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Donald J. Brown

Caterina Calsamiglia

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By

Donald J. Brown and Caterina Calsamiglia

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YALE UNIVERSITY

Box 208281

New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

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Abstract

We show that a demand function is derived from maximizing a quasilinear utility function subject to a budget constraint if and only if the demand function is cyclically monotone. On finite data sets consisting of pairs of market prices and consumption vectors, this result is equivalent to a solution of the Afriat inequalities where all the marginal utilities of income are equal.

We explore the implications of these results for maximization of a random quasilinear utility function subject to a budget constraint and for representative agent general equilibrium models.

The duality theory for cyclically monotone demand is developed using the Legendre–Fenchel transform. In this setting, a consumer's surplus is measured by the conjugate of her utility function.

1 Introduction

Hildenbrand (1983) proposed a generalization of the law of demand, i.e., downward sloping market demand curves, for multi-commodity market demand functions, where individual's income is price independent. His generalization, i.e., monotone market demand functions, was subsequently extended by Quah (2000) to individual demand functions. In particular, it is known that agents who maximize homothetic or quasilinear utility functions subject to a budget constraint have monotone demand functions. Moreover, given a finite family of observations of a consumer's demands at market prices, Afriat (1981) derived necessary and sufficient conditions such that the data set is rationalizable with a homothetic utility function (see Varian (1983) for a discussion.)

To our knowledge, there does not exist a comparable result for quasilinear rationalizations. Here we extend Afriat's non-parametric tests for models of consumer behavior to tests for quasilinearity. Our motivation derives primarily from a series of papers authored by Bewley on the permanent income hypothesis and Marshall's general equilibrium model in his famous Note XXI in the mathematical appendix to his *Principles of Economics* (1890), a model that differs in fundamental ways from the general equilibrium model of Walras (1900). In Marshall's model there are no

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explicit budget constraints for consumers, the marginal utilities of income are exogenous constants and market prices are not normalized. Moreover, he "proves" the existence of market clearing prices, as does Walras, by simply showing the equality of the number of equations and unknowns.

The permanent income hypothesis was proposed by Friedman (1975) as an explanation of a consumer's life cycle of savings and consumption decisions. Friedman's model consists of three equations. The principal equation is the demand function derived from utility maximization subject to a budget constraint, where the utility function is homothetic. As is well known, in this case the marginal utility of income depends only on prices, i.e., it is independent of income, and changes in consumer welfare can be measured by changes in consumer surplus, if only prices change and income remains constant.

Bewley (1977) reformulates this aspect of Friedman's model by assuming that consumers maximize quasilinear utility functions subject to a budget constraint. Here the marginal utility of income is independent of both prices and income, i.e., it is constant, an assumption made by Cournot (1838), Dupuit (1844) and Marshall (1890). Marshall justifies this assumption in the mathematical appendix of his *Principles of Economics*, where consumers have additively separable utilities and he assumes "consumer expenditures on any one thing is a small part of this total expenditure". Bewley's justification is similar, i.e., he makes the same assumptions for consumers maximizing additively separable utility functions over an infinite horizon.

In Bewley (1980) he considers a pure exchange economy, where each agent has a quasilinear utility function and income is price independent. He proves the existence of a unique competitive equilibrium and shows that it is globally stable with respect to tâtonnement price adjustment. It is clear that his approach can also be used to show that equilibrium prices are a monotone function of the social endowment.

Demand functions generated from quasilinear utility maximization subject to a budget constraint enjoy an additional property not possessed by monotone demand functions. That is, they are cyclically monotone, such demand functions are said to satisfy the strong law of demand. Cyclically monotone demand functions not only have downward sloping demand curves, in the sense that they are monotone functions, but also their line integrals are path-independent and measure the change in consumer's welfare for a given multi-dimensional change in market prices.

We show that a finite data set consisting of pairs of price vectors and consumption vectors can be rationalized by a quasilinear utility maximization subject to a budget constraint if and only if the data set is cyclically monotone. An equivalent condition for quasilinear rationalization is that the Afriat inequalities have a solution where all the marginal utilities of income are equal. Quasilinear rationalizations appear to be special cases of the more general consumer optimization problem considered by Bewley (1980):

$$(M) \quad \max_{x \in \mathbb{R}^n_{++}} \left\{ \frac{1}{\lambda} g(x) - p \cdot x \right\}$$

where g is a strictly concave smooth monotonic utility function on $\mathbb{R}^n_{++}$; λ is the constant exogenous marginal utility of income; p is the vector of market prices and

x is the consumption vector. In his model, there is no budget constraint and prices are not normalized. As such, this specification rationalizes the family of equations defining Marshall's general equilibrium model (absent production). We refer to (M) as the Marshallian consumer optimization problem.

(M) need not have a solution for all $p \in \mathbb{R}^n_{++}$, but as noted in Bewley (1980) the set of p such that (M) has a solution is nonempty, open and convex. Given his assumptions on g, it follows from Hadamard's Theorem — see Gordon (1972) for a discussion and proof of Hadamard's Theorem — that (M) has a solution for all $p \in \mathbb{R}^n_{++}$ if and only if the gradient map, i.e., $x \to \partial g(x)$, is a proper map. Recall that a continuous map $\ell : V \to W$ is proper if for every compact subset $K \subset W, \ell^{-1}(K)$ is a compact subset of V. In Marshall's specification of individual's utilities, where consumers have smooth additively separable utility functions, the marginal utility of consumption of each good goes to infinity as consumption goes to zero and the marginal utility of consumption of each good goes to zero as consumption goes to infinity, the gradient map is proper (e.g., $U(x) = \ln x$ has a proper gradient map, but $U(x) = x - e^{-x}$ does not). Marshall's demand functions are defined on all of $\mathbb{R}^n_{++}$. Bewley (1980) is therefore a generalization of Marshall, where Bewley drops Marshall's assumptions of separability of the utility function and properness of the gradient map.

We show for finite data sets, that the Marshallian consumer optimization problem is equivalent to the Walrasian consumer optimization problem with a quasilinear utility function. Since on finite data sets, the testable implications of Friedman's assumption of homothetic tests — see Varian (1983) — differ from those of Bewley's assumption of quasilinear preferences, we can test which permanent income hypothesis is supported by market data. That is, the equilibrium correspondence in Friedman's model is monotone, but it follows from our analysis that the equilibrium correspondence in Bewley's model is cyclically monotone.

Bewley (1986) considered an exchange economy with a continuum of traders, each endowed with a random quasilinear utility function. Random quasilinear utility functions have also been discussed by Brown and Wegkamp (2003). Their specification: $v(x, \varepsilon) = u(x) + \varepsilon \cdot x + x_0$ is a special case of the random utility model suggested originally by Brown and Matzkin (1998). Brown and Wegkamp propose a semiparametric estimator for utility functions in this class, assuming that the utility shock ε is stochastically independent of market prices and consumer income, where all random vectors have compact support, and $u(x)$ belongs to a known parametric family, e.g., Cobb–Douglass. We show that if ε lies in a compact set in $\mathbb{R}^n$, then our nonparametric characterization of quasilinear rationalization extends to maximization of a random quasilinear utility function of the form $u(x) + \varepsilon \cdot x + x_0$ subject to a budget constraint. In particular, this model is refutable on finite data sets.

2 The Strong Law of Demand

Hildenbrand's (1983) extension of the law of demand to multicommodity market demand functions requires the demand function to be monotone. He showed that it is monotone if the income distribution is price independent and has downward sloping density. Subsequently, Quah (2000) extended Hildenbrand's analysis to individual's demand functions. His sufficient condition for monotone individual demand is in terms of the income elasticity of the marginal utility of income. Assuming that the commodity space is $\mathbb{R}^{\ell}_{++}$, we denote the demand function at prices $p \in \mathbb{R}^{\ell}_{++}$ and income $I \in \mathbb{R}_{++}$ by $x(p, I)$. This demand function satisfies the law of demand or is monotone if for any pair $p, p' \in \mathbb{R}^{\ell}_{++}$ of prices and income, I:

$$(p - p') \cdot \left[x(p, I) - x(p', I)\right] < 0$$

This means, in particular, that the demand curve of any good is downward sloping with respect to its own price, i.e., satisfies the law of demand if all other prices are held constant.

We now introduce the strong law of demand, which implies not only the law of demand, i.e., monotonicity of the demand function, but also for a given change in prices, induced changes in consumer welfare is measured by the change in consumer surplus.

Definition 1 A correspondence ρ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is cyclically monotone if

$$(x_1 - x_0) \cdot x_0^* + (x_2 - x_1) \cdot x_1^* + \cdots + (x_0 - x_m) \cdot x_m^* \geq 0$$

for any set of pairs (x_i, x_i^*), $i = 0, 1, ..., m$ (m arbitrary) such that $x_i^* \in \rho(x_i)$. In particular, a demand function, $x(p) : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is cyclically monotone if for any set of pairs (p_r, x_r), $r = 1, ..., m$

$$x_0 \cdot (p_1 - p_0) + x_1 \cdot (p_2 - p_1) + \cdots + x_m \cdot (p_0 - p_m) \geq 0.$$

Definition 2 A demand function $x\,(p, I)$ satisfies the strong law of demand if it is cyclically monotone.

Definition 3 If u is a concave function on $\mathbb{R}^n$, then $\beta \in \mathbb{R}^n$ is a subgradient of u at x if for all $y \in \mathbb{R}^n : u(y) \leq u(x) + \beta \cdot (y - x)$.

Definition 4 If u is a concave function on $\mathbb{R}^n$, then $\partial u\,(x)$ is the set of subgradients of u at x.

Definition 5 Let the data (p_r, x_r), $r = 1, ..., N$ be given. The data is quasilinear Walrasian-rationalizable if for some $x_{0_r} > 0$ and $I > 0$, x_r solves

$$(I) \quad \max_{x \in \mathbb{R}^n_{++}} U(x) + x_{0_r} \quad \text{s.t. } p_r \cdot x + x_{0_r} = I$$

where U is a concave function.

Definition 6 Let the data (p_r, x_r), $r = 1, ..., N$ be given. The data is Marshallian-rationalizable if for some $\lambda > 0$, x_r solves

$$(II) \quad \max_{x \in \mathbb{R}^n_{++}} G(x) - \lambda p_r \cdot x$$

where G is a concave function, or equivalently

$$(III) \quad \max_{x \in \mathbb{R}^n_{++}} U(x) - p_r \cdot x$$

for some concave U, where $U(x) = G(x)/\lambda$.

We now prove the main result of this paper, i.e., necessary and sufficient conditions for data to be quasilinear Walrasian-rationalizable. This is an extension of Afriat's nonparametric tests for models of consumer behavior to a test for quasilinear behavior. The underlying idea is Rockafellar's characterization of the subgradient correspondence of a concave function.

Theorem 1 (Rockafellar (1970, Theorem 24.8, p. 238). *Let $\rho(x)$ be a correspondence from $\mathbb{R}^n$ to $\mathbb{R}^n$. In order that there exists a closed proper concave function U on $\mathbb{R}^n$ such that $\rho(x) \subset \partial U(x)$ for every x, it is necessary and sufficient that $\rho(x)$ be cyclically monotone.*

In his proof he constructs the following function , U, on $\mathbb{R}^n$:

$$U(x) = \inf \{x_m^* \cdot (x - x_m) + \cdots + x_0^* \cdot (x_1 - x_0)\}$$

where the infinum is taken over all finite sets of pairs (x_r^*, x_r), $r = 1, ..., m$ (m arbitrary) in the graph of $\rho(x)$. Note that if the graph of $\rho(x)$ has only a finite number of elements then the domain of U is all of $\mathbb{R}^n$. Also note that for this U, x_r^* is the subgradient of U at x_r.

Theorem 2 *The following conditions are equivalent:*

(a) *The data (p_r, x_r), $r = 1, ..., N$ is Marshallian-rationalizable by a continuous, concave, monotonic utility function U.*

(b) *The data (p_r, x_r), $r = 1, ..., N$ is quasilinear Walrasian-rationalizable by a continuous, concave, monotonic utility function U.*

(c) *The data (p_r, x_r), $r = 1, ..., N$ satisfies Afriat's inequalities with constant marginal utilities of income, i.e., there exist $G_r > 0$ and $\lambda > 0$ for $r = \ell, ..., N$ such that*

$$G_r \leq G_\ell + \lambda p_\ell \cdot (x_r - x_\ell) \quad \forall r, \ell = 1, ..., N$$

or equivalently

$$U_r \leq U_\ell + p_\ell \cdot (x_r - x_\ell) \quad \forall r, \ell = 1, ..., N$$

where $U_r = G_r/\lambda$.

(d) *The data* (x_r, p_r), $r = 1, ..., N$ *is cyclically monotone, i.e., for any set of pairs* (x_s, p_s), $s = 1, ..., m : p_0 \cdot (x_1 - x_0) + p_1 \cdot (x_2 - x_1) + \cdots + p_m \cdot (x_0 - x_m) \geq 0.$

Proof (a) $\Rightarrow$ (b): obvious.

(b) $\Rightarrow$ (c): the F.O.C. of the Walrasian problem state that $\exists \beta_r \in \partial U(x)$ and $\lambda_r > 0$ such that $\beta_r = \lambda_r p_r$, where in this case $\lambda_r = 1$ for all r. Since U is concave, $U(x_r) \leq U(x_\ell) + \beta_\ell \cdot (x_r - x_\ell)$ for $r, \ell = 1, 2, ..., N$. Let $U_r = U(x_r)$ for $r = 1, ..., N$ and we have a solution for the Afriat inequalities, where all of the marginal utilities of income are equal, i.e., $\lambda_r = 1$ for all r.

(c) $\Rightarrow$ (d): from the Afriat inequalities, we know that for every family of data points (p_r, x_r), say $r = 1, ..., m$, the following inequalities are true:

$$\begin{aligned} U_1 - U_0 &\leq p_0 \cdot (x_1 - x_0) \\ U_2 - U_1 &\leq p_1 \cdot (x_2 - x_1) \\ &\vdots \\ U_0 - U_m &\leq p_m \cdot (x_0 - x_m) \end{aligned}$$

Adding up these inequalities, we derive the condition that defines cyclical monotonicity, i.e., the left hand side sums to zero.

(d) $\Rightarrow$ (a): By Theorem 1, we obtain a concave U where p_r is a subgradient of U at x_r. Hence the F.O.C. for the Marshallian optimization problem are satisfied. ■

Bewley (1980) defines the consumer's surplus at prices p as $h(p) = \max_{x \in \mathbb{R}^n_{++}} \left\{ \frac{1}{\lambda} g(x) - p \cdot x \right\}$, i.e., the optimal value function for (M). Convex analysis has a rich theory of duality based on the Legendre–Fenchel transform of a concave function $U(x)$, denoted $U^*(p)$, called the conjugate of $U(x) : U^*(p) = \inf_{x \in \mathbb{R}^n_{++}} \{p \cdot x - U(x)\}$. Hence $U^*(p) = -h(p)$, here $h(p) = \sup_{x \in \mathbb{R}^n_{++}} \{U(x) - p \cdot x)\}, U(x) = \frac{1}{\lambda} g(x)$ and h is an extended real-valued function. The conjugate (or surplus function in Bewley's terminology) plays the same role in analysis of the Marshallian consumer optimization problem as the indirect utility function does in the Walrasian model of consumer choice.

In Theorem 2, we have shown that cyclical monotonicity of the gradient map is equivalent to Marshallian rationalizations of the data. It follows from the following proposition in Rockafellar (1970), that cyclical monotonicity of the demand function is also equivalent to Marshallian rationalizations of the data.

Theorem 3 (Rockafellar (1970), Corollary 23.5.1, p. 219) *If* f *is a continuous concave function on* $\mathbb{R}^n_{++}$, *then* $p \in \partial f(x)$ *if and only if* $x \in \partial f^*(p)$.

Theorem 4 *The following conditions are equivalent:*

(a′) *The data* (p_r, x_r), $r = 1, ..., N$ *is Marshallian-rationalizable by a continuous, concave, monotonic utility function* U.

(b′) *The data* (p_r, x_r), $r = 1, ..., N$ *is quasilinear Walrasian-rationalizable by a continuous, concave, monotonic utility function* U.

(c′) *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies Afriat's inequalities with constant marginal utilities of income, i.e., there exist* $G_r > 0$ *and* $\lambda > 0$ *for* $r = \ell, ..., N$ *such that*

$$G_r \leq G_\ell + \lambda p_\ell \cdot (x_r - x_l) \;\; \forall r, \ell = 1, ..., N$$

or equivalently

$$U_r \leq U_\ell + p_\ell \cdot (x_r - x_\ell) \;\; \forall r, \ell = 1, ..., N$$

where $Ur = G_r/\lambda$.

(d′) *The data* (p_r, x_r), $r = 1, ..., N$ *is cyclically monotone, i.e., for any set of pairs* (p_s, x_s), $s = 1, ..., m : x_0 \cdot (p_1 - p_0) + x_1 \cdot (p_2 - p_1) + \cdots + x_m \cdot (p_0 - p_m) \geq 0$.

Proof The cyclical monotonicity condition in (d′) is simply the monotonicity condition on the gradient map of $-h(p)$, the conjugate of $U(x)$. Since $[U(p)]^* (x) = U(x)$ — see Theorem 12.2, p. 104 in Rockafellar (1970) — this condition is equivalent to condition (d) in Theorem 1, by Theorem 3. ■

If we require strict inequalities in (c) and (d) in Theorem 2 or in (c′) and (d′) in Theorem 4, then it follows from Lemma (2) in Chiappori and Rochet (1987) that the rationalizations in both theorems can be chosen to be C^∞ functions. In this instance, it follows from the implicit function theorem that the Marshallian demand function $x(p) = -\partial h(p)/\partial p$. Hence for any line integral in the domain of $x(p)$, we see that $\int_{p_1}^{p_2} x(p)dp = -\int_{p_1}^{p_2} (\partial h(p)/\partial p) = -\int_{p_1}^{p_2} dh(p) = h(p_1) - h(p_2)$. That is, for smooth Marshallian demand functions consumer surplus is well-defined — the line integrals are path independent — and the change in consumer surplus induced by a change in market prices is the change in consumer's welfare.

3 Representative Agent Models

Both Bewley (1980) and Friedman (1975) formulate the permanent income hypothesis as a property of a representative agent model of a competitive market economy. The representative agent's homothetic utility function in Friedman's Walrasian model is given by Eisenberg's Theorem for aggregating homothetic consumers where the income distribution is price independent — see Eisenberg (1961). This agent's endowment is the social endowment of the economy and she maximizes her utility function subject to a budget constraint defined by market clearing, competitive prices and the social endowment. This is a Walrasian consumer optimization problem with homothetic tastes and Quah (2000) has shown that the demand function for such agents is monotone — see Theorem 2.2 in his paper and recall that the marginal utility of income for a homothetic agent is independent of income. The equilibrium correspondence for this economy is simply the inverse of the demand function of the representative agent. Assuming strict concavity of her utility function guarantees that her demand function is strictly monotone. A smooth map is strictly monotone if and only if its derivative is negative definite — see Theorem 5.4.3 in Ortega and Rheinboldt (1970) or use Roy's identity. In representative agent economies, we have the following identity: $x(p(e)) \equiv e$, where $x(p)$ is the aggregate demand function, $p(e)$ is the equilibrium map and e is the social endowment. Hence $\partial x/\partial e \equiv (\partial x/\partial p)(\partial p/\partial e) \equiv I$. That is, $\partial p/\partial e = (\partial x/\partial p)^{-1}$, but the inverse of a negative definite matrix is negative definite. Therefore, $p(e)$ is a monotone function.

If the data set is (p_r, e_r), $r = 1, ..., n$, where p_r are equilibrium market prices and e_r is the social endowment, then the following theorem of Varian (1983) characterizes the testable implications of Friedman's representative agent model, given a finite set of observations on social endowments and market clearing prices.

Definition 7 U is a Walrasian rationalization of the data if $u(x_i) \geq u(x)$ for all x such that $p_i \cdot x \leq p_i \cdot x_i$, for $i = 1, ..., N$.

Theorem 5 (Varian (1983), Theorem 2, p. 103) The following conditions are equivalent:

(1) *There exists a non-satiated homothetic utility function that is a Walrasian rationalization of the data.*

(2) *The data satisfies the Homothetic Axiom of Revealed Preference (HARP): for all distinct choices of indices $(i, j, ..., m)$ we have $(p_i \cdot x_j)(p_j \cdot x_k)...(p_m \cdot x_i) \geq 1$.*

(3) *There exists numbers $U_i > 0$, $i = 1, ..., n$ such that $U_i \leq U_j p_j x_i$ for $i, j = 1, ..., n$.*

(4) *There exists a concave, monotonic, continuous non-satiated, homothetic utility function that is a Walrasian rationalization of the data.*

The representative agent's utility function in Bewley's Marshallian model is given by the social welfare function

$$W(e) = \max_{x_t \in \mathbb{R}^n_{++}} \left[\sum_{t=1}^{T} \frac{1}{\lambda_t} G_t(x_t) \right] \quad \text{s.t.} \sum_{x=1}^{T} x_t = e$$

where G_t is a strictly concave smooth utility function on $\mathbb{R}^n_{++}$ and λ_t is consumer t's constant exogenous marginal utility of income. Bewley solves the Marshallian consumer optimization problem: $\max_{e \in \mathbb{R}^n_{++}} \{W(e) - \bar{p} \cdot e\}$. He shows that it has a solution if and only if $\bar{p}$ is the unique market clearing set of competitive prices for the exchange economy populated with consumers (G_t, λ_t) and with social endowment $\bar{e}$.

That is, fix e at $\bar{e}$ and find the supporting prices $\bar{p}$ for the Pareto optimal allocation defined by the social welfare function $W(\cdot)$ evaluated at $\bar{e}$.

Let $H(p) = \max_{e\in\mathbb{R}^n_{++}}\{W(e) - \bar{p}\cdot e\}$, then it follows from Theorem 16.4, p. 145 in Rockafellar (1970) that $H(\bar{p}) \equiv \sum_{t=1}^{T} h_t(\bar{p})$ if $\bar{p}$ is a competitive equilibrium vector of prices. Hence $-(\partial H/\partial p)|_{\bar{p}} = \sum_{t=1}^{T} -(\partial h_t/\partial p)|_{\bar{p}} = \sum_{t=1}^{T} x_t(\bar{p}) = x(p) = e$. Since $H(\cdot)$ is a strictly smooth convex function of p on its domain, we see that $-\partial^2 H/\partial p^2 = \partial x(p)/\partial p$ is negative definite.

That is, $x(p)$ is cyclically monotone. The equilibrium map $p(e)$ is again the inverse of the demand function of the representative consumer. It follows from the duality relationship of Theorem 3 that $\bar{p}$ is the unique equilibrium price vector for the social endowment $\bar{e}$ if and only if $\bar{p} = (\partial W/\partial e)|_{e=\bar{e}}$ and $-(\partial H/\partial p)|_{\bar{p}} = \bar{e}$. Since W is a smooth concave function, we know that the gradient map $\bar{e} \to (\partial W/\partial e)|_{e=\bar{e}} = \bar{p}$ is cyclically monotone. Hence, our Theorem 2 characterizes the testable implications of Bewley's representative agent model, given a finite set of observations on social endowments and market clearing prices. That is, the equilibrium correspondance is cyclically monotone.

4 Random Quasilinear Rationalizations

If consumers have random quasilinear utility functions as in Bewley (1986), then in general the testable implications cannot be derived from a representative agent model. In fact, a priori, there may be no testable implications. Can anything happen? No, not if each individual's distribution of utility shocks has compact support, and agents have random utility functions of the form $V(x,e) = U(x) + \varepsilon\cdot x + x_0$. Assuming $U(x)$ is strictly concave, smooth and monotonic, each realization of ε gives rise to a quasilinear utility function having all of the properties previously derived,. i.e, for fixed ε, the random demand function $x(p,\varepsilon)$ is cyclically monotone. Of course, a finite family of observations of such demand functions need not by cyclically monotone, since each observation can in principal be drawn from a "different" cyclically monotone demand function. It is therefore surprising that the hypothesis of random quasilinear rationalization of a data set is refutable. As a consequence, this hypothesis is testable in the sense of Brown–Matzkin (1995), i.e., there exist a finite family of polynomial inequalities involving only observations on market data that are solvable if and only if the data can be rationalized with a random quasilinear utility function. In fact, using Fourier–Motzkin elimination — see Ziegler (1995) — there exists a family of linear inequalities in the data that are solvable if and only if the data has a random quasilinear rationalization.

Definition 8 $U(x) + \varepsilon\cdot x + x_0$ is a random quasilinear rationalization of the data $(p_1, x_1), ..., (p_r, x_r)$ if $\exists \varepsilon_1, ..., \varepsilon_r$ and $I > 0$ such that x_r is the solution to

$$\begin{aligned}&\max U(x) + \varepsilon_r\cdot x + x_{0_r}\\ &\text{s.t. } p_r\cdot x + x_{0_r} = I\end{aligned}$$

The utility shock ε has compact support if there exist $\varepsilon_{\min}$ and $\varepsilon_{\max}$ such that $\varepsilon_{\min} \leq \varepsilon \leq \varepsilon_{\max}$, where the quasilinear model is a special case of the random quasilinear model if $\varepsilon_{\min} \leq 0$.

(RAI): Given the data set $(p_1, x_1), ...(p_n, x_n)$, the Afriat inequalities for a random quasilinear utility function of the form $V(x, \varepsilon) = U(x) + \varepsilon \cdot x + x_0$ are:

$$U_i \leq U_j + (p_j - \varepsilon_j) \cdot (x_i - x_j) \text{ for } i, j = 1, ..., N$$
$$\varepsilon_{\min} \leq \varepsilon_j \leq \varepsilon_{\max} \text{ for } j = 1, ..., N$$
$$U_j > 0 \text{ for } j = 1, ..., N$$

These are linear inequalities in the unknown U_j and ε_j. Hence they can be solved in polynomial time using interior-point linear programming algorithms.

Figure 1 shows that for two observations, all possible pairs of budget lines defined by the gradients of $U(x)$ at x_1 and x_2, given consumption at x_1 and x_2, violate WARP. Hence rationalizations with random quasilinear utilities of the form $U(x) + \varepsilon \cdot x + x_0$, where ε has compact support is refutable.

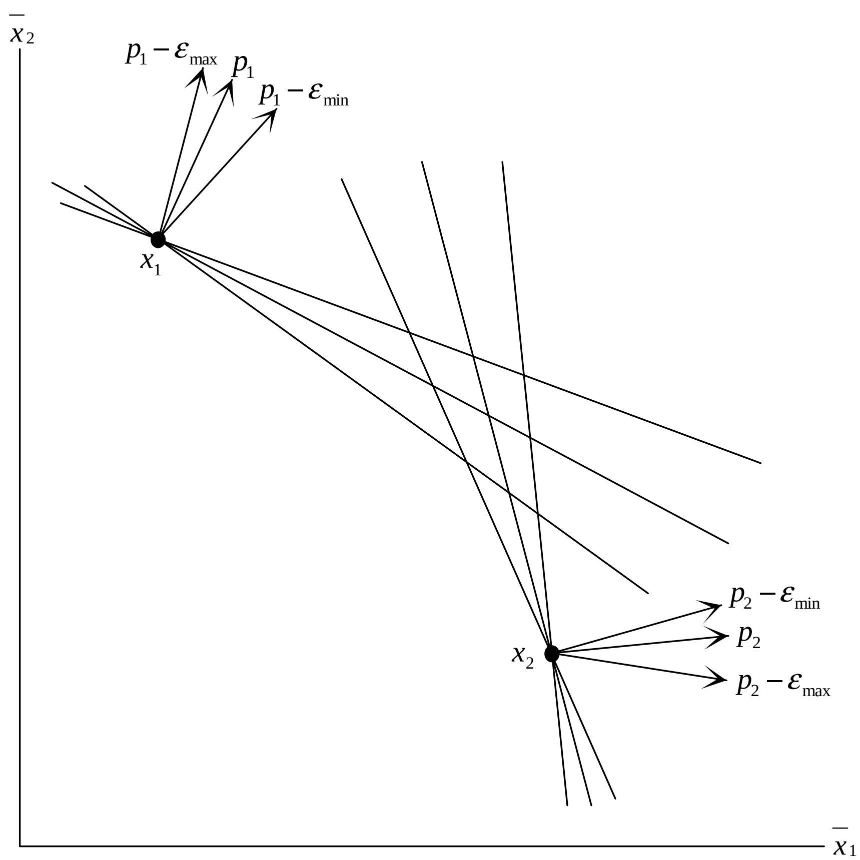

Figure 1
$DU(x) = p - \varepsilon$
$\varepsilon_{\min} \leq \varepsilon \leq \varepsilon_{\max}$

Theorem 6 *Given a finite exchange economy with T consumers and a finite set of observations consisting of allocations and market clearing prices in each observation. The data can be rationalized by random quasilinear utility functions of the form $U_t(x_t) + \varepsilon_t \cdot x_t + x_{0t}$ if and only if each consumer satisfies (RAI).*

This analysis extends to random Walrasian rationalizations, where we do not assume quasilinearity. In the homothetic case, we can derive a family of smooth convex Afriat inequalities that also can be solved in polynomial time, using interior point methods. For this model to be refutable in general, we must also assume that the marginal utilities of income lie in a compact set.

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