7-1-2002

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July 2002

COWLES FOUNDATION DISCUSSION PAPER NO. 1377

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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Best Response Equivalence*

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Abstract

Two games are best-response equivalent if they have the same best-response correspondence. We provide a characterization of when two games are best-response equivalent. The characterizations exploit a dual relationship between payoff differences and beliefs. Some "potential game" arguments (cf. Monderer and Shapley, 1996, *Games Econ. Behav.* **14**, 124–143) rely only on the property that potential games are best-response equivalent to identical interest games. Our results show that a large class of games are best-response equivalent to identical interest games, but are not potential games. Thus we show how some existing potential game arguments can be extended.

Keywords: best response equivalence; duality; Farkas' Lemma; potential games.

Suggested Running Title: Best Response Equivalence.

*We are very grateful for valuable input from Larry Blume, George Mailath and Philip Reny.

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1 Introduction

We consider three progressively stronger equivalence relations on games and characterize each of them.

- Two games are best-response equivalent if they have the same best-response correspondence.
- Two games are better-response equivalent if, for every pair of strategies, they agree when one strategy is better than the other.
- Two games are von Neumann-Morgenstern equivalent (VNM-equivalent) if, for each player, the payoff function in one game is equal to a constant times the payoff function in the other game, plus a function that depends only on the opponents' strategies.

Two games are VNM-equivalent if and only if, for each player i, there is a constant $w_i > 0$ such that the ratio of payoff differences from switching between one strategy to another strategy is always w_i. The constant w_i is thus independent of the strategies being compared.

Two games are better-response equivalent if and only if they have the same dominance relations and, for each player i and each pair of strategies a_i and a_i' such that neither strategy strictly dominates the other, there exists a constant $w_i > 0$ such that the ratio of payoff differences from switching between a_i and a_i' is always w_i. In general, this is a weaker requirement than VNM-equivalence. It is weaker both because the proportional payoff differences property is no longer required to hold between some strategy pairs, and because the weight w_i is not necessarily independent of the strategy pair. But if the game does not have dominated strategies, the weights can no longer depend on the strategies being compared, and better-response equivalence collapses to VNM-equivalence.

Two games are best-response equivalent if and only if, for each player i and each pair of strategies a_i and a_i' such that both strategies are a best response to some belief, there exists a constant $w_i > 0$ such that the ratio of payoff differences from switching between

a_i and a'_i is always w_i. Even if a game has no dominated strategies, this is a weaker requirement than VNM-equivalence. In games with diminishing marginal returns, best-response equivalence is always a strictly weaker requirement than VNM-equivalence. Examples are given in the paper.

The most extensive discussion and applications of these relations has come in the literature on potential games. Monderer and Shapley [10] said that a game was a "potential game" if there exists a potential function, defined on the strategy space, with the property that the change in any player's payoff function from switching between any two of his strategies (holding other players' strategies fixed) was equal to the change in the potential function.[1] A game is "weighted potential game," if the payoff changes are proportional for each player. Thus a game is a weighted potential game if and only if it is VNM-equivalent to a game with identical payoff functions. While some results using potential or weighted potential game arguments are using the VNM-equivalence to identical interest games, other arguments are just using the better-response equivalence and even only best-response equivalence implications of VNM-equivalence.[2] Any paper that deals only with equilibrium is using only best-response equivalence (e.g., Neyman [13], Ui [19], Morris and Ui [12]). Similarly, fictitious play only uses the best-response properties of the game (Monderer and Shapley [9]).[3] An application using only better-response equivalence but not the VNM-equivalence appears in Morris [11]. Some papers studying quantal responses or stochastic best responses in potential games use the full power of VNM-equivalence (e.g., Blume [2], Brock and Durlauf [3], Anderson *et al.* [1], Ui [20]).[4]

[1] See also Ui [18] for a characterization and examples of potential games.

[2] Arguments that exploit potential arguments to prove the existence of a pure strategy equilibrium (e.g., Rosenthal [15]) only use *ordinal* properties of payoffs. Monderer and Shapley [10] introduced ordinal potential games and Voorneveld [21] and Dubey *et al.* [4] showed how ordinal potential games can be weakened to only require pure strategy best-response equivalence.

[3] Sela [17] establishes convergence of fictitious play in a class of "One-Against-All" games. These are games best-response equivalent to identical interest games, but not potential games.

[4] More precisely, they use the full power of VNM-equivalence such that the constant w_i is the same for all the players.

The fact that VNM-equivalence is the same as better-response equivalence in the absence of dominated strategies and may be different in the presence of dominated strategies has been noted in a number of contexts (see Sela [16], Blume [2] p409, Monderer and Shapley [10] footnote 9, and Maskin and Tirole [6] p209). However, our characterizations of better-response equivalence in the presence of dominated strategies and of the significant gap between better-response equivalence and best-response equivalence fill a gap in the literature.[5]

The paper is organized as follows. In section 2, we describe our notions of equivalence and give an example illustrating the differences. In section 3, we report our characterizations. In section 4, we restrict attention to a class of games where best-response equivalence is a strictly weaker requirement than VNM-equivalence and characterize the class of games. We also discuss an extension to games with infinite strategy spaces and its application.

2 Equivalence Properties of Games

A game consists of a finite set of players N and a finite strategy set A_i for $i \in N$, and a payoff function $g_i : A \to \mathbb{R}$ for $i \in N$ where $A = \prod_{i\in N} A_i$. We write $A_{-i} = \prod_{j\neq i} A_j$ and $a_{-i} = (a_j)_{j\neq i} \in A_{-i}$. We simply denote a game by $\mathbf{g} = (g_i)_{i\in N}$. Throughout the paper, we regard $g_i(a_i, \cdot) : A_{-i} \to \mathbb{R}$ as a vector in $\mathbb{R}^{A_{-i}}$. We write $g_i(a_i, \cdot) \gg g_i(a'_i, \cdot)$ if $g_i(a_i, a_{-i}) > g_i(a'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, and $g_i(a_i, \cdot) \geq g_i(a'_i, \cdot)$ if $g_i(a_i, a_{-i}) \geq g_i(a'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

For $i \in N$, let $\Delta(A_{-i})$ denote the set of all probability distributions over A_{-i}. We call each element of $\Delta(A_{-i})$ player i's belief. For $X_i \subseteq A_i$, let $\Lambda_i(a_i, X_i|g_i) \subseteq \Delta(A_{-i})$ be a set of player i's beliefs such that player i with a payoff function g_i and a belief

[5] Mertens [8] studied various notions of best-response equivalence, but with his more abstract strategy spaces and focus on admissible best responses, there is little overlap with the material in this paper.

$\lambda_i \in \Lambda_i(a_i, X_i|g_i)$ weakly prefers a_i to any strategy in X_i:

$$\begin{aligned}&\Lambda_i(a_i, X_i|g_i)\\ &\quad = \{\lambda_i \in \Delta(A_{-i}) \mid \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \big(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})\big) \geq 0 \text{ for all } a_i' \in X_i\}.\end{aligned}$$

When X_i is a singleton, i.e., $X_i = \{a_i'\}$, we write $\Lambda_i(a_i, a_i'|g_i)$ instead of $\Lambda_i(a_i, \{a_i'\}|g_i)$.

We are interested in characterizing two equivalence relations on games captured by these sets of beliefs by which players prefer one particular strategy.

Definition 1 A game $\mathbf{g}$ is *better-response equivalent* to $\mathbf{g}' = (g_i')_{i\in N}$ if, for each $i \in N$,

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i')$$

for all $a_i, a_i' \in A_i$.

Definition 2 A game $\mathbf{g}$ is *best-response equivalent* to $\mathbf{g}' = (g_i')_{i\in N}$ if, for each $i \in N$,

$$\Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g_i')$$

for all $a_i \in A_i$.

If $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$, then $\mathbf{g}$ is best-response equivalent to $\mathbf{g}'$, since

$$\Lambda_i(a_i, A_i|g_i) = \bigcap_{a_i'\in A_i} \Lambda_i(a_i, a_i'|g_i).$$

An easy sufficient condition for better-response equivalence is the following.[6]

Definition 3 A game $\mathbf{g}$ is *VNM-equivalent* to $\mathbf{g}' = (g_i')_{i\in N}$ if, for each $i \in N$, there exists a positive constant $w_i > 0$ and a function $Q_i : A_{-i} \to \mathbb{R}$ such that

$$g_i(a_i, \cdot) = w_i g_i'(a_i, \cdot) + Q_i(\cdot).$$

[6]Blume [2] called this property "strongly best-response equivalent."

It is straightforward to see that if $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$, then

$$g_i(a_i, \cdot) - g_i(a_i', \cdot) = w_i \left(g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \right)$$

for all $a_i, a_i' \in A_i$. Conversely, if this is true, then a function $Q_i : A_{-i} \to \mathbb{R}$ such that

$$Q_i(\cdot) = g_i(a_i, \cdot) - w_i g_i'(a_i, \cdot)$$

is well defined, and thus $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$. Thus, we have the following lemma.

Lemma 1 *A game* $\mathbf{g}$ *is VNM-equivalent to* $\mathbf{g}'$ *if and only if, for each* $i \in N$*, there exists* w_i *such that*

$$g_i(a_i, \cdot) - g_i(a_i', \cdot) = w_i \left(g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \right) \tag{1}$$

for all $a_i, a_i' \in A_i$.

It is straightforward to see that VNM-equivalence is sufficient for better-response equivalence. In fact, (1) implies that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) \right)$$
$$= w_i \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}) \right)$$

for all $\lambda_i \in \Delta(A_{-i})$ and thus $\Lambda_i(a_i, a_i' | g_i) = \Lambda_i(a_i, a_i' | g_i')$ for all $a_i, a_i' \in A_i$.

Best-response, better-response, and VNM-equivalence are equivalence relations. Thus, they define an equivalence class of games. For example, weighted potential games (Monderer and Shapley [9]) with a weighted potential function $f : A \to \mathbb{R}$ are regarded as a VNM-equivalence class of an identical interest game $\mathbf{f} = (f_i)_{i \in N}$ with $f_i = f$ for all $i \in N$. This is clear by Lemma 1 and the following original definition of weighted potential games.

Definition 4 A game $\mathbf{g} = (g_i)_{i \in N}$ is a *weighted potential game* if there exists a weighted potential function $f : A \to \mathbb{R}$ and $w_i > 0$ for each $i \in N$ such that

$$g_i(a_i, \cdot) - g_i(a_i', \cdot) = w_i \left(f(a_i, \cdot) - f(a_i', \cdot) \right)$$

for all $a_i, a_i' \in A_i$. If $w_i = 1$ for all $i \in N$, $\mathbf{g}$ is called a potential game and f is called a potential function.

As the concept of VNM-equivalence leads us to the definition of weighted potential games, the concept of better-response equivalence and that of best-response equivalence lead us to the definition of the following new classes of games.

Definition 5 A game $\mathbf{g} = (g_i)_{i \in N}$ is a *better-response potential game* if it is better-response equivalent to an identical interest game $\mathbf{f} = (f_i)_{i \in N}$ with $f_i = f$ for all $i \in N$. A function f is called a better-response potential function.

Definition 6 A game $\mathbf{g} = (g_i)_{i \in N}$ is a *best-response potential game* if it is best-response equivalent to an identical interest game $\mathbf{f} = (f_i)_{i \in N}$ with $f_i = f$ for all $i \in N$. A function f is called a best-response potential function.

Voorneveld [21] called a game a best-response potential game if its best-response correspondence coincides with that of an identical interest game over the class of beliefs such that $\lambda_i(a_{-i}) = 0$ or 1. Thus, best-response potential potential games in this paper form a special class of those in Voorneveld [21].

Existing potential game results that rely only on better-response equivalence or best-response equivalence, such as those mentioned in the introduction, automatically hold for the larger class of better-response potential games or that of best-response potential games. Thus, we are interested in exactly when and to what extent better-response and best-response equivalence are weaker requirements than VNM-equivalence.

Notice that best-response and better-response equivalence are clearly weaker requirements than VNM-equivalence, because the latter imposes too many constraints on payoffs from dominated strategy. Moreover, best-response equivalence is significantly weaker than better-response equivalence, as shown by the following example.

Consider a two player, three strategy, symmetric payoff game $\mathbf{g}(x, y)$ parameterized by $(x, y) \in \mathbb{R}^2_{++}$, where each player's payoffs are given by the following payoff matrix (where the player's own strategies are represented by rows and his opponent's strategies are represented by columns).

	1	2	3
1	x	$-x$	$-2x$
2	0	0	0
3	$-2y$	$-y$	y

In the special case where $x = y = 1$, we have game $\mathbf{g}(1,1)$ with the following payoff matrix.

	1	2	3
1	1	-1	-2
2	0	0	0
3	-2	-1	1

If a row player has a belief $\lambda_i(k) = \pi_k$ for $k \in \{1,2,3\}$, he prefers strategy 1 to strategy 2 if and only if

$$\pi_1 \geq \pi_2 + 2\pi_3;$$

he prefers strategy 1 to strategy 3 if and only if

$$(x + 2y)\,\pi_1 \geq (x - y)\,\pi_2 + (2x + y)\,\pi_3;$$

he prefers strategy 3 to strategy 2 if and only if

$$\pi_3 \geq \pi_2 + 2\pi_1.$$

Thus the region of indifference between strategies 1 and 2, and between strategy 2 and 3, does not depend on x and y. Moreover, whenever strategy 1 (or 3) is preferred to strategy 2, it is also preferred to strategy 3 (or 1). Thus the best response regions for this game are as in figure 1, for *any* $(x,y) \in \mathbb{R}^2_{++}$. Thus $\mathbf{g}(x,y)$ is best-response equivalent to $\mathbf{g}(1,1)$ for any $(x,y) \in \mathbb{R}^2_{++}$. On the other hand, the region of indifference between strategies 1 and 3 does depend on x and y: in particular, $\mathbf{g}(x,y)$ is better-response equivalent to $\mathbf{g}(1,1)$ if and only if $x = y$. We will discuss this example again in section 4.

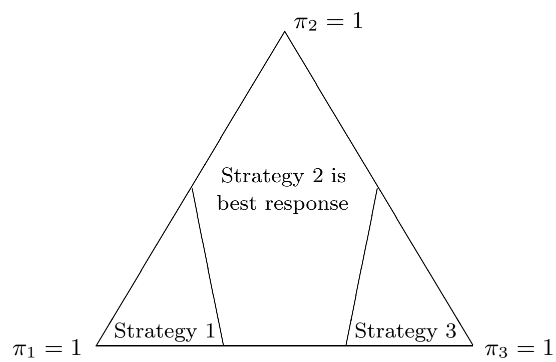

Figure 1: The best response regions

3 Results

3.1 Generic Properties of Games

We will appeal to some generic properties of games, i.e., properties that will hold for all but a Lebesgue measure zero set of payoffs.

G1: For all $i \in N$, if $g_i(a_i, \cdot) \geq g_i(a'_i, \cdot)$, then $g_i(a_i, \cdot) \gg g_i(a'_i, \cdot)$ for distinct $a_i, a'_i \in A_i$.

G2: For all $i \in N$, vectors $g_i(a_i, \cdot) - g_i(a'_i, \cdot)$ and $g_i(a_i, \cdot) - g_i(a''_i, \cdot)$ are linearly independent for distinct $a_i, a'_i, a''_i \in A_i$.

G3: For all $i \in N$, if $\Lambda_i(a_i, A_i|g_i) \cap \Lambda_i(a'_i, A_i|g_i) \neq \emptyset$, then $\Lambda_i(a_i, A_i \backslash \{a'_i\}|g_i) \backslash \Lambda_i(a_i, a'_i|g_i) \neq \emptyset$ for distinct $a_i, a'_i \in A_i$.

3.2 Better-Response Equivalence

Strategy a_i *strictly dominates* a'_i in game $\mathbf{g}$ (we write $a_i \succ_i^{\mathbf{g}} a'_i$) if $g_i(a_i, \cdot) \gg g_i(a'_i, \cdot)$, or, equivalently, $\Lambda_i(a'_i, a_i|g_i) = \emptyset$. Strategies a_i and a'_i are *better-response comparable* (we write $a_i \sim_i^{\mathbf{g}} a'_i$) if neither $a_i \succ_i^{\mathbf{g}} a'_i$ nor $a'_i \succ_i^{\mathbf{g}} a_i$.

Proposition 1 *If games $\mathbf{g}$ and $\mathbf{g}'$ satisfy generic property G1, then $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$ if and only if, for each $i \in N$, (a) they have the same dominance relations ($\succ_i^{\mathbf{g}} = \succ_i^{\mathbf{g}'}$) and (b) whenever a_i is better-response comparable to a'_i ($a_i \sim_i^{\mathbf{g}} a'_i$), there exists $w_i(a_i, a'_i) > 0$ such that*

$$g_i(a_i, \cdot) - g_i(a'_i, \cdot) = w_i(a_i, a'_i) \left(g'_i(a_i, \cdot) - g'_i(a'_i, \cdot) \right). \tag{2}$$

Farkas' Lemma[7] plays a central role in the proofs.

Lemma 2 (Farkas' Lemma) *For vectors* $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$*, the following two conditions are equivalent.*

- *If* $(\mathbf{a}_1, \mathbf{y}), \ldots, (\mathbf{a}_m, \mathbf{y}) \leq 0$ *for* $\mathbf{y} \in \mathbb{R}^n$*, then* $(\mathbf{a}_0, \mathbf{y}) \leq 0$.
- *There exists* $x_1, \ldots, x_m \geq 0$ *such that* $x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m = \mathbf{a}_0$.

Proof of Proposition 1. We first show that (a) and (b) are sufficient for the better-response equivalence of $\mathbf{g}$ and $\mathbf{g}'$. If $a_i \sim_i^{\mathbf{g}} a_i'$, then (b) implies that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})\right) = w_i(a_i, a_i') \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i})\right)$$

and thus

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i').$$

If $a_i \succ_i^{\mathbf{g}} a_i'$, then

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i') = \Delta\left(A_{-i}\right).$$

If $a_i' \succ_i^{\mathbf{g}} a_i$, then

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i') = \emptyset.$$

To prove necessity, suppose that $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$. Since

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i'),$$

we have

$$a_i \succ_i^{\mathbf{g}} a_i' \Leftrightarrow \Lambda_i(a_i', a_i|g_i) = \Lambda_i(a_i', a_i|g_i') = \emptyset \Leftrightarrow a_i \succ_i^{\mathbf{g}'} a_i'$$

[7] See a textbook of convex analysis such as recent one by Hiriart-Urruty and Lamaréchal [5], or classic one by Rockafellar [14].

and thus (a) holds.

To prove (b), suppose that $a_i \sim_i^{\mathbf{g}} a_i'$. We know that $a_i \sim_i^{\mathbf{g}'} a_i'$. Let $\lambda_i \in \Delta(A_{-i})$ be such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) \right) \geq 0.$$

Since $\lambda_i \in \Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i')$,

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}) \right) \geq 0.$$

This implies that if $(y_{a_{-i}})_{a_{-i} \in A_{-i}} \in \mathbb{R}^{A_{-i}}$ is such that

$$-\sum_{a_{-i} \in A_{-i}} y_{a_{-i}} \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i}) \right) \leq 0,$$

$$-y_{a_{-i}} \leq 0 \text{ for all } a_{-i} \in A_{-i},$$

then

$$-\sum_{a_{-i} \in A_{-i}} y_{a_{-i}} \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i}) \right) \leq 0.$$

By Farkas' Lemma, there exist $x_{a_i'}^{a_i} \geq 0$ and $z_{a_{-i}} \geq 0$ for $a_{-i} \in A_{-i}$ such that

$$-x_{a_i'}^{a_i} \left(g_i(a_i, \cdot) - g_i(a_i', \cdot) \right) - \sum_{a_{-i} \in A_{-i}} z_{a_{-i}} \delta^{a_{-i}}(\cdot) = -\left(g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \right)$$

where $\delta^{a_{-i}} : A_{-i} \to \mathbb{R}$ is such that $\delta^{a_{-i}}(a_{-i}') = 1$ if $a_{-i}' = a_{-i}$ and $\delta^{a_{-i}}(a_{-i}') = 0$ otherwise. Thus,

$$x_{a_i'}^{a_i} \left(g_i(a_i, \cdot) - g_i(a_i', \cdot) \right) \leq g_i'(a_i, \cdot) - g_i'(a_i', \cdot).$$

If $x_{a_i'}^{a_i} = 0$, then $g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \geq 0$. However, this is impossible since $a_i \sim_i^{\mathbf{g}'} a_i'$ implies that a_i does not strictly dominate a_i' in $\mathbf{g}'$ and G1 requires that if a_i does not strictly dominate a_i', then it is not the case that $g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \geq 0$. Thus, $x_{a_i'}^{a_i} > 0$.

Symmetrically, we have

$$x_{a_i}^{a_i'} \left(g_i(a_i', \cdot) - g_i(a_i, \cdot) \right) \leq g_i'(a_i', \cdot) - g_i'(a_i, \cdot)$$

where $x_{a_i}^{a_i'} > 0$. Thus,

$$\left(x_{a_i'}^{a_i} - x_{a_i}^{a_i'}\right)\left(g_i(a_i,\cdot) - g_i(a_i',\cdot)\right) \leq 0.$$

If $x_{a_i'}^{a_i} - x_{a_i}^{a_i'} > 0$, then $g_i(a_i,\cdot) - g_i(a_i',\cdot) \leq 0$, and if $x_{a_i'}^{a_i} - x_{a_i}^{a_i'} < 0$, then $g_i(a_i,\cdot) - g_i(a_i',\cdot) \geq 0$, which we already noted are impossible. Thus, $x_{a_i'}^{a_i} = x_{a_i}^{a_i'}$, which implies that

$$x_{a_i'}^{a_i}\left(g_i(a_i,\cdot) - g_i(a_i',\cdot)\right) = g_i'(a_i,\cdot) - g_i'(a_i',\cdot).$$

This proves (b). ∎

If $\mathbf{g}$ has no dominated strategy, then (2) is true for every $a_i, a_i' \in A_i$. If $w_i(a_i, a_i')$ is the same for every $a_i, a_i' \in A_i$, then better-response equivalence implies VNM-equivalence. However, Proposition 1 does not say anything about whether $w_i(a_i, a_i')$ does depend upon $a_i, a_i' \in A_i$. Thus, we are interested in when better-response equivalence implies VNM-equivalence. The following proposition provides a sufficient condition for the equivalence of better-response equivalence and VNM-equivalence.

Proposition 2 *Suppose that games $\mathbf{g}$ and $\mathbf{g}'$ satisfy generic properties G1 and G2, and that, for each $i \in N$, (a) they have the same dominance relations ($\succ_i^{\mathbf{g}} = \succ_i^{\mathbf{g}'}$), (b) $\sim_i^{\mathbf{g}}$ generates a connected graph on A_i, and (c) for any $a_i, a_i', a_i'', a_i''' \in A_i$ such that $a_i \sim_i^{\mathbf{g}} a_i'$ and $a_i'' \sim_i^{\mathbf{g}} a_i'''$ with $a_i \neq a_i'''$, there exists a sequence $\{a_i^k\}_{k=1}^m$ such that $a_i^1 = a_i, a_i^2 = a_i', a_i^{m-1} = a_i'', a_i^m = a_i'''$, $a_i^k \sim_i^{\mathbf{g}} a_i^{k+1}$ for $k = 1, \ldots, m-1$, $a_i^k \sim_i^{\mathbf{g}} a_i^{k+2}$ for $k = 1, \ldots, m-2$. Then $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$ if and only if $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$.*

Note that (c) is trivially satisfied if no strategy is dominated, i.e., $\sim_i^{\mathbf{g}}$ is the complete relation. So, the proposition immediately has the following corollary.

Corollary 3 *If $\mathbf{g}$ and $\mathbf{g}'$ satisfy generic properties G1 and G2 and have no strictly dominated strategies, then $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$ if and only if $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$.*

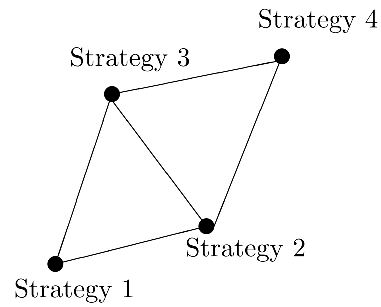

Figure 2: The graph of $\sim_i^{\mathbf{g}}$

It should be emphasized that the sufficient condition of Proposition 2 is sometimes satisfied even when there are strictly dominated strategies in the game. For example, consider the following two player game, where only the row player's payoffs are shown.

	1	2
1	4	1
2	1	3
3	2	2
4	3	0

Consider strategies of the row player. We have $1 \sim_i^{\mathbf{g}} 2$, $2 \sim_i^{\mathbf{g}} 3$, $3 \sim_i^{\mathbf{g}} 4$, $1 \sim_i^{\mathbf{g}} 3$, $2 \sim_i^{\mathbf{g}} 4$ as in figure 2, satisfying the condition of Proposition 2, while strategy 1 strictly dominates strategy 4.

Proof of Proposition 2. We show that if $\mathbf{g}$ is better-response equivalent to $\mathbf{g}'$ then $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$. Note that, by Proposition 1, if $a_i \sim_i^{\mathbf{g}} a_i'$, there exist $x_{a_i'}^{a_i} = x_{a_i}^{a_i'}$ with

$$x_{a_i'}^{a_i}\left(g_i(a_i,\cdot) - g_i(a_i',\cdot)\right) = g_i'(a_i,\cdot) - g_i'(a_i',\cdot).$$

If $|A_i| = 2$, this completes the proof by Lemma 1. Suppose that $|A_i| \geq 3$ and let $a_i \sim_i^{\mathbf{g}} a_i'$ and $a_i'' \sim_i^{\mathbf{g}} a_i'''$ with $a_i \neq a_i'''$. Then there exists a sequence $\{a_i^k\}_{k=1}^m$ satisfying the conditions in (c). Thus,

$$\begin{aligned}
x_{a_i^{k+1}}^{a_i^{k+2}} & \Big(g_i(a_i^{k+2},\cdot)-g_i(a_i^{k+1},\cdot)\Big)+x_{a_i^k}^{a_i^{k+1}}\Big(g_i(a_i^{k+1},\cdot)-g_i(a_i^{k},\cdot)\Big)\\
&=\Big(g_i'(a_i^{k+2},\cdot)-g_i'(a_i^{k+1},\cdot)\Big)+\Big(g_i'(a_i^{k+1},\cdot)-g_i'(a_i^{k},\cdot)\Big)\\
&=g_i'(a_i^{k+2},\cdot)-g_i'(a_i^{k},\cdot)\\
&=x_{a_i^k}^{a_i^{k+2}}\Big(g_i(a_i^{k+2},\cdot)-g_i(a_i^{k},\cdot)\Big)\\
&=x_{a_i^k}^{a_i^{k+2}}\Big(g_i(a_i^{k+2},\cdot)-g_i(a_i^{k+1},\cdot)\Big)+x_{a_i^k}^{a_i^{k+2}}\Big(g_i(a_i^{k+1},\cdot)-g_i(a_i^{k},\cdot)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\left(x_{a_i^{k+1}}^{a_i^{k+2}}-x_{a_i^k}^{a_i^{k+2}}\right)&\Big(g_i(a_i^{k+2},\cdot)-g_i(a_i^{k+1},\cdot)\Big)\\
&+\left(x_{a_i^k}^{a_i^{k+1}}-x_{a_i^k}^{a_i^{k+2}}\right)\Big(g_i(a_i^{k+1},\cdot)-g_i(a_i^{k},\cdot)\Big)=0.
\end{aligned}$$

By G2, $g_i(a_i^{k+2},\cdot)-g_i(a_i^{k+1},\cdot)$ and $g_i(a_i^{k+1},\cdot)-g_i(a_i^{k},\cdot)$ are linearly independent and thus it must be true that $x_{a_i^{k+1}}^{a_i^{k+2}}=x_{a_i^k}^{a_i^{k+1}}=x_{a_i^k}^{a_i^{k+2}}$ for $k=1,\dots,m-2$. Thus, it must be true that $x_{a_i}^{a_i'}=x_{a_i''}^{a_i'''}$. In other words, there exists a constant $c>0$ such that $x_{a_i}^{a_i'}=c$ for any $a_i,a_i'\in A_i$ with $a_i\sim_i^{\mathbf{g}}a_i'$.

In addition, since $\sim_i^{\mathbf{g}}$ generates a connected graph on A_i, for any $a_i,a_i'''\in A_i$ with $a_i\neq a_i'''$, there exists a_i',a_i'' and $\{a_i^k\}_{k=1}^m$ satisfying the conditions in (c). Thus,

$$\begin{aligned}
c\left(g_i(a_i''',\cdot)-g_i(a_i,\cdot)\right)&=\sum_{k=1}^{m-1}c\Big(g_i(a_i^{k+1},\cdot)-g_i(a_i^{k},\cdot)\Big)\\
&=\sum_{k=1}^{m-1}\Big(g_i'(a_i^{k+1},\cdot)-g_i'(a_i^{k},\cdot)\Big)\\
&=g_i'(a_i''',\cdot)-g_i'(a_i,\cdot).
\end{aligned}$$

To summarize, for any $a_i,a_i'\in A_i$,

$$c\left(g_i(a_i,\cdot)-g_i(a_i',\cdot)\right)=g_i'(a_i,\cdot)-g_i'(a_i',\cdot).$$

This implies that $\mathbf{g}$ is VNM-equivalent to $\mathbf{g}'$ by Lemma 1. ∎

3.3 Best-Response Equivalence

Strategies a_i and a_i' are *best-response comparable* (we write $a_i \approx_i^{\mathbf{g}} a_i'$) if both strategies are best responses at some belief, i.e., $\Lambda_i(a_i, A_i|g_i) \cap \Lambda_i(a_i', A_i|g_i) \neq \emptyset$. Note that $a_i \approx_i^{\mathbf{g}} a_i$ if and only if $\Lambda_i(a_i, A_i|g_i) \neq \emptyset$.

Proposition 4 *If games* $\mathbf{g}$ *and* $\mathbf{g}'$ *satisfy generic property G3, then* $\mathbf{g}$ *is best-response equivalent to* $\mathbf{g}'$ *if and only if, for each* $i \in N$, *(a) they have the same best-response comparability relation* ($\approx_i^{\mathbf{g}} = \approx_i^{\mathbf{g}'}$) *and (b) whenever* a_i *is best-response comparable to* a_i' ($a_i \approx_i^{\mathbf{g}} a_i'$), *there exists* $w_i(a_i, a_i') > 0$ *such that*

$$g_i(a_i, \cdot) - g_i(a_i', \cdot) = w_i(a_i, a_i') \left(g_i'(a_i, \cdot) - g_i'(a_i', \cdot) \right).$$

Proof. We first show that (a) and (b) are sufficient for the best-response equivalence of $\mathbf{g}$ and $\mathbf{g}'$. If $\Lambda_i(a_i, A_i|g_i) = \emptyset$, then $\Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g_i') = \emptyset$ because $\Lambda_i(a_i, A_i|g_i) = \emptyset$ implies that $a_i \approx_i^{\mathbf{g}} a_i$ is not true and thus (a) implies that $a_i \approx_i^{\mathbf{g}'} a_i$ is not true. If $\Lambda_i(a_i, A_i|g_i) \neq \emptyset$, then $\left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\} \neq \emptyset$, and we must have

$$\Lambda_i(a_i, A_i|g_i) = \bigcap_{a_i' \in A_i} \Lambda_i(a_i, a_i'|g_i) = \bigcap_{\left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\}} \Lambda_i(a_i, a_i'|g_i). \tag{3}$$

Clearly, (3) is true when $\left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\} = A_i$. To see that (3) is true when $\left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\} \subset A_i$, suppose otherwise. Then,

$$\bigcap_{a_i' \in A_i} \Lambda_i(a_i, a_i'|g_i) \subset \bigcap_{\left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\}} \Lambda_i(a_i, a_i'|g_i),$$

and thus there exists $a_i'' \notin \left\{a_i' | a_i \approx_i^{\mathbf{g}} a_i'\right\}$ such that

$$\bigcap_{a_i' \in A_i} \Lambda_i(a_i, a_i'|g_i) \subset \bigcap_{a_i' \in A_i \setminus \{a_i''\}} \Lambda_i(a_i, a_i'|g_i).$$

However, this implies that $a_i \approx_i^{\mathbf{g}} a_i''$, which is a contradiction. Thus, (3) must be true. If $a_i \approx_i^{\mathbf{g}} a_i'$, then (b) implies that

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})\right)$$
$$= w_i(a_i, a_i') \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i})\right),$$

and thus

$$\Lambda_i(a_i, a_i'|g_i) = \Lambda_i(a_i, a_i'|g_i'). \tag{4}$$

Therefore, by (a), (3), and (4), we have $\Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g_i')$. This completes the proof of sufficiency.

To prove necessity, suppose that $\mathbf{g}$ is best-response equivalent to $\mathbf{g}'$. Since

$$\Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g_i'),$$

we have

$$\Lambda_i(a_i, A_i|g_i) \cap \Lambda_i(a_i', A_i|g_i) = \Lambda_i(a_i, A_i|g_i') \cap \Lambda_i(a_i', A_i|g_i')$$

and thus $\approx_i^{\mathbf{g}} = \approx_i^{\mathbf{g}'}$. This proves (a).

If $a_i \approx_i^{\mathbf{g}} a_i'$, then there exists $\lambda_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i'', a_{-i})\right) \geq 0 \text{ for all } a_i'' \in A_i,$$
$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i', a_{-i}) - g_i(a_i'', a_{-i})\right) \geq 0 \text{ for all } a_i'' \in A_i \setminus \{a_i\}.$$

Since $\lambda_i \in \Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g_i')$,

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i})\right) \geq 0.$$

The above implies that, if $(y_{a_{-i}})_{a_{-i}\in A_{-i}} \in \mathbb{R}^{A_{-i}}$ is such that

$$
\begin{aligned}
-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(g_i(a_i,a_{-i}) - g_i(a_i',a_{-i})\right) &\leq 0, \\
-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(g_i(a_i,a_{-i}) - g_i(a_i'',a_{-i})\right) &\leq 0 \text{ for all } a_i'' \in A_i \backslash \left\{a_i, a_i'\right\}, \\
-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(g_i(a_i',a_{-i}) - g_i(a_i'',a_{-i})\right) &\leq 0 \text{ for all } a_i'' \in A_i \backslash \left\{a_i, a_i'\right\}, \\
-y_{a_{-i}} &\leq 0 \text{ for all } a_{-i} \in A_{-i},
\end{aligned}
$$

then

$$
-\sum_{a_{-i}\in A_{-i}} y_{a_{-i}} \left(g_i'(a_i,a_{-i}) - g_i'(a_i',a_{-i})\right) \leq 0.
$$

By Farkas' Lemma, there exist $x_{a_i'}^{a_i} \geq 0$, $\gamma_{a_i'}^{a_i} : A_{-i} \to \mathbb{R}$, and $\delta_{a_i'}^{a_i} : A_{-i} \to \mathbb{R}$ such that

$$
-x_{a_i'}^{a_i} \left(g_i(a_i,\cdot) - g_i(a_i',\cdot)\right) - \gamma_{a_i'}^{a_i}(\cdot) - \delta_{a_i'}^{a_i}(\cdot) = -\left(g_i'(a_i,\cdot) - g_i'(a_i',\cdot)\right)
$$

where

$$
\gamma_{a_i'}^{a_i}(\cdot) = \sum_{a_i''\neq a_i,a_i'} u_{a_i''}^{a_i} \left(g_i(a_i,\cdot) - g_i(a_i'',\cdot)\right) + \sum_{a_i''\neq a_i,a_i'} v_{a_i''}^{a_i'} \left(g_i(a_i',\cdot) - g_i(a_i'',\cdot)\right)
$$

with $u_{a_i''}^{a_i}, v_{a_i''}^{a_i'} \geq 0$ and

$$
\delta_{a_i'}^{a_i}(\cdot) = \sum_{a_{-i}\in A_{-i}} z_{a_{-i}} \delta^{a_{-i}}(\cdot)
$$

with $z_{a_{-i}} \geq 0$. Thus,

$$
x_{a_i'}^{a_i} \left(g_i(a_i,\cdot) - g_i(a_i',\cdot)\right) + \gamma_{a_i'}^{a_i}(\cdot) \leq g_i'(a_i,\cdot) - g_i'(a_i',\cdot).
$$

We show $x_{a_i'}^{a_i} > 0$. Suppose that $x_{a_i'}^{a_i} = 0$, i.e., $\gamma_{a_i'}^{a_i}(\cdot) \leq g_i'(a_i,\cdot) - g_i'(a_i',\cdot)$. Let

$$
\lambda_i' \in \Lambda_i(a_i, A_i\backslash\{a_i'\}|g_i)\backslash\Lambda_i(a_i,a_i'|g_i),
$$

which exists by $a_i \approx_i^{\mathbf{g}} a_i'$ and G3. Since $\lambda_i' \in \Lambda_i(a_i, A_i \backslash \{a_i'\} | g_i) \cap \Lambda_i(a_i', A_i \backslash \{a_i\} | g_i)$,

$$\begin{aligned}\sum_{a_{-i}\in A_{-i}} \lambda_i'(a_{-i})\gamma_{a_i'}^{a_i}(a_{-i}) = & \sum_{a_i''\neq a_i, a_i'} u_{a_i''}^{a_i} \sum_{a_{-i}\in A_{-i}} \lambda_i'(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i'', a_{-i})\right) \\ & + \sum_{a_i''\neq a_i, a_i'} v_{a_i''}^{a_i'} \sum_{a_{-i}\in A_{-i}} \lambda_i'(a_{-i}) \left(g_i(a_i', a_{-i}) - g_i(a_i'', a_{-i})\right) \geq 0.\end{aligned}$$

Since $\lambda_i' \in \Lambda_i(a_i', A_i | g_i) = \Lambda_i(a_i', A_i | g_i')$ and $\lambda_i' \notin \Lambda_i(a_i, A_i | g_i) = \Lambda_i(a_i, A_i | g_i')$,

$$\sum_{a_{-i}\in A_{-i}} \lambda_i'(a_{-i}) \left(g_i'(a_i, a_{-i}) - g_i'(a_i', a_{-i})\right) < 0.$$

This is a contradiction. Thus, we must have $x_{a_i'}^{a_i} > 0$.

We have

$$x_{a_i'}^{a_i} \left(g_i(a_i, \cdot) - g_i(a_i', \cdot)\right) + \gamma_{a_i'}^{a_i}(\cdot) \leq g_i'(a_i, \cdot) - g_i'(a_i', \cdot)$$

and symmetrically

$$x_{a_i}^{a_i'} \left(g_i(a_i', \cdot) - g_i(a_i, \cdot)\right) + \gamma_{a_i}^{a_i'}(\cdot) \leq g_i'(a_i', \cdot) - g_i'(a_i, \cdot)$$

where $x_{a_i'}^{a_i}, x_{a_i}^{a_i'} > 0$. Adding both,

$$\left(x_{a_i'}^{a_i} - x_{a_i}^{a_i'}\right) \left(g_i(a_i, \cdot) - g_i(a_i', \cdot)\right) + \gamma_{a_i'}^{a_i}(\cdot) + \gamma_{a_i}^{a_i'}(\cdot) \leq 0. \tag{5}$$

We show $x_{a_i'}^{a_i} - x_{a_i}^{a_i'} = 0$. Suppose that $x_{a_i'}^{a_i} - x_{a_i}^{a_i'} > 0$. Let

$$\lambda_i \in \Lambda_i(a_i', A_i \backslash \{a_i\} | g_i) \backslash \Lambda_i(a_i', a_i | g_i) \subseteq \Lambda_i(a_i, A_i \backslash \{a_i'\} | g_i) \cap \Lambda_i(a_i', A_i \backslash \{a_i\} | g_i).$$

Then, the expectation of the left-hand side of (5) is positive because

$$\left(x_{a_i'}^{a_i} - x_{a_i}^{a_i'}\right) \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})\right) > 0$$

and

$$\begin{aligned}&\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(\gamma_{a_i'}^{a_i}(a_{-i}) + \gamma_{a_i}^{a_i'}(a_{-i})\right) \\ &\quad = \sum_{a_i''\neq a_i, a_i'} (u_{a_i''}^{a_i} + v_{a_i''}^{a_i}) \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i, a_{-i}) - g_i(a_i'', a_{-i})\right) \\ &\quad + \sum_{a_i''\neq a_i, a_i'} (v_{a_i''}^{a_i'} + u_{a_i''}^{a_i'}) \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i', a_{-i}) - g_i(a_i'', a_{-i})\right) \geq 0.\end{aligned}$$

This is a contradiction. Symmetrically, if $x^{a_i}_{a'_i} - x^{a'_i}_{a_i} < 0$, then we have the symmetric contradiction. Thus, $x^{a_i}_{a'_i} - x^{a'_i}_{a_i} = 0$, and (5) is reduced to

$$\gamma^{a_i}_{a'_i}(\cdot) + \gamma^{a'_i}_{a_i}(\cdot) \leq 0. \tag{6}$$

We show $\gamma^{a_i}_{a'_i}(\cdot) = \gamma^{a'_i}_{a_i}(\cdot) = 0$. Suppose that either $\gamma^{a_i}_{a'_i}(\cdot) \neq 0$ or $\gamma^{a'_i}_{a_i}(\cdot) \neq 0$ is true. Let $\lambda_i, \lambda'_i \in \Delta(A_{-i})$ be such that

$$\lambda_i \in \Lambda_i(a'_i, A_i \backslash \{a_i\}|g_i) \backslash \Lambda_i(a'_i, a_i|g_i) \subseteq \Lambda_i(a_i, A_i \backslash \{a'_i\}|g_i) \cap \Lambda_i(a'_i, A_i \backslash \{a_i\}|g_i),$$

$$\lambda'_i \in \Lambda_i(a_i, A_i \backslash \{a'_i\}|g_i) \backslash \Lambda_i(a_i, a'_i|g_i) \subseteq \Lambda_i(a_i, A_i \backslash \{a'_i\}|g_i) \cap \Lambda_i(a'_i, A_i \backslash \{a_i\}|g_i).$$

Consider $(\lambda_i + \lambda'_i)/2 \in \Delta(A_{-i})$. Then, the expectation of the left-hand side of (6) is positive because

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \frac{\lambda_i(a_{-i}) + \lambda'_i(a_{-i})}{2} \left(\gamma^{a_i}_{a'_i}(a_{-i}) + \gamma^{a'_i}_{a_i}(a_{-i}) \right) \\
&\quad = \sum_{a''_i \neq a_i, a'_i} (u^{a_i}_{a''_i} + v^{a_i}_{a''_i}) \sum_{a_{-i} \in A_{-i}} \frac{\lambda_i(a_{-i}) + \lambda'_i(a_{-i})}{2} \left(g_i(a_i, a_{-i}) - g_i(a''_i, a_{-i}) \right) \\
&\quad + \sum_{a''_i \neq a_i, a'_i} (v^{a'_i}_{a''_i} + u^{a'_i}_{a''_i}) \sum_{a_{-i} \in A_{-i}} \frac{\lambda_i(a_{-i}) + \lambda'_i(a_{-i})}{2} \left(g_i(a'_i, a_{-i}) - g_i(a''_i, a_{-i}) \right) \\
&\quad \geq \sum_{a''_i \neq a_i, a'_i} (u^{a_i}_{a''_i} + v^{a_i}_{a''_i}) \sum_{a_{-i} \in A_{-i}} \frac{\lambda_i(a_{-i})}{2} \left(g_i(a_i, a_{-i}) - g_i(a''_i, a_{-i}) \right) \\
&\quad + \sum_{a''_i \neq a_i, a'_i} (v^{a'_i}_{a''_i} + u^{a'_i}_{a''_i}) \sum_{a_{-i} \in A_{-i}} \frac{\lambda'_i(a_{-i})}{2} \left(g_i(a'_i, a_{-i}) - g_i(a''_i, a_{-i}) \right) > 0.
\end{aligned}$$

This is a contradiction. Thus, $\gamma^{a_i}_{a'_i}(\cdot) = \gamma^{a'_i}_{a_i}(\cdot) = 0$.

Summarizing the above, we have

$$x^{a_i}_{a'_i} \left(g_i(a_i, \cdot) - g_i(a'_i, \cdot) \right) = g'_i(a_i, \cdot) - g'_i(a'_i, \cdot)$$

where $x^{a_i}_{a'_i} > 0$. This proves (b). ∎

The following proposition and corollary follow by exactly the same arguments in Proposition 2 and Corollary 3 in the previous subsection for better-response equivalence.

Proposition 5 *Suppose that games* $\mathbf{g}$ *and* $\mathbf{g}'$ *satisfy generic properties G2 and G3, and that, for each* $i \in N$, *(a) they have the same best-response comparability relation* ($\approx_i^{\mathbf{g}} = \approx_i^{\mathbf{g}'}$), *(b)* $\approx_i^{\mathbf{g}}$ *generates a connected graph on* A_i, *and (c) for any* $a_i, a_i', a_i'', a_i''' \in A_i$ *such that* $a_i \approx_i^{\mathbf{g}} a_i'$ *and* $a_i'' \approx_i^{\mathbf{g}} a_i'''$ *with* $a_i \neq a_i'''$, *there exists a sequence* $\{a_i^k\}_{k=1}^m$ *such that* $a_i^1 = a_i, a_i^2 = a_i', a_i^{m-1} = a_i'', a_i^m = a_i'''$, $a_i^k \approx_i^{\mathbf{g}} a_i^{k+1}$ *for* $k = 1, \ldots, m-1$, $a_i^k \approx_i^{\mathbf{g}} a_i^{k+2}$ *for* $k = 1, \ldots, m-2$. *Then* $\mathbf{g}$ *is best-response equivalent to* $\mathbf{g}'$ *if and only if* $\mathbf{g}$ *is VNM-equivalent to* $\mathbf{g}'$.

Corollary 6 *If* $\mathbf{g}$ *and* $\mathbf{g}'$ *satisfy generic properties G2 and G3 and* $\approx_i^{\mathbf{g}}$ *is the complete relation, then* $\mathbf{g}$ *is best-response equivalent to* $\mathbf{g}'$ *if and only if* $\mathbf{g}$ *is VNM-equivalent to* $\mathbf{g}'$.

4 Games with Own-strategy Unimodality

Best-response equivalence relation is an equivalence relation. It will be useful if, as a closed form, we can describe the best-response equivalence class of a game in which best-response equivalence is a strictly weaker requirement than VNM-equivalence.

Let A_i be linearly ordered such that $A_i = \{1, ..., K_i\}$ with $K_i \geq 3$. For $q_i : A_{-i} \to \mathbb{R}$ and $w_i : A_i \backslash \{K_i\} \to \mathbb{R}_{++}$, let $(q_i, w_i) \circ g_i : A \to \mathbb{R}$ be such that

$$(q_i, w_i) \circ g_i(1, \cdot) = q_i(\cdot),$$
$$(q_i, w_i) \circ g_i(a_i, \cdot) = q_i(\cdot) + \sum_{k=1}^{a_i - 1} w_i(k) \left(g_i(k+1, \cdot) - g_i(k, \cdot) \right) \text{ for } a_i \geq 2.$$

Let $\mathcal{D}_i(g_i)$ be a class of payoff functions of player i obtained by this transformation:

$$\mathcal{D}_i(g_i) = \{ g_i' : A \to \mathbb{R} \,|\, g_i' = (q_i, w_i) \circ g_i,\ q_i : A_{-i} \to \mathbb{R},\ w_i : A_i \to \mathbb{R}_{++} \}.$$

It is straightforward to see that $g_i' \in \mathcal{D}_i(g_i)$ if and only if there exists $w_i : A_i \backslash \{K_i\} \to \mathbb{R}_{++}$ such that

$$g_i'(a_i + 1, \cdot) - g_i'(a_i, \cdot) = w_i(a_i) \left(g_i(a_i + 1, \cdot) - g_i(a_i, \cdot) \right) \tag{7}$$

for all $a_i \in A_i \backslash \{K_i\}$. Note that $g_i \in \mathcal{D}_i(g_i)$, $g_i' \in \mathcal{D}_i(g_i)$ implies $g_i \in \mathcal{D}_i(g_i')$, and $g_i' \in \mathcal{D}_i(g_i)$ with $g_i'' \in \mathcal{D}_i(g_i')$ implies $g'' \in \mathcal{D}_i(g_i)$. Thus, $\mathcal{D}_i(g_i)$ defines an equivalence class of payoff functions of player i. We write

$$\mathcal{D}(\mathbf{g}) = \{\mathbf{g}' = (g_i')_{i\in N} \,|\, g_i' \in \mathcal{D}_i(g_i) \text{ for all } i \in N\}.$$

For example, consider a parametrized class of games $\{\mathbf{g}(x,y)\}_{(x,y)\in\mathbb{R}^2_{++}}$ discussed in section 2. We have $\{\mathbf{g}(x,y)\}_{(x,y)\in\mathbb{R}^2_{++}} \subset \mathcal{D}(\mathbf{g}(1,1))$. To see this, we write $\mathbf{g}(x,y) = (g_i(\cdot|x,y))_{i\in\{1,2\}}$. Then, for any $(x,y) \in \mathbb{R}^2_{++}$ and $i \neq j$,

$$\begin{aligned}
&g_i(1,a_j|x,y) = q_i(a_j),\\
&g_i(2,a_j|x,y) = q_i(a_j) + x\left(g_i(2,a_j|1,1) - g_i(1,a_j|1,1)\right),\\
&g_i(3,a_j|x,y) = q_i(a_j) + x\left(g_i(2,a_j|1,1) - g_i(1,a_j|1,1)\right) + y\left(g_i(3,a_j|1,1) - g_i(2,a_j|1,1)\right)
\end{aligned}$$

where $q_i : \{1,2,3\} \to \mathbb{R}$ is such that $q_i(1) = x$, $q_i(2) = -x$, and $q_i(3) = -2x$. Remember that, for any $(x,y) \in \mathbb{R}^2_{++}$, $\mathbf{g}(x,y)$ is best-response equivalent to $\mathbf{g}(1,1)$. It is easy to see that every game in $\mathcal{D}(\mathbf{g}(1,1))$ is VNM-equivalent to $\mathbf{g}(x,y)$ for some $(x,y) \in \mathbb{R}^2_{++}$. Thus, every game in $\mathcal{D}(\mathbf{g}(1,1))$ is best-response equivalent to $\mathbf{g}(1,1)$.

This observation leads us to the question when every game in $\mathcal{D}(\mathbf{g})$ is best-response equivalent to $\mathbf{g}$. We provide a necessary and sufficient condition for it.

We say that g_i is own-strategy unimodal if, for all $\lambda_i \in \Delta(A_{-i})$, there exists $k^* \in A_i$ such that,

$$\begin{aligned}
&\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(a_i,a_{-i}) - g_i(a_i-1,a_{-i})\right) \geq 0 \text{ if } a_i \leq k^*,\\
&\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(a_i,a_{-i}) - g_i(a_i+1,a_{-i})\right) \geq 0 \text{ if } a_i \geq k^*.
\end{aligned} \tag{8}$$

Note that if g_i is own-strategy unimodal, then (8) is true if and only if $\lambda_i \in \Lambda_i(k^*, A_i|g_i)$. Clearly, by (7), g_i is own-strategy unimodal if and only if $g_i' \in \mathcal{D}_i(g_i)$ is own-strategy unimodal.

We say that g_i is own-strategy concave if $g_i(\cdot, a_{-i}) : A_i \to \mathbb{R}$ is concave, i.e., $g_i(a_i + 1, a_{-i}) - g_i(a_i, a_{-i})$ is decreasing in a_i for all $a_{-i} \in A_{-i}$.

Lemma 3 *Suppose that $g_i(a_i+1, a_{-i}) \neq g_i(a_i, a_{-i})$ for all $a_i \in A_i \backslash \{K_i\}$ and $a_{-i} \in A_{-i}$, and that there is no weakly dominated strategy. Then, g_i is own-strategy unimodal if and only if there exists $\tilde{g}_i \in \mathcal{D}_i(g_i)$ such that $\tilde{g}_i$ is own-strategy concave.*

Proof. Suppose that $\tilde{g}_i \in \mathcal{D}_i(g_i)$ is own-strategy concave. Then, $\tilde{g}_i(a_i+1, a_{-i}) - \tilde{g}_i(a_i, a_{-i})$ is decreasing in a_i for all $a_{-i} \in A_{-i}$. Thus, $\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(\tilde{g}_i(a_i+1, a_{-i}) - \tilde{g}_i(a_i, a_{-i})\right)$ is also decreasing in a_i for all $\lambda_i \in \Delta(A_{-i})$. This immediately implies that $\tilde{g}_i \in \mathcal{D}_i(g_i)$ is own-strategy unimodal. Since

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(a_i+1, a_{-i}) - g_i(a_i, a_{-i})\right) \\
&\qquad = \frac{1}{w_i(a_i)} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(\tilde{g}_i(a_i+1, a_{-i}) - \tilde{g}_i(a_i, a_{-i})\right),
\end{aligned}$$

g_i is also own-strategy unimodal.

Suppose that g_i is own-strategy unimodal. We prove the existence of an own-strategy concave payoff function $\tilde{g}_i = (q_i, w_i) \circ g_i$ by construction. Later, we will show that there exists $C_k > 0$ such that

$$g_i(k+1, \cdot) - g_i(k, \cdot) \geq C_k \left(g_i(k+2, \cdot) - g_i(k+1, \cdot)\right). \tag{9}$$

For C_k satisfying (9), we let $w_i : A_i \to \mathbb{R}_{++}$ be such that $w_i(1) = 1$ and $w_i(a_i) = \prod_{k=1}^{a_i-1} C_k$ for $a_i \geq 2$, and $q_i : A_{-i} \to \mathbb{R}$ be such that $q_i(a_{-i}) = 0$ for all $a_{-i} \in A_{-i}$. Since

$$\tilde{g}_i(a_i+1, \cdot) - \tilde{g}_i(a_i, \cdot) = w_i(a_i) \left(g_i(a_i+1, \cdot) - g_i(a_i, \cdot)\right),$$

we have

$$\begin{aligned}
\tilde{g}_i(k+1, \cdot) - \tilde{g}_i(k, \cdot) &= w_i(k) \left(g_i(k+1, \cdot) - g_i(k, \cdot)\right), \\
\tilde{g}_i(k+2, \cdot) - \tilde{g}_i(k+1, \cdot) &= C_k w_i(k) \left(g_i(k+2, \cdot) - g_i(k+1, \cdot)\right).
\end{aligned}$$

By this and (9), we have

$$\tilde{g}_i(k+1, \cdot) - \tilde{g}_i(k, \cdot) \geq \tilde{g}_i(k+2, \cdot) - \tilde{g}_i(k+1, \cdot),$$

which implies that $\tilde{g}_i$ is own-strategy concave.

We prove the existence of C_k satisfying (9) by Farkas' Lemma. Before doing it, we must first observe that if

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(k+1, a_{-i}) - g_i(k, a_{-i})\right) = 0 \tag{10}$$

then

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i}) - g_i(k+1,a_{-i})\right) \leq 0.$$

To see this, suppose otherwise. Then, there exists $\lambda_i \in \Delta(A_{-i})$ satisfying both (10) and

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i}) - g_i(k+1,a_{-i})\right) > 0.$$

Since $g_i(k+1,a_{-i}) - g_i(k,a_{-i}) \neq 0$ for all $a_{-i} \in A_{-i}$, (10) implies that there exist $a'_{-i}, a''_{-i} \in A_{-i}$ such that $0 < \lambda_i(a'_{-i}) < 1$ with $g_i(k+1,a'_{-i}) - g_i(k,a'_{-i}) > 0$ and $0 < \lambda_i(a''_{-i}) < 1$ with $g_i(k+1,a''_{-i}) - g_i(k,a''_{-i}) < 0$. Let $\varepsilon > 0$ be sufficiently small. More precisely, let $\varepsilon > 0$ be such that

$$\varepsilon < \min\left\{\lambda_i(a'_{-i}), 1-\lambda_i(a''_{-i}), \frac{\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i}) - g_i(k+1,a_{-i})\right)}{2\times \max_{a_{-i}\in A_{-i}} \left|g_i(k+2,a_{-i}) - g_i(k+1,a_{-i})\right|}\right\}.$$

Let $\lambda'_i \in \Delta(A_{-i})$ be such that

$$\lambda'_i(a_{-i}) = \begin{cases} \lambda_i(a_{-i}) - \varepsilon & \text{if } a_{-i} = a'_{-i}, \\ \lambda_i(a_{-i}) + \varepsilon & \text{if } a_{-i} = a''_{-i}, \\ \lambda_i(a_{-i}) & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} &\sum_{a_{-i}\in A_{-i}} \lambda'_i(a_{-i})\left(g_i(k+1,a_{-i}) - g_i(k,a_{-i})\right) \\ &\qquad = \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+1,a_{-i}) - g_i(k,a_{-i})\right) \\ &\qquad\quad + \varepsilon\left(g_i(k+1,a''_{-i}) - g_i(k,a''_{-i})\right) - \varepsilon\left(g_i(k+1,a'_{-i}) - g_i(k,a'_{-i})\right) \\ &\qquad = \varepsilon\left(g_i(k+1,a''_{-i}) - g_i(k,a''_{-i})\right) - \varepsilon\left(g_i(k+1,a'_{-i}) - g_i(k,a'_{-i})\right) < 0, \end{aligned}$$

$$
\begin{aligned}
&\sum_{a_{-i}\in A_{-i}} \lambda_i'(a_{-i})\left(g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right) \\
&\qquad = \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right) \\
&\qquad\quad + \varepsilon\left(g_i(k+2,a''_{-i})-g_i(k+1,a''_{-i})\right) - \varepsilon\left(g_i(k+2,a'_{-i})-g_i(k+1,a'_{-i})\right) \\
&\qquad \geq \sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right) \\
&\qquad\quad - 2\varepsilon \max_{a_{-i}\in A_{-i}} \left|g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right| > 0,
\end{aligned}
$$

which contradicts to the assumption that g_i is own-strategy unimodal.

Now, we know that, if g_i is own-strategy unimodal and satisfies the assumptions, then it must be true that if

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+1,a_{-i})-g_i(k,a_{-i})\right) \leq 0,$$

then

$$\sum_{a_{-i}\in A_{-i}} \lambda_i(a_{-i})\left(g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right) \leq 0.$$

This implies that if $(y_{a_{-i}})_{a_{-i}\in A_{-i}} \in \mathbb{R}^{A_{-i}}$ is such that

$$
\begin{aligned}
\sum_{a_{-i}\in A_{-i}} y_{a_{-i}}\left(g_i(k+1,a_{-i})-g_i(k,a_{-i})\right) \leq 0,& \\
-y_{a_{-i}} \leq 0 \text{ for all } a_{-i}\in A_{-i},&
\end{aligned}
$$

then

$$\sum_{a_{-i}\in A_{-i}} y_{a_{-i}}\left(g_i(k+2,a_{-i})-g_i(k+1,a_{-i})\right) \leq 0.$$

By Farkas' Lemma, there exist $x_k \geq 0$ and $z_{a_{-i}} \geq 0$ for $a_{-i}\in A_{-i}$ such that

$$x_k\left(g_i(k+1,\cdot)-g_i(k,\cdot)\right) - \sum_{a_{-i}\in A_{-i}} z_{a_{-i}}\delta^{a_{-i}}(\cdot) = g_i(k+2,\cdot)-g_i(k+1,\cdot).$$

Thus,

$$x_k\left(g_i(k+1,\cdot)-g_i(k,\cdot)\right)\geq g_i(k+2,\cdot)-g_i(k+1,\cdot). \tag{11}$$

If $x_k=0$, then $g_i(k+2,\cdot)-g_i(k+1,\cdot)\leq 0$. However, this is impossible since there is no weakly dominated strategy. Thus, $x_k>0$. By letting $C_k=1/x_k$, (11) implies (9). ∎

Consider again $\{\mathbf{g}(x,y)\}_{(x,y)\in\mathbb{R}^2_{++}}\subset\mathcal{D}(\mathbf{g}(1,1))$. In general, $g_i(\cdot|x,y)$ is not always own-strategy concave. However, $g_i(\cdot|1,1)$ is own-strategy concave. Thus, Lemma 3 says that $g_i(\cdot|x,y)$ is own-strategy unimodal.

We claim that, generically, $\mathcal{D}(\mathbf{g})$ is a best-response equivalence class if and only if g_i is own-strategy unimodal for all $i\in N$.

Proposition 7 *Suppose that* $\mathbf{g}$ *has no dominated strategy. Every game in* $\mathcal{D}(\mathbf{g})$ *is best-response equivalent to* $\mathbf{g}$ *if and only if* g_i *is own-strategy unimodal for all* $i\in N$*. If* g_i *is own-strategy unimodal for all* $i\in N$ *and* $\mathbf{g}$ *satisfies generic property G3, then every game best-response equivalent to* $\mathbf{g}$ *and satisfying G3 is in* $\mathcal{D}(\mathbf{g})$*.*

Proof. Suppose that g_i is own-strategy unimodal for all $i\in N$. We show that if $\mathbf{g}'\in\mathcal{D}(\mathbf{g})$ then $\mathbf{g}'$ is best-response equivalent to $\mathbf{g}$. Let $\lambda_i\in\Lambda_i(a_i^*,A_i|g_i)$. Then, (8) implies that

$$\begin{aligned}\sum_{a_{-i}\in A_{-i}}\lambda_i(a_{-i})\left(g_i(a_i,a_{-i})-g_i(a_i-1,a_{-i})\right)\geq 0 \text{ if } a_i\leq a_i^*,\\ \sum_{a_{-i}\in A_{-i}}\lambda_i(a_{-i})\left(g_i(a_i,a_{-i})-g_i(a_i+1,a_{-i})\right)\geq 0 \text{ if } a_i\geq a_i^*.\end{aligned} \tag{12}$$

By (7), this is true if and only if

$$\begin{aligned}\sum_{a_{-i}\in A_{-i}}\lambda_i(a_{-i})\left(g_i'(a_i,a_{-i})-g_i'(a_i-1,a_{-i})\right)\geq 0 \text{ if } a_i\leq a_i^*,\\ \sum_{a_{-i}\in A_{-i}}\lambda_i(a_{-i})\left(g_i'(a_i,a_{-i})-g_i'(a_i+1,a_{-i})\right)\geq 0 \text{ if } a_i\geq a_i^*.\end{aligned} \tag{13}$$

Thus, $\lambda_i\in\Lambda_i(a_i^*,A_i|g_i')$. Conversely, let $\lambda_i\in\Lambda_i(a_i^*,A_i|g_i')$. Since g_i' is own-strategy unimodal, we have (13), which is true if and only if (12) is true. Thus, $\lambda_i\in\Lambda_i(a_i^*,A_i|g_i)$. Therefore, $\Lambda_i(a_i^*,A_i|g_i)=\Lambda_i(a_i^*,A_i|g_i')$ and thus $\mathbf{g}'$ is best-response equivalent to $\mathbf{g}$.

Conversely, suppose that every game in $\mathcal{D}(\mathbf{g})$ is best-response equivalent to $\mathbf{g}$. We show that g_i is own-strategy unimodal for all $i \in N$. Seeking a contradiction, suppose otherwise. Then, there exist $a_i^*, \tilde{a}_i \in A_i$ and $\lambda_i \in \Lambda_i(a_i^*, A_i|g_i)$ such that either of the following is true:

$$a_i^* < \tilde{a}_i \text{ and } \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\tilde{a}_i, a_{-i}) - g_i(\tilde{a}_i - 1, a_{-i})\right) > 0, \tag{14}$$

$$a_i^* > \tilde{a}_i \text{ and } \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\tilde{a}_i, a_{-i}) - g_i(\tilde{a}_i + 1, a_{-i})\right) > 0. \tag{15}$$

When (14) is true, let $g_i' = (q_i, w_i) \circ g_i \in \mathcal{D}_i(g_i)$ be such that $q_i(\cdot) = 0$ and

$$w_i(a_i) = \begin{cases} L & \text{if } a_i = \tilde{a}_i - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(g_i'(\tilde{a}_i, a_{-i}) - g_i'(a_i^*, a_{-i})\big) \\
&\qquad = \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(g_i'(\tilde{a}_i, a_{-i}) - g_i'(\tilde{a}_i - 1, a_{-i})\big) \\
&\qquad\quad + \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(g_i'(\tilde{a}_i - 1, a_{-i}) - g_i'(a_i^*, a_{-i})\big) \\
&\qquad = L \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\tilde{a}_i, a_{-i}) - g_i(\tilde{a}_i - 1, a_{-i})\right) \\
&\qquad\quad + \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left(g_i(\tilde{a}_i - 1, a_{-i}) - g_i(a_i^*, a_{-i})\right).
\end{aligned}$$

By choosing very large $L > 0$, we have

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \big(g_i'(\tilde{a}_i, a_{-i}) - g_i'(a_i^*, a_{-i})\big) > 0$$

and thus $\Lambda_i(a_i^*, A_i|g_i) \neq \Lambda_i(a_i^*, A_i|g_i')$. When (15) is true, we also have $\Lambda_i(a_i^*, A_i|g_i) \neq \Lambda_i(a_i^*, A_i|g_i')$ by the similar argument. This implies that some game in $\mathcal{D}(\mathbf{g})$ is not best-response equivalent to $\mathbf{g}$, which completes the proof of the first half of the proposition.

We prove the last half of the proposition. Suppose that g_i is own-strategy unimodal for all $i \in N$ and that $\mathbf{g}$ satisfies generic property G3. Let $\mathbf{g}'$ be best-response equivalent to $\mathbf{g}$ and satisfy G3. We show $\mathbf{g}' \in \mathcal{D}(\mathbf{g})$.

We first observe that $a_i \approx_i^{\mathbf{g}} a_i + 1$ for all $a_i \in A_i \backslash \{K_i\}$. To see this, let $\lambda_i^k \in \Lambda_i(k, A_i | g_i)$ for $k \in A_i$, which exists since $\mathbf{g}$ has no dominated strategy. Note that if $\lambda_i = \lambda_i^k$ or $\lambda_i = \lambda_i^{k+1}$ then

$$\begin{aligned} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(k, a_{-i}) &\geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \text{ for all } a_i \leq k, \\ \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(k+1, a_{-i}) &\geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \text{ for all } a_i \geq k+1. \end{aligned} \tag{16}$$

Let $t \in [0,1]$ and $\lambda_i^{k,t} = t\lambda_i^k + (1-t)\lambda_i^{k+1} \in \Delta(A_{-i})$ be such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(k, a_{-i}) = \sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(k+1, a_{-i}). \tag{17}$$

Then, (16) implies that

$$\begin{aligned} \sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(k, a_{-i}) &\geq \sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(a_i, a_{-i}) \text{ for all } a_i \leq k, \\ \sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(k+1, a_{-i}) &\geq \sum_{a_{-i} \in A_{-i}} \lambda_i^{k,t}(a_{-i}) g_i(a_i, a_{-i}) \text{ for all } a_i \geq k+1. \end{aligned}$$

By (17), we have $\lambda_i^{k,t} \in \Lambda_i(k, A_i | g_i) \cap \Lambda_i(k+1, A_i | g_i)$. This implies that $a_i \approx_i^{\mathbf{g}} a_i + 1$ for all $a_i \in A_i \backslash \{K_i\}$.

Since $\mathbf{g}$ and $\mathbf{g}'$ satisfy G3 and are best-response equivalent, we can use Proposition 4, which says that there exists $w_i : A_i \backslash \{K_i\} \to \mathbb{R}_{++}$ such that

$$g_i'(a_i + 1, \cdot) - g_i'(a_i, \cdot) = w_i(a_i) \left(g_i(a_i + 1, \cdot) - g_i(a_i, \cdot) \right).$$

This implies that $g_i' \in \mathcal{D}_i(g_i)$ and thus $\mathbf{g}' \in \mathcal{D}(\mathbf{g})$. ∎

A weaker, but similar claim is true for games such that strategy sets are intervals of real numbers and payoff functions are differentiable, which has a couple of applications. In the remainder of this section, we discuss this issue.

Abusing notations, we give a definition of best-response equivalence of the class of games. Let A_i be a closed interval of $\mathbb{R}$ for all $i \in N$. Assume that $g_i : A \to \mathbb{R}$ is bounded and continuously differentiable. Let $\Delta(A_{-i})$ be the set of all probability measures over A_{-i} and $\Lambda_i(a_i, X_i|g_i)$ be such that

$$\begin{aligned}
&\Lambda_i(a_i, X_i|g_i) \\
&\quad = \{\lambda_i \in \Delta(A_{-i}) \,|\, \int_{A_{-i}} \big(g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})\big)\, d\lambda_i(a_{-i}) \geq 0 \text{ for all } a'_i \in X_i\}.
\end{aligned}$$

The definition of best-response equivalence is the same as that for finite games: we say that $\mathbf{g}$ is best-response equivalent to $\mathbf{g}'$ if, for each $i \in N$, $\Lambda_i(a_i, A_i|g_i) = \Lambda_i(a_i, A_i|g'_i)$ for all $a_i \in A_i$.

We say that g_i is own-strategy unimodal if, for any $\lambda_i \in \Delta(A_{-i})$, there exists x^* such that

$$\begin{aligned}
\frac{\partial}{\partial a_i} \int_{A_{-i}} g_i(a_i, a_{-i}) d\lambda_i(a_{-i}) \geq 0 \text{ if } a_i \leq x^*, \\
\frac{\partial}{\partial a_i} \int_{A_{-i}} g_i(a_i, a_{-i}) d\lambda_i(a_{-i}) \leq 0 \text{ if } a_i \geq x^*.
\end{aligned} \tag{18}$$

Note that if g_i is own-strategy unimodal, then (18) is true if and only if $\lambda_i \in \Lambda_i(x^*, A_i|g_i)$. Since

$$\frac{\partial}{\partial a_i} \int_{A_{-i}} g_i(a_i, a_{-i}) d\lambda_i(a_{-i}) = \int_{A_{-i}} \frac{\partial g_i(a_i, a_{-i})}{\partial a_i} d\lambda_i(a_{-i}),$$

g_i is own-strategy unimodal if g_i is own-strategy concave, i.e., $\partial g_i(a_i, a_{\_i})/\partial a_i$ is decreasing in a_i for all $a_{-i} \in A_{-i}$.

For measurable functions $q_i : A_{-i} \to \mathbb{R}$ and $w_i : A_i \to \mathbb{R}_{++}$, let $(q_i, w_i) \circ g_i : A \to \mathbb{R}$ be such that, for $a_i \in A_i$ and $a_{-i} \in A_{-i}$,

$$(q_i, w_i) \circ g_i(a_i, a_{-i}) = q_i(a_{-i}) + \int_{x \leq a_i} w_i(x) \frac{\partial g_i(x, a_{-i})}{\partial x} dx.$$

Let

$$\begin{aligned}
\mathcal{D}_i(g_i) &= \{g'_i : A \to \mathbb{R} \,|\, g'_i = (q_i, w_i) \circ g_i,\ q_i : A_{-i} \to \mathbb{R},\ w_i : A_i \to \mathbb{R}_{++}\}, \\
\mathcal{D}(\mathbf{g}) &= \{\mathbf{g}' = (g'_i)_{i \in N} \,|\, g'_i \in \mathcal{D}_i(g_i)\}.
\end{aligned}$$

Proposition 8 *Suppose that g_i is own-strategy unimodal for all $i \in N$. Then, every game in $\mathcal{D}(\mathbf{g})$ is best-response equivalent to $\mathbf{g}$.*

Proof. Let $\mathbf{g}' \in \mathcal{D}(\mathbf{g})$. Since g_i is own-strategy unimodal, for all $\lambda_i \in \Delta(A_i)$, there exists $a_i^* \in A_i$ such that

$$\begin{aligned} \frac{\partial}{\partial a_i} \int_{A_{-i}} g_i(a_i, a_{-i}) d\lambda_i(a_{-i}) \geq 0 \text{ if } a_i \leq a_i^*, \\ \frac{\partial}{\partial a_i} \int_{A_{-i}} g_i(a_i, a_{-i}) d\lambda_i(a_{-i}) \leq 0 \text{ if } a_i \geq a_i^*. \end{aligned} \tag{19}$$

Since

$$\frac{\partial g_i'(a_i, a_{-i})}{\partial a_i} = w_i(a_i) \frac{\partial g_i(a_i, a_{-i})}{\partial a_i},$$

(19) is true if and only if

$$\begin{aligned} \frac{\partial}{\partial a_i} \int_{A_{-i}} g_i'(a_i, a_{-i}) d\lambda_i(a_{-i}) \geq 0 \text{ if } a_i \leq a_i^*, \\ \frac{\partial}{\partial a_i} \int_{A_{-i}} g_i'(a_i, a_{-i}) d\lambda_i(a_{-i}) \leq 0 \text{ if } a_i \geq a_i^*. \end{aligned} \tag{20}$$

Thus, g_i' is also own-strategy unimodal. Since (19) is true if and only if $\lambda_i \in \Lambda_i(a_i^*, A_i|g_i)$ and (20) is true if and only if $\lambda_i \in \Lambda_i(a_i^*, A_i|g_i')$, we must have $\Lambda_i(a_i^*, A_i|g_i) = \Lambda_i(a_i^*, A_i|g_i')$, which completes the proof. ∎

This proposition has a useful application concerning the uniqueness of correlated equilibria. Neyman [13] showed that if $\mathbf{g}$ has a continuously differentiable and strictly concave potential function,[8] then the potential maximizer is the unique correlated equilibrium of $\mathbf{g}$. The set of correlated equilibria is the same for two games if the two games are best-response equivalent. Thus, we claim the following.

Corollary 9 *Suppose that $\mathbf{g}$ has a continuously differentiable and strictly concave potential function f. Then, the potential maximizer is the unique correlated equilibrium of every game in $\mathcal{D}(\mathbf{g})$.*

Note that a game in $\mathcal{D}(\mathbf{g})$ is not necessarily a potential game and payoff functions are not necessarily concave.

[8]The definition of potential functions of this class of games is the same as those of finite games.

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