occur when the variance under the spectrum between $k_0$ and $k_1$ ($k_0 < k_1$) has a characteristic length scale $k_0^{-1}$ as $k_1 \to \infty$. Otherwise its length scale is $k_1^{-1} \to 0$, which we define to be representative of a noisy distribution. The borderline spectrum has a power law behavior $\propto k^{-1}$, just the equilibrium spectrum in the inertial subrange of a passive, nongrowing (conserved) scalar in two-dimensional turbulent flow. Thus, for our derived spectra to represent patchy distributions, they must be steeper or "redder" than for a passive nongrowing scalar.

We have considered mesoscale turbulence and the related patchiness on length scales from the energy-containing eddies at $10^2$ km down to scales of 1 km, with an assumed r.m.s. turbulent velocity of $10^{-1}$ ms$^{-1}$. Time scales considered range from $10^5$ s ($\sim 1$ d) to $10^7$ s ($\sim 100$ d; see Table 1). Although both the time scale $<r>^{-1}$ and the standard deviation $R^{-1}$ associated with the specific growth rate $r$ are considered, it is the decorrelation time scale $T_r$, associated with temporal changes in $r$ itself, that is the crucial biological time scale in the patchiness problem.

We have obtained solutions to the general equation (2.5): $\partial C/\partial t + u \cdot \nabla C = rF(C)$ where the "growth function" $rF(C)$ consists of an exponential growth term $r = r(x, t)$ that may vary in space and time and a term $F(C)$ that is a function of concentration $C$. $F(C)$ usually will assume a form that causes self-limiting growth (as $C$ increases) thus eliminating unbounded exponential growth; e.g., the logistic equation for $F(C) = C(1 - C/C_p)$. We were able to linearize the growth equation (rendering it simpler mathematically) by the variable transformation $L = L(C) = \int C(1/F(C')) dC'$. Both the $L$ field and the $C$ field have the same contour shapes although they differ in magnitude. Furthermore, for the logistic model: $L = ln C - ln (1 - C/C_p)$. In the limit of large carrying capacity $C_p \to \infty$, $L \to ln C$ and the simple unbounded exponential growth equation $\partial C/\partial t + u \cdot \nabla C = rC$ is recovered.

The behavior of the variance spectrum of $L$ was deduced from dimensional and scale analysis for the cases of small elapsed time and intermediate elapsed time (the equilibrium case). For small elapsed time $t \ll T_u$ (or $\ll T_r$), the eddy (or growth rate) decorrelation timescale, the variance spectra of $L$ and $r$ are related as $E_L(k, t) = t^2E_r(k, 0)$. The initial patchiness in $L$ develops on the same scales as the initial growth rate $r(x, t)$, that is the variance is concentrated at $k_0 \sim S^{-1}$. In fact, the patterns themselves are the same (by Eq. 4.2) since $L(x, t) = r(x, 0)t$. For these small times, the motion field has little effect on the distribution of $L$, as the eddies are only just beginning to "stir" the patterns in $L$.

For times large relative to the eddy and growth decorrelation times, $T_u$ and $T_r$, an equilibrium spectrum for the $L$ field should be approached as $t \to T_e$. By dimensional analysis, this spectrum is identical to that for a nongrowing scalar, namely it obeys a $k^{-1}$ power law which we have designated as the borderline case between "patchy" and "noisy" distributions. We have estimated the relaxation time $T_r$ for approach to this
equilibrium by considering the time for the net variance in $L$ input at wavenumber $k_0$ to equal the total variance between $k_0$ and $k$, at equilibrium (see Eq. 2.10). For $(k_1/k_0)$ large, $T_r \gg T_u$; that is, many eddies must be traversed by a typical phytoplankter before the stirring is sufficient for the variance of $L$ to approach statistical equilibrium. We find $T_r \sim 50–100$ days for the scales we have used, suggesting that the equilibrium distribution in $L$, i.e. a $k^{-1}$ spectral shape, may be approached before seasonal variations in the growth rate $r$ make our solutions invalid. The remaining results of Section 4 considered the time evolution of the variance spectrum of $L$ for intermediate times $T_u \sim 10 \text{ d} \ll t \ll T_r \sim 100 \text{ d}$.

Two types of random spatial variation in growth rate have been considered. They are first an *advected* growth rate conserved along particle paths (as might occur from an inexhaustible nutrient field swept along with the phytoplankton), and second a *local* growth rate specified at each position and time independently of the velocity field (as might occur from a nonuniform vertical supply of nutrients induced by variable bottom topography). One might expect the former case (advected growth rate) to induce greater patchiness in phytoplankton concentration because the transformed phytoplankton and growth rate (or nutrient) fields are contiguous: they are advected (and consequently stirred) together by the horizontal flow field.

The time evolution of the phytoplankton variance spectrum $E_L(k)$ for the case of advected growth rate was presented in Figure 3. There is a solitary maximum at $k = k_0$. Also a region of the curve behaves as $k^{-1}$; this region $\rightarrow \infty$ as $t \rightarrow \infty$. But most importantly, the magnitude of the maximum at $k = k_0$ increases until $t \sim T_u$; i.e., one eddy turnover time, after which it decreases exponentially with time, indicating variance cascading to higher wavenumbers. The final result is a "noisy" distribution in the $L$ field. Our detailed results are a consequence of having adopted a log-normal $pdf$ for particle-pair separations. Numerical simulations of particle separation in two-dimensional turbulence (Holloway, personal communication) indicate exponentially growing second and fourth moments of separation. This is entirely consistent with a log-normal $pdf$.

We considered two cases for a local growth rate field: one for $T_r \ll T_u$ where the growth rate decorrelates well before particles have been "stirred once" by the turbulent eddies, and one for $T_r \gg T_u$ where the growth rate stays correlated while the particles are "stirred many times" by the eddy field. For both cases we obtained the approximate spectrum for transformed phytoplankton concentration $E_L(k)$ shown in Figure 5, which, strictly speaking, is valid only for $k \gg k_0$. $E_L(k)$ has a peak at $k = k_0$ and a maximum (reached as $t \rightarrow \infty$) which for $k \gg k_0$ is asymptotic to $k^{-1}$. Whereas the $L$ variance for advected growth increased as $t^2$, here it increases linearly with time $t$, more slowly because the $r$ and $L$ fields are not advected together by the flow field.

For $T_r \ll T_u$, the total variance in $L$ grows as $T_r t$. This case is essentially that of Levins (1969) who considered the random growth of phytoplankton in a motionless ocean. In other words, since $T_r \ll T_u$, the growth rate itself decorrelates before sig-
significant stirring occurs; hence, $T_u$ and the turbulent motion field are not relevant to the problem. For $T_u \gg T_r$, the total variance in $L$ grows as $T_r(\epsilon T_u / T_r)t$. ($\epsilon$ is of order 1.) As $T_u / T_r \to 0$, i.e. as the stirring becomes more intense relative to the random decorrelation of $r$, the rate of growth of total variance in $L$ also decreases. Essentially the stirring is so vigorous that the phytoplankton pass through regions of high (or low) local growth rate too quickly for each region to impart a significant anomaly to the accumulated $L$ concentration. Also, any two neighboring phytoplankters at some given time have been stirred by many different eddies; hence they have passed through many different growth rate patches and probably had wide divergent particle paths. There has been little chance for growth rate patches of scale $k_0$ to create variance in the $L$ field at similar scales.

Our results concerning phytoplankton patchiness have been expressed in terms of wavenumber variance spectra; more realistic descriptions of the eddy and phytoplankton concentration fields will require numerical simulation. Although we used nondiffusive particle statistics, it may be shown (Bennett, 1985) that inclusion of small scale turbulent diffusion of $L$ modifies our conclusions negligibly. Numerical simulation of the various models discussed will require first a numerical simulation of the two-dimensional turbulence. These tend to produce kinetic energy spectra somewhat steeper than $k^{-3}$ (see, e.g. Bennett and Haidvogel, 1983; Bennett and Middleton, 1983, and their references). However the separation statistics should still be log normal for separations initially much smaller than an energetic eddy (Bennett, 1984). In particular the numerical simulations could be examined, in the case of an advected growth rate field, for evidence of a temporal peak in the patchiness in an advected growth rate field, support for the estimated relaxation time, and the tendency to $k^{-1}$ spectra. In fact Holloway and Kristmannsson (1984) have simulated such a tendency for a passive scalar conserved in the enstrophy subrange, like $L$. They, and other authors, have found that in the field realizations, where the scalar has a $k^{-1}$ spectrum, it is stretched and distorted by the eddy field into long convoluted streaks or tendrils. In the case of a local growth rate field, the sensitivity of the amplitude of total variance to the ratio $T_r / T_u$ is also of interest.

Simulations could be carried out for more realistic situations. The spinup of the advected growth rate case for different initial and boundary conditions could be investigated. An advected growth rate field with an exhaustible nutrient supply would also be of interest. Finally, a numerical simulation of local growth with $T_r = T_u$ may be relevant to the situation of patches being generated by nutrient enrichment at certain locations due to flow disruptions over features in bottom topography such as seamounts, or canyons and bumps on continental shelves.

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