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Oceanic property transport, Lagrangian particle statistics, and their prediction

by R. E. Davis

ABSTRACT

The relationship between the transport of scalar properties and the statistics of material particle motion is examined. It is shown that evolution of the mean concentration field is determined by the statistics of single particles while two particle statistics describe the typical dispersion of individual property clouds. It is argued that oceanic observations of quasi-Lagrangian floats provide a useful and direct description of lateral advection and eddy dispersal. A simple model for predicting statistics of particle dispersal from Eulerian statistics of velocity is advanced. This model is tested against simulations of particle motion in random two-dimensional velocity fields with prescribed Eulerian statistics in which there is no mean velocity. The model is found to provide satisfactory description of the one and two particle statistics in statistically stationary, homogeneous, and joint normally distributed velocity fields. Adequately predicted are (i) mean particle velocity and mean square particle speed, (ii) the Lagrangian frequency spectrum of single particles from which the single particle diffusivity can be computed, and (iii) the mean square separation between particles from which the two particle diffusivity can be computed. A simple generalization of the theory to velocity fields with weakly inhomogeneous statistics describes the evolution of the single particle density, or equivalently the mean concentration field of a conserved scalar property, and describes particle migration induced by spatial variation of dispersion.

1. Introduction

The physical analogy between the movement of an oceanic property $\theta$, such as heat or salt, and Lagrangian particles invites the use of drifting buoys for characterizing oceanic transport processes. Complete descriptions of velocity fields obtained from Lagrangian sampling following fluid particles or from Eulerian sampling at prescribed positions and times are equivalent since there is only one velocity field. But realizable observations are always incomplete and thus must be interpreted at least partially in a statistical way. It is here that Lagrangian and Eulerian sampling differ fundamentally and it becomes important to determine which approach most efficiently answers the questions of interest. For example, Eulerian sampling may be preferred for mapping a velocity field or determining energetic scales because the sampling locations may be specified so as to uniformly sample space or separation.
space and there is maximum likelihood that the sampling locations are typical because, in contrast to Lagrangian sampling, the measurement locations are chosen without regard to the flow itself.

One purpose of this paper is to show how Lagrangian sampling addresses directly those features of the velocity field most important in describing oceanic transport of scalar properties. Within the Eulerian framework, property transport is conventionally treated as the result of the advective flux $U\Theta$ and the eddy flux $\bar{u}'\bar{\theta}'$, where $U$ and $\Theta$ are the resolvable mean fields, $\theta' = \theta - \bar{\theta}$ and $u'$ are fluctuations which are considered unresolvable and thus to be described statistically, and the overbar represents the Eulerian average used to define the mean fields. Defining this average is a subtle point (see Bryan, 1975), but for present purposes it will be considered an ensemble average over similarly prepared realizations; in practice the realizations averaged would be sequentially repeated occurrences in which time is reckoned from the randomly selected start time. While both advective and eddy fluxes are directly observable, generalization of the results to scalar fields other than that measured is difficult. Frequently this generalization is approached by assuming that there is a local relation between the eddy flux and $\nabla\Theta$, that is through an analogy between eddy dispersion and molecular diffusion.

There are theoretical reasons (see Batchelor and Townsend (1956) and Roberts (1961)) to believe that, in general, no local relation between the eddy flux and the $\Theta$ field exists. As is discussed in Section 2, evolution of the mean field follows a flux-gradient law only over times long compared with the decorrelation time of Lagrangian velocity variability. Since the magnitude of this decorrelation time (or even its existence) in the ocean has not been determined observationally, eddy diffusion modelling of oceanic property transport must be regarded as a bold assumption. The point emphasized here is that this assumption is not necessary if single particle Lagrangian statistics are known. This is so because, as is discussed in Section 2, the probability density function of particle displacement is the Green's function for the determination of the mean concentration of scalar property resulting from arbitrary initial conditions and/or source and sink distributions.

Many questions about oceanic property transport involve more than evolution of the mean field. For example, the dispersion or stirring of individual property clouds is of particular interest with respect to localized man-made sources. This is an important additional description of transport because, in contrast to the precepts of the mixing length description of eddy diffusion, oceanic dispersion can smooth the mean field $\Theta$ without stirring the individual realizations from which the mean is constructed; random displacement of the initial cloud in each realization will lead to a dispersed mean field even if the individual clouds are not stirred. This is why $\Theta$ can be predicted from single particle statistics alone. No satisfactory method for predicting characteristic property cloud dispersal from Eulerian velocity statistics has been developed. As pointed out by Stommel (1949) and demonstrated further
by Okubo (1971), eddy diffusion approaches are unsatisfactory, at least with respect to lateral dispersion, because the apparent diffusivity increases with the size of the dispersing cloud. Various characterizations of property field stirring can be obtained from the statistics of multi-particle clusters. As is discussed in Section 2, the space lagged covariance of concentration is determined by the joint probability density of two particles and the typical size of dispersing property clouds is determined by the mean square separation between particle pairs.

A large quantity of information is required to fully describe particle statistics, particularly multi-particle statistics. It is therefore important that the information to be determined through observation be minimized. A major thesis of this paper is that in statistically homogeneous fields credible descriptions of property transport can be obtained by assuming particle displacements to be joint normally distributed. This greatly reduces the information required to describe transport since joint normal distributions are completely specified by mean values and covariances. Extension of the hypothesis of normally distributed particle displacements to fields with statistics varying slowly in space is also considered.

While Lagrangian statistics provide the most direct and concise description of property transport, there is need to relate such statistics to Eulerian observations so that the available observational base be maximized and so that a physical intuition about the Eulerian and Lagrangian views of the ocean be obtained. In Section 3 and Section 4 the hypothesis of joint normally distributed particle displacements is combined with an extension of Corrsin's conjecture and a statistically optimized estimator of particle density in order to predict one and two particle statistics in statistically homogeneous flows. These predictions are tested against simulations of particle motion in random velocity fields and are found to be remarkably accurate, thus lending some support to the underlying hypotheses. In Section 5 simulations of particle motion in Eulerian fields with inhomogeneous statistics are compared with theoretical predictions based on an extension of the single particle theory of Section 3.

2. Particle statistics and property transport

In what follows the equations and semantics pertain to particles and a property field \( O(x,t) \) confined to a two-dimensional surface in which the Eulerian velocity is \( u(x,t) \); extension to one or three dimensions is trivial. In the oceanographic context the two-dimensional surface might be one of constant pressure or constant potential density. The Lagrangian description of motion in terms of the particle position \( r \) and velocity \( v \) in this surface is

\[
v(t,r_0) = \partial r(t,r_0) = u[r(t,r_0), t], \quad r_0 = r(0,r_0).
\]  

Identification of a particle by its position at time \( t = 0 \) is arbitrary, but \( t = 0 \) will have a special meaning with respect to Lagrangian statistics.
It is assumed that Eulerian statistics are stationary and thus may be defined as either time or ensemble averages denoted by an overbar. The Eulerian mean velocity and velocity covariance are

\[ U(x) = \overline{u(x,t)}, \quad E_{nm}(x,y,t) = \overline{u'(x,t+\tau) u'(y,\tau)}, \]

respectively, where \( u' = u - U \). In statistically homogeneous fields, \( U \) is independent of \( x \) and \( E_{nm} \) depends only on \( x - y \).

Lagrangian statistics are defined with respect to an average, denoted by \( <\cdot> \), over an ensemble of particles. Care must be exercised in specifying how the particles in this ensemble are chosen. Here the ensemble includes particles deployed at time \( t = 0 \) and position \( x = r_0 \) in different realizations of the flow. In the statistically stationary velocity fields considered here, this corresponds to a series of realizations made by deploying particles at \( r_0 \) at random times (without knowledge of the flow) and initializing the time coordinate at that instant. The Lagrangian mean velocity and velocity covariance are

\[ V(t,r_0) = <u(t,r_0), t>, \quad <v'(t,r_0) v'(t,r_0)>, \]

respectively. In statistically homogeneous fields \( V \) will not depend on \( r_0 \) and the covariance will depend only on \( r_0 - r_0 \).

It is important to contrast the random deployment average \( <\cdot> \) with the alternative procedure of averaging an ensemble of realizations begun when a previously deployed particle is found at \( r_0 \). The example of Stokes Drift in wave fields (Phillips (1977) discusses this for surface waves in Section 3.3) illustrates the problem. Imagine a three-dimensional wave field in which particles are confined to a relatively flat two-dimensional surface such as the interface in a gravity wave field, a constant density surface for internal waves, or a surface of constant potential vorticity in a field of barotropic Rossby waves. Imagine further that the flow statistics are homogeneous within this surface. Then the random deployment average \( <\cdot> \) can be obtained by averaging over particles deployed at \( t = 0 \), uniformly over a surface of infinite extent. The probability of finding a particle at a point is simply the particle density there. Since the velocity tangential to the surface is generally divergent, this density changes and correlations between particle density and flow parameters can develop. Because the deployment is random, particle density is initially uncorrelated with the flow and the Lagrangian mean velocity is equal to the Eulerian mean over the surface. But as time passes a difference \( V - U \) develops because particles remain longer in particular parts of the flow or, equivalently, because particle density is higher in certain regions. In surface waves particles remain longer under crests, where particle motion is parallel to phase propagation, than under crests. Equivalently, the density of particles is highest under crests.

The example above shows how particles preferentially sample certain regions of the flow, where particle density is high, and how Lagrangian velocity statistics may
not be stationary even though the Eulerian statistics are stationary and homogeneous. However, in this case the Lagrangian statistics defined in (2.3) will generally approach a stationary limit as $t \to \infty$ and the correlation between particle density and flow parameters becomes constant. Lagrangian statistics in this stationary limit will be called "long-after-deployment" averages, denoted by $\langle \cdot \rangle_\infty$, and are equivalent to the average over an ensemble of particles encountered randomly long after their deployment. The approach to this limit was discussed in Davis (1982).

The Lagrangian mean velocity and covariance of (2.3) are only partial descriptions of particle motion. The complete description of single particles is the probability density

$$P_1(x,t,r_0) = \langle \delta[x-r(t,r_0)] \rangle,$$

(2.4)

where $\delta$ is the Dirac delta. Similarly, the complete description of the simultaneous positions of two particles deployed at time $t = 0$ is the joint probability density

$$P_2(x,\hat{x},t,r_0,\hat{r}_0) = \langle \delta[x-r(t,r_0)] \delta[\hat{x}-r(t,\hat{r}_0)] \rangle.$$

(2.5)

A subset of the description in $P_2$ is the probability density of the separation, $y$, between particle pairs

$$P_\delta(y,t,r_0,\hat{r}_0) = \langle \delta[y-r(t,\hat{r}_0) + \hat{r}(t,r_0)] \rangle = \int dx \ P_2(y+x,t,r_0,\hat{r}_0).$$

(2.6)

The remainder of this section is devoted to demonstrating how $P_1$ can be used to describe evolution of the mean concentration field $\Theta$ resulting from prescribed initial/source/sink distributions and how $P_2$ and $P_\delta$ describe the typical dispersal of individual fields of $\theta$. Much of this material is review (see Csanady (1973)) but I believe a unified discussion is of value. Before proceeding to this it is well to note that the "particles" in (2.4)-(2.6) can be regarded as either molecules or continuum particles. If they are taken as molecules then the probability densities describe both turbulent dispersion and molecular diffusion. In the presence of reasonable levels of oceanic turbulence the two statistics are unlikely to differ significantly on any but the smallest time and space scales so molecular diffusion will be neglected. For conceptual purposes, however, it is helpful to bear both points of view in mind.

The unaveraged transport equation for $\theta$ may be written

$$\frac{D\theta}{Dt} = q, \quad \theta(x,0) = \theta_0(x),$$

(2.7)

where the substantive time derivative is taken following a fluid particle, that is holding $r_0$ constant. Thus the solution is simply

$$\theta(x,t) = \int dx_0 \int_0^t d\tau \ \delta[x-r(t-\tau,r_0)] [q(r_0,\tau) + \delta(\tau)\theta_0(r_0)].$$

(2.8)
So long as the initial/source/sink data are deterministic or statistically independent of velocity, averaging this yields

\[ \Theta(x,t) = \langle \theta(x,t) \rangle = \int dx_0 \int_0^t d\tau P_1(x,t-\tau, x_0) [q(\tau) + \delta(\tau)\theta(0)] . \]  

(2.9)

This is a generalization of a well-known result (see Monin and Yaglom, 1971, ch. 10) in which account is taken of sources and sinks. It shows that \( P_1 \) is, in fact, the Green's function for the averaged transport equation. Thus even though there exists no accepted closed form Eulerian equation for the evolution of \( \Theta \), the simplest particle probability density provides a solution.

The nature of (2.9) shows that it is fundamentally incorrect to associate eddy dispersion smoothing of \( \Theta \) with the kind of stirring which dramatically distorts the individual fields \( \theta \) from which the mean field is obtained. Smoothing of \( \Theta \) is the result of growing uncertainty in the position of Lagrangian particles and it is this growing unpredictability which introduces the irreversible nature of eddy dispersion. While particle position uncertainty may be produced by complex fields capable of vigorous stirring, it can also result from random displacement by large scale fields which distort very little individual \( \theta \) distributions. In order to smooth an initially localized \( \Theta \) distribution, a velocity field need only have Lagrangian velocity variability with time scales long compared with the time since release of \( \Theta \). This is demonstrated further in Section 3 where it is found that purely wavelike velocity fields, without low frequency Eulerian energy, can produce finite energy at low frequency in the Lagrangian velocity spectrum and can, therefore, disperse mean fields to arbitrarily large scales.

The quantity of information required to determine \( P_1 \) is substantial so that (2.9) offers little practical advantage over phenomenological transport models involving eddy diffusivities. It is a thesis of this paper that accurate approximations for \( P_1 \) can and should be developed for exploration of oceanic property distributions. It is here argued that \( P_1 \) is a physically meaningful quantity which, when described in terms of a few parameters, provides a more sound basis for such exploration than do eddy diffusivities, which demonstrably fail to describe eddy dispersal in the ocean. Two main avenues of approximation present themselves. First, the shape of \( P_1 \) as a function of \( x \) can be modeled and, secondly, the information needed to specify the time evolution of \( P_1 \) can be minimized.

Modelling the shape of \( P_1 \) is explored below, starting from the hypothesis that the shape is approximately Gaussian in statistically homogeneous flows and is therefore determined by the mean particle displacement and the displacement variance

\[ R(t,x_0) = \langle r(t,x_0)^2 \rangle , \quad \langle r_0(t,x_0) r_0'(t,x_0) \rangle , \]  

(2.10)

respectively. A Gaussian shaped \( P_1 \) is consistent with both eddy diffusivity modelling and observations in laboratory turbulence (see Hinze (1959), ch. 5).

A possible second step toward simplifying description of \( P_1 \) involves determining
the decorrelation time, \( t_D \), over which sequential particle displacements are effectively statistically independent. If, for example, \( \mathbf{r}(t+\tau_0, \mathbf{r}_0) - \mathbf{r}(t_0, \mathbf{r}_0) \) is independent of \( \mathbf{r}(t_0, \mathbf{r}_0) \) for \( t > t_D \), then the displacement covariance \( \langle r_n(t) r'_m(t) \rangle \) grows linearly with \( t \) for \( t > t_D \), and

\[
P_1(x, t+t_0, \mathbf{r}_0) = \int dy P_1(x, t, y) \cdot P_1(y, t_0, \mathbf{r}_0) \tag{2.11}
\]

for \( t > t_D \). A Gaussian shaped \( P_1 \) with a displacement covariance which grows linearly with time for all \( t \) is assumed in eddy diffusivity modelling, and results in (2.11). So long as the eddy dispersing medium has a finite time scale, two sequential displacements can never be strictly independent, regardless of the length of time over which the displacements occur. But if there truly is a decorrelation time for oceanic Lagrangian velocities (a hope as yet unproven by observation), then it is reasonable to expect (2.11) to hold for \( t > t_D \), so that \( P_1 \) can be computed for all times once it was known for times of the order \( t_D \). Eddy diffusivity modelling can only be justified if the decorrelation time \( t_D \) is much smaller than the time scales over which oceanic sources and sinks persist and if features of \( \Theta \) smaller than \( \langle |r'(t_D)|^2 \rangle^{1/2} \) are unimportant. In view of the central role gradient diffusion plays in our perception of oceanic transport, it seems that demonstration of a decorrelation time and its determination should be primary goals of ocean observation programs.

The ability to predict the mean field \( \Theta \) does not constitute a complete understanding of oceanic property transport, although it would seem a primary step in general circulation studies. Description of the typical evolution of individual \( \theta \) fields, including how they are distorted and stirred, is another part of the understanding needed. Characteristics of stirring can be developed from multi-particle statistics. In view of the difficulty of measuring the statistics of more than two particles, it is fortunate that significant information can be gained from the particle pair statistics \( P_2 \) and \( P_S \) of (2.5) and (2.6). The simplest circumstance is when \( \theta \) evolves from the deterministic initial condition \( \theta_0 \) in the absence of sources and sinks, in which case \( q = 0 \) in (2.7) and the solution (2.8) simplifies to

\[
\theta(x, t) = \int d\mathbf{r}_0 \delta(x - \mathbf{r}(t, \mathbf{r}_0)) \theta(\mathbf{r}_0). \tag{2.12}
\]

Extension of the following discussion to nonvanishing sources and sinks is straightforward but leads to the necessity of knowing the analog of \( P_2 \) for particles deployed at different times.

A useful description of the scales introduced in the \( \theta \) field by stirring is the space lagged mean product which, from (2.12), is

\[
\overline{\theta(x, t) \theta(\mathbf{x}, t)} = \int d\mathbf{r}_0 \int d\mathbf{r}_0 P_2(\mathbf{x}, \mathbf{x}, \mathbf{r}_0, \mathbf{r}_0) \theta_0(\mathbf{r}_0) \theta_0(\mathbf{r}_0). \tag{2.13}
\]
The equality in (2.13) follows from the equivalence of $\overline{\cdot}$ and $<\cdot>$ averages. The space lagged covariance of $\theta$ is easily obtained by combining (2.9) and (2.13). Since neither $P_2$ nor $\overline{\theta(x)\theta(y)}$ is readily observable, it is fortunate that much of the stirring description provided by $P_2$ can be gained from the associated, and much more easily measured, particle separation statistic $P_s$ of (2.6). In particular, the energetic scales of $\theta$ may be described by the distance-neighbor function introduced by Richardson (1926) (see also Batchelor (1952)), which is related to $P_s$ through

$$Q^{-2} \int d\mathbf{x} \, \theta(y+x,t)\theta(x,t) = \int d\mathbf{r}_0 \int d\mathbf{r}_1 \, P_s(y,t,\mathbf{r}_0,\mathbf{r}_1) \, Q^{-2} \theta_0(\mathbf{r}_0)\theta_0(\mathbf{r}_1) \, ,$$

(2.14)

where $Q = \int dx \theta(x)$ is the total quantity of $\theta$ stuff. This follows directly from (2.13). The related distance neighbor construct based on the fluctuations $\theta'$ introduced by Chatwin and Sullivan (1980) may be related to $P_s$ and $P_2$ by combining (2.9) and (2.14). It will be noted from (2.14) that if the property field is homogeneous then $P_s$ determines the wavenumber spectrum of $\theta$.

Equations (2.13) and (2.14) serve the same function in describing variance of $\theta$ as (2.9) plays in describing the mean $\Theta$: in each case the Eulerian statistics are the convolution of initial/source/sink prescription data with the description of dispersion as provided by particle statistics. It is this structure which makes difficult inversion of Eulerian statistics to obtain characterizations of the dispersing medium and argues for direct measurement of the particle statistics.

In order for the transport descriptions involving $P_2$ or $P_s$ to be useful the considerable information in these probability densities must be measured or inferred. Various conjectures on the structure of these functions may be advanced but it seems unlikely that anything simple will well describe the structure from the dissipating scales to the energetic scales. Batchelor (1952) discussed the nature of $P_s$ for well-developed turbulence using dimensional arguments and hypotheses about the dependence of separation on its initial value. In Section 4 the prediction of two particle statistics is approached using a model based on the proposition that in statistically homogeneous flows $P_s$ has a Gaussian shape. In this case $P_s$ is fully specified by the mean of the particle separation $s(t,s_0) = r(t,s_0) - r(t,0)$ between particles initially separated by $s_0$ and the mean products

$$\sigma^2_{nm}(t,s_0) = <s_n(t,s_0) s_m(t,s_0)> = \int dy y_n y_m P_s(y,t,\mathbf{r}_0,0) \, .$$

(2.15)

The success of the model based on a Gaussian $P_s$ in predicting statistics obtained from simulations of particle behavior lends some support to the hypothesis, but the test is a weak one involving primarily the energetic scales of relative Lagrangian motion and a very special class of velocity fields.

Some insight into the meaning of particle displacement and separation statistics
is gained from describing the typical behavior of initially concentrated clouds of \( \theta \) released from the same location at time \( t = 0 \). A measure of the position of each cloud is the centroid

\[
e(t) = Q^{-1} \int dx \, \theta(x, t) .
\]

The motion of the mean centroid \( \bar{c} \) provides information analogous to, but not identical to, what would be called mean advection in the Eulerian frame. Thus

\[
\frac{d}{dt} \bar{c}(t) = \frac{d}{dt} \frac{1}{Q} \int dx \, \bar{\theta}(x, t) = \frac{1}{Q} \int d\mathbf{r}_0 \, \mathbf{V}(t, \mathbf{r}_0) \, \theta_0(\mathbf{r}_0) .
\]

As is discussed in Section 3 and Section 5, the Lagrangian mean velocity is determined by the Eulerian mean velocity, the degree of correlation between velocity and particle density, and spatial variations of Eulerian velocity statistics. To me it seems that \( \mathbf{V} \) is a more direct description of "mean transport" than is the Eulerian mean velocity.

A measure of the spread of the mean concentration field \( \theta \) about the mean centroid is

\[
\frac{1}{Q} \int dx \, (x_n - c_n)^2 \theta(x) = \frac{1}{Q} \int d\mathbf{r}_0 \, <[\mathbf{r}'_n(t, \mathbf{r}_0)]^2> \theta_0(\mathbf{r}_0) = \frac{1}{Q} \int dx \, (x_n - c_n)^2 \theta(x) + (c_n - \bar{c})^2 .
\]

This statistic, a measure of absolute dispersion, is determined by the single particle displacement variance. From the second line of (2.16) it is seen that absolute dispersion is the sum of the relative dispersion of each cloud around its own centroid plus a contribution from centroid dispersion. In molecular diffusion the latter term is negligible. But in turbulent dispersion much of the spread of the mean field can result from centroid dispersion, a process which can occur in the absence of what would usually be called stirring. The measure of relative dispersion in (2.16) is determined by particle pair separation statistics through

\[
Q^{-1} \int dx \, \frac{(x_n - c_n)(x_m - c_m)}{\theta(x)} = \frac{1}{2} \int dy \, y_n y_m Q^{-2} \int dx \, \frac{\theta(t, y + x)}{\theta(t, x)} ,
\]

\[
= \frac{1}{2} \int d\mathbf{r}_0 \int d\mathbf{r}_0 \, \sigma_{nm}(t, \mathbf{r}_0, \mathbf{r}_0) Q^{-2} \theta_0(\mathbf{r}_0) \theta_0(\mathbf{r}_0) .
\]

The first equality in (2.17) follows by direct manipulation using the definition of \( \theta \) (see Batchelor (1952)) and the second follows from (2.14) and (2.15).

The probability densities \( P_1, P_2, \) and especially \( P_2 \) are essentially unobservable in the ocean. In order to properly model oceanic transport it appears most promising
to represent these probability densities with prescribed functional forms involving a few observable parameters. In this light, eddy diffusivity modelling may be regarded as a first step in which statistically homogeneous velocities lead to Gaussian shaped $P_1$, the displacement variance grows linearly with time, $P_2$ is the product of two single particle probability densities, and the only parameter to be measured or predicted is the eddy diffusivity. The models introduced below are more accurate representations of real particle statistics but, as a consequence, require more information to be measured or predicted. The quantities required are mean velocity and the mean squares of particle displacement and the separation between particle pairs. These are readily measured and, as discussed below, in some cases they can be estimated if the Eulerian statistics of velocity are known.

3. Single particles in homogeneous flows

In this section attention is directed to predicting the stationary long-after-deployment statistics of single particles in flows with homogeneous Eulerian statistics and $U = 0$. The development parallels that of Davis (1982) in which the foundation was more fully discussed and some attention was given to the approach to stationary Lagrangian statistics. Because only the stationary statistics of a single particle in a homogeneous field are considered, some notational economies and conceptual simplifications can be achieved. The Lagrangian mean can be obtained by averaging over particles initially deployed uniformly over an infinite two-dimensional surface. The Lagrangian mean $\langle \cdot \rangle$ is then equal to the Eulerian mean weighted by the probability of encountering a particle, which is simply particle density per unit area, $\rho(x,t)$.

It is assumed that at some time long after deployment the particles are relocated, relabelled with their position $r_o$, and time is reset to $t = 0$. The statistics of particle displacement, $\delta r = r - r_o$, are homogeneous so that, with $\bar{\rho} = 1$, the Lagrangian velocity mean and mean product are

$$V_\infty = \frac{d}{dt} \langle \delta r \rangle = \rho(x,t)u(x,t), \quad (3.1a)$$

and

$$\langle v_n(0) v_m(t) \rangle = \rho(x,0) u_n(x,0) u_m[x+\delta r(t,x),t], \quad (3.1b)$$

respectively. These are stationary for all $t \geq 0$. If the flow is nondivergent, single time statistics like $V$ are equal to the equivalent Eulerian average over the surface. The complete single particle statistic is $P_6$, the probability density function of displacement which is related to $P_1$ of (2.4) through

$$P_6(\delta r, t) = P_1(\delta r + r_o, t, r_o). \quad (3.2)$$

Prediction of these is explored below. The development involves two reasonably
distinct elements. The first, and more novel, is a statistically optimized predictor for
the particle density $\rho$. This determines quantities like particle drift and mean square
particle speed. The second element is a pair of assumptions about the statistics of
$\delta r$, one first advanced by Corrsin (1960). This is the primary element in predicting
the shape of the Lagrangian velocity spectrum and the single particle diffusivity.

Development of the statistically optimized density estimator was discussed more ex-
tensively in Davis (1982). Basically, the procedure is to specify $\rho$ in terms of $u$, whose
statistics are known, using a functional involving adjustable parameters. These
parameters are then selected to minimize the mean square error when the functional
is substituted into the particle conservation equation

$$
(\partial_t + u \cdot \nabla) \ln(\rho) = - \nabla \cdot u .
$$

(3.3)
The approach of employing statistically optimized approximate solutions to prob-
lems involving many degrees of freedom appears a powerful, if rarely employed,
tactic. It should be regarded as an augmentation to more classical approximation
schemes, such as asymptotic expansions. The functionals to be optimized can, and
should, be developed from analytic approximations and then modified, through in-
troduction of parameters which are adjusted to minimize the statistical error measure.
If the original approximation cannot be improved upon, the optimization procedure
will return it unmodified.

The particular estimator proposed here is

$$
\rho = \rho_0 e^{-\alpha \zeta} [1 + \beta |u|^2]^{-\gamma}, \quad \zeta = \int_{-\infty}^t \nabla \cdot u \, dt'.
$$

(3.4)
The constants $\alpha, \beta, \gamma$ are selected to minimize the mean square of the error $\epsilon$
when $\rho$ is substituted into the particle conservation equation (3.3). The form of $\rho$
was selected as a convenient relationship which models the two basic tendencies for
particle concentration. The exponential term describes behavior in weak flow, where
the advective term in (3.3) is relatively small. The remaining term serves to model
behavior in strong flow, when the particle conservation equation is essentially steady
and the effect on $\rho$ of $\nabla \cdot u$ (here modelled by $\exp(-\alpha \zeta)$) is exaggerated if $|u|$
is small. An obvious example of this tendency is the trapping of particles where $\nabla \cdot u$
< 0 and $u = 0$. Experience with alternates to (3.4) suggests that as long as these
two basic tendencies are accounted for the results are satisfactory.

The calculations involved in finding $\bar{\epsilon}^2$, the mean square of

$$
\epsilon = (\alpha - 1) \partial_t \zeta + \alpha u \cdot \nabla \zeta + \frac{\beta \gamma}{1 + \beta |u|^2} (\partial_t + u \cdot \nabla) |u|^2 ,
$$

are straightforward, if tedious, and require knowledge of the joint probabilities of
$u, \zeta$, and their time and space derivatives. The model described above has been
tested only for joint normal Eulerian velocity fields. Although in principle this is not
a limitation of the method, the unavailability of knowledge about the higher cum-
mulants of oceanic velocity fields probably makes it a practical necessity. When the
velocity field is joint normally distributed, $\bar{e}^2$ can be related to the Eulerian velocity
spectrum and the constants $\alpha$, $\beta$, and $\gamma$ selected to minimize $\bar{e}^2$; $\rho_0$ is selected to
make $\bar{p} = 1$. The minimization requires an exhaustive search over $\beta$ but the other
constants are found from a system of linear equations. Selection of the four con-
stants in (3.4) specifies the density estimator and allows prediction of the Lagrangian
mean velocity and velocity variance by substitution of $\bar{p}$ into (3.1). Predictions of $\mathbf{V}$
and $< \nu'_n, \nu'_m >$ are compared with simulations of particle motion below.

The second element of the theory advanced here is a statistical hypothesis which
permits prediction of the temporal variation of the Lagrangian mean product in
(3.1b), and thus particle displacement statistics. As shown by Taylor (1921), the
covariance matrix of displacement in the long-after-deployment limit, when La-
grangian statistics are stationary, is related to the velocity covariance derived from
(3.1) through

$$\frac{d}{dt} [\mu_{nm}(t) = <\delta\rho_n(t) \delta\rho_m(t)>] = \int_0^t \int < \nu'_n(\tau) \nu'_m(0) + \nu'_m(\tau) \nu'_n(0)> \, \tau ,$$

where

$$\nu' = \mathbf{v} - \mathbf{V} \quad \text{and} \quad \delta\rho'(t) = \delta\rho(t) - \delta\rho t .$$

The hypotheses proposed for the Lagrangian velocity covariance are (i) the vector
components of $\delta\rho$ are joint normally distributed, so that

$$P_\delta(\delta\rho) = \frac{1}{2\pi} |\mu|^{-\frac{1}{2}} \exp \left[ - \frac{1}{2} \sum_{n,m} \mu_{nm}^{-1} \delta\rho'_n \delta\rho'_m \right]$$

where $|\mu|$ is the determinant of the displacement covariance matrix of (3.5) and
$\mu_{nm}^{-1}$ is an element of the inverse of that matrix; (ii) the effect of variable $\rho$
on the Lagrangian velocity covariance (3.1b) may be accounted for through $< \nu'_n, \nu'_m >$
and $\mathbf{V}$ so that

$$< \nu'_n(0) \nu'_m(t) > \propto \frac{< \nu'_n \nu'_m >}{u_n (0,0) u_m (\delta\rho, t)} \propto < u_n (0,0) u_m (\delta\rho, t) > \propto$$

(iii) $\delta\rho$ may be treated as uncorrelated with $\mathbf{u}(x)\mathbf{u}(x+\delta\rho)$ so that

$$< \nu'_n(0) \nu'_m(t) > \propto \frac{< \nu'_n \nu'_m >}{u_n u_m} \int d\delta\rho' u'_n(0,0) u'_m(\delta\rho, t) P_\delta(\delta\rho, t) .$$

The third hypothesis was first advanced by Corrsin (1960) and has become known
as Corrsin's conjecture.

Once the density estimator is determined and $< \mathbf{V} >$ and $< \mathbf{v}'\mathbf{v}' >$ are found
from (3.1), the Lagrangian velocity covariance in (3.5) is specified as a function of
\[ \mu_{nm} \text{ through (3.6).} \] Then (3.5) provides a differential equation from which \( \mu \) may be found and the Lagrangian time lagged covariance of velocity, or the associated frequency spectrum \( \Phi_{nm}(\omega) \), calculated. Davis (1982) examined this model for one-dimensional flows and found it quite successful in comparison with other approaches based on weak interaction expansion, successive approximation, and statistical closure hypotheses. The bold element in this model is Corrsin's conjecture, which might be regarded as resulting from the simplest possible statistical estimator for particle displacement, namely that \( \delta r \) is unpredictable. In fact, Davis (1982) explored the statistical estimator

\[ \delta \bar{r}(t, r_0) = \alpha(t) u(r_0, 0) + n, \]

where \( n \) is a normally distributed random variable and \( \alpha \) was selected to minimize the mean square of \( \frac{d}{dt} (\delta r - \delta \bar{r}) \). Although this estimator is more accurate than taking \( \delta r \) to be completely unpredictable, it is testimony to the power of Corrsin's conjecture that the resulting predictions of the Lagrangian velocity covariance were not significantly improved.

From (3.5) it is evident that \( \mu_{nm} \) will initially grow as \( t^2 \). For times long compared with \( t_D \), the decorrelation time of the Lagrangian velocity covariance, \( \mu \) will become proportional to \( t \). In this large \( t \) limit, when displacement is the sum of many nearly independent steps, the probability density \( P_\delta \) is expected to approach a Gaussian form so that gradient-diffusion behavior, with the diffusivity tensor (Batchelor, 1949)

\[ \kappa_{nm} = \lim_{t \to \infty} \frac{1}{2} \frac{d}{dt} \frac{\mu_{nm}}{t}, \]

will obtain. The cause of this diffusive behavior is made clearer by noting that the diffusivity \( \kappa \) results from zero-frequency Lagrangian velocity variability since

\[ \kappa_{nm} = \frac{\pi}{2} [\Phi_{nm}(0) + \Phi_{nm}(0)] \cdot \]

Because it is the zero-frequency energy in the Lagrangian spectrum (rather than the Eulerian one) which leads to diffusion, it is possible for wave motion to produce dispersion of mean property fields (cf. Rhines, 1977). The nonlinear relation between Eulerian velocity and particle position transfers energy across the Lagrangian spectrum so, as is shown below, relatively weak velocity fields with no zero-frequency energy in the Eulerian spectrum can lead to significant diffusivities.

The above methods for predicting Lagrangian statistics from Eulerian statistics have been compared with direct simulation of particle motion in joint normally distributed random velocity fields. Random two-dimensional velocity fields which were either irrotational or nondivergent were generated from independent, normally distributed Fourier coefficients chosen to produce the Eulerian covariance
\[ u_n(0,0) u_m(x,t) = \frac{Q}{2\theta_c} \int_{\omega_c}^{1} d\omega \int_{-\theta_c}^{\theta_c} d\theta \gamma_\alpha \gamma_m \cos(k \cdot x - \omega t), \] (3.7)

where \( \theta \) is the angle between \( k \) and the \( x_1 \) axis and the dispersion relation \( \omega^2 = |k| \) is obeyed. For the irrotational case \( \gamma_1 = \cos(\theta) \) and \( \gamma_2 = \sin(\theta) \) while for the non-divergent case \( \gamma_1 = \sin(\theta) \) and \( \gamma_2 = -\cos(\theta) \). Eliminating low frequency energy from the Eulerian spectrum by choosing \( \omega_c = 0.2 \) makes it easy to observe nonlinear interactions transfer energy to zero frequency in the Lagrangian spectrum and, hence, see how significant eddy diffusion can be produced by a field of waves. Since the phase speeds fall in \( 1 \leq \omega/k \leq 5 \), the mean square Eulerian speed \( Q \) provides a measure of the degree of nonlinearity in the Eulerian description of advection.

In each simulation a velocity field was constructed from 300 Fourier components. Each component's frequency and direction were selected randomly from uniform distributions over the appropriate ranges and the amplitude was selected as a normally distributed random variable with variance in accord with the wavenumber frequency spectrum in (3.7). The fundamental kinematic equation (2.1) was integrated using a third order Runge-Kutta scheme and complete evaluation of the velocity Fourier series at each time step. Particle statistics were accumulated from time series started after the mean particle velocity and mean particle speed appeared to become stationary. \( \langle v \rangle, \langle v'_n v'_m \rangle, \) and the Lagrangian velocity spectrum \( \Phi_{nm}(\omega) \) were computed from 225 such time series. Evaluation of the theoretical predictions required numerical integration of (3.5) using a third order scheme and involved numerical evaluation of various integrals of the Eulerian spectrum, but these were considerably less time consuming than the simulations.

Predicted and observed values of mean Lagrangian velocity and velocity covariance are depicted in Figure 1 for the irrotational velocity field with an isotropic model spectrum \( (\theta_c = \pi \) in (3.7)) and a halfplane spectrum \( (\theta_c = \pi/2) \). Results are not shown for the nondivergent spectrum since, because \( \rho \) is constant, \( V = 0 \) and \( \langle v' v'_m \rangle = u_n u_m \). The behavior seen in Figure 1 is qualitatively similar to that found by Davis (1982) for particles in one-dimensional flow: for weakly nonlinear flow both \( V/Q \) and \( \langle v'_n v'_m \rangle/Q \) increase as the flow strength increases, but near \( Q = 1 \) these normalized statistics reach a maximum and then decrease abruptly. The strong flow behavior is evidence of particle trapping at locations where \( |u| \) is small and \( \nabla \cdot u < 0 \). Examination of the constants \( \beta \) and \( \gamma \) of (3.4) confirms this behavior. Behavior in weak flows is easily understood from a weak interaction expansion of particle motion in a single wave propagating in the +x direction. In the first approximation convergence \( -\partial_x u \) leads to density variation in phase with \( u \); this is responsible for mean drift in anisotropic velocity fields. This first approximation to particle density results in a flux \( \rho u \) which is in phase with \( u^2 \) and this, in turn, leads to a second approximation to the density which is also in phase with \( u^2 \). It is this second approximation which causes mean square particle speed to exceed the
Figure 1. Comparison of predicted (curves) and observed (symbols) mean particle velocity, 
\( \langle v_n \rangle \), and velocity variance, \( \langle v'_n \rangle \), for the irrotational form of the Eulerian spectrum (3.7).
The Eulerian mean square velocity is \( Q \) and the values plotted are \( \langle v \rangle / Q \) and \( \langle v'_n \rangle / Q \).
The subscript \( I \) corresponds to the isotropic case (when \( \langle v \rangle = 0 \) and \( \langle v'_n \rangle = \langle v'_n \rangle \)) and
the subscript 1/2 corresponds to Fourier components restricted to propagate in the +x, half-plane. Sampling uncertainty of \( \langle v \rangle \) is approximately 1% of \( Q^1 \) and uncertainty of \( \langle v'_n \rangle \) is approximately 4%.

Eulerian mean square speed \( Q \) for weak flows. Examination of the density estimator shows that for weak flows \( \alpha \approx 1 \) and \( \beta \) is small, confirming that the weak interaction expansion for \( \rho \) is reasonably accurate.

For the isotropic spectrum, denoted by the subscript I in Figure 1, there is close
close agreement between predicted and observed particle speeds, with the discrepancies
within sampling errors from the simulations. For the half-plane spectrum, denoted
by the subscript 1/2, there is qualitative agreement but the differences found for
\( Q > 0.5 \) are larger than can be explained by sampling errors. The general features
\( \langle v'_1^2 \rangle > \langle v'_2^2 \rangle \), the maxima of \( V/Q \) and \( \langle |v'|^2 \rangle / Q \) near \( Q = 1 \), and the
decrease of these measures for large \( Q \) are predicted. But quantitative agreement
for \( Q > 1 \) is not good; \( V \) is significantly underestimated for \( Q > 0.5 \) and velocity
variance is overestimated for \( Q > 5 \). It may be concluded, therefore, that the funda-
mental processes leading to particle density variation are modelled by the density
Figure 2. Comparison of predicted (dashed) and observed (solid) Lagrangian frequency spectra, $\Phi_n(\omega)$, for the isotropic, nondivergent form of the Eulerian spectrum (3.7). The curves correspond to $Q = 0.05, 0.2, 1, 5$, and $20$. Observed spectra are computed with 450 degrees of freedom and have a sampling uncertainty of approximately 7%.

estimator (3.4), but a more elaborate form is required to accurately represent the relations between $p$ and $u$ in strongly nonlinear anisotropic flows.

Figures 2 and 3 present the Lagrangian frequency spectra $\Phi_{11}(\omega)$ obtained from simulations using isotropic forms of (3.7) for nondivergent and irrotational velocities, respectively. These spectra are similar to each other and to the analogous results obtained by Davis (1982) for one-dimensional flows. Most obvious is how the band-limited nature of the Eulerian frequency spectrum is preserved only for weak flows and almost all evidence of the Eulerian time scale is lost for the strong flows. This demonstrates the power of the nonlinear interactions involved in the relationship between Eulerian and Lagrangian velocities and points out the difficulty of relating Eulerian and Lagrangian temporal statistics through any simple scaling. Similarly, it can be seen that any estimate of the diffusivity $\kappa_{11} = \pi\Phi_{11}(0)$ made from characteristic velocities and Eulerian time or space scales is unlikely to be generally accurate; since $\kappa$ increases much more rapidly than $Q$, any such estimate must be in error for some range of $Q$. This same rapid increase of $\kappa/Q$ indicates how the suggestion that
a significant diffusivity indicates eddy motion rather than waves can be true only for the most restrictive definition of "waves."

The Lagrangian spectra in Figures 2 and 3 might be regarded as broadened versions of the Eulerian frequency spectra. This is just the reverse of the situation found by Freeland et al. (1975) in their analysis of current meter and SOFAR float observations of meso-scale eddies. Whether this difference arises because the velocity fields employed here are dynamically different from meso-scale eddies, or simply because the Eulerian spectrum here is so highly localized, cannot be determined. The theoretical framework proposed here is certainly capable of producing Lagrangian frequency spectra with more structure than the associated Eulerian spectrum.

Also shown in Figures 2 and 3 are the Lagrangian frequency spectra predicted by applying (3.5) and (3.6) to the isotropic forms of (3.7). In these cases the equation to be solved for $\mu = \mu_{11} = \mu_{22}$ is

$$\frac{d^2}{dt^2} \mu = 2 \frac{\langle v_1^2 \rangle}{u_1^2} \int dx \frac{u_1(x,t)u_1(0,0)}{P_\theta(x)}$$

$$= 2 \frac{\langle v_1^2 \rangle}{(2\pi \mu)^2} \int dx e^{-|x|^2/2\mu^2} \int_0^1 d\omega \int_{-\pi}^{\pi} d\theta \gamma_1^2 \cos(\omega^2 \cos \theta x_1 + \omega^2 \sin \theta x_2 - \omega t).$$

(3.8)
The appropriate initial conditions are $\mu = \frac{d\mu}{dt} = 0$ at $t = 0$. The integrals on the right of (3.8) can be found analytically, yielding

$$\frac{d^2}{dt^2} \mu = <v_1^2> \int_{\omega_c}^{1} d\omega \cos(\omega t)s^{-\omega \mu^2/2}.$$

Integration over $\omega$ was performed numerically and the differential equation was solved using a third order Runge-Kutta algorithm.

Figure 2 applies to the nondivergent case when the present theory simplifies to Corrsin's conjecture and the hypothesis that particle displacements are normally distributed. Direct examination of histograms made from the simulations gave no evidence of significant differences from a normal distribution (see Section 5) but neither was the number of realizations sufficient for a definite conclusion. Agreement between predicted and observed spectra in Figure 2 is good, sufficient for quantitative predictions generally accurate to a factor of 2. The most notable differences are a tendency to overestimate the small energy density at high frequencies and to overestimate the zero-frequency energy density, and hence $\kappa$, for the weakest flows.

Figure 3 applies to the irrotational case and exhibits good agreement between prediction and observation except at the lowest frequencies. The spectrum is similar to the nondivergent case but significant differences exist at large $Q$, and these are predicted. The tendency for the observed spectra to rise rather abruptly near $\omega = 0$ for intermediate values of $Q$ is not seen in the predicted spectra and this leads to errors in the predicted eddy diffusivity nearly as large as a factor of 3. Thus it may be concluded that, while the hypotheses upon which the theory is based are adequate to predict the qualitative features of the relation between Eulerian and Lagrangian statistics, the accuracy of the procedure is no better than a factor of 2 or 3. It must also be pointed out that predictive ability may be significantly less in velocity fields with statistics other than the joint normal distribution employed in these tests. Tests with other Eulerian spectra give no indication that the exact form of the spectrum (3.7), or the fact that a single mode dispersion relation was imposed, is important to comparison of predicted and observed statistics. In Section 4, which deals with particle pair separations, some additional comparisons between observed and predicted single particle dispersion $\mu(t)$ are presented for a different spectrum and dispersion relation than used here.

4. Particle separation in homogeneous flows

As pointed out in Section 2, single particle statistics describe the evolution of mean property distributions but multi-particle statistics are needed to describe the characteristics of stirring. In this section attention is directed toward the statistics of the separation between two particles initially deployed with separation $s$. Discussion
is limited to nondivergent velocity fields with isotropic, homogeneous statistics and vanishing Eulerian mean. As a consequence of these restrictions the Lagrangian mean velocity vanishes, Lagrangian single particle velocity statistics are stationary from the moment of deployment, and two particle statistics depend only on time since deployment and the initial particle separation vector. Aligning the \( x_1 \) axis with \( s \) makes the coordinates \( x_1 \) and \( x_2 \) principal axes for two particle statistics.

Discussion requires introduction of the two time particle separation

\[
p(t_1, t_2, s) = r(t_1, s) - r(t_2, 0) .
\]

Since \( V = 0 \), the mean of \( p \) is the initial separation, \( s \), and the variance of \( p \) is

\[
\lambda_{nn}(t_1, t_2, s) = \langle [p_n(t_1, t_2, s) - s_n]^2 \rangle . \tag{4.1}
\]

The simultaneous mean square particle separation is

\[
\sigma^2_{nn}(t, s) = \lambda_{nn}(t, t, s) + \delta_{n1} |s|^2 , \tag{4.2}
\]

where \( \delta \) is the Kronecker delta and \( \sigma^2 \) has the same meaning as in (2.15).

It follows directly from the fundamental kinematic relation (2.1) that

\[
\partial_{t_1} \partial_{t_2} \lambda_{nn}(t_1, t_2, s) = -2 \langle v_n(t_1, s) v_n(t_2, 0) \rangle , \tag{4.3}
\]

and from (4.1) the appropriate boundary conditions are

\[
\lambda_{nn}(t, 0) = \lambda_{nn}(0, t) = \langle \delta r_1^2(t, 0) \rangle = \mu_{nn}(t) ,
\]

where \( \mu_{nn} \) is the particle displacement variance of (3.5). Because the two particle covariance in (4.3) is not stationary, it is not possible to reduce that equation to an ordinary differential form, as was done in Taylor’s (1921) development of its single particle analog (3.5). The reason for this nonstationarity is easily appreciated: the separation between two particles will, on average, increase with time so that, even though the variance of each particle’s velocity might be stationary, the correlation between their velocities will decrease with time.

We turn now to predicting mean square particle separation from Eulerian statistics using hypotheses like those used in developing (3.6) for the single particle velocity covariance. Like hypothesis (i) leading to (3.6a), it is proposed that particle separation may be taken as joint normally distributed, so that the probability of \( p \) is

\[
\langle \delta [y - r(t_1, s) + r(t_2, 0)] \rangle = \frac{1}{2 \pi \sqrt{1/\lambda_{11} \lambda_{22}}} \exp \left[ -\frac{1}{2} \left( \frac{(y_1 - s_1)^2}{\lambda_{11}} - \frac{(y_2 - s_2)^2}{\lambda_{22}} \right) \right] , \tag{4.4a}
\]

where \( \lambda_{nn} = \lambda_{nn}(t_1, t_2, s) \). Like hypothesis (iii) leading to (3.6c), it is proposed that in evaluating the two particle covariance, particle position may be treated as completely unpredictable so that

\[
\langle v_n(t_1, s) v_n(t_2, 0) \rangle = \langle u_n[r(t_1, s), t_1] u_n[r(t_2, 0), t_2] \rangle
\]
\begin{equation}
\int dx \int d\xi <u_n(x,t_1) u_n(x,t_2)> \cdot <\delta[x-r(t_1,s)] \delta[x-r(t_2,0)]> \\
= \int dy E_{nn}(y,t_1-t_2) <\delta[y-r(t_1,s) + r(t_2,0)]>
\end{equation}

where \(E_{nn}\) is the Eulerian velocity covariance of (2.2). The combination of (4.3) and (4.4) provides a prescription for finding \(\lambda_{nn}\) and thereby specifying the probability density of the particle separation \(p\). It may be noted that in the limiting case of infinite \(s\), when the two particles move independently, these equations admit the solution \(\lambda_{nn}(t_1,t_2) = \mu_{nn}(t_1) + \mu_{nn}(t_2)\) and in the limit \(s = 0\), when the particles are essentially one, \(\lambda_{nn}(t_1,t_2) = \mu_{nn}(t_1-t_2)\) is a solution as expected.

The predictions of (4.3) and (4.4) were tested against numerical simulations of particle motion. Particle pairs were tracked in randomly selected realizations of nondivergent, joint normally distributed velocity fields consistent with the Eulerian covariance

\begin{equation}
E_{nn}(x,t) = Q \pi^{-3/2} \int_0^\infty d\omega \exp(-\omega^2) \int_0^{2\pi} d\theta \gamma_n \gamma_m \cos(k \cdot x - \omega t),
\end{equation}

where \(\theta\) is the angle between \(k\) and the \(x_1\) axis, \(\gamma_1 = \sin(\theta), \gamma_2 = \cos(\theta)\), and the dispersion relation \(|k| = \omega\) is obeyed. This corresponds to a time lagged covariance \(E_{nn}(0,t) = \frac{1}{2} Q \exp(-t^2/4)\) and a space lagged covariance of the order \(Q \exp(-|x|^2)\). This Eulerian covariance was chosen in place of (3.7) to ease analytic evaluations in (4.4).

The simulation procedure was the same as described in Section 3 except that 800 Fourier coefficients were used in each realization to insure a rich spectrum at low frequency and wavenumber. The statistics of particle separation were accumulated over 1000 realizations for each initial particle separation \(s\). Evaluation of the predictions was accomplished using a first order explicit scheme for the integration over \(t_1\) and \(t_2\) in (4.3) combined with analytic integrations over \(y\) and \(\omega\) and numerical trapezoid rule integration over \(\theta\) in (4.4). From a practical point of view it should be noted that, even though the Eulerian covariance was selected to simplify evaluation of (4.4), the computational effort involved in the theoretical calculations approached that of the direct simulations.

Figure 4 presents a comparison of predicted and observed mean square particle separation for the case \(Q = 1\) and initial separations \(|s| = 1/4, 3/4, 9/4\) and \(27/4\). Plotted is the simultaneous separation variance \(\lambda(t) = <r(t,s) - r(t,0) - s^2> = \lambda_{11}(t,t) + \lambda_{22}(t,t)\) as a function of time since particle deployment; the mean square particle separation is \(\sigma^2 = \lambda + |s|^2\). Included are curves for infinite initial separation, when the two particles move independently. These were determined from single particle statistics and predicted from the model of Section 3 using \(\lambda = 4\mu\). Agreement between theory and simulation is good, particularly for large initial separations.
Figure 4. Comparison of predicted (dashed) and observed (solid) mean square separation between pairs of particles for the Eulerian spectrum (4.5) with $Q = 1$. The mean square particle separation is $\langle |r(t,s) - r(t,0)|^2 \rangle = \lambda(t) + |s|^2$ where $s$ is the initial separation. The curves of $\lambda$ are labelled by the corresponding value of $|s|$. The curve for $|s| = \infty$ corresponds to two independent particles and is derived from single particle statistics. Sampling uncertainty is about 5% and is highly correlated in time.

From the results reported in Section 3 it is known that this level of agreement is better than is typical, but neither is it trivial since the Lagrangian and Eulerian spectra differ by a factor of 2 at low frequency.

In analogy with the separation between particles in pure diffusion, a two particle diffusivity may be defined as $K = \frac{1}{8} \frac{d}{dt} \sigma^2$. Figures 5 and 6 are plots of $K$ vs. $\sigma$ for the cases of $Q = 1$ (corresponding to Fig. 4) and $Q = 20$, respectively. In these figures there is a plot of $K$ for each initial separation $|s|$, which is the value of $\sigma$ at $t = 0$ when $K = 0$. There are four noteworthy features of these curves which are important to the general problem of predicting and measuring two particle diffusivities. First, there is a general tendency for $K$ to increase with increasing $\sigma$, but the curves for different initial separations do not coincide, so it is not gen-
Figure 5. Two particle diffusivity vs. root mean square separation for the results in Figure 4.

The mean square separation is $\sigma^2 = \lambda + |s|^2$ and the two particle diffusivity is $K = \frac{1}{8} \frac{d}{dt} \sigma^2$. Predicted (dashed) and observed (solid) values are shown for $|s| = 1/4$, $3/4$, $9/4$, and $27/4$. Independent motion of two particles corresponds to predicted and observed values of $\frac{d}{dt} \sigma^2$ of 10.3 and 9.8, respectively.

 Generally possible to relate $K$ to $\sigma$ alone. Second, in neither figure does $K$ reach the value for infinite particle separation, even for values of $\sigma$ an order of magnitude greater than the decorrelation length of the Eulerian covariance. Third, the shapes of the curves in the two different figures are rather different and could not easily be related using any simple scaling based on Eulerian length, time and velocity scales. Finally, agreement of theory and simulations are generally good, lending some support for the hypotheses leading to (4.4).

In Richardson's (1926) pioneering work it was suggested that the two particle diffusivity $K$ could be regarded as primarily a function of the root mean square particle separation $\sigma$. From Figures 5 and 6 it is evident that this suggestion is true only of particles long after they are deployed, that is long after they are randomly placed with separations exactly equal to $\sigma(0) = |s|$. Thus two ensembles of particles with the same mean square separation will have quite different diffusivities if one
ensemble was recently deployed and the other was deployed long before. In part, this may result from the smaller spread of separations about the mean separation for the more recently deployed ensemble. But it must also be recognized that the relative velocity between particles has a finite time scale so that particle separation is not determined by instantaneous conditions alone. This is why $K$ in Figure 4 is not a monotonic function of time or $\sigma$ and why the theory outlined above leads to a differential evolution equation for the diffusivity, rather than an algebraic equation. The fact that $K$ depends on both $\sigma$ and time since deployment clearly leads to confusion in interpretation of plots of $K$ against $\sigma$ since the value of $K$ obtained depends on the distribution of time since deployment within the collection of particles contributing to the diffusivity at any particular $\sigma$.

It is at first paradoxical that even for particle separations much greater than the Eulerian length scale the two particle diffusivity is much less than that for infinitely separated particles. In the strong flow case of Figure 5 the increase of $K$ with $\sigma$ continues for $\sigma$ an order of magnitude greater than the Eulerian length scale, and in the weak flow case in Figure 4 the rate of increase is very much less. Some insight into this behavior is obtained by recalling that diffusive behavior results from
zero-frequency variability in the Lagrangian spectrum. Thus two particle dispersion is the result of low frequency velocity components which are incoherent between the two particles. For the velocity fields described by (4.5) the zero-frequency Eulerian coherence of velocities at points separated by $\mathbf{x}$ is unity for all separations and the variance of the incoherent velocity is of the order $Q\omega^2|x|^2$. Therefore the velocity variability leading to two particle dispersion results primarily from the nonlinear relation between Eulerian and Lagrangian velocity statistics, and it is quite possible for two particles with very great separations not to behave independently and for the "infinite separation" diffusivity not to pertain. The dependence on nonlinearity is apparently the reason that diffusivity increases more slowly with $\sigma$ in the $Q = 1$ case than in the $Q = 20$ case.

It is evident from the above discussion of the decorrelation length of particle velocities and the dissimilarity of the curves in Figures 4 and 5, that predictions of $K$ based on simple scaling laws are not apt to be generally accurate. The difficulty is that the degree of nonlinearity in the relation of Eulerian and Lagrangian velocity, the spatial coherence of the low frequency part of the velocity spectrum, and the recent history of particles all have strong influences on the diffusivity for all particle separations. Since the predictions of equations (4.3) and (4.4) seem to be at least qualitatively accurate, it may be profitable to explore simplified models for estimating $K$ based on analytic simplifications of those equations. Short of this, reliable predictions of two particle diffusivity are probably best obtained directly from particle motion simulations since the computational effort is little more than is required to solve (4.3) and (4.4).

5. Single particles in inhomogeneous flows

When the Eulerian statistics of velocity vary in space new aspects of particle behavior arise and some almost paradoxical transport phenomena are found. The purpose of this section is to address prediction of single particle statistics in velocity fields with inhomogeneous but stationary Eulerian statistics. The approach is to generalize the model of Section 3 to account for slow spatial variation of statistics and it is found that a first correction for inhomogeneity can be obtained very simply. Interest is restricted to the case of nondivergent velocity fields with stationary Eulerian statistics and a vanishing mean velocity, $U = 0$. Eulerian stationarity insures that particle statistics depend only on time since deployment and nondivergence insures that there is no difference between particles deployed at $r_0$, $t = 0$ and particles encountered there. The single particle probability density defined in (2.4)

$$P_1(x,t,r_0) = \langle \delta[x - r(t,r_0,t)] \rangle.$$ 

Unlike the statistically homogeneous case considered in Section 3, in general $P_1$ does not reduce to a function of $x - r_0$, but certain weaker symmetry relations can be deduced.
A temporal symmetry of $P_1$ in stationary, nondivergent fields is

$$P_1(x,t,r_0) = P_1(r_0,-t,x). \quad (5.1)$$

To verify this, note that the joint probability density of some particle being found near both $x_1$ at $t_1$ and near $x_2$ at $t_2$ is

$$P_2(x_2,t_2-t_1,x_1) = N^{-1} \int dr_0 < \delta[x_2-r(t_2,r_0)]\delta[x_1-r(t_1,r_0)] > ,$$

where the integral includes the entire area occupied by particles and $N$ is the total number of particles. Clearly, $P_T$ is symmetric with respect to exchange of $x_1,t_1$ and $x_2,t_2$ so that

$$P_1(x,t,r_0) = \int dx_1 \delta(x_1-r_0) P_T(x,t,x_1)$$

is unaltered by exchanging $r_0$ and $x$ while changing the sign of $t$, thus demonstrating (5.1). Note that this relation connects the probability of a particle going from $r_0$, $t = 0$ to $x,t$ with the probability of a particle found at $x,t$ having come from $r_0,t = 0$. The relation is, however, not as useful as it might seem. For example, if $X^P$ is the mean position of particles passing through $x = 0$, $t = 0$, it is not possible to relate $X^P(t)$ and $X^P(-t)$ using only (5.1).

In the case considered in Section 3, with homogeneous statistics and $U = 0$, the single particle density depends on $|x-r_0|$ and from (5.1) it follows that the temporal dependence is on $|t|$. Thus (3.6a) describes the density of previous particle positions if $\mu_{nn}$ of (3.5) is taken to be an even function of $t$. This underscores the fact that the diffusive smoothing of the mean field described by $P_1$ is the result of uncertainty in the position of material particles. This uncertainty grows with $|t|$ whether $t$ is the time before or after particles are located. This reversibility of uncertainty is entirely consistent with the irreversibility of the Second Law of Thermodynamics.

A second symmetry of $P_1$ can be obtained from conservation of fluid mass. Suppose that velocity statistics are homogeneous in the $x_2$ direction and that the mass density of particles is $\rho$. The mass flux through the line $x_1 = c$ into $x_1 > c$ is the result of the net flux of particles into $x_1 > c$ and is equal to

$$\rho U_1 = \rho \frac{d}{dt} \int d\hat{x}_2 \left[ \int_0^\infty dx_1 \int_{-\infty}^c d\hat{x}_1 P_1(x,t,\hat{x}) - \int_{-\infty}^0 dx_1 \int_0^\infty d\hat{x}_1 P_1(x,t,\hat{x}) \right].$$

In the present case, where $U = 0$, it follows that

$$\int_0^\infty dx_1 \int_{-\infty}^c d\hat{x}_1 [P_1(x,t,\hat{x}) - P_1(\hat{x},t,x)] = 0 . \quad (5.2)$$

Consider now particle dispersion in velocity fields whose statistics are inhomogeneous on a scale $\varepsilon^{-1}$ which is large compared with the velocity field's scale, here
taken as unity. Without loss of generality, the Eulerian covariance of (2.2) and \( P_1 \) can be written as

\[
E_{nm}(x,\xi,t) = E_{nm}[(x-\xi,t,\frac{1}{2}(x+\xi))],
\]

and

\[
P_1(x,t,\xi) = P_1[(x-\xi,t,\frac{1}{2}(x+\xi))],
\]

respectively; by construction these depend on \( \epsilon \) through \( x + \xi \). If \( \epsilon \) is sufficiently small, velocity fields in any local region are indistinguishable from statistically homogeneous. Then the theory of Section 3 would require the displacement \( r - r_0 \) to be normally distributed with variance \( \beta_{nm}(t,r_0) \) obtained from

\[
\frac{\partial^2}{\partial t^2} \beta_{nm}(t,y) = 2 \int dx' E_{nm}(x',y) P_1(x',y). \tag{5.3}
\]

This is (3.5) and (3.6) with \( y \) appearing as a parameter and simply means that the dependence of \( P_1 \) on \( \epsilon \) is regular as \( \epsilon \to 0 \). Then, like (3.6a),

\[
P_1(x,t,r_0) = \frac{1}{2\pi} |\hat{\mu}|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \sum_{n,m} \beta_{nm}^{-1} (x-r_0)_n (x-r_0)_m \right], \tag{5.4a}
\]

where \( \hat{\mu} = \hat{\mu}(r_0) \).

In the limit \( \epsilon \to 0 \), (5.4a) is accurate so long as \( \hat{\mu} \) is \( \hat{\mu}(y) \) evaluated at any \( y \) near \( r_0 \). It is here proposed that a first order correction for the effect of statistical inhomogeneity can be obtained from the simplest possible modification of this form which is also consistent with the symmetry principles (5.1) and (5.2), namely (5.4a) with

\[
\hat{\mu} = \hat{\mu} \left( \frac{x+r_n}{2} \right) + O(\epsilon^2) = \hat{\mu}(r_0) + \frac{x-r_0}{2} \cdot \nabla \hat{\mu} + O(\epsilon^2). \tag{5.4b}
\]

It is worth noting that if \( \beta_{nm} = \delta_{nm} 2Kt \), the density \( P_1 \) given by (5.4) is a solution of the diffusion equation

\[
\partial_t P_1(x,t,r_0) = \nabla \cdot \kappa(x) \cdot \nabla P_1 \tag{5.5}
\]

to \( O(\epsilon) \). Thus (5.4) includes, but is not restricted to, gradient diffusion with an inhomogeneous diffusivity.

The single particle density (5.4) represents a slightly distorted Gaussian shape. A somewhat easier to use form, also accurate to \( O(\epsilon) \), can be obtained by translating the \( x \) coordinates so that the deployment location \( r_0 \) is the origin, rotating them to coincide with the principal axes of \( \beta(x=0) \), expanding \( \hat{\mu} \) about \( x = 0 \), and dropping \( O(\epsilon^2) \) terms to obtain

\[
P_1(x,0,t) = \frac{1}{2\pi\sigma_1\sigma_2} \left[ 1 + \frac{1}{4} \sum_{n,m} \frac{x \cdot \nabla \beta_{nm}}{\sigma_n\sigma_m} \left( \frac{x_n x_m}{\sigma_n\sigma_m} - \delta_{nm} \right) \right]
\]
where \( \sigma_n = \sqrt{\mu_{nn}(0)} \) and \( \nabla \mu \) is evaluated at \( x = 0 \). From (5.6) it is straightforward to compute various descriptions of particle motion, and hence mean concentration evolution, in velocity fields with weakly inhomogeneous statistics.

The mean displacement of particles released from \( r_0 = 0 \) is \( X^D(t) \) given by

\[
X^D(t) = \int dx_n P_1(x_0, t) = \frac{1}{2} \sum_m \frac{\partial}{\partial x_m} \mu_{nm}(t) .
\]

This is equivalent to the proposition advanced by Freeland et al. (1975), which was expressed in terms of the diffusivity tensor \( \kappa_{nm} = \frac{1}{2} \partial_t <r_n r_m> \). When the principal axes of \( \mu \) do not rotate (i.e. \( \nabla \mu = 0 \)), the migration of \( X^D \) described by (5.7) can be rationalized by saying that particles moving toward areas of high dispersion are apt to be swept farther from their point of origin than are particles moving into regions of low dispersion. This rationalization is consistent with the observation that the peak of \( P_1 \), that is the most probable particle position or the maximum of the mean concentration field, is at \( X^P(t) \) where

\[
X^P(t) = - \frac{1}{4} \mu_{nn} \frac{\partial}{\partial x_n} \ln|\mu| .
\]

Thus while particles in areas of high dispersion are apt to be found far from their origin, particles tend to remain in regions of low dispersion and their concentration there is large.

It is interesting to note how the mean motion indicated by \( X^D \) is very different from that associated with a mean flow. In particular, \( X^D \) (and \( X^P \)) are even functions of time whereas mean Eulerian currents lead to displacements which are odd functions of time. Thus if \( U = 0 \), the center of mass of particles which pass through \( x = 0 \), \( t = 0 \) in an inhomogeneous field moves from the region of high dispersion toward the origin until \( t = 0 \) and then reverses its course and returns to the energetic region.

In light of the fact that (5.6) satisfies the vanishing particle flux condition (5.2), it is at first paradoxical that there is a flux of particles through their point of release. The fraction of the particles released from \( r_0 = 0 \) which are found in the half-plane \( x_1 > 0 \) is

\[
\phi = \frac{1}{2} + \frac{1}{2\sqrt{2\pi}} \left[ \frac{\partial}{\partial x_1} \mu_{11}^{1/2} + \frac{\partial}{\partial x_2} \mu_{12}^{1/2} \right] r_0 = 0 .
\]

If the principal axes of \( \mu \) do not rotate with position, then there is a flux of particles released at \( x = 0 \) across \( x_1 = 0 \) toward increasing \( \mu_{11} \), but there is no net flux of particles. The rationalization of this rests on considering particles deployed at all
locations. At each point there is a flux of particles deployed at that point in the direction of increasing $\mu_{11}$, which we imagine to be toward $+x_1$. The net flux across $x_1 = 0$ is the difference in the number of highly mobile particles moving from positive to negative $x_1$ minus the number of less mobile particles moving in the opposite direction. Even though a majority of the particles released at any point move toward positive $x_1$, the extra mobility of the particles originating from $x_1 > 0$ allows their flux to balance this tendency.

Simulated particle motions were used to test the model (5.6). Random, joint normally distributed, isotropic and homogeneous streamfunctions, $\psi_H$, were generated to be consistent with the Eulerian covariance of velocity (3.7) with $\omega_0 = 0.2$, $\theta_0 = \pi$, $|k| = \omega$, and $Q = 20$. This is the homogeneous isotropic test case in Section 3 except that the dispersion relation is linear rather than quadratic. Statistically inhomogeneous velocity fields were generated from the streamfunction $\psi_l = (1 + \epsilon x_1) \psi_H$ and used for particle motion simulation. The associated velocity fields are statistically inhomogeneous at and anisotropic at $O(\epsilon)$, with the mean square velocities.

$$q_1 = E_{11}(x,x,t) = (1 + \epsilon x_1)^2 \frac{Q}{2} , \quad q_2 = E_{22}(x,x,t) = [(1 + \epsilon x_1)^2 + 2 \epsilon^2] \frac{Q}{2} .$$

Thus $\epsilon$ is related to the scale of inhomogeneity through $\partial_{x_1} \ln q_1 = 2 \epsilon$ while the characteristic scale of the velocity field is $O(1)$. 

Figure 7. Sections of the single particle density $P_s(x,0,t)$ for a case with weak inhomogeneity in the $x_1$ direction. The solid curve is developed from simulations of 4500 particle tracks in velocity fields described in the text. Eulerian energy increases toward $+x_1$, $\epsilon = 0.05$, and $t = 5$. The dashed curve corresponds to the model (5.6).
Figure 8. As Figure 7 for a strongly inhomogeneous case with $\epsilon = 0.25$, and $t = 0.5$.

Sections of $P_1(x,0,t)$ through the deployment site $x = 0$ are shown in Figures 7 and 8. Both figures correspond to $Q = 20$ and each figure is based on 4500 particles. Figure 7 is a weakly varying case with $\epsilon = 0.05$. In this case the asymmetry of $P_1$ develops slowly and the figure corresponds to $t = 5$, by which time the displacement variances are growing approximately linearly with time. Thus the example in Figure 7 approximates a case of gradient diffusion. Figure 8 is for a much more inhomogeneous example with $\epsilon = 0.25$ and the figure is for $t = 0.5$. At this time the diffusivity $\frac{1}{4} \partial_t |\mu|$ is about half its large time value; thus this is a small time case for which gradient diffusion does not apply.

In order to test the model of $P_1$ with a minimum of confusion, the parameters in (5.6) were obtained from particle simulations in statistically homogeneous fields constructed by setting $\epsilon = 0$. This minimizes the influences of inaccuracies in (5.3) which could otherwise be used to estimate the homogeneous statistics $\bar{\mu}$; the errors in this estimation were documented in Section 3. From homogeneous runs with differing values of $Q$ it is possible to relate the isotropic matrix $\mu$ to $Q = q_1 + q_2$. The parameters in (5.6) were then obtained from

$$\bar{\mu}_{nn}(x) = \mu_{nn}(q_n=10) + [q_n(x)-10] \left[ \frac{\partial \mu_{nn}}{\partial q_n} \right]_{q_n=10}.$$

The results of using this $\bar{\mu}$ to evaluate $P_1$ from (5.6) are plotted in Figures 7 and 8. Agreement with the simulations is excellent in both cases. Figure 7 may be regarded as verification that the gradient diffusion equation (5.5) pertains when the single particle diffusivity has equilibrated at its large time value. Agreement in Figure 8 shows that the model above, based on weak inhomogeneity, applies in a case where $\epsilon$ is not so small and where gradient diffusion is inapplicable.
6. Strategies for describing property transport

An important observation in Section 2 is that the probability density of single material particles provides the Green's function for evolution of mean property fields. From this function, $P_1$, the mean concentration resulting from prescribed initial, source and sink data can be found from (2.9) just as it would be from an advection and gradient diffusion model. But $P_1$ applies to cases where property fluxes are not determined locally and gradient diffusion does not apply, and there is considerable evidence that oceanic fluxes are not of the gradient type (Okubo, 1971), at least for time scales of a year or less. Description of the typical dispersal of individual property clouds is provided by multiple particle statistics. When gradient diffusion does not apply, this description contains fundamentally new information beyond that needed to describe evolution of the mean field.

In Sections 3 and 5 the single particle density was explored using simulated particle motion in joint-normally distributed velocity fields with vanishing Eulerian mean. It was found that $P_1$ could be adequately described by a Gaussian form in statistically homogeneous flows and by a related simple form in weakly inhomogeneous fields. From the practical standpoint, the importance of this result is that $P_1$ is determined by the Lagrangian mean velocity, $V(t,r_0)$ and the covariance $\langle v_n'(t,r_0) v_m'(t,r_0) \rangle$ of (2.3). While this result is far from conclusive, it suggests that reasonably accurate descriptions of property transport might be developed in terms of relatively simple Lagrangian statistics without introducing the unfounded assumption of gradient diffusion. Clearly, more study is needed to test this idea, including determination of the effects of mean shear and non-normal velocity distributions, but there is reason for optimism.

Even if it is found that dispersal by unresolved velocity components can be simply described in terms of Lagrangian statistics, it is not clear how these statistics can best be determined. In the simple cases examined in Sections 3 and 4 it was found that mean particle velocity and Lagrangian covariances of one and two particles can be adequately predicted from Eulerian statistics of velocity. This involved prediction of the relation between particle density and velocity, accomplished through a statistically optimized model of particle density, and use of Corrsin's conjecture relating the statistics of particle displacement and products of velocity. I have been unable to find a very convincing justification for the success of Corrsin’s conjecture. In some sense, it is based on the maximally random particle distribution subject to the constraint that displacement covariances develop in accord with velocity covariances. This may be a powerful principle of importance beyond the present context or it may be empirical magic which applies only in special cases like those examined here.

Rather than appeal to theories relating Eulerian and Lagrangian statistics, it is possible to deduce some particle statistics from observations of quasi-Lagrangian drifters. Important steps toward characterizing meso-scale lateral dispersion have
been made by Freeland et al. (1975) and by Price (1982). These studies indicate some fundamental limitations of the direct measurement approach to Lagrangian statistics. First, drifters are not true material particles. This discrepancy may not be serious since it is not necessary that drifters follow exactly true material particles, only that their statistics resemble, or permit determination of, Lagrangian statistics. Thus, for example, vertical motion of isobaric floats is unlike water motion but it is likely that on time scales greater than a few days float horizontal motion is representative of material particles. Perhaps more serious is the tendency for constant level floats to collect at points of convergence in the horizontal velocity field, unlike particles which are free to move vertically. A second observational problem is the sheer volume of data required to establish statistics accurately over representative regions of the ocean. This problem is equally serious with Eulerian observations. A third difficulty is that single particle statistics are defined with respect to repeated random deployments at the same site. Estimation of these statistics from a few float tracks, by assuming stationarity, can lead to significant under-estimates of dispersal while repeated deployments are expensive. Despite these limitations, it seems that quasi-Lagrangian floats will provide less ambiguous results than inferences from Eulerian observations or inversions based on observed property distributions.

Even if a simple description of particle dispersal can be specified from a combination of observation and theory, it remains necessary to assimilate this description into our conceptual and quantitative models of ocean transport. One approach, based on (2.9), involves purely Lagrangian statistical information. The Lagrangian mean velocity takes the place of advection and dispersal is characterized by the spread of $P_1$ about the mean particle position. This involves a conceptually more complex description of mean motion than is obtained in the Eulerian frame because the mean velocity depends on both initial location and time, whereas the Eulerian mean velocity depends only on position. An alternate approach is to develop a representation of material particle dispersal which can be employed in the Eulerian frame. The difficulty here is that property fluxes supported by velocity variability are not generally determined locally. This is evidenced by the observed dependence of diffusivity on time since deployment or, as is sometimes said, on the scale to which dispersal has advanced. Okubo (1976) has examined a formulation of scale-dependent diffusion in the Eulerian frame. This formulation seems inadequate since it violates both the principle that in a homogeneous field the diffusivity should not depend on position and the observation that $P_1$ has a Gaussian shape even as the diffusivity varies. Further, the results of Section 4 cast some doubt on the appropriateness of taking the diffusivity to depend only on spatial scale. It remains to be seen if a consistent formulation of eddy flux can be developed in the Eulerian frame. At a minimum it must describe the (i) observed dependence of diffusivity on time since deployment, (ii) the difference between single particle and particle pair diffusivities, and (iii) the tendency toward a Gaussian shaped $P_1$. 


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