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GENERALIZED POTENTIALS AND ROBUST SETS OF EQUILIBRIA

By

Stephen Morris and Takashi Ui

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Generalized Potentials and Robust Sets of Equilibria*

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Abstract

This paper introduces generalized potential functions of complete information games and studies the robustness of sets of equilibria to incomplete information. A set of equilibria of a complete information game is robust if every incomplete information game where payoffs are almost always given by the complete information game has an equilibrium which generates behavior close to some equilibrium in the set. This paper provides sufficient conditions for the robustness of sets of equilibria in terms of argmax sets of generalized potential functions and shows that the sufficient conditions generalize the existing sufficient conditions for the robustness of equilibria.

Journal of Economic Literature Classification numbers: C72, D82.

Key Words: incomplete information; potential; refinements; robustness.

*This paper combines an earlier paper by Ui of the same title that introduced generalized potentials; and a paper by Morris on "Potential Methods in Interaction Games" that introduced characteristic potentials and local potentials. Some of the results on local potentials have also been reported in Frankel *et al.* (2001).

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1 Introduction

Outcomes of a game with common knowledge of payoffs may be very different from outcomes of the game with a "small" departure from common knowledge, as demonstrated by Rubinstein (1989) and Carlsson and van Damme (1993). This observation lead Kajii and Morris (1997a) to study what equilibria of complete information games are not much affected by weakening of the assumption of common knowledge; they studied the robustness of equilibria to incomplete information. An equilibrium of a complete information game is robust if every incomplete information game with payoffs almost always given by the complete information game has an equilibrium which generates behavior close to the equilibrium.

Kajii and Morris (1997b) demonstrated that robustness can be seen as a very strong refinement of Nash equilibria. The refinements literature examines what happens to a given Nash equilibrium in perturbed version of the complete information game. A weak class of refinements requires only that the Nash equilibrium continues to be equilibrium in some nearby perturbed game. The notion of perfect equilibria by Selten (1975) is the leading example of this class. A stronger class requires that the Nash equilibrium continues to be played in all perturbed nearby games. The notion of stable equilibria by Kohlberg and Mertens (1986) or that of strictly perfect equilibria by Okada (1981) are leading examples of this class. Robustness belongs to the latter, stronger class of refinements. Moreover, robustness to incomplete information allows an extremely rich set of perturbed games. In particular, while Kohlberg and Mertens (1986) allowed only independent action trembles across players, the definition of robustness leads to highly correlated trembles and thus an even stronger refinement. Indeed, Kajii and Morris (1997a) constructed an example in the spirit of Rubinstein (1989) to show that even a game with a unique Nash equilibrium, which is strict, may fail to have any robust equilibrium.

Kajii and Morris (1997a) and Ui (2001) provided sufficient conditions for the robustness of equilibria. Kajii and Morris (1997a) introduced the concept of p-dominance where $\mathsf{p} = (p_1, \ldots, p_n)$ is a vector of probabilities. An action profile is a p-dominant equilibrium if each player's action is a best response whenever he assigns probability at least p_i to his opponents choosing actions according to the action profile. Kajii and Morris (1997a) showed that a p-dominant equilibrium with $\sum_i p_i < 1$ is robust. Ui

(2001) considered robust equilibria of potential games, a class of complete information games possessing potential functions. As considered by Monderer and Shapley (1996), a potential function is a function on the action space such that it incorporates information about players' preferences over the action space that is sufficient to determine all the equilibria. Ui (2001) showed that the action profile that uniquely maximizes a potential function is robust.

The purpose of this paper is to provide a new sufficient condition for the robustness. The condition unifies and generalizes the sufficient conditions provided by Kajii and Morris (1997a) and Ui (2001).[1] Furthermore, the condition applies not only to the robustness of equilibria but also the robustness of sets of equilibria. This paper introduces generalized potential functions and provides the condition in terms of argmax sets of generalized potential functions.

We start by defining the robustness concept as a set valued one,[2] the robustness of sets of equilibria to incomplete information. A set of equilibria of a complete information game is robust if every incomplete information game with payoffs almost always given by the complete information game has an equilibrium which generates behavior close to some equilibrium in the set. If a robust set is a singleton then the equilibrium is robust in the sense of Kajii and Morris (1997a, 1997b). Because some games have no robust equilibria, it is natural to ask if a set of equilibria is robust.

We then introduce generalized potential functions. A generalized potential function is a function on a covering of the action space, a collection of subsets of the action space such that the union of the subsets is the action space. It incorporates some information about players' preferences over the collection of subsets. We call each element of the domain of a generalized potential function an action subspace. If an action subspace maximizes a generalized potential function and the generalized potential function has a unique maximum then we call the action subspace a generalized potential maximizer (GP-maximizer).

The main results state that there exists a correlated equilibrium assigning probability 1 to a GP-maximizer and that the set of such correlated equilibria is robust. This

[1] This unification of conditions based on potential arguments and conditions based on p-dominance may be of interest in other contexts. For example, potential arguments are widely used in evolutionary contexts and Sandholm (2001) has a p-dominance sufficient condition for almost global convergence.

[2] Kohlberg and Mertens (1986) were the first to propose making sets of equilibria the objects of a theory of equilibrium refinements.

immediately implies that if a GP-maximizer consists of one action profile then the action profile is a robust equilibrium. It should be noted that a robust set induced by the GP-maximizer condition is not always minimal. A robust set is minimal if no robust set is a proper subset of the robust set. In this paper, we do not explore the problem of how to identify minimal robust sets.

It is not so straightforward to find GP-maximizers from the definition. One reason is that, as we will see later, a complete information game may have multiple generalized potential functions with different domains. We restrict attention to generalized potential functions with two special classes of domains. One class of domains are unordered partitions of action spaces. We introduce best-response potential functions as functions over the partitions such that the best response correspondence of the function defined over the partition coincides with that of a complete information game. Potential functions of Monderer and Shapley (1996) form a special class of best-response potential functions with the finest partitions.[3] We show that a best-response potential function is a generalized potential function. The other class of domains are those induced by ordered partitions of action spaces. We introduce monotone potential functions as functions over the partitions such that the best response correspondence of the function defined over the partition and that of a complete information game has some monotonic relationship with respect to the order relation of the partition. We show that a monotone potential function naturally induces a generalized potential function where the domain consists of intervals of the ordered partition. We then show that a p-dominant equilibrium with $\sum_i p_i < 1$ is the induced GP-maximizer, by which the discussion of Kajii and Morris (1997a) and that of Ui (2001) are unified.

Rosenthal (1973) was the first to use potential functions in noncooperative game theory.[4] He used potential functions as tools for finding pure-action Nash equilibria.[5] Recent studies such as Blume (1993, 1997), Ui (1997, 2001), and Hofbauer and Sorger

[3] Morris and Ui (2002) demonstrated that the class of best-response potential functions with the finest partitions are much larger than the class of potential functions.

[4] In cooperative game theory, Hart and Mas-Colell (1989) introduced potential functions. The potential functions of Monderer and Shapley (1996) can be regarded as an extension of the potential functions of Hart and Mas-Colell (1989) to noncooperative games, as demonstrated by Ui (2000).

[5] In traffic network theory, non-atomic games similar to the finite games of Rosenthal (1973) are studied and non-atomic potential functions are used to calculate pure-action Nash equilibria. See Oppenheim (1995), for example.

(1999) used potential functions as tools for finding Nash equilibria satisfying some criteria for equilibrium selection. Since a narrow class of games admit potential functions, attempts have been made to introduce tools for a broader class of games. Monderer and Shapley (1996) introduced ordinal potential functions[6] and generalized ordinal potential functions. Voorneveld (2000) introduced best-response potential functions,[7] which are different from best-response potential functions in this paper. These functions inherit ordinal aspects of potential functions and serve as tools for the former use (finding pure-action equilibria). They are in clear contrast to generalized potential functions in this paper, which serve as tools for the latter use (refining equilibria).

The organization of this paper is as follows. Section 2 defines robust sets of equilibria. Section 3 introduces generalized potential functions. Section 4 provides the main results. Section 5 discusses best-response potential functions and Section 6 discusses monotone potential functions. Section 7 concludes the paper.

2 Robust Sets

A complete information game consists of a finite set of players N, a finite action set A_i for $i \in N$, and a payoff function $g_i : A \to \mathsf{R}$ for $i \in N$ where $A = \prod_{i \in N} A_i$. We write $A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$. We also write, for $S \in 2^N$, $A_S = \prod_{i \in S} A_i$ and $a_S = (a_i)_{i \in S} \in A_S$. Because we will fix N and A throughout the paper, we simply denote a complete information game by $\mathsf{g} = (g_i)_{i \in N}$.

An action distribution $\mu \in \Delta(A)$ is a correlated equilibrium of g if, for each $i \in N$,

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i}) g_i(a'_i, a_{-i})$$

for all $a_i, a'_i \in A_i$.[8] An action distribution $\mu \in \Delta(A)$ is a Nash equilibrium of g if it is a correlated equilibrium and, for all $a \in A$, $\mu(a) = \prod_{i \in N} \mu_i(a_i)$ where $\mu_i \in \Delta(A_i)$. We also say that $a \in A$ is a Nash equilibrium if $\mu \in \Delta(A)$ with $\mu(a) = 1$ is a Nash equilibrium.

Consider an incomplete information game with the set of players N and the action space A. Let T_i be a countable set of types of player $i \in N$. The state space is $T = \prod_{i \in N} T_i$. We write $T_{-i} = \prod_{j \neq i} T_j$ and $t_{-i} = (t_j)_{j \neq i} \in T_{-i}$. Let $P \in \Delta(T)$ be

[6] See also Kukushkin (1999).

[7] Ui (1997) considered similar functions in the context of stochastic evolutionary games.

[8] For any finite or countable set S, $\Delta(S)$ denotes the set of all probability distributions on S.

the prior probability distribution on T with $\sum_{t_{-i}\in T_{-i}} P(t_i,t_{-i}) > 0$ for all $i \in N$ and $t_i \in T_i$. A payoff function of player $i \in N$ is a bounded function $u_i : A \times T \to \mathsf{R}$. Because we will fix T, N, and A throughout the paper, we simply denote an incomplete information game by (u, P) where $\mathsf{u} = (u_i)_{i\in N}$.

A (mixed) strategy of player $i \in N$ is a mapping $\sigma_i : T_i \to \Delta(A_i)$. We write Σ_i for the set of strategies of player i. The strategy space is $\Sigma = \prod_{i\in N} \Sigma_i$. We write $\Sigma_{-i} = \prod_{j\neq i} \Sigma_j$ and $\sigma_{-i} = (\sigma_j)_{j\neq i} \in \Sigma_{-i}$. We write $\sigma_i(a_i|t_i)$ for the probability of $a_i \in A_i$ given $\sigma_i \in \Sigma_i$ and $t_i \in T_i$. For $\sigma \in \Sigma$ and $\sigma_{-i} \in \Sigma_{-i}$, we write $\sigma(a|t) = \prod_{i\in N} \sigma_i(a_i|t_i)$ and $\sigma_{-i}(a_{-i}|t_{-i}) = \prod_{j\neq i} \sigma_j(a_j|t_j)$ respectively. Let $\sigma_P \in \Delta(A)$ be such that $\sigma_P(a) = \sum_{t\in T} P(t)\sigma(a|t)$ for all $a \in A$. We call σ_P an action distribution generated by σ.

A strategy profile $\sigma \in \Sigma$ is a Bayesian Nash equilibrium of (u, P) if, for each $i \in N$,

$$\sum_{t_{-i}\in T_{-i}} \sum_{a\in A} P(t_{-i}|t_i)\sigma(a|t)u_i(a,t) \geq \sum_{t_{-i}\in T_{-i}} \sum_{a_{-i}\in A_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})u_i((a'_i, a_{-i}), t)$$

for all $t_i \in T_i$ and $a'_i \in A_i$ where $P(t_{-i}|t_i) = P(t_i,t_{-i})/\sum_{t_{-i}\in T_{-i}} P(t_i,t_{-i})$. Let $U_i(\sigma) = \sum_{t\in T}\sum_{a\in A} P(t)\sigma(a|t)u_i(a,t)$ be the payoff of strategy profile $\sigma \in \Sigma$ to player $i \in N$. Then, $\sigma \in \Sigma$ is a Bayesian Nash equilibrium of (u, P) if and only if, for each $i \in N$, $U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i})$ for all $\sigma'_i \in \Sigma_i$.

For given g, consider the following subset of T_i:

$$T_i^{u_i} = \{t_i \in T_i \,|\, u_i(a,(t_i,t_{-i})) = g_i(a) \text{ for all } a \in A,\ t_{-i} \in T_{-i} \text{ with } P(t_i,t_{-i}) > 0\}.$$

When $t_i \in T_i^{u_i}$ is realized, payoffs of player i are given by g_i and he knows his payoffs. We write $T^{\mathsf{u}} = \prod_{i\in N} T_i^{u_i}$.

Definition 1 An incomplete information game (u, P) is an ε-elaboration of g if $P(T^{\mathsf{u}}) = 1-\varepsilon$ for $\varepsilon \in [0,1]$.

Payoffs of a 0-elaboration are given by g with probability 1 and every player knows his payoffs. It is straightforward to see that if a 0-elaboration has a Bayesian Nash equilibrium $\sigma \in \Sigma$ then an action distribution generated by σ, $\sigma_P \in \Delta(A)$, is a correlated equilibrium of g. Kajii and Morris (1997a, Corollary 3.5) showed the following property of ε-elaborations, which we will use later.

Lemma 1 Let $\{(\mathsf{u}^k, P^k)\}_{k=1}^{\infty}$ be such that (u^k, P^k) is an ε^k-elaboration of g and $\varepsilon^k \to 0$ as $k \to \infty$. Let σ^k be a Bayesian Nash equilibrium of (u^k, P^k) and let σ_P^k be an action distribution generated by σ^k. Then $\{\sigma_P^k\}_{k=1}^{\infty}$ has a subsequence which converges to some correlated equilibrium of g.

We say that a set of correlated equilibria of g is robust if, for small $\varepsilon > 0$, every ε-elaboration of g has a Bayesian Nash equilibrium $\sigma \in \Sigma$ such that $\sigma_P \in \Delta(A)$ is close to some equilibrium in the set.

Definition 2 A set of correlated equilibria of g, $\mathcal{E} \subseteq \Delta(A)$, is robust to all elaborations in g if, for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every ε-elaboration of g has a Bayesian Nash equilibrium $\sigma \in \Sigma$ such that $\max_{a \in A} |\mu(a) - \sigma_P(a)| \leq \delta$ for some $\mu \in \mathcal{E}$.

If $\mathcal{E}$ is a singleton then the equilibrium in $\mathcal{E}$ is robust in the sense of Kajii and Morris (1997a).

Kajii and Morris (1997b) considered a weaker version of the robustness of equilibria than that of Kajii and Morris (1997a).[9] We consider the corresponding version of the robustness of sets of equilibria. A type $t_i \in T_i \backslash T_i^{u_i}$ is committed if player i of this type has a strictly dominant action $a_i^{t_i} \in A_i$ such that $u_i((a_i^{t_i}, a_{-i}), (t_i, t_{-i})) > u_i((a_i, a_{-i}), (t_i, t_{-i}))$ for all $a_i \in A_i \backslash \{a_i^{t_i}\}$, $a_{-i} \in A_{-i}$, and $t_{-i} \in T_{-i}$ with $P(t_i, t_{-i}) > 0$. An ε-elaboration of g is canonical if every $t_i \in T_i \backslash T_i^{u_i}$ is committed for all $i \in N$.

Definition 3 A set of correlated equilibria of g, $\mathcal{E} \subseteq \Delta(A)$, is robust to canonical elaborations in g if, for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every canonical ε-elaboration of g has a Bayesian Nash equilibrium $\sigma \in \Sigma$ such that $\max_{a \in A} |\mu(a) - \sigma_P(a)| \leq \delta$ for some $\mu \in \mathcal{E}$.

If $\mathcal{E}$ is a singleton then the equilibrium in $\mathcal{E}$ is robust in the sense of Kajii and Morris (1997b).

In Section 4, we will provide two sufficient conditions for the robustness of sets of equilibria, one for the robustness to all elaborations and the other for the robustness to canonical elaborations respectively.

For either of the robustness concepts, if $\mathcal{E}$ is robust then a set of correlated equilibria $\mathcal{E}'$ with $\mathcal{E} \subseteq \mathcal{E}'$ is also robust. A robust set $\mathcal{E}$ is minimal if no robust set is a proper subset of $\mathcal{E}$. In this paper, we do not explore the problem of how to identify minimal robust sets.

[9] The difference between them is an open question.

3 Generalized Potentials

Monderer and Shapley (1996) defined weighted potential functions of complete information games.

Definition 4 A function $f : A \to \mathsf{R}$ is a weighted potential function of g if there exists $w_i > 0$ such that

$$g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) = w_i \big(f(a_i, a_{-i}) - f(a'_i, a_{-i})\big) \tag{1}$$

for all $i \in N$, $a_i, a'_i \in A_i$, and $a_{-i} \in A_{-i}$. A complete information game g is a weighted potential game if it has a weighted potential function. When $w_i = 1$ for $i \in N$, we call f a potential function and g a potential game.

Suppose that g has a weighted potential function f. Then

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})\big) = w_i \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(f(a_i, a_{-i}) - f(a'_i, a_{-i})\big)$$

for all $i \in N$, $a_i, a'_i \in A_i$, and $\lambda_i \in \Delta(A_{-i})$. Thus, we have

$$\arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) = \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) f(a_i, a_{-i}) \tag{2}$$

for all $i \in N$ and $\lambda_i \in \Delta(A_{-i})$. We generalize (2) to define generalized potential functions.[10]

Before providing a formal definition, we present an example. Let $A_i = \{0, 1, 2\}$ for $i \in N \equiv \{1, 2\}$. We define a collection of subsets of A_i, $\mathcal{A}_i = \{\{0, 1\}, \{0, 1, 2\}\}$ for $i \in N$, and define $\mathcal{A} = \{X_1 \times X_2 \,|\, X_1 \in \mathcal{A}_1,\, X_2 \in \mathcal{A}_2\}$. Consider g and $F : \mathcal{A} \to \mathsf{R}$ given by the following tables.

g

	0	1	2
0	3, 2	2, 3	0, 0
1	2, 3	3, 2	0, 0
2	0, 0	0, 0	1, 1

F

	$\{0,1\}$	$\{0,1,2\}$
$\{0,1\}$	2	0
$\{0,1,2\}$	0	1

[10] The existence of a function f such that property (2) is satisfied is in fact a necessary but not a sufficient condition for g to be a weighted potential game. See the discussion in Section 5 and Morris and Ui (2002).

The function F has the following property: for $\Lambda_i \in \Delta(\mathcal{A}_j)$ and $\lambda_i \in \Delta(A_j)$ with $\lambda_i(0)+\lambda_i(1) \geq \Lambda_i(\{0,1\})$,

$$X_i \cap \arg\max_{a_i \in A_i} \sum_{a_j \in A_j} \lambda_i(a_j) g_i(a_i, a_j) \neq \emptyset \text{ for all } X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_j \in \mathcal{A}_j} \Lambda_i(X_j) F(X_i' \times X_j)$$

where $i \neq j$. As we will see later, F is a generalized potential function of g.

To provide the formal definition, we first introduce the domain of a generalized potential function denoted by $\mathcal{A}$. For each $i \in N$, let $\mathcal{A}_i \subseteq 2^{A_i} \backslash \emptyset$ be a covering of A_i. That is, $\mathcal{A}_i$ is a collection of nonempty subsets of A_i such that $\bigcup_{X_i \in \mathcal{A}_i} X_i = A_i$. The domain of a generalized potential function is $\mathcal{A} = \{\prod_{i \in N} X_i \,|\, X_i \in \mathcal{A}_i \text{ for } i \in N\}$. We write $\mathcal{A}_{-i} = \{\prod_{j \neq i} X_j \,|\, X_j \in \mathcal{A}_j \text{ for } j \neq i\}$ and $X_{-i} = \prod_{j \neq i} X_j \in \mathcal{A}_{-i}$. Note that $\mathcal{A}$ and $\mathcal{A}_{-i}$ are coverings of A and A_{-i} respectively. We call $X \in \mathcal{A}$ an action subspace.

We then introduce, for $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, a corresponding subset of $\Delta(A_{-i})$ denoted by $\Delta_{\Lambda_i}(A_{-i})$. Imagine that player i believes that $a_{-i} \in A_{-i}$ is chosen in two steps: first, $X_{-i} \in \mathcal{A}_{-i}$ is chosen according to $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, and then, $a_{-i} \in X_{-i}$ is chosen according to some $\lambda_i^{X_{-i}} \in \Delta(A_{-i})$ such that $\lambda_i^{X_{-i}}$ assigns probability 1 to X_{-i}, i.e., $\sum_{a_{-i} \in X_{-i}} \lambda_i^{X_{-i}}(a_{-i}) = 1$. Then, the induced belief of player i over A_{-i} is $\lambda_i \in \Delta(A_{-i})$ such that

$$\lambda_i(a_{-i}) = \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i(X_{-i}) \lambda_i^{X_{-i}}(a_{-i})$$

for all $a_{-i} \in \mathcal{A}_{-i}$. We write $\Delta_{\Lambda_i}(A_{-i})$ for the set of the beliefs of player i over A_{-i} induced by the above rule:

$$\begin{aligned}\Delta_{\Lambda_i}(A_{-i}) = \{\lambda_i \in \Delta(A_{-i}) \,|\, \lambda_i(a_{-i}) = \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i(X_{-i}) \lambda_i^{X_{-i}}(a_{-i}) \text{ for } a_{-i} \in A_{-i}, \\ \lambda_i^{X_{-i}} \in \Delta(A_{-i}) \text{ with } \sum_{a_{-i} \in X_{-i}} \lambda_i^{X_{-i}}(a_{-i}) = 1 \text{ for } X_{-i} \in \mathcal{A}_{-i}\}.\end{aligned}$$

Definition 5 A function $F : \mathcal{A} \to \mathsf{R}$ is a generalized potential function of g if, for all $i \in N$, $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, and $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$,

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i', a_{-i}) \neq \emptyset$$

for every

$$X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i(X_{-i}) F(X_i' \times X_{-i})$$

such that X_i is maximal in the argmax set ordered by the set inclusion relation. An action subspace $X^* \in \mathcal{A}$ is a generalized potential maximizer (GP-maximizer) if $F(X^*) > F(X)$ for all $X \in \mathcal{A} \backslash \{X^*\}$.

It is clear that $F : \mathcal{A} \to \mathsf{R}$ in the above example is a generalized potential function because $\Delta_{\Lambda_i}(A_j) \subseteq \{\lambda_i \in \Delta(A_j) \mid \lambda_i(0) + \lambda_i(1) \geq \Lambda_i(\{0,1\})\}$ where $i \neq j$.

At the extreme, consider $F : \mathcal{A} \to \mathsf{R}$ such that $\mathcal{A}_i = \{A_i\}$ for all $i \in N$. Note that $\mathcal{A} = \{A\}$. Clearly, every complete information game has a generalized potential function of this type. At the other extreme, consider $F : \mathcal{A} \to \mathsf{R}$ such that $\mathcal{A}_i = \{\{a_i\} \mid a_i \in A_i\}$ for all $i \in N$. Note that $\mathcal{A} = \{\{a\} \mid a \in A\}$. A weighted potential game has a generalized potential function of this type, which we prove in Section 5.

Lemma 2 If g is a weighted potential game with a weighted potential function f then g has a generalized potential function $F : \mathcal{A} \to \mathsf{R}$ such that $\mathcal{A}_i = \{\{a_i\} \mid a_i \in A_i\}$ for all $i \in N$ and $F(\{a\}) = f(a)$ for all $a \in A$.

Before closing this section, we give a characterization of $\Delta_{\Lambda_i}(A_{-i})$.

Lemma 3 For all $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$ if and only if

$$\sum_{a_{-i} \in B_{-i}} \lambda_i(a_{-i}) \geq \sum_{\substack{X_{-i} \in \mathcal{A}_{-i} \\ X_{-i} \subseteq B_{-i}}} \Lambda_i(X_{-i})$$

for all $B_{-i} \in 2^{A_{-i}}$.

This lemma is an immediate consequence of the result of Strassen (1964), which is well known in the study of Dempster-Shafer theory.[11] Dempster-Shafer theory considers non-additive probability functions called belief functions. Every $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, called a basic probability assignment, defines a corresponding belief function $v_i^{\Lambda_i} : 2^{A_{-i}} \to [0,1]$ such that

$$v_i^{\Lambda_i}(B_{-i}) = \sum_{\substack{X_{-i} \in \mathcal{A}_{-i} \\ X_{-i} \subseteq B_{-i}}} \Lambda_i(X_{-i})$$

for all $B_{-i} \in 2^{A_{-i}}$. It is known that the correspondence between Λ_i and $v_i^{\Lambda_i}$ is one-to-one. An additive probability function $\lambda_i \in \Delta(A_{-i})$ is said to be compatible with a belief function $v_i^{\Lambda_i}$ if

$$\lambda_i(B_{-i}) \geq v_i^{\Lambda_i}(B_{-i})$$

[11] Dempster (1967, 1968) and Shafer (1976).

for all $B_{-i} \in 2^{A_{-i}}$.[12] Strassen (1964) proved that, for all $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, λ_i is compatible with $v_i^{\Lambda_i}$ if and only if $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$, which is exactly Lemma 3.

4 Main Results

Suppose that g has a generalized potential function $F : \mathcal{A} \to \mathsf{R}$ with a GP-maximizer X^*. Let $\mathcal{E}_{X^*}$ be the set of correlated equilibria of g that assign probability 1 to X^*:

$$\mathcal{E}_{X^*} = \{\mu \in \Delta(A) \,|\, \mu \text{ is a correlated equilibrium of } \mathsf{g} \text{ such that } \sum_{a \in X^*} \mu(a) = 1\}.$$

Our main results state that $\mathcal{E}_{X^*}$ is nonempty and robust. We present two theorems below. In Theorem 1, we consider all generalized potential functions and provide a sufficient condition for the robustness to canonical elaborations. In Theorem 2, we consider a special class of generalized potential functions such that $A_i \in \mathcal{A}_i$ for all $i \in N$ and provide a sufficient condition for the robustness to all elaborations.

Theorem 1 If g has a generalized potential function $F : \mathcal{A} \to \mathsf{R}$ with a GP-maximizer X^*, then $\mathcal{E}_{X^*}$ is nonempty and robust to canonical elaborations in g.

Theorem 2 If g has a generalized potential function $F : \mathcal{A} \to \mathsf{R}$ with a GP-maximizer X^* such that $A_i \in \mathcal{A}_i$ for all $i \in N$, then $\mathcal{E}_{X^*}$ is nonempty and robust to all elaborations in g.

If $\mathcal{E}_{X^*}$ is a singleton, then it is a minimal robust set and the equilibrium in $\mathcal{E}_{X^*}$ is robust in the sense of Kajii and Morris (1997a, 1997b). Clearly, if a GP-maximizer consists of one action profile, then $\mathcal{E}_{X^*}$ is a singleton. It is straightforward to see that $\mathcal{E}_{X^*}$ of the example in the previous section is also a singleton where the GP-maximizer consists of four action profiles.

It should be noted that $\mathcal{E}_{X^*}$ is not always a minimal robust set. For example, if a generalized potential function is such that $\mathcal{A}_i = \{A_i\}$ for all $i \in N$, then $\mathcal{E}_{X^*}$ is the set of all correlated equilibria.[13] The above theorems are useful only when we have nontrivial generalized potential functions.

[12] In literature of non-additive probabilities written by economists, λ_i is called a core of $v_i^{\Lambda_i}$ because it is a core when we regard $B_{-i} \in 2^{A_{-i}}$ as a coalition.

[13] Kajii and Morris (1997a) remarked the robustness of the set of all correlated equilibria.

In the remainder of this section, we prove Theorem 1 and Theorem 2 simultaneously. The proof is presented in four steps.

For the first step, let (u, P) be an ε-elaboration of g and consider collections of mappings

$$\Xi_i = \{\xi_i : T_i \to \mathcal{A}_i \mid \text{ for all } t_i \in T_i \backslash T_i^{u_i},\ \xi_i(t_i) \in \mathcal{A}_i \text{ contains}$$
$$\text{every undominated action of type } t_i\},$$
$$\Xi = \{\xi : T \to \mathcal{A} \mid \xi(t) = \prod_{i \in N} \xi_i(t_i) \text{ for all } t \in T \text{ where } \xi_i \in \Xi_i \text{ for all } i \in N\}$$

where we say that $a_i \in A_i$ is an undominated action of type t_i if it is not a strictly dominated action of type t_i. We say that $a_i \in A_i$ is a strictly dominated action of type t_i if there exists $a_i' \in A_i$ such that $u_i((a_i', a_{-i}), (t_i, t_{-i})) > u_i((a_i, a_{-i}), (t_i, t_{-i}))$ for all $a_{-i} \in A_{-i}$ and $t_{-i} \in T_{-i}$ with $P(t_i, t_{-i}) > 0$. Note that Ξ is nonempty if and only if, for all $i \in N$ and $t_i \in T_i \backslash T_i^{u_i}$, there exists $X_i \in \mathcal{A}_i$ such that X_i contains every undominated action of type t_i. As considered in Theorem 1, if (u, P) is canonical and player i of type $t_i \in T_i \backslash T_i^{u_i}$ has a strictly dominant action $a_i^{t_i} \in A_i$ then Ξ is nonempty because $\mathcal{A}_i$ is a covering of A_i and there exists $X_i \in \mathcal{A}_i$ such that $a_i^{t_i} \in X_i$. As considered in Theorem 2, if $A_i \in \mathcal{A}_i$ for all $i \in N$ then Ξ is nonempty because A_i contains every action. To summarize, we have the following lemma.

Lemma 4 If (u, P) is canonical then Ξ is nonempty. If $A_i \in \mathcal{A}_i$ for all $i \in N$ then Ξ is nonempty.

For the second step, let $V : \Xi \to \mathsf{R}$ be such that

$$V(\xi) = \sum_{t \in T} P(t) F(\xi(t))$$

for all $\xi \in \Xi$ and consider the set of its maximizers $\Xi^* = \arg\max_{\xi \in \Xi} V(\xi)$.

Lemma 5 If Ξ is nonempty then Ξ^* is nonempty. If $\xi^* \in \Xi^*$ then

$$\sum_{t \in T,\, \xi^*(t) = X^*} P(t) \geq 1 - \varepsilon\kappa$$

where κ is a positive constant.

Proof. Let $\{\xi^k \in \Xi\}_{k=1}^{\infty}$ be such that

$$\lim_{k\to\infty} V(\xi^k) = \sup_{\xi\in\Xi} V(\xi).$$

Let $Q^k \in \Delta(T\times\mathcal{A})$ be such that $Q^k(t,X) = P(t)\delta(\xi^k(t),X)$ for all $(t,X)\in T\times\mathcal{A}$ where $\delta : \mathcal{A}\times\mathcal{A} \to \{0,1\}$ is such that $\delta(X',X) = 1$ if $X' = X$ and $\delta(X',X) = 0$ otherwise. Then

$$\sum_{(t,X)\in T\times\mathcal{A}} Q^k(t,X)F(X) = \sum_{(t,X)\in T\times\mathcal{A}} P(t)\delta(\xi^k(t),X)F(X) = \sum_{t\in T} P(t)F(\xi^k(t)) = V(\xi^k).$$

We regard $\{Q^k\}_{k=1}^{\infty}$ as a sequence of probability measures on a discrete metric space $T\times\mathcal{A}$. Note that, for every $\varepsilon > 0$, there exists a finite subset $S_\varepsilon \subset T$ such that $\sum_{(t,X)\in S_\varepsilon\times\mathcal{A}} Q^k(t,X) = P(S_\varepsilon) > 1-\varepsilon$ for all $k \geq 1$. This implies that $\{Q^k\}_{k=1}^{\infty}$ is tight because $S_\varepsilon\times\mathcal{A}$ is finite and thus compact. Accordingly, by Prohorov's theorem,[14] $\{Q^k\}_{k=1}^{\infty}$ has a weakly convergent subsequence $\{Q^{k_l}\}_{l=1}^{\infty}$ such that $Q^{k_l} \to Q^*$ as $l\to\infty$. It is straightforward to see that there exists $\xi^* \in \Xi$ such that

$$Q^*(t,X) = \lim_{l\to\infty} Q^{k_l}(t,X) = P(t)\lim_{l\to\infty}\delta(\xi^{k_l}(t),X) = P(t)\delta(\xi^*(t),X)$$

for all $(t,X)\in T\times\mathcal{A}$. Then

$$\begin{aligned}
\sup_{\xi\in\Xi} V(\xi) &= \lim_{l\to\infty} V(\xi^{k_l}) \\
&= \lim_{l\to\infty} \sum_{(t,X)\in T\times\mathcal{A}} Q^{k_l}(t,X)F(X) \\
&= \sum_{(t,X)\in T\times\mathcal{A}} Q^*(t,X)F(X) \;=\; V(\xi^*).
\end{aligned}$$

Therefore, $\xi^* \in \Xi^*$ and thus Ξ^* is nonempty.

Let $F^* = F(X^*)$, $F' = \max_{X\in\mathcal{A}\setminus\{X^*\}} F(X)$, and $F'' = \min_{X\in\mathcal{A}} F(X)$. Note that $F^* > F' \geq F''$. Let $\xi \in \Xi$ be such that $\xi_i(t_i) = X_i^*$ for all $t_i \in T_i^{u_i}$ and $i \in N$. We have

$$\begin{aligned}
V(\xi^*) &\geq V(\xi) \\
&= \sum_{t\in T^{\mathsf{u}}} P(t)F(\xi(t)) + \sum_{t\in T\setminus T^{\mathsf{u}}} P(t)F(\xi(t)) \\
&\geq P(T^{\mathsf{u}})F^* + (1-P(T^{\mathsf{u}}))\,F'' = (1-\varepsilon)F^* + \varepsilon F''.
\end{aligned}$$

[14] See Billingsley (1968), for example.

We also have

$$V(\xi^*) = \sum_{t\in T,\,\xi^*(t)=X^*} P(t)F(\xi^*(t)) + \sum_{t\in T,\,\xi^*(t)\neq X^*} P(t)F(\xi^*(t))$$
$$\leq \sum_{t\in T,\,\xi^*(t)=X^*} P(t)F^* + \left(1 - \sum_{t\in T,\,\xi^*(t)=X^*} P(t)\right) F'.$$

Combining the above inequalities, we have

$$(1-\varepsilon)F^* + \varepsilon F'' \leq \sum_{t\in T,\,\xi^*(t)=X^*} P(t)F^* + \left(1 - \sum_{t\in T,\,\xi^*(t)=X^*} P(t)\right) F'$$

and thus

$$\sum_{t\in T,\,\xi^*(t)=X^*} P(t) \geq 1 - \varepsilon\kappa$$

where $\kappa = (F^* - F'')/(F^* - F') > 0$. ∎

For the third step, let Ξ be partially ordered by the relation $\subseteq$ such that $\xi \subseteq \xi'$ for $\xi, \xi' \in \Xi$ if and only if $\xi_i(t_i) \subseteq \xi'_i(t_i)$ for all $t_i \in T_i$ and $i \in N$.

Lemma 6 If $\Xi^* \subseteq \Xi$ is nonempty, then it contains at least one maximal element. If ξ^* is a maximal element of Ξ^*, then (u, P) has a Bayesian Nash equilibrium $\sigma^* \in \Sigma$ such that $\sigma^*(t) \in \Delta(A)$ assigns probability 1 to the action subspace $\xi^*(t) \in \mathcal{A}$ for all $t \in T$, i.e., $\sum_{a\in\xi^*(t)} \sigma^*(a|t) = 1$ for all $t \in T$.

Proof. If every linearly ordered subset of Ξ^* has an upper bound in Ξ^*, then Ξ^* contains at least one maximal element by Zorn's Lemma. Let $\Xi' \subseteq \Xi^*$ be linearly ordered. Fix $t = (t_i)_{i\in N} \in T$. For each $i \in N$, observe that

$$\{X_i \,|\, X_i = \xi'_i(t_i),\, \xi' \in \Xi'\} \subseteq \mathcal{A}_i$$

is linearly ordered by the set inclusion relation. Since this set is finite, it has a maximum element, which is equal to $\bigcup_{\xi'_i\in\Xi'_i} \xi'_i(t_i) \in \mathcal{A}_i$. Clearly, there exists $\xi^{(i,t)} \in \Xi'$ such that $\xi^{(i,t)}_i(t_i) = \bigcup_{\xi'_i\in\Xi'_i} \xi'_i(t_i)$. Consider $\{\xi^{(i,t)} \,|\, i \in N\} \subseteq \Xi'$. Since this set is linearly ordered and finite, it has a maximum element $\xi^{(j,t)}$. Simply denote it by $\xi^{\langle t\rangle}$, which satisfies $\xi^{\langle t\rangle}_i(t_i) = \bigcup_{\xi'_i\in\Xi'_i} \xi'_i(t_i)$ for all $i \in N$. For $\varepsilon > 0$, consider $\{\xi^{\langle t\rangle} \,|\, t \in T,\, P(t) > \varepsilon\} \subseteq \Xi'$. Since this set is linearly ordered and finite, it has a maximum element $\xi^{\langle s\rangle}$. Simply denote it by ξ^ε, which satisfies $\xi_i^\varepsilon(t_i) = \bigcup_{\xi_i'\in\Xi_i'} \xi_i'(t_i)$ for all $t_i \in T_i$ and $i \in N$ such that $P(t) > \varepsilon$. Let $\tilde{\xi} \in \Xi$ be such that $\tilde{\xi}_i(t_i) = \bigcup_{\xi_i'\in\Xi_i'} \xi_i'(t_i)$ for all $t_i \in T_i$ and $i \in N$. Note that $\tilde{\xi}$ is an upper bound of Ξ'. Since $\xi^\varepsilon(t) = \tilde{\xi}(t)$ for $t \in T$ with $P(t) > \varepsilon$, it must be true that

$$|V(\tilde{\xi}) - V(\xi^\varepsilon)| \leq \max_{X,X'\in\mathcal{A}} |F(X) - F(X')| \times \sum_{t\in T,\, P(t)\leq\varepsilon} P(t).$$

This implies that $\lim_{\varepsilon\to 0} |V(\tilde{\xi}) - V(\xi^\varepsilon)| = 0$. Note that $V(\xi^\varepsilon) = \max_{\xi\in\Xi} V(\xi)$ because $\xi^\varepsilon \in \Xi^*$. Therefore, $V(\tilde{\xi}) = \max_{\xi\in\Xi} V(\xi)$ and thus $\tilde{\xi} \in \Xi^*$, which completes the proof of the first half of the lemma.

We prove the second half. Let $\xi^* \in \Xi^*$ be a maximal element. Let $\xi_i^* \in \Xi_i$ be such that $\xi^*(t) = \prod_{i\in N} \xi_i^*(t_i)$ for all $t \in T$ and write $\xi_{-i}^*(t_{-i}) = \prod_{j\neq i} \xi_j^*(t_j)$.

We write $\Sigma_i^* = \{\sigma_i \in \Sigma_i \,|\, \sum_{a_i\in\xi_i^*(t_i)} \sigma_i(a_i|t_i) = 1 \text{ for all } t_i \in T_i\}$, $\Sigma^* = \prod_{i\in N} \Sigma_i^*$, and $\Sigma_{-i}^* = \prod_{j\neq i} \Sigma_j^*$. We show that there exists a Bayesian Nash equilibrium $\sigma^* \in \Sigma^*$.

Let $\beta_i : \Sigma_{-i}^* \to 2^{\Sigma_i^*}$ be such that $\beta_i(\sigma_{-i}) = \arg\max_{\sigma_i\in\Sigma_i} U_i(\sigma_i, \sigma_{-i}) \cap \Sigma_i^*$ for all $\sigma_{-i} \in \Sigma_{-i}^*$ and $\beta : \Sigma^* \to 2^{\Sigma^*}$ be such that $\beta(\sigma) = \prod_{i\in N} \beta_i(\sigma_{-i})$ for all $\sigma \in \Sigma^*$. Note that β is the best response correspondence of (u, P) restricted to Σ^*.

We show that β has nonempty values. This is true if and only if, for all $i \in N$, $\sigma_{-i} \in \Sigma_{-i}^*$, and $t_i \in T_i$,

$$\xi_i^*(t_i) \cap \arg\max_{a_i\in A_i} \sum_{t_{-i}\in T_{-i}} \sum_{a_{-i}\in A_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})u_i((a_i, a_{-i}), t) \neq \emptyset. \tag{3}$$

Suppose that $t_i \in T_i \backslash T_i^{u_i}$. Then (3) is true because $\xi_i^*(t_i)$ contains every undominated action of type t_i.

Suppose that $t_i \in T_i^{u_i}$. Rewrite the left-hand side of (3) as

$$\begin{aligned}
&\xi_i^*(t_i) \cap \arg\max_{a_i\in A_i} \sum_{t_{-i}\in T_{-i}} \sum_{a_{-i}\in A_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})u_i((a_i, a_{-i}), t) \\
&= \xi_i^*(t_i) \cap \arg\max_{a_i\in A_i} \sum_{a_{-i}\in A_{-i}} \left(\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i}) \right) g_i(a_i, a_{-i}) \qquad (4) \\
&= \xi_i^*(t_i) \cap \arg\max_{a_i\in A_i} \sum_{a_{-i}\in A_{-i}} \lambda_i^{t_i}(a_{-i}) g_i(a_i, a_{-i})
\end{aligned}$$

where $\lambda_i^{t_i} \in \Delta(A_{-i})$ is such that

$$\lambda_i^{t_i}(a_{-i}) = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})$$

for all $a_{-i} \in A_{-i}$. Because ξ^* is a maximal element of Ξ^*,

$$\begin{aligned}
\xi_i^*(t_i) &\in \arg\max_{X_i \in \mathcal{A}_i} \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) F(X_i \times \xi_{-i}^*(t_{-i})) \\
&= \arg\max_{X_i \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} \left(\sum_{\substack{t_{-i} \in T_{-i} \\ \xi_{-i}^*(t_{-i}) = X_{-i}}} P(t_{-i}|t_i) \right) F(X_i \times \xi_{-i}^*(t_{-i})) \\
&= \arg\max_{X_i \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i^{t_i}(X_{-i}) F(X_i \times X_{-i})
\end{aligned}$$

and $\xi_i^*(t_i)$ is maximal in the argmax set where $\Lambda_i^{t_i} \in \Delta(\mathcal{A}_{-i})$ is such that

$$\Lambda_i^{t_i}(X_{-i}) = \sum_{\substack{t_{-i} \in T_{-i} \\ \xi_{-i}^*(t_{-i}) = X_{-i}}} P(t_{-i}|t_i)$$

for all $X_{-i} \in \mathcal{A}_{-i}$. This implies that if $\lambda_i^{t_i} \in \Delta_{\Lambda_i^{t_i}}(A_{-i})$ then

$$\xi_i^*(t_i) \cap \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i^{t_i}(a_{-i}) g_i(a_i, a_{-i}) \neq \emptyset \tag{5}$$

by the definition of generalized potential functions. To see that $\lambda_i^{t_i} \in \Delta_{\Lambda_i^{t_i}}(A_{-i})$, rewrite $\lambda_i^{t_i}(a_{-i})$ as

$$\begin{aligned}
\lambda_i^{t_i}(a_{-i}) &= \sum_{X_{-i} \in \mathcal{A}_{-i}} \sum_{\substack{t_{-i} \in T_{-i} \\ \xi_{-i}^*(t_{-i}) = X_{-i}}} P(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}) \\
&= \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i^{t_i}(X_{-i}) \lambda_i^{t_i, X_{-i}}(a_{-i})
\end{aligned}$$

where

$$\lambda_i^{t_i, X_{-i}}(a_{-i}) = \begin{cases} \displaystyle\sum_{\substack{t_{-i} \in T_{-i} \\ \xi_{-i}^*(t_{-i}) = X_{-i}}} \frac{P(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i})}{\Lambda_i^{t_i}(X_{-i})} & \text{if } \Lambda_i^{t_i}(X_{-i}) \neq 0, \\ \dfrac{1}{|X_{-i}|} & \text{if } \Lambda_i^{t_i}(X_{-i}) = 0 \text{ and } a_{-i} \in X_{-i}, \\ 0 & \text{if } \Lambda_i^{t_i}(X_{-i}) = 0 \text{ and } a_{-i} \notin X_{-i}. \end{cases}$$

Because $\sigma_{-i} \in \Sigma_{-i}^*$ and thus $\sum_{a_{-i} \in \xi_{-i}^*(t_{-i})} \sigma_{-i}(a_{-i}|t_{-i}) = 1$ for all $t_{-i} \in T_{-i}$, we have $\lambda_i^{t_i, X_{-i}} \in \Delta(A_{-i})$ with $\sum_{a_{-i} \in X_{-i}} \lambda_i^{t_i, X_{-i}}(a_{-i}) = 1$. This implies that $\lambda_i^{t_i} \in \Delta_{\Lambda_i^{t_i}}(A_{-i})$ and thus (5). Therefore, (3) is true by (4) and (5).

We have shown that β has nonempty values. We can show that Σ^* is compact[15] and convex and that β has a closed graph and convex values. By Kakutani-Fan-Glicksberg fixed point theorem, β has a fixed point $\sigma^* \in \Sigma^*$, which is a Bayesian Nash equilibrium of (u, P). ∎

We now report the fourth and final step. An immediate implication of the above lemmas is the following. If (u, P) is canonical (the case considered in Theorem 1), or if $A_i \in \mathcal{A}_i$ for all $i \in N$ (the case considered in Theorem 2), then (u, P) has a Bayesian Nash equilibrium $\sigma^* \in \Sigma$ such that $\sum_{a \in \xi^*(t)} \sigma^*(a|t) = 1$ for all $t \in T$ and

$$\begin{aligned} \sum_{a \in X^*} \sigma_P^*(a) &= \sum_{a \in X^*} \sum_{t \in T} P(t)\sigma^*(a|t) \\ &\geq \sum_{t \in T,\, \xi^*(t) = X^*} P(t) \sum_{a \in X^*} \sigma^*(a|t) \\ &= \sum_{t \in T,\, \xi^*(t) = X^*} P(t) \geq 1 - \varepsilon\kappa \end{aligned} \tag{6}$$

where ξ^* is a maximal element of Ξ^*. Thus, to complete the proof, it is enough to show that, for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$ and every ε-elaboration with a Bayesian Nash equilibrium σ^* satisfying (6), there exists $\mu \in \mathcal{E}_{X^*}$ such that $\max_{a \in A} |\mu(a) - \sigma_P^*(a)| \leq \delta$.

Seeking a contradiction, suppose otherwise. Then, for some $\delta > 0$, there exists a sequence $\{(\mathsf{u}^k, P^k)\}_{k=1}^\infty$ such that:

- (u^k, P^k) is an ε^k-elaboration of g and $\varepsilon^k \to 0$ as $k \to \infty$.
- (u^k, P^k) has a Bayesian Nash equilibrium σ^{*k} with $\sum_{a \in X^*} \sigma_P^{*k}(a) \geq 1 - \varepsilon^k\kappa$.
- $\max\limits_{a \in A} |\mu(a) - \sigma_P^{*k}(a)| > \delta$ for all $\mu \in \mathcal{E}_{X^*}$ or $\mathcal{E}_{X^*} = \emptyset$.

By Lemma 1, $\{\sigma_P^{*k}\}_{k=1}^\infty$ has a subsequence $\{\sigma_P^{*k_l}\}_{l=1}^\infty$ such that

$$\lim_{l \to \infty} \max_{a \in A} |\mu(a) - \sigma_P^{*k_l}(a)| = 0$$

where $\mu \in \Delta(A)$ is a correlated equilibrium of g. Because

$$\sum_{a \in X^*} \mu(a) = \lim_{l \to \infty} \sum_{a \in X^*} \sigma_P^{*k_l}(a) \geq \lim_{l \to \infty} (1 - \varepsilon^{k_l}\kappa) = 1,$$

we have $\mu \in \mathcal{E}_{X^*}$. This is a contradiction, which completes the proof of the theorems.

[15] A strategy subspace Σ^* is compact with the topology of weak convergence defined in $\{\rho_\sigma \in \Delta(T \times A) \,|\, \sigma \in \Sigma^*,\ \rho_\sigma(t,a) = P(t)\sigma(a|t) \text{ for all } (t,a) \in T \times A\}$.

5 Unordered Domains

We restrict attention to the class of generalized potential functions such that domains are partitions of action spaces. Let $\mathcal{P}_i \subseteq 2^{A_i} \backslash \emptyset$ be a partition of A_i. We write $\mathcal{P} = \{\prod_{i \in N} X_i \,|\, X_i \in \mathcal{P}_i \text{ for } i \in N\}$ and $\mathcal{P}_{-i} = \{\prod_{j \neq i} X_j \,|\, X_j \in \mathcal{P}_j \text{ for } j \neq i\}$, which are partitions of A and A_{-i}, respectively. The partition element of $\mathcal{P}_i$ containing $a_i \in A_i$ is denoted by $P_i(a_i)$. Similarly, the partition element of $\mathcal{P}$ containing a and that of $\mathcal{P}_{-i}$ containing a_{-i} are denoted by $P(a)$ and $P_{-i}(a_{-i})$, respectively. We say that a function $v : A \to \mathsf{R}$ is $\mathcal{P}$-measurable if $v(a) = v(a')$ for $a, a' \in A$ with $a' \in P(a)$.

Definition 6 A $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ is a best-response potential function of g if, for each $i \in N$,

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i', a_{-i}) \neq \emptyset$$

for all $X_i \in \mathcal{P}_i$ and $\lambda_i \in \Delta(A_{-i})$ such that

$$X_i \subseteq \arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i', a_{-i}).$$

A partition element $X^* \in \mathcal{P}$ is a best-response potential maximizer (BRP-maximizer) if $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$.

For example, consider the special case where $\mathcal{P}_i$ is the finest partition, i.e., $\mathcal{P}_i = \{\{a_i\}\}_{a_i \in A_i}$ for all $i \in N$. Then, it is straightforward to see that a function $v : A \to \mathsf{R}$ is a best-response potential function of g if and only if

$$\arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i', a_{-i}) \subseteq \arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i', a_{-i})$$

for all $i \in N$ and $\lambda_i \in \Delta(A_{-i})$.[16] For example, a weighted potential function is a best-response potential function by (2). However, a best-response potential function is not always a weighted potential function, even if there are no dominated actions, as demonstrated by Morris and Ui (2002). Thus the class of best-response potential functions is much larger than the class of weighted potential functions.

[16]A best-response potential function considered by Voorneveld (2000) is a function satisfying this condition for the class of beliefs such that $\lambda_i(a_{-i}) = 0$ or 1. Thus, best-response potential functions in this paper form a special class of those in Voorneveld (2000).

A best-response potential function v induces a generalized potential function. Let $F : \mathcal{A} \to \mathsf{R}$ be such that $\mathcal{A} = \mathcal{P}$ and $F(P(a)) = v(a)$ for all $a \in A$. Note that $\mathcal{P}$-measurability of v implies that F is well defined. Since $\mathcal{A}_{-i}$ is a partition of A_{-i}, $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$ if and only if $\sum_{a_{-i} \in X_{-i}} \lambda_i(a_{-i}) = \Lambda_i(X_{-i})$ for all $X_{-i} \in \mathcal{A}_{-i}$ by Lemma 3. Thus, for $\Lambda_i \in \Delta(\mathcal{A}_{-i})$ and $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$,

$$\sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i(X_{-i}) F(X'_i \times X_{-i}) = \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a'_i, a_{-i})$$

if $X'_i = P_i(a'_i)$. This implies that, if

$$X_i \in \arg\max_{X'_i \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} \Lambda_i(X_{-i}) F(X'_i \times X_{-i}),$$

then

$$X_i \subseteq \arg\max_{a'_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a'_i, a_{-i})$$

and thus

$$X_i \cap \arg\max_{a'_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a'_i, a_{-i}) \neq \emptyset$$

for all $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$ by the definition of best-response potential functions. Therefore, $F : \mathcal{A} \to \mathsf{R}$ is a generalized potential function. This proves Lemma 2 and immediately implies the following result by Theorem 1.

Proposition 1 If g has a best-response potential function $v : A \to \mathsf{R}$ with a BRP-maximizer X^*, then $\mathcal{E}_{X^*}$ is nonempty and robust to canonical elaborations in g.

This proposition generalizes the result of Ui (2001), who showed that the action profile that uniquely maximizes a potential function is robust to canonical elaborations.

6 Ordered Domains

Let $\mathcal{P}_i$ be a partition of A_i such that $\mathcal{P}_i$ is linearly ordered by the order relation $\leq_i$ for $i \in N$. Let $\underline{Z}_i$ and $\overline{Z}_i$ be the smallest and the largest elements of $\mathcal{P}_i$, respectively. The corresponding product order relation over $\mathcal{P}$ is denoted by $\leq_N$, and that over $\mathcal{P}_{-i}$ is denoted by $\leq_{-i}$, respectively. If $P_i(a_i) \leq_i Z_i$ for $a_i \in A_i$ and $Z_i \in \mathcal{P}_i$, we simply write $a_i \leq_i Z_i$. For $X_i \subseteq A_i$, we say that $a_i \in X_i$ is minimal in X_i if $a_i \leq_i P_i(x_i)$ for all $x_i \in X_i$ and that $a_i \in X_i$ is maximal in X_i if $a_i \geq_i P_i(x_i)$ for all $x_i \in X_i$.

Definition 7 Let $X^* \in \mathcal{P}$ be given. A $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ with $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$ is a monotone potential function of g if, for all $i \in N$ and $\lambda_i \in \Delta(A_{-i})$, there exists

$$a_i \in \arg\max_{a_i' \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i', a_{-i}),$$
$$\underline{a}_i \in \arg\max_{a_i' \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i', a_{-i})$$

such that $P_i(a_i) \geq_i P_i(\underline{a}_i)$, and symmetrically, there exists

$$a_i \in \arg\max_{a_i' \geq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i', a_{-i}),$$
$$\overline{a}_i \in \arg\max_{a_i' \geq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i', a_{-i})$$

such that $P_i(a_i) \leq_i P_i(\overline{a}_i)$. A partition element $X^* \in \mathcal{P}$ is called a monotone potential maximizer (MP-maximizer).

We restrict attention to a complete information game g satisfying strategic complementarities or a monotone potential function v satisfying strategic complementarities in the following sense.

Definition 8 A complete information game g satisfies strategic complementarities if, for each $i \in N$,

$$g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) \geq g_i(a_i, a_{-i}') - g_i(a_i', a_{-i}')$$

for all $a_i, a_i' \in A_i$ and $a_{-i}, a_{-i}' \in A_{-i}$ such that $P_i(a_i) >_i P_i(a_i')$ and $P_{-i}(a_{-i}) >_{-i} P_{-i}(a_{-i}')$. A function $v : A \to \mathsf{R}$ satisfies strategic complementarities if an identical interest game g with $g_i = v$ for all $i \in N$ satisfies strategic complementarities.

Note that if the partition $\mathcal{P}_i$ is the finest one, then the order relation $\leq_i$ naturally induces an order relation over the action set A_i and the above definition of strategic complementarities reduces to the standard one.

A monotone potential function v with an MP-maximizer X^* induces a generalized potential function with a GP-maximizer X^* if g or v satisfies strategic complementarities. Let $\mathcal{A}$ be such that

$$\mathcal{A}_i = \{[Z_i', Z_i''] \,|\, Z_i', Z_i'' \in \mathcal{P}_i,\ Z_i' \leq_i X_i^* \leq_i Z_i''\}$$

for $i \in N$ where $[Z'_i, Z''_i] \subseteq A_i$ is such that

$$[Z'_i, Z''_i] = \bigcup_{Z'_i \leq_i Z_i \leq_i Z''_i} Z_i.$$

Note that $[\underline{Z}_i, \overline{Z}_i] = A_i \in \mathcal{A}_i$. For $Z'_{-i}, Z''_{-i} \in \mathcal{P}_{-i}$ with $Z'_{-i} \leq_{-i} Z''_{-i}$ and $Z', Z'' \in \mathcal{P}$ with $Z' \leq_N Z''$, we write

$$[Z'_{-i}, Z''_{-i}] = \prod_{j \neq i} [Z'_j, Z''_j] = \bigcup_{Z'_{-i} \leq_{-i} Z_{-i} \leq_{-i} Z''_{-i}} Z_{-i},$$
$$[Z', Z''] = \prod_{i \in N} [Z'_i, Z''_i] = \bigcup_{Z' \leq_N Z \leq_N Z''} Z.$$

Then, we have

$$\mathcal{A}_{-i} = \{[Z'_{-i}, Z''_{-i}] \mid Z'_{-i}, Z''_{-i} \in \mathcal{P}_{-i},\ Z'_{-i} \leq_{-i} X^*_{-i} \leq_{-i} Z''_{-i}\},$$
$$\mathcal{A} = \{[Z', Z''] \mid Z', Z'' \in \mathcal{P},\ Z' \leq_N X^* \leq_N Z''\}.$$

Note that, for $[Z'_i, Z''_i] \in \mathcal{A}_i$ and $[Z'_{-i}, Z''_{-i}] \in \mathcal{A}_{-i}$, $[Z', Z''] = [Z'_i, Z''_i] \times [Z'_{-i}, Z''_{-i}] \in \mathcal{A}$. Let $F : \mathcal{A} \to \mathsf{R}$ be such that

$$F([Z', Z'']) = V(Z') + V(Z'')$$

where $V : \mathcal{P} \to \mathsf{R}$ is such that $V(P(a)) = v(a)$ for all $a \in A$, which is well defined by $\mathcal{P}$-measurability of v. Note that $F(X^*) > F(X)$ for all $X \in \mathcal{A} \setminus \{X^*\}$. By showing that F is a generalized potential function, we claim the following result.

Proposition 2 Suppose that g has a monotone potential function $v : A \to \mathsf{R}$ with an MP-maximizer X^*. If g or v satisfies strategic complementarities, then $\mathcal{E}_{X^*}$ is nonempty and robust to all elaborations in g.

Proof. By Theorem 2, it is enough to show that $F : \mathcal{A} \to \mathsf{R}$ given above is a generalized potential function of g with a GP-maximizer X^*.

For $\Lambda_i \in \Delta(\mathcal{A}_{-i})$, let $Z^*_i, Z^{**}_i \in \mathcal{P}_i$ be such that

$$[Z^*_i, Z^{**}_i] \in \arg \max_{[Z'_i, Z''_i] \in \mathcal{A}_i} \sum_{[Z'_{-i}, Z''_{-i}] \in \mathcal{A}_{-i}} \Lambda_i([Z'_{-i}, Z''_{-i}]) F([Z'_i, Z''_i] \times [Z'_{-i}, Z''_{-i}])$$

and $[Z^*_i, Z^{**}_i]$ is maximal in the argmax set ordered by the set inclusion relation. We prove that

$$[Z^*_i, Z^{**}_i] \cap \arg \max_{x_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i}) \neq \emptyset \tag{7}$$

for all $\lambda_i \in \Delta_{\Lambda_i}(\mathcal{A}_{-i})$.

First, we calculate

$$\begin{aligned}
&\sum_{[Z'_{-i},Z''_{-i}]\in\mathcal{A}_{-i}} \Lambda_i([Z'_{-i},Z''_{-i}])F([Z'_i,Z''_i]\times[Z'_{-i},Z''_{-i}]) \\
&\qquad = \sum_{[Z'_{-i},Z''_{-i}]\in\mathcal{A}_{-i}} \Lambda_i([Z'_{-i},Z''_{-i}])V(Z'_i\times Z'_{-i}) \\
&\qquad\quad + \sum_{[Z'_{-i},Z''_{-i}]\in\mathcal{A}_{-i}} \Lambda_i([Z'_{-i},Z''_{-i}])V(Z''_i\times Z''_{-i}) \\
&\qquad = \sum_{Z'_{-i}\le_{-i}X^*_{-i}} \left(\sum_{Z''_{-i}\ge_{-i}X^*_{-i}} \Lambda_i([Z'_{-i},Z''_{-i}])\right) V(Z'_i\times Z'_{-i}) \\
&\qquad\quad + \sum_{Z''_{-i}\ge_{-i}X^*_{-i}} \left(\sum_{Z'_{-i}\le_{-i}X^*_{-i}} \Lambda_i([Z'_{-i},Z''_{-i}])\right) V(Z''_i\times Z''_{-i}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
Z^*_i &= \min\left(\arg\max_{Z'_i\le_i X^*_i} \sum_{Z_{-i}\in\mathcal{P}_{-i}} \Gamma'_i(Z_{-i})V(Z'_i\times Z_{-i})\right), \\
Z^{**}_i &= \max\left(\arg\max_{Z''_i\ge_i X^*_i} \sum_{Z_{-i}\in\mathcal{P}_{-i}} \Gamma''_i(Z_{-i})V(Z''_i\times Z_{-i})\right)
\end{aligned}$$

where $\Gamma'_i,\Gamma''_i \in \Delta(\mathcal{P}_{-i})$ are such that

$$\Gamma'_i(Z_{-i}) = \begin{cases} \displaystyle\sum_{Z''_{-i}\ge_{-i}X^*_{-i}} \Lambda_i([Z_{-i},Z''_{-i}]) & \text{if } Z_{-i}\le_{-i} X^*_{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Gamma''_i(Z_{-i}) = \begin{cases} \displaystyle\sum_{Z'_{-i}\le_{-i}X^*_{-i}} \Lambda_i([Z'_{-i},Z_{-i}]) & \text{if } Z_{-i}\ge_{-i} X^*_{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

Next, consider $\lambda_i \in \Delta_{\Lambda_i}(A_{-i})$. Let $\Gamma_i \in \Delta(\mathcal{P}_{-i})$ be such that

$$\Gamma_i(Z_{-i}) = \sum_{a_{-i}\in Z_{-i}} \lambda_i(a_{-i})$$

for all $Z_{-i} \in \mathcal{P}_{-i}$. We show that Γ''_i first order stochastically dominates Γ_i and Γ_i first order stochastically dominates Γ'_i. We say that $\mathcal{Q}_{-i} \subseteq \mathcal{P}_{-i}$ is a decreasing subset of

$\mathcal{P}_{-i}$ if $Z_{-i} \in \mathcal{Q}_{-i}$ and $Z'_{-i} \leq_{-i} Z_{-i}$ together imply $Z'_{-i} \in \mathcal{Q}_{-i}$. The definition of the stochastic dominance relation says that Γ''_i first order stochastically dominates Γ_i if, for any decreasing subset $\mathcal{Q}_{-i} \subseteq \mathcal{P}_{-i}$,

$$\sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma_i(Z_{-i}) \geq \sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma''_i(Z_{-i}). \tag{8}$$

It is known that Γ''_i first order stochastically dominates Γ_i if and only if, for any increasing function[17] $G_i : \mathcal{P}_{-i} \to \mathsf{R}$,

$$\sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i(Z_{-i}) G_i(Z_{-i}) \leq \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma''_i(Z_{-i}) G_i(Z_{-i}).$$

We show (8) for two cases separately, $X^*_{-i} \notin \mathcal{Q}_{-i}$ and $X^*_{-i} \in \mathcal{Q}_{-i}$. If $X^*_{-i} \notin \mathcal{Q}_{-i}$, then $Z_{-i} \geq_{-i} X^*_{-i}$ is false for all $Z_{-i} \in \mathcal{Q}_{-i}$ and thus

$$\sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma_i(Z_{-i}) \geq \sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma''(Z_{-i}) = 0$$

because $\Gamma''_i(Z_{-i}) = 0$ unless $Z_{-i} \geq_{-i} X^*_{-i}$. If $X^*_{-i} \in \mathcal{Q}_{-i}$, Lemma 3 implies that

$$\begin{aligned}
\sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma_i(Z_{-i}) &= \sum_{Z_{-i} \in \mathcal{Q}_{-i}} \left(\sum_{a_{-i} \in Z_{-i}} \lambda_i(a_{-i}) \right) \\
&= \sum_{a_{-i} \in \bigcup_{Z_{-i} \in \mathcal{Q}_{-i}} Z_{-i}} \lambda_i(a_{-i}) \\
&\geq \sum_{\substack{[Z'_{-i}, Z''_{-i}] \in \mathcal{A}_{-i} \\ [Z'_{-i}, Z''_{-i}] \subseteq \bigcup_{Z_{-i} \in \mathcal{Q}_{-i}} Z_{-i}}} \Lambda_i([Z'_{-i}, Z''_{-i}]) \\
&= \sum_{\substack{Z''_{-i} \geq_{-i} X^*_{-i} \\ Z''_{-i} \in \mathcal{Q}_{-i}}} \left(\sum_{Z'_{-i} \leq_{-i} X^*_{-i}} \Lambda_i([Z'_{-i}, Z''_{-i}]) \right) \\
&= \sum_{\substack{Z_{-i} \geq_{-i} X^*_{-i} \\ Z_{-i} \in \mathcal{Q}_{-i}}} \Gamma''_i(Z_{-i}) \\
&= \sum_{Z_{-i} \in \mathcal{Q}_{-i}} \Gamma''_i(Z_{-i}).
\end{aligned}$$

Therefore, Γ''_i first order stochastically dominates Γ_i. Symmetrically, we can show that Γ_i first order stochastically dominates Γ'_i.

[17] We say that $G_i : \mathcal{P}_{-i} \to \mathsf{R}$ is increasing if $G_i(Z_{-i}) \geq G_i(Z'_{-i})$ for $Z_{-i} \geq_{-i} Z'_{-i}$.

Using the stochastic dominance relation, we show that

$$[Z_i^*, X_i^*] \cap \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i}) \neq \emptyset, \tag{9}$$

$$[X_i^*, Z_i^{**}] \cap \arg\max_{x_i \geq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i}) \neq \emptyset, \tag{10}$$

which imply (7). For $Z_{-i} \in \mathcal{P}_{-i}$, let $\lambda_i^{Z_{-i}} \in \Delta(A_{-i})$ be such that

$$\lambda_i^{Z_{-i}}(a_{-i}) = \begin{cases} \dfrac{\lambda_i(a_{-i})}{\Gamma_i(Z_{-i})} & \text{if } \Gamma_i(Z_{-i}) > 0 \text{ and } a_{-i} \in Z_{-i}, \\ \dfrac{1}{|Z_{-i}|} & \text{if } \Gamma_i(Z_{-i}) = 0 \text{ and } a_{-i} \in Z_{-i}, \\ 0 & \text{if } a_{-i} \notin Z_{-i}. \end{cases}$$

Note that $\sum_{a_{-i} \in Z_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) = 1$ and $\lambda_i(a_{-i}) = \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i(Z_{-i}) \lambda_i^{Z_{-i}}(a_{-i})$ for all $a_{-i} \in A_{-i}$. Thus,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i}) = \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) g_i(x_i, a_{-i}).$$

Let $\lambda_i' \in \Delta(A_{-i})$ be such that

$$\lambda_i'(a_{-i}) = \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i'(Z_{-i}) \lambda_i^{Z_{-i}}(a_{-i}).$$

Then, we have

$$\begin{aligned} \sum_{a_{-i} \in A_{-i}} \lambda_i'(a_{-i}) v(x_i, a_{-i}) &= \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i'(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) v(x_i, a_{-i}) \\ &= \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i'(Z_{-i}) V(P_i(x_i) \times Z_{-i}). \end{aligned}$$

This implies that $Z_i^* = P_i(\underline{a}_i')$ where

$$\underline{a}_i' \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i'(a_{-i}) v(x_i, a_{-i})$$

is minimal in the argmax set. Let

$$\underline{a}_i \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(x_i, a_{-i})$$

be minimal in the argmax set and let

$$\underline{b}_i \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i}),$$

$$\underline{b}'_i \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda'_i(a_{-i}) g_i(x_i, a_{-i})$$

be maximal in the argmax sets, respectively. Since v is a monotone potential function, it must be true that $P_i(\underline{a}_i) \leq_i P_i(\underline{b}_i)$ and $P_i(\underline{a}'_i) \leq_i P_i(\underline{b}'_i)$. Suppose that g satisfies strategic complementarities. For any $x_i \in A_i$ with $P_i(x_i) <_i P_i(\underline{b}'_i)$,

$$g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \geq g_i(\underline{b}'_i, a'_{-i}) - g_i(x_i, a'_{-i})$$

whenever $P_{-i}(a_{-i}) >_{-i} P_{-i}(a'_{-i})$. This implies that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z'_{-i}}(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right)$$

whenever $Z_{-i} >_{-i} Z'_{-i}$. In other words, $\sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right)$ is increasing in Z_{-i}. Since Γ_i first order stochastically dominates Γ'_i, it must be true that

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right) \\
&\qquad = \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right) \\
&\qquad \geq \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma'_i(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right) \\
&\qquad = \sum_{a_{-i} \in A_{-i}} \lambda'_i(a_{-i}) \left(g_i(\underline{b}'_i, a_{-i}) - g_i(x_i, a_{-i}) \right) \geq 0.
\end{aligned}$$

This implies that $P_i(\underline{b}'_i) \leq_i P_i(\underline{b}_i)$. Therefore, $Z_i^* = P_i(\underline{a}'_i) \leq_i P_i(\underline{b}'_i) \leq_i P_i(\underline{b}_i)$ and thus (9) is true. Suppose that v satisfies strategic complementarities. By the similar discussion, for any $x_i \in A_i$ with $P_i(x_i) <_i P_i(\underline{a}'_i)$,

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(v(\underline{a}'_i, a_{-i}) - v(x_i, a_{-i}) \right) \\
&\qquad = \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma_i(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(v(\underline{a}'_i, a_{-i}) - v(x_i, a_{-i}) \right) \\
&\qquad \geq \sum_{Z_{-i} \in \mathcal{P}_{-i}} \Gamma'_i(Z_{-i}) \sum_{a_{-i} \in A_{-i}} \lambda_i^{Z_{-i}}(a_{-i}) \left(v(\underline{a}'_i, a_{-i}) - v(x_i, a_{-i}) \right) \\
&\qquad = \sum_{a_{-i} \in A_{-i}} \lambda'_i(a_{-i}) \left(v(\underline{a}'_i, a_{-i}) - v(x_i, a_{-i}) \right) > 0.
\end{aligned}$$

This implies that $P_i(\underline{a}_i') \leq_i P_i(\underline{a}_i)$. Therefore, $Z_i^* = P_i(\underline{a}_i') \leq_i P_i(\underline{a}_i) \leq_i P_i(\underline{b}_i)$ and thus (9) is true.

To summarize, if either g or v satisfies strategic complementarities, (9) is true. Similarly, we can show that (10) is true. Therefore, we obtain (7). ∎

We can obtain the simpler form of the MP-maximizer condition if a complete information game satisfies diminishing marginal returns. We say that a complete information game satisfies diminishing marginal returns if every player's payoff function is concave with respect to his own action. Let $Z_i^+ \in \mathcal{P}_i$ be the smallest element larger than $Z_i \neq \overline{Z}_i$, and $Z_i^- \in \mathcal{P}_i$ be the largest element smaller than $Z_i \neq \underline{Z}_i$.

Definition 9 A complete information game g satisfies diminishing marginal returns if, for each $i \in N$ and $a_{-i} \in A_{-i}$,

$$g_i(a_i^+, a_{-i}) - g_i(a_i, a_{-i}) \leq g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})$$

for $a_i \notin \underline{Z}_i \cup \overline{Z}_i$, $a_i^+ \in P_i(a_i)^+$, and $a_i^- \in P_i(a_i)^-$.

In the case of diminishing marginal returns, we will see that the MP-maximizer condition reduces to the following simpler condition.

Definition 10 Let $X^* \in \mathcal{P}$ be given. A $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ with $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$ is a local potential function of g if, for each $i \in N$, $a_i \in Z_i$ with $Z_i >_i X_i^*$, and $a_i^- \in Z_i^-$,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i^-, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i})$$

for all $\lambda_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i^-, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i, a_{-i}),$$

and symmetrically, for each $i \in N$, $a_i \in Z_i$ with $Z_i <_i X_i^*$, and $a_i^+ \in Z_i^+$,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i^+, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i})$$

for all $\lambda_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i^+, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i, a_{-i}).$$

A partition element $X^* \in \mathcal{P}$ is called a local potential maximizer (LP-maximizer).

We show that if a complete information game satisfies diminishing marginal returns, then a local potential function is a monotone potential function, by which we claim the following result.

Proposition 3 Suppose that g has a local potential function $v : A \to \mathrm{R}$ with an LP-maximizer X^*. If g satisfies diminishing marginal returns, and if g or v satisfies strategic complementarities, then $\mathcal{E}_{X^*}$ is nonempty and robust to all elaborations in g.

Proof. By Proposition 2, it is enough to show that if g satisfies diminishing marginal returns, then a local potential function v is a monotone potential function. Let

$$a_i \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i})$$

be maximal in the argmax set and let

$$\underline{a}_i \in \arg\max_{x_i \leq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(x_i, a_{-i})$$

be minimal in the argmax set. We prove that $P_i(a_i) \geq_i P_i(\underline{a}_i)$. If $P_i(\underline{a}_i) = \underline{Z}_i$, then $P_i(a_i) \geq_i P_i(\underline{a}_i)$. If $P_i(\underline{a}_i) \neq \underline{Z}_i$, then $P_i(\underline{a}_i)^-$ exists, and it must be true that, for all $\underline{a}_i^- \in P_i(\underline{a}_i)^-$,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(v(\underline{a}_i, a_{-i}) - v(\underline{a}_i^-, a_{-i}) \right) > 0.$$

Since v is a local potential function,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\underline{a}_i, a_{-i}) - g_i(\underline{a}_i^-, a_{-i}) \right) \geq 0$$

for all $\underline{a}_i^- \in P_i(\underline{a}_i)^-$. Since g satisfies diminishing marginal returns, we must have

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(x_i, a_{-i}) - g_i(x_i^-, a_{-i}) \right) \geq 0$$

for all $x_i \leq_i P_i(\underline{a}_i)^-$ and $x_i^- \in P_i(x_i)^-$. This implies that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(\underline{a}_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i})$$

for all $x_i \leq_i P_i(\underline{a}_i)^-$. Therefore, it must be true that $P_i(a_i) \geq_i P_i(\underline{a}_i)$.

Symmetrically, let

$$\underline{a}_i \in \arg\max_{x_i \geq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(x_i, a_{-i})$$

be minimal in the argmax set and let

$$\overline{a}_i \in \arg\max_{x_i \geq_i X_i^*} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(x_i, a_{-i})$$

be maximal in the argmax set. By the symmetric argument, we can prove that $P_i(\underline{a}_i) \leq_i P_i(\overline{a}_i)$.

Combining the above arguments, we conclude that a local potential function v is a monotone potential function. ∎

Proposition 3 has an important implication in the special case where

$$\mathcal{P}_i = \{\{a_i^*\}, A_i \backslash \{a_i^*\}\}$$

with $\{a_i^*\} \leq_i A_i \backslash \{a_i^*\}$ for all $i \in N$ and an LP-maximizer is $\{a^*\}$. Note that a complete information game satisfies diminishing marginal returns in the trivial sense. It is straightforward to see that a function $v : A \to \mathsf{R}$ is a local potential function with an LP-maximizer $\{a^*\}$ if and only if

- $v(a^*) > v(a)$ for $a \neq a^*$,
- for all $i \in N$, $v(a_i, a_{-i}) = v(a_i', a_{-i})$ for $a_i, a_i' \in A_i \backslash \{a_i^*\}$ and $a_{-i} \in A_{-i}$,
- for all $i \in N$, if

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) v(a_i, a_{-i}), \tag{11}$$

then

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \tag{12}$$

for $a_i \neq a_i^*$.

One can show that if g has a p-dominant equilibrium a^* with $\sum_{i \in N} p_i < 1$, then g has a local potential function v of this type. Let $\mathsf{p} = (p_i)_{i \in N} \in [0,1]^N$. Kajii and

Morris (1997a) defined $a^* \in A$ to be a p-dominant equilibrium of g if, for all $i \in N$ and $\lambda_i \in \Delta(A_{-i})$ with $\lambda_i(a^*_{-i}) \geq p_i$,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i})$$

for $a_i \in A_i$. Kajii and Morris (1997a) showed that a p-dominant equilibrium with $\sum_{i \in N} p_i < 1$ is robust to all elaborations. This is an immediate consequence of Proposition 3, the above discussion and the following lemma.

Lemma 7 If g has a p-dominant equilibrium a^* with $\sum_{i \in N} p_i < 1$, then g has a local potential function $v : A \to \mathsf{R}$ with an LP-maximizer $\{a^*\}$ such that

$$v(a) = \begin{cases} 1 - \sum_{i \in N} p_i & \text{if } a = a^*, \\ -\sum_{i \in S} p_i & \text{if } a_i = a_i^* \text{ for } i \in S \text{ and } a_i \neq a_i^* \text{ for } i \notin S. \end{cases}$$

In addition, v satisfies strategic complementarities.

Proof. Note that v is $\mathcal{P}$-measurable and $v(a^*) > v(a)$ for $a \neq a^*$. Note also that

$$v(a_i^*, a_{-i}) - v(a_i, a_{-i}) = \begin{cases} 1 - p_i & \text{if } a_{-i} = a^*_{-i}, \\ -p_i & \text{otherwise} \end{cases}$$

for $a_i \neq a_i^*$. Thus, v satisfies strategic complementarities.

Suppose that $\lambda_i \in \Delta(A_{-i})$ satisfies (11). Then, for $a_i \neq a_i^*$,

$$\begin{aligned} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(v(a_i^*, a_{-i}) - v(a_i, a_{-i}) \right) &= \lambda_i(a^*_{-i})(1 - p_i) + \sum_{a_{-i} \neq a^*_{-i}} \lambda_i(a_{-i})(-p_i) \\ &= \lambda_i(a^*_{-i}) - p_i \geq 0. \end{aligned}$$

Because a^* is a p-dominant equilibrium, (12) is true, which completes the proof. ∎

Local potential functions have the following dual characterization, which is easier to apply in finding local potential functions. We use it when we discuss examples. The dual characterization translates the condition with respect to beliefs to the condition with respect to payoff differences.[18] Remember that, in weighted potential functions, the payoff difference condition (1) leads to the belief condition (2). The following lemma provides the payoff difference condition corresponding to the belief condition in Definition 10.

[18] See Morris and Ui (2002) for the duality argument between beliefs and payoff differences.

Lemma 8 Let $X^* \in \mathcal{P}$ be given. A $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ with $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$ is a local potential function of g if and only if, for each $i \in N$, there exists $\mu_i(a_i^-, a_i) \geq 0$ for $a_i \in Z_i$ with $Z_i >_i X_i^*$ and $a_i^- \in Z_i^-$ such that

$$g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i}) \geq \mu_i(a_i^-, a_i) \left(v(a_i^-, a_{-i}) - v(a_i, a_{-i}) \right)$$

for all $a_{-i} \in A_{-i}$, and symmetrically, there exists $\mu_i(a_i^+, a_i) \geq 0$ for $a_i \in Z_i$ with $Z_i <_i X_i^*$ and $a_i^+ \in Z_i^+$ such that

$$g_i(a_i^+, a_{-i}) - g_i(a_i, a_{-i}) \geq \mu_i(a_i^+, a_i) \left(v(a_i^+, a_{-i}) - v(a_i, a_{-i}) \right)$$

for all $a_{-i} \in A_{-i}$.

Proof. Suppose that v satisfies the condition in the lemma. Then,

$$\begin{aligned} &\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i}) \right) \\ &\qquad \geq \mu_i(a_i^-, a_i) \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(v(a_i^-, a_{-i}) - v(a_i, a_{-i}) \right). \end{aligned}$$

Clearly, if

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(v(a_i^-, a_{-i}) - v(a_i, a_{-i}) \right) \geq 0,$$

then

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i}) \right) \geq 0.$$

Thus, v satisfies the first half of the condition in Definition 10. By the symmetric argument, we can show that v also satisfies the second half. Therefore, v is a local potential function.

Suppose that v is a local potential function. To show that $\mu_i(a_i^-, a_i)$ and $\mu_i(a_i^+, a_i)$ exist, we use Farkas' Lemma.[19] Farkas' Lemma says that, for finite dimensional vectors $\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_m \in \mathsf{R}^n$, the following two conditions are equivalent.

- If $(\mathsf{a}_1, \mathsf{y}), \ldots, (\mathsf{a}_m, \mathsf{y}) \leq 0$ for $\mathsf{y} \in \mathsf{R}^n$, then $(\mathsf{a}_0, \mathsf{y}) \leq 0$.
- There exists $x_1, \ldots, x_m \geq 0$ such that $x_1\mathsf{a}_1 + \cdots + x_m\mathsf{a}_m = \mathsf{a}_0$.

[19] See textbooks of convex analysis such as Rockafellar (1970).

Since v is a local potential function, if

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(v(a_i^-, a_{-i}) - v(a_i, a_{-i})\right) \geq 0,$$

then

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i})\right) \geq 0.$$

This implies that, if $(y_{a_{-i}})_{a_{-i}\in A_{-i}} \in \mathsf{R}^{A_{-i}}$ is such that

$$-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(v(a_i^-, a_{-i}) - v(a_i, a_{-i})\right) \leq 0,$$

$$-y_{a_{-i}} \leq 0 \text{ for } a_{-i} \in A_{-i},$$

then

$$-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i})\right) \leq 0.$$

By Farkas' Lemma, there exist $x \geq 0$ and $x_{a_{-i}} \geq 0$ for $a_{-i} \in A_{-i}$ such that

$$-x\left(v(a_i^-, a_{-i}) - v(a_i, a_{-i})\right) - \sum_{a'_{-i}\in A_{-i}} x_{a'_{-i}} \delta^{a'_{-i}}(a_{-i}) = -\left(g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i})\right)$$

for all $a_{-i} \in A_{-i}$ where $\delta^{a'_{-i}} : A_{-i} \to \mathsf{R}$ is such that $\delta^{a'_{-i}}(a_{-i}) = 1$ if $a_{-i} = a'_{-i}$ and $\delta^{a'_{-i}}(a_{-i}) = 0$ otherwise. Thus,

$$g_i(a_i^-, a_{-i}) - g_i(a_i, a_{-i}) \geq x\left(v(a_i^-, a_{-i}) - v(a_i, a_{-i})\right)$$

and we can choose $\mu_i(a_i^-, a_i) = x$. Symmetrically, we can show the existence of $\mu_i(a_i^+, a_i)$, which completes the proof. ∎

We report a couple of generalized potential functions using the local potential function characterization.

Example 1

For $i \in N = \{1, \ldots, n\}$, let $A_i = \{1, 2\}$ and $\mathcal{P}_i = \{\{1\}, \{2\}\}$ where $\mathcal{P}_i$ is linearly ordered by the rule $\{1\} \leq_i \{2\}$. Note that g satisfies diminishing marginal returns in the trivial sense. By Lemma 8, $v : A \to \mathsf{R}$ is a local potential function with an LP-maximizer $\{\mathsf{1}\} = \{(1, \ldots, 1)\}$ if and only if $v(\mathsf{1}) > v(a)$ for all $a \neq \mathsf{1}$ and there exists $\mu_i \geq 0$ such that $g_i(1, a_{-i}) - g_i(2, a_{-i}) \geq \mu_i\left(v(1, a_{-i}) - v(2, a_{-i})\right)$ for all $a_{-i} \in A_{-i}$ and $i \in N$.

To illustrate the condition, consider a unanimity game g such that

$$g_i(a) = \begin{cases} y_i & \text{if } a = \mathbf{1}, \\ z_i & \text{if } a = \mathbf{2}, \\ 0 & \text{otherwise} \end{cases}$$

where $y_i, z_i > 0$ for all $i \in N$. Note that g satisfies strategic complementarities. A function $v : A \to \mathsf{R}$ is a local potential function with an LP-maximizer $\{\mathbf{1}\}$ if and only if $v(\mathbf{1}) > v(a)$ for all $a \neq \mathbf{1}$ and there exists $\mu_i \geq 0$ such that $y_i \geq \mu_i (v(\mathbf{1}) - v(2, \mathbf{1}_{-i}))$, $-z_i \geq \mu_i (v(1, \mathbf{2}_{-i}) - v(\mathbf{2}))$, and $0 \geq \mu_i (v(1, a_{-i}) - v(2, a_{-i}))$ for $a_{-i} \neq \mathbf{1}_{-i}, \mathbf{2}_{-i}$, for all $i \in N$. Because $z_i > 0$, we must have $\mu_i > 0$ and $v(1, \mathbf{2}_{-i}) - v(\mathbf{2}) < 0$. Then, we can show that the above condition implies that $y_i/\mu_i > z_j/\mu_j$ for all $i \neq j$. In other words, $\{\mathbf{1}\}$ is an LP-maximizer only if there exists $\mu_i > 0$ for $i \in N$ such that $y_i/\mu_i > z_j/\mu_j$ for all $i \neq j$. We show this when $i = 1$ and $j = n$. Let $\{a^k \in A\}_{k=0}^n$ be such that, for each k, $a_i^k = 1$ if $i > k$ and $a_i^k = 2$ if $i \leq k$. Note that $a^0 = \mathbf{1}$ and $a^n = \mathbf{2}$. We have

$$\begin{aligned} y_1/\mu_1 &\geq v(a^0) - v(a^1), \\ 0 &\geq v(a^{k-1}) - v(a^k) \text{ for } k \in \{2, \ldots, n-1\}, \\ -z_n/\mu_n &\geq v(a^{n-1}) - v(a^n). \end{aligned}$$

Thus, $y_1/\mu_1 - z_n/\mu_n \geq \sum_{k=1}^n \left(v(a^{k-1}) - v(a^k)\right) = v(a^0) - v(a^n) = v(\mathbf{1}) - v(\mathbf{2}) > 0$.

It should be noted that there exist an open set of games that do not have any local potential function. For example, all games in the neighborhood of the following unanimity game do not have a local potential function with an LP-maximizer $\{\mathbf{1}\}$ or $\{\mathbf{2}\}$. Let $N = \{1, 2, 3\}$, $y_1 = 6$, $y_2 = y_3 = 1$, $z_1 = z_2 = z_3 = 2$. If $\{\mathbf{1}\}$ is an LP-maximizer, then it must be true that $1/\mu_2 > 2/\mu_3$ and $1/\mu_3 > 2/\mu_2$, which implies that $1 > 4$. Thus, $\{\mathbf{1}\}$ is not an LP-maximizer. If $\{\mathbf{2}\}$ is an LP-maximizer, then it must be true that $2/\mu_2 > 6/\mu_1$ and $2/\mu_1 > 1/\mu_2$, which implies that $4 > 6$. Thus, $\{\mathbf{2}\}$ is not an LP-maximizer.

Example 2

For $i \in N = \{1, \ldots, n\}$, let $A_i = \{0, 1, 2\}$ and $\mathcal{P}_i = \{\{0, 1\}, \{2\}\}$ where $\mathcal{P}_i$ is linearly ordered by the rule $\{0, 1\} \leq_i \{2\}$. Note that g satisfies diminishing marginal returns in the trivial sense. By Lemma 8, a $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ is a local potential function with an LP-maximizer $X^* = \{0,1\}^N$ if and only if $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$, and there exists $\mu_i^0, \mu_i^1 \geq 0$ such that

$$g_i(0, a_{-i}) - g_i(2, a_{-i}) \geq \mu_i^0 \left(v(0, a_{-i}) - v(2, a_{-i})\right),$$
$$g_i(1, a_{-i}) - g_i(2, a_{-i}) \geq \mu_i^1 \left(v(1, a_{-i}) - v(2, a_{-i})\right)$$

for all $a_{-i} \in A_{-i}$ and $i \in N$.

For example, consider the following game:

$$g_i(a) = \begin{cases} y_i(a) & \text{if } a \in X^*, \\ z_i & \text{if } a = \mathsf{2}, \\ 0 & \text{otherwise} \end{cases}$$

where $y_i : X^* \to \mathsf{R}$ is such that $y_i(a) > 0$ for all $a \in X^*$ and $z_i > 0$. Note that g satisfies strategic complementarities. A $\mathcal{P}$-measurable function $v : A \to \mathsf{R}$ is a local potential function with an LP-maximizer X^* if and only if $v(a^*) > v(a)$ for all $a^* \in X^*$ and $a \notin X^*$, and there exists $\mu_i^{a_i} \geq 0$ for $a_i \in \{0,1\}$ such that $y_i(a) \geq \mu_i^{a_i} \left(v(a) - v(2, a_{-i})\right)$ for $a_{-i} \in X_{-i}^*$, $-z_i \geq \mu_i^{a_i} \left(v(a_i, \mathsf{2}_{-i}) - v(\mathsf{2})\right)$, and $0 \geq \mu_i^{a_i} \left(v(a) - v(2, a_{-i})\right)$ for $a_{-i} \notin X_{-i}^* \cup \{\mathsf{2}_{-i}\}$, for all $i \in N$. Note that $\mu_i^{a_i} > 0$ and $v(a_i, \mathsf{2}_{-i}) - v(\mathsf{2}) < 0$ because $z_i > 0$.

In general, a robust set induced by the LP-maximizer, $\mathcal{E}_{X^*}$, is not a singleton. For example, let $N = \{1,2,3\}$ and $z_i = 1$ for all $i \in N$. Let the restricted game $(y_i)_{i \in N}$ be the cyclic matching pennies game; each player's payoffs depend only on his own action and the action of his "adversary." Player 3's adversary is player 2, player 2's adversary is player 1, and player 1's adversary is player 3. Thus, for example, player 1's payoffs are completely independent of player 2's action. Every player tries to choose action different from his adversary's. Player 1's restricted payoff function is such that $y_1(1,0,a_3) = y_1(0,1,a_3) = 3$ and $y_1(1,1,a_3) = y_1(0,0,a_3) = 2$ for all $a_3 \in \{0,1\}$. The other players' restricted payoff functions are given similarly. Then, $v : A \to \mathsf{R}$ such that

$$v(a) = \begin{cases} 2 & \text{if } a \in X^*, \\ 1 & \text{if } a = \mathsf{2}, \\ 0 & \text{otherwise} \end{cases}$$

is a local potential function and X^* is an LP-maximizer. Thus, $\mathcal{E}_{X^*}$ is a robust set. By the discussion of Example 3.1 of Kajii and Morris (1997a), $\mathcal{E}_{X^*}$ is not a singleton and any single correlated equilibrium in $\mathcal{E}_{X^*}$ is not robust.

Frankel *et al.* (2001) report further discussion of singleton GP-maximizers for games with strategic complementarities and diminishing marginal returns, using the LP-maximizer condition. For example, they show that a two player, three action, symmetric payoff game in that class always has an LP-maximizer and give an example of a two player, four action, symmetric payoff game with no LP-maximizer.

7 Concluding Remarks

This paper introduces generalized potential functions and provides sufficient conditions for the robustness of sets of equilibria. The special cases of the conditions unify the sufficient conditions for the robustness of equilibria provided by Kajii and Morris (1997a) and Ui (2001).

There are several open questions concerning the robustness of equilibria to incomplete information. Our "potential" technique could help with investigating them. One of the basic questions is when robust equilibria are unique if they exist. Kajii and Morris (1997a) showed that a strictly p-dominant equilibrium with $\sum_{i \in N} p_i < 1$ is the unique robust equilibrium. We do not find examples of generic games with multiple robust equilibria. However, we do not yet conclude whether or not robust equilibria of generic games are unique if they exist. Using generalized potential functions, we can make examples of robust equilibria of various games, which could help with investigating the question.

This paper is the first step in studying the robustness of sets of equilibria. Topics for the future work include the problem of how to identify minimal robust sets.

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