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**By**

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# Nonlinear Log-Periodogram Regression for Perturbed Fractional Processes*


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## ABSTRACT

This paper studies fractional processes that may be perturbed by weakly dependent time series. The model for a perturbed fractional process has a components framework in which there may be components of both long and short memory. All commonly used estimates of the long memory parameter (such as log periodogram (LP) regression) may be used in a components model where the data are affected by weakly dependent perturbations, but these estimates can suffer from serious downward bias. To circumvent this problem, the present paper proposes a new procedure that allows for the possible presence of additive perturbations in the data. The new estimator resembles the LP regression estimator but involves an additional (nonlinear) term in the regression that takes account of possible perturbation effects in the data. Under some smoothness assumptions at the origin, the bias of the new estimator is shown to disappear at a faster rate than that of the LP estimator, while its asymptotic variance is inflated only by a multiplicative constant. In consequence, the optimal rate of convergence to zero of the asymptotic MSE of the new estimator is faster than that of the LP estimator. Some simulation results demonstrate the viability and the bias-reducing feature of the new estimator relative to the LP estimator in finite samples. A test for the presence of perturbations in the data is given.

*JEL Classification: C13; C14; C22; C51*

*Keywords:* Asymptotic bias; Asymptotic normality; Bias reduction; Fractional components model; Perturbed fractional process; Rate of convergence; Testing perturbations.

# 1 Introduction

Fractional processes have been gaining increasing popularity with empirical researchers in economics and finance. In part, this is because fractional processes can capture forms of long run behavior in economic variables that elude other models, a feature that has proved particularly important in modelling inter-trade durations and the volatility of financial asset returns. In part also, fractional processes are attractive to empirical analysts because they allow for varying degrees of persistence, including a continuum of possibilities between weakly dependent and unit root processes.

For a pure fractional process, short run dynamics and long run behavior are driven by the same innovations. This may be considered restrictive in that the innovations that drive long run behavior may arise from quite different sources and therefore differ from those that determine the short run fluctuations of a process. To accommodate this possibility, the model we consider in the present paper allows for perturbations in a fractional process and has a components structure that introduces different sources and types of variation. Such models provide a mechanism for simultaneously capturing the effects of persistent and temporary shocks on the realized observations. They seem particularly realistic in economic and financial applications when there are many different sources of variation in the data and both long run behavior and short run fluctuations need to be modeled.

Specifically, a perturbed fractional process $(z_t)$ is defined as a fractional process $(y_t)$ that is perturbed by a weakly dependent process $(u_t)$ as follows

$$z_t = y_t + \mu + u_t, t = 1, 2, ..., n, \tag{1}$$

where $\mu$ is a constant and

$$y_t = (1-L)^{-d_0} w_t = \sum_{k=0}^{\infty} \frac{\Gamma(d_0+k)}{\Gamma(d_0)\Gamma(k+1)} w_{t-k},\ 0 < d_0 < 1/2. \tag{2}$$

Here, $y_t$ is a pure fractional process and $u_t$ and $w_t$ are independent Gaussian processes with zero means and continuous spectral densities $f_u(\lambda)$ and $f_w(\lambda)$, respectively. We confine attention to the case where the memory parameter $d_0 \in (0, \frac{1}{2})$ largely for technical reasons that will become apparent later. The case is certainly the most relevant in empirical practice, at least for stationary series, but the restriction is an important one. To maintain generality in the short run components of $z_t$ we do not impose specific functional forms on $f_u(\lambda)$ and $f_w(\lambda)$. Instead, we allow them to belong to a family that is characterized only by regularity conditions near the zero frequency. This formulation corresponds to the conventional semiparametric approach to modelling long range dependence.

By allowing for the presence of two separate stochastic components, the model (1) captures mechanisms in which different factors may come into play in determining long run and short run behaviors. Such mechanisms may be expected to occur in the generation of macroeconomic and financial data for several reasons. For example, time series observations of macroeconomic processes often reflect short run competitive forces as well as long run growth determinants. Additionally, economic and financial time series frequently arise from processes of aggregation and involve errors of measurement, so that the presence of an additive, short memory disturbance is quite realistic. For instance, if the underlying volatility of stock returns follows a fractional process, then realized volatility may follow a perturbed fractional process because the presence of a bid-ask bounce adds a short memory component to realized returns, with consequent effects on volatility.

Some empirical models now in use are actually special cases of perturbed fractional processes. Among these, the long memory stochastic volatility model (LMSV) is growing in popularity for modelling the volatility of financial time series (see Anderson and Bollerslev, 1997, Breidt, Crato and De Lima, 1998, and Deo and Hurvich, 2001). This model assumes that $\log r_t^2 = y_t + \mu + u_t$, where $r_t$ is the return, $y_t$ is an underlying fractional process and $u_t = iid(0, \sigma^2)$, thereby coming within the framework of (1). Another example is a rational expectation model in which the ex ante variable follows a fractional process, so that the corresponding ex post variable follows (1) with $u_t$ being a martingale difference sequence. Sun and Phillips (2000) used this framework to model the real rate of interest and inflation as perturbed fractional processes and found that this model helped explain the empirical incompatibility of memory parameter estimates of the components in the ex post Fisher identity. The study by Granger and Marmol (1997) provides a third example, addressing the frequently observed property of financial time series that the autocorrelogram can be low but positive for many lags. Granger and Marmol explained this phenomenon by considering time series that consist of a long memory component combined with a white noise component that has a much larger variance, again coming within the framework of (1).

The main object in the present paper is to develop a suitable estimation procedure for the memory parameter $d_0$ in (1). As we will show, existing procedures for estimating $d_0$ typically suffer from serious downward bias in models where there are additive perturbations like (1 ). The present paper therefore proposes a new procedure that allows for the possible presence of such perturbations in the data.

The spectral density $f_z(\lambda)$ of $z_t$ can be written as $f_z(\lambda) = (2\sin\frac{\lambda}{2})^{-2d_0} f^*(\lambda)$, where $f^*(\lambda) = f_w(\lambda) + (2\sin\frac{\lambda}{2})^{2d_0} f_u(\lambda)$ is a continuous function over $[0, \pi]$. So, $f_z(\lambda)$ satisfies a power law around the origin of the form $f_z(\lambda) \sim G_0 \lambda^{-2d_0}$ as $\lambda \to 0+$, for some positive constant $G_0$. Therefore, we can estimate $d_0$ by using the linear log-periodogram (LP) regression introduced by Geweke and Porter-Hudak (1983). Building on the earlier work of Künsch (1986), Robinson (1995a) established the asymptotic normality of the LP estimator. Subsequently, Hurvich, Deo and Brodsky (1998) (hereafter HDB) computed the mean square error of the LP estimator and provided an MSE-optimal rule for bandwidth selection.

The LP estimator has undoubted appeal. It is easy to implement in practice and has been commonly employed in applications. However, when the spectral density of $u_t$ dominates that of $w_t$ in a neighborhood of the origin, the estimator may be biased downward substantially, especially in small samples. One source of the bias is the error of approximating the logarithm of $f^*(\lambda)$ by a constant in a shrinking neighborhood of the origin. This crude approximation also restricts the rate of convergence. The rate of convergence of the LP estimator will be shown to be $n^{-2d_0/(4d_0+1)}$, which is quite slow, especially when $d_0$ is close to zero.

To alleviate these problems, we take advantage of the structure of our model and propose to estimate the logarithm of $f^*(\lambda)$ locally by $c + \beta\lambda^{2d_0}$. Our new estimator is defined as the minimizer of the average-squared-errors (ASE) in a nonlinear log periodogram regression of the form

$$\log I_{zj} = \alpha - 2d \log \lambda_j + \beta \lambda_j^{2d} + error, j = 1, 2, ..., m, \quad (3)$$

where

$$I_{zj} = I_z(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=0}^{n-1} z_t \exp(it\lambda_j)|^2, \ \lambda_j = \frac{2\pi j}{n}, \tag{4}$$

and $m$ is a positive integer smaller than the sample size $n$. We will call our estimator Nonlinear Log Periodogram (NLP) estimator hereafter.

The NLP estimator can be seen as a way of utilizing parametric information in a nonparametric setting. We approximate the unknown function locally by a nonlinear function instead of a constant. From a broad perspective, the NLP estimator has at least superficial similarity to the local nonlinear estimator (Linton and Gozalo, 2000) in the nonparametric literature. Linton and Gozalo (2000) found that the local nonlinear estimator had superior performance compared to the local constant kernel estimator when the local nonlinear parameterization is close to the unknown function. Analogously, we expect the nonlinear log periodogram regression estimator to work well in the presence of perturbations, especially when the perturbations are relatively large.

Let $(\widehat{d}, \widehat{\beta})$ denote the NLP estimator that minimizes the concentrated ASE in which the intercept $\alpha$ has been concentrated out. We show the consistency of $\widehat{d}$ by proving that the concentrated ASE converges uniformly over $(d, \beta)' \in \Theta$ to a function which has a unique minimizer $d_0$, where $\Theta$ is the parameter space to be defined later. To establish the asymptotic normality of $\widehat{d}$, a typical argument would first establish the consistency of $\widehat{\beta}$. But showing that $\widehat{\beta}$ is consistent is not straightforward, because the concentrated ASE becomes flat as a function of $\beta$ as $n \to \infty$. To circumvent this problem, we first show that $\widehat{d}$ converges to $d_0$ at some rate $k_n$ (meaning $\widehat{d} - d_0 = O_p(k_n)$) without using the consistency of $\widehat{\beta}$. We then show that, when $|d - d_0| \le Ck_n$, the flatness problem disappears if the ASE is recentered and normalized by $k_n^2$.

The flatness problem also appeared in Andrews and Sun (2000). It seemed that they had difficulty in establishing a sufficient rate of convergence for their estimator $\widehat{d}(r)$ without resort to the consistency of $\widehat{\theta}$, the subvector of the estimator that causes the problem. Their way to overcome this problem is to define the estimator as the solution to the first order conditions which comes closest to the minimizer of the criterion function. The present paper proposes a new and simple approach to overcome the flatness problem that avoids having to redefine the estimator.

We investigate both the asymptotic and finite sample properties of $\widehat{d}$. Asymptotic bias, variance, asymptotic mean squared error (AMSE), and asymptotic normality are determined. We find that the asymptotic bias of $\widehat{d}$ is of order $m^{4d_0}/n^{4d_0}$, provided that $f_w(\cdot)$ and $f_u(\cdot)$ are boundedly differentiable around the origin, whereas that of the LP estimator $\widehat{d}_{LP}$ has the larger order $m^{2d_0}/n^{2d_0}$. The asymptotic variances of $\widehat{d}$ and $\widehat{d}_{LP}$ are both of order $m^{-1}$. In consequence, the optimal rate of convergence to zero of $\widehat{d}$ is of order $n^{-4d_0/(8d_0+1)}$, whereas that of $\widehat{d}_{LP}$ is of the larger order $n^{-2d_0/(4d_0+1)}$. But when $d_0$ is close to zero, the rate of convergence of $\widehat{d}$ will still be quite slow. We find that $\widehat{d}$ is asymptotically normal with mean zero, provided that $m^{8d_0+1}/n^{8d_0} \to 0$, whereas $\widehat{d}_{LP}$ is asymptotically normal only under the more stringent condition $m^{4d_0+1}/n^{4d_0} \to 0$.

When the underlying process is a pure fractional process, we encounter a nonstandard estimation problem as the true parameter $\beta_0$ is on the boundary of the parameter set. In this case, the limiting distribution of $\widehat{\beta}$ is truncated normal, and the limiting distribution of $\widehat{d}$ is more complicated, involving a mixture distribution with the mixing probabilities that depend on the component distributions. We find that the asymptotic bias and variance of $\widehat{d}$ are of the same orders as those of $\widehat{d}_{LP}$. However, it is difficult to obtain exact expressions for the asymptotic bias and variance. In consequence, it is hard to evaluate the performance of $\widehat{d}$ relative to that of $\widehat{d}_{LP}$ in this case.

Some Monte Carlo simulations show that the asymptotic results of the paper capture the finite sample properties of the NLP estimator quite well. For the fractional component processes considered in the simulations, the NLP estimator $\widehat{d}$ has a lower bias, a higher standard deviation, and a lower RMSE compared to the LP estimator $\widehat{d}_{LP}$, as the asymptotic results suggest. The lower bias leads to better coverage probabilities for $\widehat{d}$ over a wider range of $m$ than for $\widehat{d}_{LP}$. On the other hand, the lower standard deviation of $\widehat{d}_{LP}$ leads to shorter confidence intervals than confidence intervals based on $\widehat{d}$.

The properties of the NLP estimator are investigated under the assumption of Gaussian errors. Gaussianity is usually assumed in the log-periodogram regression literature (e.g., Robinson, 1995a, and Andrews and Guggenberger, 1999) and the present paper is no exception. Nevertheless, Gaussianity is restrictive in some empirical applications and could be relaxed following the lines of recent work by Velasco (2000) and Deo and Hurvich (2001), although we have not done so here.

The paper by Andrews and Guggenberger (1999) is most related to our work. They considered the conventional fractional model (i.e., $var(u_t) = 0$) and proposed to approximate $\log f_w(\lambda)$ by a constant plus a polynomial of even order. Andrews and Sun (2000) investigated the same issue in the context of a local Whittle estimator. Other related papers include Henry and Robinson (1996), Hurvich and Deo (1999) and Henry (1999). These papers consider approximating $\log f^*(\lambda)$ by a more sophisticated function than a constant for the purpose of obtaining a data-driven choice of $m$. The present paper differs from those papers in that a nonlinear approximation is used in order to achieve bias reduction and to increase the rate of convergence in the estimation of $d_0$. Also, the nonlinear polynomial function used here depends on the memory parameter $d_0$ (whereas this is not so in the work just mentioned) and the estimation procedure for $d_0$ utilizes this information.

The rest of the paper is organized as follows. Section 2 formally defines the NLP estimator. Section 3 outlines the asymptotics of discrete Fourier transforms and log-periodogram ordinates, which are used extensively in later sections. Section 4 establishes consistency and derives limiting distribution results for the NLP estimator. This section also proposes a test for the pure fractional process against a perturbed fractional process. Section 5 investigates the finite sample performance of the NLP estimator by simulations. Section 6 concludes. Proofs are collected in the Appendix.

Throughout the paper, $\{E\}$ is defined to be the indicator function for event $E$. $C$ is a generic constant.

## 2 Nonlinear Log Periodogram Regression

This section motivates the NLP estimator that explicitly accounts for the additive perturbations in (1). Throughout, (1) is taken as the data generating process and then

$$f_z(\lambda) = (2\sin\frac{\lambda}{2})^{-2d_0} f^*(\lambda). \tag{5}$$

Taking the logarithms of (5) leads to

$$\log(f_z(\lambda)) = -2d_0 \log \lambda + \log f^*(\lambda) - 2d_0 \log(2\lambda^{-1} \sin(\frac{\lambda}{2})). \tag{6}$$

Replacing $f_z(\lambda)$ by periodogram ordinates $I_z(\lambda)$ evaluated at the fundamental frequencies $\lambda_j, j = 1, 2, ..., m$ yields

$$\log(I_{zj}) = -c_0 - 2d_0 \log \lambda_j + \log f^*(\lambda_j) + U_j + O(\lambda_j^2), \tag{7}$$

where $c_0 = 0.577216...$ is the Euler constant and $U_j = \log[I_z(\lambda_j)/f_z(\lambda_j)] + c_0$.

By virtue of the continuity of $f^*(\lambda)$, we can approximate $\log f^*(\lambda_j)$ by a constant over a shrinking neighborhood of the zero frequency. This motivates log-periodogram regression on the equation

$$\log(I_{zj}) = \text{constant} - 2d \log \lambda_j + error. \tag{8}$$

The LP estimator $\widehat{d}_{LP}$ is then given by the least squares estimator of $d$ in this regression. If $\{U_j\}_{j=1}^m$ behave asymptotically like independent and identically distributed random variables, then the LP estimator is a reasonable choice. In fact, under assumptions to be stated below, we establish that $\sqrt{m}(\widehat{d}_{LP} - d_0) \sim N(b_{LP}, \frac{\pi^2}{24})$ where $b_{LP} = O(m^{2d_0+1/2}/n^{2d_0})$ and '$\sim$' signifies 'asymptotically distributed.' The 'asymptotic bias' of $\widehat{d}_{LP}$ itself is therefore of order $O(m^{2d_0}/n^{2d_0})$, which can be quite large. To reduce the bias, we can approximate $\log f^*(\lambda_j)$ by a simple nonlinear function of frequency under the following assumptions:

**Assumption 1**: *Either (a)* $\sigma_u = var^{1/2}(u_t) = 0$ *for all* $t$*, so* $f_u(\lambda) \equiv 0$*, for* $\lambda \in [-\pi, \pi]$ *or: (b)* $\sigma_u > 0$ *and* $f_u(\lambda)$ *is continuous on* $[-\pi, \pi]$*, bounded above and away from zero with bounded first derivative in a neighborhood of zero.*

**Assumption 2:** $f_w(\lambda)$ *is continuous on* $[-\pi, \pi]$*, bounded above and away from zero. When* $\sigma_u = 0$*,* $f_w(\lambda)$ *is three times differentiable with bounded third derivative in a neighborhood of zero. When* $\sigma_u > 0$*,* $f_w(\lambda)$ *is differentiable with bounded derivative in a neighborhood of zero.*

Assumptions 1(b) and 2 are local smoothness conditions and hold for many models in current use, including ARMA models. They allow us to develop a Taylor expansion of $\log f^*(\lambda)$ about $\lambda = 0$ with an error of the order of the first omitted term. Specifically, when $\sigma_u = 0$,

$$\log f^*(\lambda_j) = \log f_w(0) + O(\lambda_j^2). \tag{9}$$

When $\sigma_u > 0$,

$$\begin{aligned} \log f^*(\lambda_j) &= \log f_w(\lambda_j) + \log[1 + (2 \sin \frac{\lambda_j}{2})^{2d_0} \frac{f_u(\lambda_j)}{f_w(\lambda_j)}] \\ &= \log f_w(\lambda_j) + \log \left\{ 1 + \lambda_j^{2d_0}(1 + O(\lambda_j^2)) \left( \frac{f_u(0)}{f_w(0)} + O(\lambda_j^2) \right) \right\} \\ &= \log f_w(0) + \frac{f_u(0)}{f_w(0)} \lambda_j^{2d_0} + O(\lambda_j^{4d_0}). \end{aligned} \tag{10}$$

So, in either case

$$\log f^*(\lambda_j) = \log f_w(0) + \frac{f_u(0)}{f_w(0)} \lambda_j^{2d_0} + O(\lambda_j^r) \tag{11}$$

where $O(\cdot)$ holds uniformly over $j = 1, 2, ..., m$, $r = 4d_0\{\sigma_u > 0\} + 2\{\sigma_u = 0\}$.

Combining (7) with (11) produces the nonlinear LP regression model:

$$\log(I_{zj}) = -2d_0 \log \lambda_j + \alpha_0 + \lambda_j^{2d_0}\beta_0 + U_j + \varepsilon_j, \tag{12}$$

where

$$\alpha_0 = \log f_w(0) - c_0, \beta_0 = f_u(0)/f_w(0), \text{ and}$$
$$\varepsilon_j = \log f^*(\lambda_j) - \log f_w(0) - \beta_0\lambda_j^{2d_0} - 2d_0[\log(2\sin\frac{\lambda_j}{2}) - \log\lambda_j]. \tag{13}$$

The NLP estimator is then defined as the minimizer of the average-squared-errors in this model, i.e.

$$(\widehat{\alpha}, \widehat{d}, \widehat{\beta}) = \arg\min_{\alpha,d,\beta} ASE(\alpha, d, \beta), \tag{14}$$

where

$$ASE(\alpha, d, \beta) = \frac{1}{m}\sum_{j=1}^{m}[\log(I_{zj}) - \alpha + 2d\log\lambda_j - \lambda_j^{2d}\beta]^2. \tag{15}$$

Concentrating (15) with respect to $\alpha$, we obtain

$$(\widehat{d}, \widehat{\beta}) = \arg\min_{d\in D,\beta\in B} Q(d, \beta), \tag{16}$$

with

$$\begin{aligned} Q(d,\beta) &= \frac{1}{m}\sum_{j=1}^{m}\{(\log I_{zj} - \frac{1}{m}\sum_{k=1}^{m}\log I_{zk}) \\ &\quad +2d(\log\lambda_j - \frac{1}{m}\sum_{k=1}^{m}\log\lambda_k) - \beta(\lambda_j^{2d} - \frac{1}{m}\sum_{k=1}^{m}\lambda_k^{2d})\}^2. \end{aligned} \tag{17}$$

where $B$ and $D$ are parameter sets. We write $\theta = (d, \beta)'$, $\Theta = D \times B$ for convenience and make the following assumption on the parameter space:

**Assumption 3:** *(a)* $D = [d_1, d_2]$ *where* $0 < d_1 < d_2 < 1/2$ *and* $B = [0, \bar{b}]$ *where* $\bar{b} > 0$; *(b) The true parameter* $(d_0, \beta_0) \in (d_1, d_2) \times [0, \bar{b})$.

In the above assumption, $d_1$ and $d_2$ can be chosen arbitrarily close to 0 and 1/2, respectively, and $\bar{b}$ can be chosen arbitrarily large. When $\beta_0 = 0$, the model becomes nonstandard in the sense that the true parameter is on the boundary of the parameter set. Section 4 explores the implication of the boundary problem.

# 3 Log-periodogram Asymptotics and Useful Lemmas

To establish the asymptotic properties of the NLP estimator, we need to characterize the asymptotic behavior of the log-periodogram ordinates $U_j = \log[I_z(\lambda_j)/f_z(\lambda_j)] + c_0$. Define

$$A_{zj} = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} z_t \cos\lambda_j t \text{ and } B_{zj} = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} z_t \sin\lambda_j t, \tag{18}$$

then

$$U_j = \ln\left(\frac{A_{zj}^2}{f_{zj}} + \frac{B_{zj}^2}{f_{zj}}\right) + c_0, j = 1, ..., m. \tag{19}$$

In view of the Gaussianity of $A_{zj}$ and $B_{zj}$, we can evaluate the means, variances, and covariances of $U_j$, if the asymptotic behavior of the vector $\left(A_{zj}/f_{zj}^{1/2}, B_{zj}/f_{zj}^{1/2}, A_{zk}/f_{zk}^{1/2}, B_{zk}/f_{zk}^{1/2}\right)$ is known. The properties of this vector depend in turn on those of the discrete Fourier transforms of $z_t$, defined as $w(\lambda) = (2\pi n)^{-1/2}\sum_1^n z_t e^{it\lambda}$.

The asymptotic behavior of $w(\lambda)$ is given in the following lemma, which is a variant of results given earlier by several other authors (Robinson, 1995a, HDB, 1998, Andrews and Guggenberger, 1999).

**Lemma 1** *Let Assumptions 1 and 2 hold. Then, uniformly over $j$ and $k$, $1 \leq k < j \leq m$, $m/n \to 0$,*

**(a)** $E\left[w(\lambda_j)\overline{w}(\lambda_j)/f_z(\lambda_j)\right] = 1 + O\left(j^{-1}\log j\right)$,

**(b)** $E\left[w(\lambda_j)w(\lambda_j)/f_z(\lambda_j)\right] = O\left(j^{-1}\log j\right)$,

**(c)** $E\left[w(\lambda_j)\overline{w}(\lambda_k)/\left(f_z(\lambda_j)f_z(\lambda_k)\right)^{1/2}\right] = O\left(k^{-1}\log j\right)$,

**(d)** $E\left[w(\lambda_j)w(\lambda_k)/\left(f_z(\lambda_j)f_z(\lambda_k)\right)^{1/2}\right] = O\left(k^{-1}\log j\right)$.

It follows directly from Lemma 1 that for $1 \leq k < j \leq m$,

$$EA_{zj}^2/f_{zj} = \frac{1}{2} + O(\frac{\log j}{j}),\ EB_{zj}^2/f_{zj} = \frac{1}{2} + O(\frac{\log j}{j}),$$
$$EA_{zj}B_{zj}/f_{zj} = O(\frac{\log j}{j}),\ EA_{zj}B_{zk}/(f_{zj}f_{zk})^{1/2} = O(\frac{\log j}{k}). \tag{20}$$

Using these results and following the same line of derivation as in HDB (1998), we can prove Lemma 2 below. Since the four parts of this lemma are proved in a similar way to Lemmas 3, 5, 6 and 7 in HDB, the proofs are omitted here.

**Lemma 2** *Let Assumptions 1 and 2 hold. Then*

**(a)** $Cov(U_j, U_k) = O\left(\log^2 j/k^2\right)$, *uniformly for* $\log^2 m \leq k < j \leq m$,

**(b)** $\lim_n \sup_{1\leq j\leq m} EU_j^2 < \infty$,

**(c)** $E(U_j) = O(\log j/j)$, *uniformly for* $\log^2 m \leq j \leq m$,

**(d)** $Var(U_j) = \pi^2/6 + O(\log j/j)$, *uniformly for* $\log^2 m \leq j \leq m$.

With the asymptotic behavior of $U_j$ in hand, we can proceed to show that the normalized sums $m^{-1}\sum_{j=1}^m c_j U_j$ are uniformly negligible under certain conditions on the coefficients $c_j$. Quantities of this form appear in the normalized Hessian matrix below.

**Lemma 3** *Let* $\{c_j(d,\beta)\}_{j=1}^m$ *be a sequence of functions such that, for some* $p \geq 0$,

$$\sup_{(d,\beta)' \in \Theta} |c_j| = O(\log^p m) \text{ uniformly for } 1 \leq j \leq m, \tag{21}$$

*and for some* $q \geq 0$,

$$\sup_{(d,\beta)' \in \Theta} |c_j - c_{j-1}| = O(j^{-1} \log^q m) \text{ uniformly for } 1 \leq j \leq m. \tag{22}$$

*Then*

$$\sup_{(d,\beta)' \in \Theta} \left| \frac{1}{m} \sum_{j=1}^m c_j U_j \right| = O_p\left(\frac{\log^{\max(p,q)} m}{\sqrt{m}}\right). \tag{23}$$

We can impose additional conditions to get a tighter bound. For example, if we also require that $\sup_{(d,\beta)' \in \Theta} |c_m| = O(1)$, then $\sup_{(d,\beta)' \in \Theta} \left| \frac{1}{m} \sum_{j=1}^m c_j U_j \right| = O_p(\log^q m / \sqrt{m})$, as is readily seen from the proof of the lemma. Further, the lemma remains valid if we remove the 'sup' operator from both the conditions and the conclusion.

Let $V_j(d,\beta) = 2(d - d_0) \log \lambda_j - \beta \lambda_j^{2d} + \beta_0 \lambda_j^{2d_0}$, and $\overline{V}(d,\beta) = 1/m \sum_{j=1}^m V_j(d,\beta)$. We can use an argument similar to the proof of Lemma 3 to establish the following corollary, which will be used extensively in the consistency proof.

**Corollary 1** *Let* $D^0 = \{d : d \in D, |d - d_0| \leq C(m/n)^\gamma\}$, *for some constants* $C \in R^+$, $\gamma \in [0, 2d_0]$, *then*

$$\sup_{(d,\beta)' \in D^0 \times B} \left| \frac{1}{m} \sum_{j=1}^m U_j \left( V_j(d,\beta) - \bar{V}(d,\beta) \right) \right| = O_p\left( \left(\frac{m}{n}\right)^\gamma \frac{1}{\sqrt{m}} \right) \tag{24}$$

*as* $1/n + m/n \to 0$.

The following lemma assists in establishing the asymptotic normality of the nonlinear log-periodogram regression estimator.

**Lemma 4** *Let* $a_{kn} = a_k$ *be a triangular array for which*

$$\max_k |a_k| = o(m), \qquad \sum_{k=[1+m^{0.5+\delta}]}^m a_k^2 \sim \rho m, \qquad \sum_{k=[1+m^{0.5+\delta}]}^m |a_k|^p = O(m), \tag{25}$$

*for all* $p \geq 1$, *and* $0 < \delta < 0.5$. *Then,*

$$\frac{1}{\sqrt{m}} \sum_{k=[1+m^{0.5+\delta}]}^m a_k U_k \xrightarrow{d} N\left(0, \frac{\pi^2}{6}\rho\right), \tag{26}$$

*where* $[\cdot]$ *denotes the integer part.*

The proof of this lemma is based on the method of moments and involves a careful exploration of the dependence structure of the discrete Fourier transforms. Robinson's argument (1995a, pp. 1067-70) forms the basis of this development and can be used here with some minor modifications to account for differences in the models. Details are omitted here and are available upon request.

# 4 Asymptotic Properties of the LP and NLP Estimators

## 4.1 Asymptotic Properties of the LP Estimator

We establish the asymptotic properties for the LP estimator in the context of the components model (1). Theorem 1 gives the limit theory and provides a benchmark for later comparisons.

**Theorem 1** *Let Assumptions 1 and 2 hold. Let* $m = m(n) \to \infty$ *and*

$$\frac{m^{r'+1/2}}{n^{r'}} \to K'_\sigma\{\sigma_u > 0\} + K'_0\{\sigma_u = 0\} \tag{27}$$

*as* $n \to \infty$*, where* $r' = 2d_0\{\sigma_u > 0\} + 2\{\sigma_u = 0\}$ *and* $K'_\sigma, K'_0 > 0$ *are positive constants. Then*

$$\sqrt{m}(\widehat{d}_{LP} - d_0) \Rightarrow N(b_{LP}, \frac{\pi^2}{24}), \tag{28}$$

*where*

$$b_{LP} = -(2\pi)^{2d_0}\frac{f_u(0)}{f_w(0)}\frac{d_0}{(2d_0+1)^2}K'_\sigma\{\sigma_u > 0\} - \frac{2\pi^2}{9}\left(\frac{f''_w(0)}{f_w(0)} + \frac{d_0}{6}\right)K'_0\{\sigma_u = 0\}. \tag{29}$$

When $\sigma_u > 0$, the ratio $m^{r'+1/2}/n^{r'} = m^{2d_0+1/2}/n^{2d_0} \to K'_\sigma$ in (27). This delivers an upper bound of order $O(n^{4d_0/(1+4d_0)})$ on the rate at which $m$ can increase with $n$ and allows for larger choices of $m$ for larger values of $d_0$. Intuitively, as $d_0$ increases, the contamination from perturbations at frequencies away from the origin becomes relatively smaller and we can expect to be able to employ a wider bandwidth in the regression. To eliminate the asymptotic bias $b_{LP}$ in (28) altogether, we use a narrower band and set $m = o(n^{4d_0/(1+4d_0)})$ in place of (27). Deo and Hurvich (2001) established a similar result under the assumption that $u_t$ is iid, but not necessarily Gaussian. Their assumption that $m^{4d_0+1}\log^2 m/n^{4d_0} = o(1)$ is slightly stronger than the assumption made here.

When $\sigma_u > 0$, the limit distribution (28) involves the bias

$$b_{LP} = -(2\pi)^{2d_0}\frac{f_u(0)}{f_w(0)}\frac{d_0}{(2d_0+1)^2}K'_\sigma < 0, \tag{30}$$

which is always negative, as one would expect, because of the effect of the short memory perturbations. Correspondingly, the dominating bias term of $\widehat{d}_{LP}$ has the form

$$b_{n,LP} = -(2\pi)^{2d_0}\frac{f_u(0)}{f_w(0)}\frac{d_0}{(2d_0+1)^2}\frac{m^{2d_0}}{n^{2d_0}} < 0. \tag{31}$$

The magnitude of the bias obviously depends on the quantity $f_w(0)/f_u(0)$, which is the ratio of the long run variance of the short memory input of $y_t$ to that of the perturbation component $u_t$. The ratio can be interpreted as a long run signal-noise ratio (SNR), measuring the strength in the long run of the signal from the $y_t$ inputs relative to the long run signal in the perturbations. The stronger the long run signal in the perturbations, the greater the downward bias and the more difficult it becomes to estimate the memory parameter accurately. One might expect these effects to be exaggerated in small samples where the capacity of the data to discriminate between long run and short run effects is reduced.

When $\sigma_u > 0$, the asymptotic mean-squared error (AMSE) of $\widehat{d}_{LP}$ satisfies

$$AMSE(\widehat{d}_{LP}) = O_p(\frac{m}{n})^{4d_0} + O_p(\frac{1}{m}). \tag{32}$$

So the AMSE-optimal bandwidth has the form $m_{LP}^{opt} = C_{LP} n^{4d_0/(4d_0+1)}$ for some constant $C_{LP}$. When $m = m_{LP}^{opt}$, $AMSE(\widehat{d}_{LP}) = O_p(n^{-4d_0/(4d_0+1)})$. In contrast, in the case $\sigma_u = 0$, it is well known that when $m = m_{LP}^{opt}$, $AMSE(\widehat{d}_{LP}) = O_p(n^{-4/5})$. Due to the presence of the perturbations, the optimal AMSE of $\widehat{d}_{LP}$ converges to zero at a slower rate.

When $\sigma_u = 0$, the theorem contains essentially the same results proved in HDB. In this case, the dominating bias of $\widehat{d}_{LP}$ is given by $b_{n,LP} = -2\pi^2/9 \left(f_w''(0) f_w^{-1}(0) + d_0/6\right) m^2/n^2$. HDB showed that the dominating bias of $\widehat{d}_{LP}$ in the case of pure fractional process regression is given by the expression $-2\pi^2/9 \left(f_w''(0) f_w^{-1}(0)\right) m^2/n^2$. The presence of the additional factor $d_0/6$ in the second term of our expression arises from the use of a slightly different regressor in the LP regression. In particular, we employ $-2\log\lambda_j$ as one of the regressors in (3), while HDB use $-2\log(2\sin\lambda_j/2)$. These regressors are normally considered to be asymptotically equivalent. However, while the use of $-2\log\lambda_j$ rather than $-2\log(2\sin\lambda_j/2)$ has no effect on the asymptotic variance, it does affect the asymptotic bias.

### 4.2 Consistency of the NLP estimator

To establish the limiting distribution of the NLP estimator, we first prove the consistency of the NLP estimator.

**Theorem 2** *Let Assumptions 1 and 2 hold.*

**(a)** *If* $\frac{1}{m} + \frac{m}{n} \to 0$ *as* $m, n \to \infty$, *then* $\widehat{d} - d_0 = o_p(1)$.

**(b)** *If for some arbitrary small* $\Delta > 0$, $\frac{m}{n} + \frac{n^{4d_0(1+\Delta)}}{m^{4d_0(1+\Delta)+1}} \to 0$, *as* $m, n \to \infty$, *then* $\widehat{d} - d_0 = O_p\left((\frac{m}{n})^{2d_0}\right)$ *and* $\widehat{\beta} - \beta_0 = o_p(1)$.

Theorem 2 shows that $\widehat{d}$ is consistent under mild conditions. All that is needed is that $m$ approaches infinity slower than the sample size $n$. As shown by HDB, trimming out low frequencies is not necessary. This point is particularly important in the present case because, in seeking to reduce contamination from the perturbations, the lowest frequency ordinates are the most valuable in detecting the long memory effects.

It is not straightforward to establish the consistency of $\widehat{\beta}$, because, as $n \to \infty$, the objective function becomes flat as a function of $\beta$. The way we proceed is, in fact, to show first that $\widehat{d}$ converges to $d_0$ at some slower rate, more precisely, $\widehat{d} - d_0 = O_p((m/n)^{2d_0})$. We prove this rate of convergence stepwise. We start by showing that $\widehat{d} - d_0 = o_p((m/n)^{d_1/2})$ for $0 < d_1 < d_0$, using the fact that $\beta\lambda_j^{2d} = O(m/n)^{2d_1}$ uniformly in $(d, \beta)' \in \Theta$. We can then deduce that $\widehat{d} - d_0 = o_p((m/n)^{d_0(1+\Delta)})$. With this faster rate of convergence, we have better control over some quantities and can obtain an even faster rate of convergence for

$\widehat{d}$. Repeating this procedure leads to $\widehat{d} - d_0 = O_p((m/n)^{2d_0})$, as desired. With this result, we observe that $(m/n)^{-4d_0}(Q(d,\beta) - Q(d_0,\beta_0))$ is no longer flat as a function of $\beta$ for any value of $d$ such that $|d - d_0| \le C(m/n)^{2d_0}$. This observation can be readily seen from the proof of the theorem. This approach to overcoming the problem of apparent flatness in the objective function is likely to be applicable in other nonlinear estimation contexts when the involved variables are integrated of different orders or have different stochastic orders.

We also prove the rate of convergence of $\widehat{d}$ without using the consistency of $\widehat{\beta}$. This is unusual because in most nonlinear estimation problems it is common to prove the consistency of all parameters first in order to establish rates of convergence. The approach is successful in the present case because when $d$ is close to $d_0$, the regressor $\lambda_j^{2d}$ evaporates as $n \to \infty$ and approaches zero approximately at the rate of $(m/n)^{2d_0}$.

### 4.3 Asymptotic Distribution of the NLP Estimator

The asymptotic distribution of the NLP estimator depends on whether $\beta_0$ is on the boundary of the parameter set. In this section, we first establish the asymptotic properties of the gradient and Hessian functions. These asymptotic results hold for any value of $\beta_0 \in B$. Using these results, we then investigate the asymptotic distribution of the NLP estimator for the cases $0 < \beta_0 < \overline{b}$ and $\beta_0 = 0$, respectively. We impose a somewhat stronger assumption:

**Assumption 4:** $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} \to 0$ *for some arbitrary small* $\Delta > 0$ *and*

$$m^{r+1/2}/n^r \to K_\sigma\{\sigma_u > 0\} + K_0\{\sigma_u = 0\} \tag{33}$$

*as* $m, n \to \infty$, *where* $r = 4d_0\{\sigma_u > 0\} + 2\{\sigma_u = 0\}$.

The two conditions in Assumption 4 are always compatible because $r \ge 4d_0$ and $\Delta$ is arbitrarily small. The lower bound on the growth rate of $m$ ensures the consistency of $\widehat{d}$ and $\widehat{\beta}$. The upper bound on the growth rate of $m$ guarantees that the normalized gradient of $Q(d,\beta)$ is $O_p(1)$, which is required for deriving the asymptotic distribution of $(\widehat{d},\widehat{\beta})$.

When $\sigma_u = 0$, the upper bound becomes $m = O(n^{4/5})$, which is the same as the upper bound for asymptotic normality of the LP estimator for a pure fractional process. When $\sigma_u > 0$, the upper bound becomes $m^{8d_0+1}/n^{8d_0} = O(1)$, which is less stringent than the upper bound given in Theorem 1. It therefore allows us to take $m$ larger than in conventional LP regression applied to the fractional components model. In consequence, by an appropriate choice of $m$, we have asymptotic normality for $\widehat{d}$ with a faster rate of convergence than is possible in LP regression. However, for any $0 < d_0 < 1/2$, the upper bound is more stringent than $m = O(n^{4/5})$, the upper bound for asymptotic normality of LP regression in a pure fractional process model. Hence, the existence of the weakly dependent perturbations in (1) requires the use of a narrower bandwidth than LP regression for a pure fractional process. Interestingly, as $d_0$ approaches $1/2$, the upper bound becomes arbitrarily close to $m = O(n^{4/5})$.

We now proceed to establish the asymptotic distribution of the NLP estimator. The consistency result and Assumption 3 ensure that we only need to consider the constraint $\beta \ge 0$. Therefore, the first order conditions for (16) are:

$$S_n(d,\beta) = (0,\Lambda)' \tag{34}$$

$$\Lambda\beta = 0, \tag{35}$$

where $\Lambda$ is the Lagrangian multiplier for the constraint $\beta \geq 0$,

$$S_n(d,\beta) = -\sum_{j=1}^{m} \begin{pmatrix} x_{1j}(d,\beta) - \bar{x}_1(d,\beta) \\ x_{2j}(d,\beta) - \bar{x}_2(d,\beta) \end{pmatrix} e_j(d,\beta), \tag{36}$$

$$\begin{aligned} x_{1j}(d,\beta) &= -2\log\lambda_j(1-\beta\lambda_j^{2d}), \quad \bar{x}_1(d,\beta) = \frac{1}{m}\sum_{k=1}^{m} x_{1k}, \\ x_{2j}(d,\beta) &= \lambda_j^{2d}, \quad \bar{x}_2(d,\beta) = \frac{1}{m}\sum_{k=1}^{m} x_{2k}, \text{ and} \end{aligned} \tag{37}$$

$$e_j(d,\beta) = \log I_{zj} - \frac{1}{m}\sum_{k=1}^{m}\log I_{zk} + 2d(\log\lambda_j - \frac{1}{m}\sum_{k=1}^{m}\log\lambda_k) - \beta\left(x_{2j}(d,\beta) - \bar{x}_2(d,\beta)\right). \tag{38}$$

Expanding $S_n(\widehat{d},\widehat{\beta})$ about $S_n(d_0,\beta_0)$, we have

$$(0,\widehat{\Lambda})' = S_n(d_0,\beta_0) + H_n(d^*,\beta^*)(\widehat{d}-d_0,\widehat{\beta}-\beta_0)', \tag{39}$$

where $H_n(d,\beta)$ is the Hessian matrix, $(d^*,\beta^*)$ is between $(d_0,\beta_0)$ and $(\widehat{d},\widehat{\beta})$. The elements of the Hessian matrix are:

$$\begin{aligned} H_{n,11}(d,\beta) &= \sum_{j=1}^{m}(x_{1j}-\bar{x}_1)^2 - \beta\sum_{j=1}^{m} e_j\left(\log\lambda_j^2\right)^2\lambda_j^{2d}, \\ H_{n,12}(d,\beta) &= \sum_{j=1}^{m}(x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2) - \sum_{j=1}^{m} e_j\left(\log\lambda_j^2\right)\lambda_j^{2d}, \\ H_{n,22}(d,\beta) &= \sum_{j=1}^{m}(x_{2j}-\bar{x}_2)^2. \end{aligned} \tag{40}$$

Define the diagonal matrix $D_n = diag(\sqrt{m}, \lambda_m^{2d_0}\sqrt{m})$. We show in the following lemma that the normalized Hessian $D_n^{-1}H_n(d_0,\beta_0)D_n^{-1}$ converges in probability to a $2\times 2$ matrix defined by

$$\Omega = \begin{pmatrix} 4 & -4d_0/(2d_0+1)^2 \\ -4d_0/(2d_0+1)^2 & 4d_0^2/\left((4d_0+1)(2d_0+1)^2\right) \end{pmatrix}, \tag{41}$$

and the 'asymptotic bias' of the normalized score $D_n^{-1}S_n(d_0,\beta_0)$ is $-b$, where

$$b = \{\sigma_u > 0\}b_\sigma + \{\sigma_u = 0\}b_0, \tag{42}$$

and

$$\begin{aligned} b_\sigma &= \frac{(2\pi)^{4d_0} f_w^2(0) K_\sigma}{2f_u^2(0)}\left(\frac{8d_0}{(4d_0+1)^2}, \ -\frac{8d_0^2}{(2d_0+1)(4d_0+1)(6d_0+1)}\right)', \\ b_0 &= (2\pi)^2 K_0\left(\frac{f_w''(0)}{f_w(0)} + \frac{d_0}{6}\right)\left(-\frac{2}{9}, \ \frac{2d_0}{3\,(2d_0+3)\,(2d_0+1)}\right)'. \end{aligned} \tag{43}$$

Before stating the lemma, we need the following notation. Let $J_n(d,\beta)$ be a $2\times 2$ matrix whose $(i,j)$-th element is

$$J_{n,ij} = \sum_{k=1}^{m} \left(x_{ik}(d,\beta) - \bar{x}_i(d,\beta)\right)\left(x_{jk}(d,\beta) - \bar{x}_j(d,\beta)\right), \tag{44}$$

and let $\Theta_n$ be a set defined by

$$\Theta_n = \{(d,\beta)' : |\lambda_m^{-d_0}(d-d_0)| < \varepsilon \text{ and } |\beta - \beta_0| < \varepsilon\}. \tag{45}$$

**Lemma 5** *Let Assumptions 1-4 hold. Then*

**(a)** $\sup_{(d,\beta)'\in\Theta_n} ||D_n^{-1}(H_n(d,\beta) - J_n(d,\beta))D_n^{-1}|| = o_p(1)$,

**(b)** $\sup_{(d,\beta)'\in\Theta_n} ||D_n^{-1}[J_n(d,\beta) - J_n(d_0,\beta_0)]D_n^{-1}|| = o_p(1)$,

**(c)** $D_n^{-1}J_n(d_0,\beta_0)D_n^{-1} \to \Omega$,

**(d)** $D_n^{-1}S_n(d_0,\beta_0) \Rightarrow N(-b, \frac{\pi^2}{6}\Omega)$.

We now consider the asymptotic distribution when $\sigma_u > 0$. In this case, the true parameter $(d_0,\beta_0)'$ is an interior point of the parameter space. Hence $\widehat{\Lambda} = 0$ and $(\widehat{d},\widehat{\beta})$ is asymptotically normal.

**Theorem 3** *Let Assumptions 1-4 hold. If $\sigma_u > 0$, then*

$$D_n \begin{pmatrix} \widehat{d} - d_0 \\ \widehat{\beta} - \beta_0 \end{pmatrix} \Rightarrow N(b_{NLP}, \frac{\pi^2}{6}\Omega^{-1}) \tag{46}$$

*where $b_{NLP} = \Omega^{-1}b_\sigma$ and*

$$\Omega^{-1} = \begin{pmatrix} (2d_0+1)^2 / \left(16d_0^2\right) & (2d_0+1)^2(4d_0+1) / \left(16d_0^3\right) \\ (2d_0+1)^2(4d_0+1) / \left(16d_0^3\right) & (4d_0+1)(2d_0+1)^4 / \left(16d_0^4\right) \end{pmatrix}. \tag{47}$$

**Remark 1** From the above theorem, we deduce immediately that when $\sigma_u > 0$, the asymptotic variance of $\sqrt{m}(\widehat{d} - d_0)$ is $\pi^2 C_d/24$, where $C_d = 1 + (4d_0+1)/(4d_0^2) > 1$. Approximating $\log f^*(\cdot)$ locally by a nonlinear function instead of a constant therefore inflates the usual asymptotic variance of the LP regression estimator in a pure fractional model by the factor $C_d$. This is to be expected, as adding more variables in regression usually inflates variances.

**Remark 2** When $\sigma_u > 0$, the limiting distribution (46) involves the bias $b_{NLP}$. The dominating bias term of $(\widehat{d},\widehat{\beta})'$ is thus equal to

$$D_n^{-1}\Omega^{-1}b_n = -\frac{(2\pi)^{4d_0} f_w^2(0)}{f_u^2(0)} (\frac{m}{n})^{4d_0} \begin{pmatrix} d_0(2d_0+1) / \left((4d_0+1)^2(6d_0+1)\right) \\ 2(2d_0+1)^2 / ((4d_0+1)(6d_0+1)) \end{pmatrix}. \tag{48}$$

**Remark 3** When $\sigma_u > 0$, according to (48) the asymptotic bias of $\widehat{d}$ is of order $m^{4d_0}/n^{4d_0}$. In contrast, the asymptotic bias of the LP estimator is of order $m^{2d_0}/n^{2d_0}$, as shown above in (31). The asymptotic bias of the NLP estimator is therefore smaller than that of the LP estimator by order $m^{2d_0}/n^{2d_0}$.

**Remark 4** Following the previous remarks, the asymptotic mean-squared error (AMSE) of $\widehat{d}$ has the form $AMSE(\widehat{d}) = K^2(m/n)^{8d_0} + \pi^2 C_d/(24m)$, where

$$K = (2\pi)^{4d_0}\beta_0^2 \frac{d_0\,(2d_0+1)}{(4d_0+1)^2\,(6d_0+1)}. \tag{49}$$

Straightforward calculations yield the value of $m$ that minimizes $AMSE(\widehat{d})$, viz.

$$m^{opt} = [(\frac{\pi^2 C_d}{192 d_0 K^2})^{1/(8d_0+1)} n^{8d_0/(8d_0+1)}], \tag{50}$$

where $[\cdot]$ denotes the integer part. When $m = m^{opt}$, the AMSE of $\widehat{d}$ converges to zero at the rate of $n^{-8d_0/(8d_0+1)}$. In contrast, when $m = m_{LP}^{opt}$, the AMSE of $\widehat{d}_{LP}$ converges to zero only at the rate of $n^{-4d_0/(4d_0+1)}$. Thus, the optimal AMSE of $\widehat{d}$ converges faster to zero than that of $\widehat{d}_{LP}$.

**Remark 5** When $d_0$ is close to zero, the asymptotic bias of $\widehat{d}$ is of order $m^{4d_0}/n^{4d_0}$, which is close to the order of the asymptotic bias of $\widehat{d}_{LP}$. In addition, when $d_0$ is close to zero, the asymptotic variance of the $\widehat{d}$ will be large. Therefore, as $d_0$ approaches zero, the advantage of $\widehat{d}$ over $\widehat{d}_{LP}$ diminishes. This is expected as when $d_0$ is close to zero, the downward bias of $\widehat{d}_{LP}$ will be small and there is not much scope for $\widehat{d}$ to manifest its bias-reducing capacity.

**Remark 6** $\widehat{\beta}$ converges more slowly by a rate of $(m/n)^{-2d_0}$ than $\widehat{d}$. Heuristically, the excitation levels of the two regressors ($\log\lambda_j$ and $\lambda_j^{2d_0}$) and thus their information content are different. More specifically, we have $\sum_{j=1}^m(\log\lambda_j - \sum_{k=1}^m \log\lambda_k/m)^2 = O(m)$ whereas $\sum_{j=1}^m(\lambda_j^{2d_0} - \sum_{k=1}^m \lambda_k^{2d_0}/m)^2 = O(m\lambda_m^{2d_0})$.

Next, we consider the asymptotic distribution when $\sigma_u = 0$. In this case, the parameter $\beta_0$ lies on the boundary of the parameter space. As a consequence, $\Lambda$ may not equal zero and we have a different limiting distribution.

**Theorem 4** *Let Assumptions 1-4 hold. If $\sigma_u = 0$, then*

$$\begin{aligned}\sqrt{m}(\widehat{d} - d_0) \;&\Rightarrow\; -\left(\widetilde{\Omega}_{11}\eta_1 + \widetilde{\Omega}_{12}\eta_2\right)\left\{\widetilde{\Omega}_{12}\eta_1 + \widetilde{\Omega}_{22}\eta_2 \le 0\right\} \\ &\qquad -\Omega_{11}^{-1}\eta_1\left\{\widetilde{\Omega}_{12}\eta_1 + \widetilde{\Omega}_{22}\eta_2 > 0\right\}, \qquad (51)\\ \sqrt{m}\lambda_m^{2d_0}\left(\widehat{\beta} - \beta_0\right) \;&\Rightarrow\; -\left(\widetilde{\Omega}_{12}\eta_1 + \widetilde{\Omega}_{22}\eta_2\right)\left\{\widetilde{\Omega}_{12}\eta_1 + \widetilde{\Omega}_{22}\eta_2 \le 0\right\}, \qquad (52)\end{aligned}$$

*where $\widetilde{\Omega} = (\widetilde{\Omega}_{ij}) = \Omega^{-1}$, and $\eta = (\eta_1, \eta_2)' \sim N(-b_0, \pi^2\Omega/6)$.*

**Remark 7** Theorem 4 shows that when the true parameter $\beta_0 = 0$, i.e. in the case of a pure fractional process, the limiting distribution of $\widehat{\beta}$ is truncated normal. The truncation arises because the true parameter is on the boundary of the parameter set. Since the Hessian matrix is not diagonal, the truncation also affects the limiting distribution of $\widehat{d}$, which becomes a mixture distribution with the mixing probabilities depending on the component distributions.

**Remark 8** It follows easily from Theorem 4 that the dominating bias term of $\widehat{d}$ is of order $O_p(m^2/n^2)$, the same order as that of $\widehat{d}_{LP}$. Since its limiting distribution is a complicated function of normal random variables, it is not easy to derive an exact expression for the dominating bias term. Hence, it is quite difficult to compare the dominating bias term of $\widehat{d}$ with that of $\widehat{d}_{LP}$ in this case.

The following corollary follows from Theorem 4 by using a narrower frequency band. The proof is straightforward and is omitted.

**Corollary 2** *Let Assumptions 1-3 hold . If* $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} + m^5/n^4 \to 0$ *for some arbitrary small* $\Delta > 0$, *then*

$$\sqrt{m}\lambda_m^{2d_0}\left(\widehat{\beta} - \beta_0\right) \Rightarrow v(d_0)\tau\{\tau \geq 0\} \tag{53}$$

*where* $\tau \equiv N(0,1)$ *and* $v(d_0) = \pi d_0^{-2}\left(2d_0+1\right)^2\sqrt{\left(4d_0+1\right)/96}$.

## 4.4 A Test for Perturbations

The properties of the LP and NLP estimators depend on whether short memory perturbations are present in the data. The limit theory can be used to construct a test for the presence of perturbations, which can be formulated in terms of the hypotheses

$$H_0 : \beta_0 = 0 \qquad vs. \qquad H_1 : \beta_0 > 0,$$

with no perturbations under $H_0$, and with short memory perturbations present under $H_1$ whose intensity increases with $\beta_0$.

Using Corollary 2, we can construct the $t$-statistic

$$t_{\widehat{\beta}} = \sqrt{m}\lambda_m^{2\widehat{d}}\widehat{\beta}/v(\widehat{d}). \tag{54}$$

If Assumptions 1-3 hold and $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} + m^5/n^4 \to 0$, then under the null hypothesis,

$$t_{\widehat{\beta}} \Rightarrow \tau\{\tau \geq 0\}. \tag{55}$$

If Assumptions 1-3 hold and $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} + m^{4d_0+1/2}/n^{4d_0} \to 0$, then under the alternative hypothesis that $\beta_0 = \beta_A > 0$, we have

$$\begin{aligned} t_{\widehat{\beta}} &= \sqrt{m}\lambda_m^{2\widehat{d}}\left(\widehat{\beta} - \beta_A\right)/v(\widehat{d}) + \sqrt{m}\lambda_m^{2\widehat{d}}\beta_A/v(\widehat{d}) \\ &= O_p(1) + \sqrt{m}\lambda_m^{2\widehat{d}}\beta_A/v(\widehat{d}) \end{aligned} \tag{56}$$

by Theorem 3. Since $\sqrt{m}\lambda_m^{2\widehat{d}} \to \infty$ when $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} \to 0$, we deduce that $t_{\widehat{\beta}} \to \infty$ in probability and the test is therefore consistent. Note that $m^5/n^4 \to 0$ implies that $m^{4d_0+1/2}/n^{4d_0} \to 0$. We collect the results in the following corollary:

**Corollary 3** *If Assumptions 1-3 hold and* $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} + m^5/n^4 \to 0$, *then*

$$t_{\widehat{\beta}} \Rightarrow \tau\{\tau \geq 0\} \text{ under } H_0 \text{ and } t_{\widehat{\beta}} \to \infty \text{ in probability under } H_1.$$

Simulations, not reported here, indicate that for sample sizes less than 2048 the power of the test is quite low.

# 5 Simulations

## 5.1 Experimental Design

This section investigates the finite sample performance of the NLP estimator in comparison with conventional LP regression. The chosen data generating process is

$$z_t = (1-L)^{-d_0} w_t + u_t, \tag{57}$$

where $\{w_t : t = 1, 2, ..., n\}$ are iid $N(0,1)$, $\{u_t : t = 1, 2, ..., n\}$ are iid $N(0, \sigma_u^2)$ and $\{w_t\}$ are independent of $\{u_t\}$.

We consider the following constellation of parameter combinations

$$d_0 = 0.25, 0.45, 0.65, 0.85, \text{ and} \tag{58}$$

$$\sigma_u^2 = 0, 4, 8, 16. \tag{59}$$

In view of the fact that the LP estimator is consistent for both stationary fractional processes ($d_0 < 0.5$) and nonstationary fractional processes ($0.5 \leq d_0 < 1$) (see Kim and Phillips, 2000), we expect the NLP estimator to work well for nonstationary fractional component processes for this range of values of $d_0$ as well as for stationary fractional component processes over ($0 < d_0 < 0.5$). Hence it is of interest to include some values of $d_0$ that fall in the nonstationary zone.

The value of $\sigma_u^2$ determines the strength of the noise from the perturbations. The long run SNR increases as $\sigma_u^2$ decreases. When $\sigma_u^2 = 0$, $z_t$ is a pure fractional process with an infinite long-run SNR. The inverse of the long run SNR, viz. $f_u(0)/f_w(0)$, takes the values $0, 4, 8, 16$. These are close to the values in Deo and Hurvich (2001). In their simulation study, the ratio $f_u(0)/f_w(0)$ takes the values 6.17 and 13.37.

We consider sample sizes $n = 128$, 512, and 2048. Because $n$ has the composite form $2^k$ ($k$ integer) for these choices, zero-padding is not a concern when we use the fast Fourier transform to compute the periodogram. For each sample size and parameter combination, 2000 replications are performed from which we calculate the biases, standard deviations and root mean square errors of $\widehat{d}$ and $\widehat{d}_{LP}$, for different selections of the bandwidth $m$. Then, for each parameter combination, we graph each of these quantities as functions of $m$. The results are shown in panels (a)-(c) of Figs. 1–6.

In addition, we compute the coverage probabilities, as functions of $m$, of the nominal 90% confidence intervals (CI) that are obtained using the asymptotic normality results of Theorems 1 and 3. When constructing these confidence intervals, we estimate the standard errors of $\widehat{d}$ and $\widehat{d}_{LP}$ using finite sample expressions rather than the limit expressions, because the former yield better finite sample performance for all parameter combinations and for both estimators. Specifically, the standard error of $\widehat{d}$ is estimated by $SE_{HJ} = SE_J + (SE_H - SE_J)\{H(\widehat{d}, \widehat{\beta}) > 0\}$ where $\{H(\widehat{d}, \widehat{\beta}) > 0\} = 1$ if $H(\widehat{d}, \widehat{\beta})$ is positive definite, and

$$SE_H = \pi/\sqrt{6}H_{22,n}^{1/2}(\widehat{d}, \widehat{\beta})\left(H_{11}(\widehat{d}, \widehat{\beta})H_{22}(\widehat{d}, \widehat{\beta}) - H_{12}^2(\widehat{d}, \widehat{\beta})\right)^{-1/2}, \quad (60)$$

$$SE_J = \pi/\sqrt{6}J_{22,n}^{1/2}(\widehat{d}, \widehat{\beta})\left(J_{11}(\widehat{d}, \widehat{\beta})J_{22}(\widehat{d}, \widehat{\beta}) - J_{12}^2(\widehat{d}, \widehat{\beta})\right)^{-1/2}. \quad (61)$$

The standard errors of $\widehat{d}_{LP}$ is estimated by

$$\pi/\sqrt{24}\left(\sum_{j=1}^{m}\left(\log\lambda_j - \frac{1}{m}\sum_{k=1}^{m}\log\lambda_k\right)^2\right)^{-1/2}. \quad (62)$$

We calculate the average lengths of the confidence intervals as functions of $m$. For some data generating processes, the coverage probabilities and the average lengths are graphed against $m$ in panels (d) and (e) of Figs. 1–2.

The idea of using the finite sample expression instead of the asymptotic expression has been used in many papers (e.g. Andrews and Guggenberger, 1999, Andrews and Sun 2000). The approximation (62) was originally suggested by Geweke and Porter-Hudak (1983) and was used in Deo and Hurvich (2001). To verify that $SE_{HJ}$ is more accurate, we simulate the mean and standard deviation of $SE_{HJ}$ for different parameter combinations, and compare them with those of the asymptotic expression, i.e. $SE_A = \pi\left(2d_0 + 1\right)/(\sqrt{96}d_0 m)$. To save space, we only report the simulation results in Table 1 for the case $n = 512$, $\sigma_u^2 = 8$, $d_0 = 0.45$ and $m = [n^{1/2}]$, $[n^{2/3}]$ and $[n^{3/4}]$. The last row of the table presents the simulation standard deviations of $\widehat{d}$. It is seen from Table 1 that $SE_{HJ}$ provides a much better approximation than both $SE_J$ and $SE_A$. We find that $SE_J$ is quite unreliable and $SE_A$ is even worse. This is because when $\widehat{d}$ goes to zero, both $SE_J$ and $SE_A$ approach infinity.

Table 1. Accuracy Comparison of Several Expressions for Standard Errors

| | | $m = n^{1/2}$ | $m = n^{2/3}$ | $m = n^{3/4}$ |
|---|---|---|---|---|
| $SE_{HJ}$ | Mean | 0.385 | 0.191 | 0.145 |
| | Standard Error | 0.255 | 0.081 | 0.061 |
| $SE_J$ | Mean | 0.898 | 0.527 | 0.246 |
| | Standard Error | 7.461 | 6.186 | 1.082 |
| $SE_A$ | Mean | 4835 | 2103 | 716 |
| | Standard Error | 37172 | 22415 | 11996 |
| $SE_S$ | | 0.247 | 0.183 | 0.181 |

## 5.2 Results

We report results for the cases $d_0 = 0.45$ and $d_0 = 0.85$ in details, since these are representative of the results found in the other two cases, $d_0 = 0.25$ and $0.65$, respectively. Also, for each value of $d_0$, we discuss only the cases $\sigma_u^2 = 0$ and $\sigma_u^2 = 8$, as the results for the other values of $\sigma_u^2$ were qualitatively similar. We will concentrate on the case $n = 512$.

We first discuss the results when $d_0 = 0.45$ and $\sigma_u^2 = 0$. In this case, $z_t$ is a pure fractional process. Fig. 1(a) shows that the bias of $\widehat{d}$ is positive and larger than that of $\widehat{d}_{LP}$. The positiveness of the bias of $\widehat{d}$ is not surprising. Intuitively, the two regressors $\lambda_j^{2d}$ and $\log \lambda_j$ in the nonlinear log periodogram regression move together. When $\sigma_u = 0$, we have $\beta_0 = 0$. But $\widehat{\beta}$ is constrained to be positive, we thus expect $\widehat{d}$ to be biased upward. Fig. 1(b) shows that the variance of $\widehat{d}$ is larger than that of $\widehat{d}_{LP}$. Comparing RMSE's in Fig. 1(c), we see that the RMSE of $\widehat{d}$ is larger than that of $\widehat{d}_{LP}$. The inferior performance of $\widehat{d}$ in this case is not surprising since the LP estimator is designed for pure fractional processes, whereas our estimator $\widehat{d}$ allows for additional noise in the system and is designed for perturbed fractional processes. However, it is encouraging that the LP estimator outperforms the NLP estimator only by a small margin. Apparently, the cost of including the additional regressor, even when it is not needed, is small.

Next, we discuss the results when $d_0 = 0.45$ and $\sigma_u^2 = 8$. Fig. 2(a) shows that the LP estimator $\widehat{d}_{LP}$ has a large downward bias in this case, whereas the NLP estimator $\widehat{d}$ has a much smaller bias. Apparently, the bias-reducing feature of $\widehat{d}$ established in the asymptotic theory is manifest in finite samples. Fig. 2(b) shows that the standard error of $\widehat{d}_{LP}$ is less than that of $\widehat{d}$ for all values of $m$, again consistent with the asymptotic results. For each estimator, the standard error declines at the approximate rate $1/\sqrt{m}$ as $m$ increases, because $m$ is the effective sample size in the estimation of $d_0$. Fig. 2(c) shows that the RMSE of $\widehat{d}$ is smaller than that of $\widehat{d}_{LP}$ over a wide range of $m$ values. Fig. 2(d) shows that the coverage probability of $\widehat{d}$ is fairly close to the nominal value of 0.9, provided that $m$ is not taken too large. In contrast, $\widehat{d}_{LP}$ has a true coverage probability close to 0.9 only for very small values of $m$. This is due to the large bias of $\widehat{d}_{LP}$. However, the larger standard error of $\widehat{d}$ leads to longer confidence intervals on average, and this is apparent in Fig. 2(e).

The qualitative comparisons and conclusions made for the case $d_0 = 0.45$ remain valid for the case $d_0 = 0.25$. For brevity, we do not present the figures but we comment on these figures briefly. When $d_0 = 0.25$ and $\sigma_u^2 = 0$, the bias and standard deviation of $\widehat{d}_{LP}$ remain more or less the same as in Fig. 1 (a) and (b). Comparing with Fig. 1, the bias curve of $\widehat{d}$ remains the same, but the standard deviation curve moves up, meaning that variance inflation is more serious. When $d_0 = 0.25$ and $\sigma_u^2 = 8$, the bias reduction of $\widehat{d}$ is slightly less effective and the variance inflation is slightly larger than was shown in Fig. 2. Nevertheless, the RMSE of $\widehat{d}$ is still smaller than that of $\widehat{d}_{LP}$ for a wide range of the $m$ values.

We now turn to the results when $d_0 = 0.85$ and $\sigma_u^2 = 0$. To save space, we only present the bias, standard deviation and RMSE graphs analogous to graphs (a), (b) and (c) in Fig. 1. Fig. 3 shows that both $\widehat{d}_{LP}$ and $\widehat{d}$ work reasonably well for nonstationary fractional processes ($1/2 \leq d_0 < 1$). Compared with Fig. 1, we find that the difference in the standard errors of these two estimators becomes smaller while the difference in the biases remains more or less the same. Although $\widehat{d}_{LP}$ is still a better estimator than $\widehat{d}$ in this case, the advantage of $\widehat{d}_{LP}$ has clearly diminished with the increase in $d_0$.

Fig. 4 provides results for the case $d_0 = 0.85$ and $\sigma_u^2 = 8$. Fig. 4(a) shows that the bias reduction from using $\widehat{d}$ is substantial. For example, when $m = 40$, the bias of $\widehat{d}_{LP}$ is $-0.18$, while that of $\widehat{d}$ is only $-0.02$. The evidence seems to suggest that $\widehat{d}$ is effective in reducing bias for stationary fractional component models as well as for nonstationary models. Fig. 4(b) shows that the standard error of $\widehat{d}$ is only slightly larger than that of $\widehat{d}_{LP}$. The large bias reduction and small variance inflation lead to a smaller RMSE for $\widehat{d}$ over a wide range of $m$ values, as shown in Fig. 4(c). Other simulations results (not reported in Figure 4) show that the coverage probability based on $\widehat{d}_{LP}$ decreases very rapidly as $m$ increases, whereas that based on $\widehat{d}$ decreases much more slowly. In fact, the coverage probability based on $\widehat{d}$ is close to 0.9 over a wide range of $m$ values.

The simulation results for $d_0 = 0.65$ are qualitatively similar to those found for the case $d_0 = 0.85$. We omit the detailed discussion. Comparing the simulation results for different values of $d_0$, we find that $\widehat{d}$ is more effective in bias reduction for larger values of $d_0$. Intuitively, when $d_0$ is small, the bias of $\widehat{d}_{LP}$ is small no matter what value $\sigma_u$ may take. For a large value of $\sigma_u$, the perturbation component dominates the fractional component, so that $\widehat{d}_{LP}$ would be around 0. In this case, the bias of $\widehat{d}_{LP}$ is small only because the true value of $d_0$ itself is small. Also, for small values of $\sigma_u$, the bias from contamination is naturally going to be small. Therefore, in both cases, the bias of $\widehat{d}_{LP}$ will be small when $d_0$ is small and there is not much scope for $\widehat{d}$ to manifest its bias-reducing capacity.

We present a representative figure for the cases $n = 2048$ and $n = 512$. The qualitative comparisons made and conclusions reached for the $n = 512$ sample size continue to apply to $n = 2048$ and $n = 128$. The simulation results show that $\widehat{d}$ is more effective in bias reduction when the sample size is smaller. This is because a smaller sample size implies a larger finite sample bias of $\widehat{d}_{LP}$ and there is some scope for $\widehat{d}$ to manifest its bias-reducing capacity.

To sum up, the simulations show that, for fractional component processes, the NLP estimator $\widehat{d}$ has a lower bias, a higher standard deviation, and a lower RMSE in comparison to the LP estimator $\widehat{d}_{LP}$, corroborating the asymptotic theory. The lower bias generally leads to improved coverage probability in confidence intervals based on $\widehat{d}$ over a wide range of $m$. On the other hand, the lower standard deviation of $\widehat{d}_{LP}$ leads to shorter confidence intervals than those based on $\widehat{d}$.

## 6 Conclusion

In empirical applications it has become customary practice to investigate the order of integration of the variables in a model when nonstationarity is suspected. This practice is now being extended to include analyses of the degree of persistence using fractional models and estimates of long memory parameters. Nonetheless, for many time series, and particularly macroeconomic variables for which there is limited data, the actual degree of persistence in the data continues to be a controversial issue. The empirical resolution of this problem inevitably relies on our capacity to separate low-frequency behavior from high-frequency fluctuations and this is particularly difficult when short run fluctuations have high variance. Actual empirical results often depend critically on the discriminatory power of the statistical techniques being employed to implement the separation.

The model used in the present paper provides some assistance in this regard. It allows for an explicit components structure in which there are different sources and types of variation, thereby accommodating a separation of short and long memory components and allowing for fractional processes that are perturbed by weakly dependent effects. Compared to the conventional formulation of a pure fractional process like (2), perturbed fractional processes allow for multiple sources of high-frequency variation and, in doing so, seem to provide a richer setting for uncovering latent persistence in an observed time series. In particular, the model provides a mechanism for simultaneously capturing the effects of persistent and temporary shocks and seems realistic in economic and financial applications when there are many different sources of variation in the data. The new econometric methods we have introduced for estimating the fractional parameter in such models take account of the presence of additive disturbances, and help to achieve bias reduction and attain a faster rate of convergence. The asymptotic theory is easy to use and seems to work reasonably well in finite samples.

The methods of the paper can be extended in a number of directions. First, the nonlinear approximation approach can be used in combination with other estimators, such as the local Whittle estimator (Robinson 1995b), which seems natural in the present context because the procedure already uses optimization methods. Second, the idea of using a nonlinear approximation can be applied to nonstationary fractional component models and used to adapt the methods which have been suggested elsewhere (e.g., Phillips, 1999, Shimotsu and Phillips, 2001) for estimating the memory parameter in such models to cases where there are fractional components.

# 7 Appendix of Proofs

**Proof of Lemma 1**. A spectral density satisfying Assumptions 1 and 2 also satisfies Assumptions 1 and 2 of Robinson (1995a). In consequence, the lemma follows from Theorem 2 of Robinson (1995a). Since we normalize the discrete Fourier transform by the spectral density $f_z^{1/2}(\lambda)$ instead of the power function $C_g^{-1/2}\lambda^{-d}$, (4.2) of Robinson (1995a) is always zero and the extra term $(\frac{j}{n})^{\min(\alpha,\beta)}$ in Robinson (1995a) does not arise in our case. □

**Proof of Lemma 3**. Note that

$$m^{-1}\sum_{j=1}^{m} c_j U_j = m^{-1}\sum_{j=1}^{[\log^2 m]} c_j U_j + m^{-1}\sum_{j=[\log^2 m]+1}^{m} c_j U_j \equiv F_1 + F_2. \tag{A.1}$$

But $E\sup_{(d,\beta)'\in\Theta}|F_1|$ is less than

$$Em^{-1}\sum_{j=1}^{[\log^2 m]}\sup_{(d,\beta)'\in\Theta}|c_j||U_j| \le m^{-1}\log^p m\sum_{j=1}^{[\log^2 m]}(EU_j^2)^{1/2} = O(\log^{p+2} m/m) \tag{A.2}$$

by Lemma 2(b). Hence

$$\sup_{(d,\beta)'\in\Theta}|F_1| = O_p(\log^{p+2} m/m) = O_p(\log^p m/\sqrt{m}). \tag{A.3}$$

Let $s_r = \sum_{k=[\log^2 m]+1}^{r} U_r, r = [\log^2 m]+1, ..., m$ and $s_{[\log^2 m]} = 0$. Then, from Lemma 2(a), (c) and (d), it follows that

$$\begin{aligned} Es_r^2 &= \sum_{k=[\log^2 m]+1}^{r} EU_k^2 + 2 \sum_{[\log^2 m]+1 \le k < j \le r} EU_j U_k \\ &= \sum_{k=[\log^2 m]+1}^{r} (\frac{\pi^2}{6} + k^{-1} \log k) + 2 \sum_{[\log^2 m+1] \le k < j < r} O(k^{-2} \log^2 j) \qquad \text{(A.4)} \\ &= O(r) + O(r \log^2 r / \log^2 m), \end{aligned}$$

which implies $s_r = O_p(r^{1/2})$. Using this result and partial summation, we have:

$$\begin{aligned} \sup_{(d,\beta)' \in \Theta} |F_2| &\le \sup_{(d,\beta)' \in \Theta} \left| m^{-1} \sum_{j=[\log^2 m]+1}^{m} c_j U_j \right| \\ &= \sup_{(d,\beta)' \in \Theta} m^{-1} \left| \sum_{j=[\log^2 m]+1}^{m} s_{j-1}(c_{j-1} - c_j) \right| + \sup_{(d,\beta)' \in \Theta} m^{-1} |s_m c_m| \\ &= m^{-1} \sum_{j=[\log^2 m]+1}^{m} O_p(j^{1/2}) O(j^{-1} \log^q m) + O_p(\log^p m / \sqrt{m}) \\ &= \log^q m / m \sum_{j=[\log^2 m]+1}^{m} O_p(j^{-1/2}) + O_p(\log^p m / \sqrt{m}) \\ &= O_p(\log^q m / \sqrt{m}) + O_p(\log^p m / \sqrt{m}) \\ &= O_p\left( \left( \log^{\max(p,q)} m \right) / \sqrt{m} \right). \qquad \text{(A.5)} \end{aligned}$$

Combine (A.3) with (A.5) to complete the proof. □

**Proof of Corollary 1.** Following the same steps as in the proof of Lemma 3, we compute the orders of $|V_j(d,\beta) - \bar{V}(d,\beta)|$, $|V_m(d,\beta) - \bar{V}(d,\beta)|$ and $|V_j(d,\beta) - V_{j-1}(d,\beta)|$ as follows. First,

$$\begin{aligned} &\sup_{(d,\beta)' \in D^0 \times B} |V_j(d,\beta) - \bar{V}(d,\beta)| \\ \le\ & 2 \sup_{(d,\beta)' \in D^0 \times B} |d - d_0| \, | \log \lambda_j - \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j | + 2 \sup_{(d,\beta)' \in D^0 \times B} |\beta| |\lambda_j^{2d} - \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d}| \\ =\ & 2 \sup_{(d,\beta)' \in D^0 \times B} |d - d_0| \log m + O(\lambda_m^{2d_0})\{\gamma > 0\} + O(1)\{\gamma = 0\} \\ =\ & O((m/n)^\gamma \log m) \text{ uniformly over } j, \qquad \text{(A.6)} \end{aligned}$$

where we have used the fact that when $d \in D^0$ and $\gamma > 0$,

$$\begin{aligned} \lambda_m^{2d} &= \lambda_m^{2d_0} \lambda_m^{2d-2d_0} = \lambda_m^{2d_0} \exp((2d - 2d_0) \log \lambda_m) \\ &\le C \lambda_m^{2d_0} \exp(\lambda_m^\gamma |\log \lambda_m|) = O(\lambda_m^{2d_0}). \qquad \text{(A.7)} \end{aligned}$$

Second,

$$
\begin{aligned}
& \sup_{(d,\beta)' \in D^0 \times B} |V_j(d,\beta) - V_{j-1}(d,\beta)| \\
\leq \;& 2 \sup_{(d,\beta)' \in D^0 \times B} |d - d_0| \, |\log(1 - \frac{1}{j})| + 2 \sup_{(d,\beta)' \in D^0 \times B} |\beta \lambda_j^{2d} (1 - (1 - \frac{1}{j})^{2d})| \\
= \;& O((m/n)^{\gamma} \frac{1}{j}) + O_p\left( (m/n)^{2d_0} \frac{1}{j} \right) \{\gamma > 0\} + O(\frac{1}{j}) \{\gamma = 0\} \\
= \;& O((m/n)^{\gamma} \frac{1}{j}) \text{ for all } j, \qquad \text{(A.8)}
\end{aligned}
$$

where the final line follows from the fact that $\sup_{(d,\beta)' \in \Theta} |1 - (1 - \frac{1}{j})^{2d}| = O(\frac{1}{j})$. Finally,

$$
\begin{aligned}
& |V_m(d,\beta) - \bar{V}(d,\beta)| \\
= \;& 2 \sup_{(d,\beta)' \in D^0 \times B} |d - d_0| |\log \lambda_m - \frac{1}{m} \sum_{j=1}^{m} \log \lambda_m| + 2 \sup_{(d,\beta)' \in D^0 \times B} |\beta| |\lambda_m^{2d} - \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d}| \\
= \;& 2 \sup_{(d,\beta)' \in D^0 \times B} |d - d_0| |\log m - \frac{1}{m} \sum_{j=1}^{m} \log j| + O(\lambda_m^{2d_0}) \{\gamma > 0\} + O(1) \{\gamma = 0\}) \\
= \;& O((m/n)^{\gamma}). \qquad \text{(A.9)}
\end{aligned}
$$

The Corollary is now proved by invoking the same argument as in the proof of Lemma 3. □

**Proof of Theorem 1.** When $\sigma_u = 0$, the theorem is essentially the same as results already established in HDB. Only one modification is needed. HDB use $-2\log(2 \sin \lambda_j / 2)$ as one of the regressors while we employ $-2 \log \lambda_j$. The use of $-2 \log \lambda_j$ rather than $-2\log(2 \sin \lambda_j / 2)$ has no effect on the asymptotic variance, but it does affect the asymptotic bias. This is because the asymptotic bias comes from the dominating term in $\varepsilon_j$ and this term is different for different regressors. Using $-2\log(2 \sin \lambda_j / 2)$ as the regressor yields

$$
\varepsilon_j = \log f_w(\lambda_j) - \log f_w(0) = \left( \frac{f_w''(0)}{2 f_w'(0)} \right) \lambda_j^2 (1 + o(1)). \qquad \text{(A.10)}
$$

In contrast, using $-2 \log \lambda_j$ as the regressor yields

$$
\begin{aligned}
\varepsilon_j \;&=\; \log f_w(\lambda_j) - \log f_w(0) - 2 d_0 \left( \log(2 \sin \frac{\lambda_j}{2}) - \log \lambda_j \right) \\
&=\; \left( \frac{f_w''(0)}{2 f_w'(0)} + \frac{d_0}{12} \right) \lambda_j^2 (1 + o(1)). \qquad \text{(A.11)}
\end{aligned}
$$

With this adjustment, the arguments in HDB go through without further change.

Now consider the case $\sigma_u > 0$. Rewrite the spectral density of $z_t$ as $f_z(\lambda) = \lambda^{-2d_0} g(\lambda)$, where $g(\lambda) = (\lambda^{-1} 2 \sin \lambda / 2)^{-2d_0} f^*(\lambda)$. Since

$$
g(\lambda) - g(0) = (1 + O\left(\lambda^2\right)) \left( f_w(0) + \lambda^{2d_0} f_u(0) + O(\lambda^2) \right) - f_w(0) = O(\lambda^{2d_0}) \qquad \text{(A.12)}
$$

as $\lambda \to 0+$, $g(\lambda)$ is smooth of order $2d_0$. Combining this with our assumption that $m \to \infty$ and $m^{4d_0+1}/n^{4d_0} = O(1)$ verifies Assumptions 1 and 2 of Andrews and Guggenberger (1999). Hence their Theorem 1 is valid with $r = 0$, $s = 2d_0$ and $q = 2d_0$. It is easy to show that the term $O(m^q/n^q)$ in their theorem is actually $-f_u(0)/f_w(0)d_0(2d_0+1)^{-2}\lambda_m^{2d_0}$. Andrews and Guggenberger established asymptotic normality under their Assumption 3 that $m^{4d_0+1}/n^{4d_0} = o(1)$. In fact, asymptotic normality holds under our assumption $m^{4d_0+1}/n^{4d_0} = O(1)$ as long as an asymptotic bias of order $O(1)$ is allowed. □

**Proof of Theorem 2.** Decompose $Q(d,\beta) - Q(d_0,\beta_0)$ into two parts as follows:

$$Q(d,\beta) - Q(d_0,\beta_0) = \frac{1}{m}\sum_{j=1}^{m}(V_j - \bar{V})^2 + \frac{2}{m}\sum_{j=1}^{m}(U_j + \varepsilon_j)(V_j - \bar{V}) \tag{A.13}$$

where the dependence on $(d,\beta)$ has been suppressed for notational simplicity.

**Part (a)** We prove part (a) by showing that $1/m\sum_{j=1}^{m}(U_j+\varepsilon_j)(V_j-\bar{V}) = o_p(1)$ uniformly in $(d,\beta)'$ and $1/m\sum_{j=1}^{m}(V_j-\bar{V})^2$ converges uniformly to a function, which has a unique minimizer $d_0$.

First, using Corollary 1 with $\gamma = 0$, we have

$$\sup_{(d,\beta)'\in\Theta}\left|1/m\sum_{j=1}^{m}U_j(V_j-\bar{V})\right| = O_p(1/\sqrt{m}). \tag{A.14}$$

Next, we show that $\sup_{(d,\beta)'\in\Theta}|\frac{1}{m}\sum_{j=1}^{m}\varepsilon_j(V_j(d,\beta)-\bar{V}(d,\beta))| = O_p(\lambda_m^{4d_0})$. Under Assumptions 1 and 2,

$$\varepsilon_j = O(\lambda_j^r) = O(\lambda_j^{4d_0}), \tag{A.15}$$

so we have, using (A.8) and (A.9) with $\gamma = 0$,

$$\begin{aligned}
&\sup_{(d,\beta)'\in\Theta}|\frac{1}{m}\sum_{j=1}^{m}\varepsilon_j(V_j(d,\beta)-\bar{V}(d,\beta))|\\
\leq\;& \sup_{(d,\beta)'\in\Theta}\left(\frac{1}{m}\left|\sum_{j=1}^{m}\sum_{r=1}^{j-1}\varepsilon_r(V_{j-1}(d,\beta)-V_j(d,\beta))\right| + \frac{1}{m}|\sum_{j=1}^{m}\varepsilon_j||V_m(d,\beta)-\bar{V}(d,\beta)|\right)\\
=\;& \lambda_m^{4d_0}\frac{1}{m}\left|\sum_{j=1}^{m}\sum_{r=1}^{j-1}O_p(\frac{r}{m})^{4d_0}(\frac{1}{j})\right| + O_p(\lambda_m^{4d_0}) = O_p(\lambda_m^{4d_0}) = o_p(1). 
\end{aligned}\tag{A.16}$$

Finally,

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}(V_j-\bar{V})^2 &= \frac{1}{m}\sum_{j=1}^{m}\left(2(d-d_0)(\log(\frac{j}{m}) - \frac{1}{m}\sum_{k=1}^{m}\log(\frac{k}{m})) + o(1)\right)^2\\
&= 4(d-d_0)^2\left(\frac{1}{m}\sum_{j=1}^{m}\log^2(\frac{j}{m}) - (\frac{1}{m}\sum_{k=1}^{m}\log(\frac{k}{m}))^2\right) + o(1)\\
&= 4(d-d_0)^2(1+o(1)),
\end{aligned}\tag{A.17}$$

where $o(\cdot)$ holds uniformly over $(d, \beta)' \in \Theta$.

In view of (A.14), (A.16) and (A.17), we can complete the proof by using a standard textbook argument.

**Part (b)** Compared with $\log \lambda_j$, $\lambda_j^{2d_0}$ is negligible since $d_0 > 0$. Due to the difference in the orders of magnitude of the regressors, it is not straightforward to establish the consistency of $\widehat{\beta}$. In fact, we proceed by showing first that $\widehat{d}$ converges to $d_0$ at some preliminary rate and then go on to show that $\widehat{d} - d_0 = O_p((m/n)^{2d_0})$. We obtain this rate sequentially.

First, we show that $\widehat{d} - d_0 = o_p((m/n)^{d_1/2})$, where $d_1$ is the lower bound of the interval $D$. From $Q(\widehat{d}, \widehat{\beta}) - Q(d_0, \beta_0) \leq 0$, we get

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}(V_j(\widehat{d},\widehat{\beta}) - \bar{V}(\widehat{d},\widehat{\beta}))^2 && \text{(A.18)}\\
\leq \quad & -\frac{2}{m}\sum_{j=1}^{m}(U_j + \varepsilon_j)(V_j(\widehat{d},\widehat{\beta}) - \bar{V}(\widehat{d},\widehat{\beta})) \\
= \quad & O_p(\frac{1}{\sqrt{m}}) + O_p(\lambda_m^{4d_0}) = o_p\left((\frac{m}{n})^{2d_1}\right), && \text{(A.19)}
\end{aligned}$$

where the last equality follows from the assumptions that $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} = o(1)$ and that $d \geq d_1 > 0$. But $\frac{1}{m}\sum_{j=1}^{m}(V_j(\widehat{d},\widehat{\beta}) - \bar{V}(\widehat{d},\widehat{\beta}))^2$ equals

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}\left(2(\widehat{d} - d_0)(\log(\frac{j}{m}) - \frac{1}{m}\sum_{j=1}^{m}\log(\frac{j}{m})) + O(\lambda_m^{2\widehat{d}}) + O(\lambda_m^{2d_0})\right)^2 \\
= \quad & 4(\widehat{d} - d_0)^2(1 + o(1)) + O(\lambda_m^{2d_0}) + O(\lambda_m^{2\widehat{d}}) \\
= \quad & 4(\widehat{d} - d_0)^2(1 + o(1)) + O\left((\frac{m}{n})^{2d_1}\right). && \text{(A.20)}
\end{aligned}$$

Therefore,

$$4(\widehat{d} - d_0)^2(1 + o(1)) + O_p\left((\frac{m}{n})^{2d_1}\right) \leq o_p\left((\frac{m}{n})^{2d_1}\right), \quad \text{(A.21)}$$

which implies that $\widehat{d} - d_0$ is at most $O_p((\frac{m}{n})^{d_1})$. Thus $\widehat{d} - d_0 = o_p((\frac{m}{n})^{d_1/2})$.

Second, we show that $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{d_0(1+\Delta)}\right)$. Since $\widehat{d} - d_0 = o_p((\frac{m}{n})^{d_1/2})$, we only need consider $d \in D_n' = \{d : |d - d_0| < \varepsilon(\frac{m}{n})^{d_1/2}\}$ for some small $\varepsilon > 0$. Approximating sums by integrals, we deduce that, for $d \in D_n'$,

$$\frac{1}{m}\sum_{j=1}^{m}V_j^2(d,\beta) - \left(\bar{V}(d,\beta)\right)^2 = \mathcal{I}_1 + \mathcal{I}_2 \quad \text{(A.22)}$$

where

$$\mathcal{I}_1 = \left(4(d - d_0)^2 + \left(\frac{2d\beta\lambda_m^{2d}}{(2d+1)\sqrt{4d+1}} - \frac{2d_0\beta_0\lambda_m^{2d_0}}{(2d_0+1)\sqrt{4d_0+1}}\right)^2\right)(1 + o(1)), \quad \text{(A.23)}$$

and

$$\mathcal{I}_2 = \frac{8dd_0\beta\beta_0\lambda_m^{2d+2d_0}}{(2d+1)(2d_0+1)}\left(\frac{1}{\sqrt{(4d+1)(4d_0+1}} - \frac{1}{2d+2d_0+1}\right)(1+o(1)). \qquad \text{(A.24)}$$

Therefore

$$\frac{1}{m}\sum_{j=1}^{m} V_j^2(d,\beta) - \left(\bar{V}(d,\beta)\right)^2 = 4(d-d_0)^2 + O(\lambda_m^{4d_0}), \qquad \text{(A.25)}$$

where the $o(\cdot)$ and $O(\cdot)$ term in the above three equations hold uniformly over $(d,\beta)' \in D_n' \times B$. Using $Q(\widehat{d},\widehat{\beta}) - Q(d_0,\beta_0) \leq 0$ again, we have

$$\frac{1}{m}\sum_{j=1}^{m}(V_j(\widehat{d},\widehat{\beta}) - \bar{V}(\widehat{d},\widehat{\beta}))^2 \leq O_p(\frac{1}{\sqrt{m}}) + O_p(\lambda_m^{4d_0}) = o_p\left((\frac{m}{n})^{2d_0(1+\Delta)}\right), \qquad \text{(A.26)}$$

where the equality follows from the assumption $n^{4d_0(1+\Delta)}/m^{4d_0(1+\Delta)+1} = o(1)$. Combining (A.25) and (A.26), we get

$$4(\widehat{d}-d_0)^2 + o(\lambda_m^{2d_0(1+\Delta)}) \leq o_p\left((\frac{m}{n})^{2d_0(1+\Delta)}\right). \qquad \text{(A.27)}$$

Hence $\widehat{d} - d_0 = o_p((\frac{m}{n})^{d_0(1+\Delta)})$.

Next, we show that $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{3d_0(1+\Delta)/2}\right)$. From Corollary 1 and Equation (A.16), we know that $\left|\frac{1}{m}\sum_{j=1}^{m} U_j(V_j - \bar{V})\right| = O_p\left((\frac{m}{n})^{d_0(1+\Delta)}/\sqrt{m}\right) = O_p\left((\frac{m}{n})^{3d_0(1+\Delta)}\right)$ and $\left|\frac{1}{m}\sum_{j=1}^{m} \varepsilon_j(V_j - \bar{V})\right| = o_p\left((\frac{m}{n})^{3d_0(1+\Delta)}\right)$ uniformly in $(d,\beta)' \in D_n'' \times B$, where $D_n'' = \{d : |d-d_0| < \varepsilon(\frac{m}{n})^{d_0(1+\Delta)}\}$. In addition, it follows from (A.25) that when $d \in D_n''$, $\frac{1}{m}\sum_{j=1}^{m}(V_j - \bar{V})^2 = 4(d-d_0)^2(1+o(1)) + o(\lambda_m^{3d_0(1+\Delta)})$. Applying the same argument as before, we get

$$4(\widehat{d}-d_0)^2(1+o(1)) + o(\lambda_m^{3d_0(1+\Delta)}) \leq o_p\left((\frac{m}{n})^{3d_0(1+\Delta)}\right), \qquad \text{(A.28)}$$

and so $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{3d_0(1+\Delta)/2}\right)$.

Repeating the procedure again we obtain $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{7d_0(1+\Delta)/4}\right)$ if $7(1+\Delta)/4 < 2$. Further iterations of this procedure lead to $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{(2-2^{-k})(1+\Delta)}\right)$, $k = 0,1,2,3,...$ if $(2-2^{-k})(1+\Delta) < 2$. We stop the iteration if we obtain $\widehat{d} - d_0 = o_p\left((\frac{m}{n})^{(2-2^{-k_0})(1+\Delta)}\right)$ for some $k_0 \geq 0$ such that $(2-2^{-k_0})(1+\Delta) < 2$ and $(4-2^{-k_0})(1+\Delta) \geq 4$. In this case, we have

$$\left|\frac{1}{m}\sup_{(d,\beta)' \in D_n^* \times B}\sum_{j=1}^{m} U_j(V_j - \bar{V})\right| = O_p\left((\frac{m}{n})^{(2-2^{-k_0})(1+\Delta)d_0}\frac{1}{\sqrt{m}}\right) = o_p\left((\frac{m}{n})^{4d_0}\right), \qquad \text{(A.29)}$$

and

$$\sup_{(d,\beta)' \in D_n^* \times B}\left|\frac{1}{m}\sum_{j=1}^{m} \varepsilon_j(V_j - \bar{V})\right| = o_p\left((\frac{m}{n})^{4d_0}\right), \qquad \text{(A.30)}$$

where $D_n^* = \{d : |d - d_0| < \varepsilon(\frac{m}{n})^{(2-2^{-k_0})(1+\Delta)}\}$. Applying the same argument as before, we deduce

$$4(\widehat{d} - d_0)^2(1 + o(1)) + O(\lambda_m^{4d_0}) \leq o_p\left((\frac{m}{n})^{4d_0}\right). \quad \text{(A.31)}$$

In consequence, $\widehat{d} - d_0 = O_p\left((\frac{m}{n})^{2d_0}\right)$.

Now, since $(2d + 2d_0 + 1)^2 - (4d + 1)(4d_0 + 1) = 4d^2 - 8dd_0 + 4d_0^2 = 4(d - d_0)^2 > 0$, we deduce from (A.22) that

$$\frac{1}{m}\sum_{j=1}^{m}(V_j - \bar{V})^2 \geq \left(\frac{2d\beta\lambda_m^{2d}}{(2d+1)\sqrt{4d+1}} - \frac{2d_0\beta_0\lambda_m^{2d_0}}{(2d_0+1)\sqrt{4d_0+1}}\right)^2 (1 + o(1)). \quad \text{(A.32)}$$

for $d \in D$ such that $|d - d_0| \leq C(\frac{m}{n})^{2d_0}$. In view of $\frac{1}{m}\sum_{j=1}^{m}(V_j(\widehat{d}, \widehat{\beta}) - \bar{V}(\widehat{d}, \widehat{\beta}))^2 \leq o_p(\lambda_m^{4d_0})$, we obtain

$$\left(\frac{2\widehat{d}\widehat{\beta}\lambda_m^{2\widehat{d}}}{(2\widehat{d}+1)\sqrt{4\widehat{d}+1}} - \frac{2d_0\beta_0\lambda_m^{2d_0}}{(2d_0+1)\sqrt{4d_0+1}}\right)^2 (1 + o(1)) \leq o_p(\lambda_m^{4d_0}). \quad \text{(A.33)}$$

Some algebraic manipulations show that when $\widehat{d} - d_0 = O_p((\frac{m}{n})^{2d_0})$,

$$\frac{2d\widehat{\beta}\lambda_m^{2\widehat{d}}}{(2\widehat{d}+1)\sqrt{4\widehat{d}+1}} - \frac{2d_0\beta_0\lambda_m^{2d_0}}{(2d_0+1)\sqrt{4d_0+1}} = \frac{2\widehat{d}\lambda_m^{2\widehat{d}}}{(2\widehat{d}+1)\sqrt{4\widehat{d}+1}}\left(\widehat{\beta} - \beta_0\right) + o_p(\lambda_m^{3d_0}), \quad \text{(A.34)}$$

So

$$\left(\frac{2\widehat{d}\lambda_m^{2\widehat{d}}}{(2\widehat{d}+1)\sqrt{4\widehat{d}+1}}\left(\widehat{\beta} - \beta_0\right) + o_p(\lambda_m^{3d_0})\right)^2 \leq o_p(\lambda_m^{4d_0}). \quad \text{(A.35)}$$

This implies that

$$\frac{4\widehat{d}\lambda_m^{4(\widehat{d}-d_0)}}{(2\widehat{d}+1)^2\left(4\widehat{d}+1\right)}(\widehat{\beta} - \beta_0)^2 \leq o_p(\lambda_m^{4d_0}), \quad \text{(A.36)}$$

from which we deduce that $\widehat{\beta} - \beta_0 = o_p(1)$. □

**Proof of Lemma 5**

**Part (a)** The (2,2) element of $\sup_{\theta\in\Theta_n} ||D_n^{-1}(H_n(d, \beta) - J_n(d, \beta))D_n^{-1}||$ is zero, so it suffices to consider the (1,1) and (1,2) elements. Since

$$I_{z_j} + 2d\log\lambda_j - \beta\lambda_j^{2d} = \alpha_0 + U_j + \varepsilon_j + (d - d_0)\log\lambda_j^2 + \beta_0\lambda_j^{2d_0} - \beta\lambda_j^{2d},$$

$\sup_{\theta\in\Theta_n} |\frac{\beta}{m}\sum_{j=1}^{m} e_j(\log\lambda_j^2)^2\lambda_j^{2d}|$, the (1,1) element, is bounded by $L_1 + L_2 + L_3 + L_4$, where

$$L_1 = \sup_{\theta\in\Theta_n}\left|\frac{\beta}{m}\sum_{j=1}^{m}\left((\log\lambda_j^2)^2\lambda_j^{2d} - \frac{1}{m}\sum_{k=1}^{m}(\log\lambda_k^2)^2\lambda_k^{2d}\right)U_j\right|,$$

$$L_2 = \sup_{\theta\in\Theta_n}\left|\frac{\beta}{m}\sum_{j=1}^{m}\left((\log\lambda_j^2)^2\lambda_j^{2d} - \frac{1}{m}\sum_{k=1}^{m}(\log\lambda_k^2)^2\lambda_k^{2d}\right)\varepsilon_j\right|,$$

$$
\begin{aligned}
L_3 &= \sup_{\theta\in\Theta_n}\left|\frac{\beta}{m}\sum_{j=1}^{m}\left((\log\lambda_j^2)^2\lambda_j^{2d}-\frac{1}{m}\sum_{k=1}^{m}(\log\lambda_k^2)^2\lambda_k^{2d}\right)(d-d_0)\log\lambda_j^2\right|, \text{ and}\\
L_4 &= \sup_{\theta\in\Theta_n}\left|\frac{\beta}{m}\sum_{j=1}^{m}\left((\log\lambda_j^2)^2\lambda_j^{2d}-\frac{1}{m}\sum_{k=1}^{m}(\log\lambda_k^2)^2\lambda_k^{2d}\right)\left(\beta_0\lambda_j^{2d_0}-\beta\lambda_j^{2d}\right)\right|.
\end{aligned}
\tag{A.37}
$$

We first show that $L_1 = o_p(1)$. Note that $\log^2(\lambda_j^2)\lambda_j^{2d} - \frac{1}{m}\sum_{k=1}^{m}\log^2(\lambda_k^2)\lambda_k^{2d}$ equals

$$
\begin{aligned}
&4\log^2\lambda_m\left(\lambda_j^{2d}-\frac{1}{m}\sum_{k=1}^{m}\lambda_k^{2d}\right)+8\log\lambda_m\left(\log(\frac{j}{m})\lambda_j^{2d}-\frac{1}{m}\sum_{k=1}^{m}\log(\frac{k}{m})\lambda_k^{2d}\right)\\
&+4\log^2(\frac{j}{m})\lambda_j^{2d}-\frac{4}{m}\sum_{k=1}^{m}\log^2(\frac{k}{m})\lambda_k^{2d}.
\end{aligned}
\tag{A.38}
$$

$L_1$ is thus bounded by $\sup_{\theta\in\Theta_n}|4\beta\lambda_m^{2d}|(\log^2\lambda_m L_{11} + 2|\log\lambda_m| L_{12} + L_{13})$, where

$$
L_{1i+1} = \sup_{\theta\in\Theta_n}\left|\frac{1}{m}\sum_{j=1}^{m}\left((\frac{j}{m})^{2d}\log^i(\frac{j}{m})-\frac{1}{m}\sum_{k=1}^{m}(\frac{k}{m})^{2d}\log^i(\frac{k}{m})\right)U_j\right|, i=0,1,2. \tag{A.39}
$$

It follows from Lemma 3 that $L_{1i+1} = O_p(\log^i m/\sqrt{m})$. The first condition is satisfied because

$$
\sup_{\theta\in\Theta_n}\left|(\frac{j}{m})^{2d}\log^i\left(\frac{j}{m}\right)-\frac{1}{m}\sum_{k=1}^{m}(\frac{k}{m})^{2d}\log^i\left(\frac{k}{m}\right)\right| = O(\log^i(m)). \tag{A.40}
$$

The second condition is satisfied because

$$
\begin{aligned}
&\sup_{\theta\in\Theta_n}\left|(\frac{j}{m})^{2d}\log^i\left(\frac{j}{m}\right)-(\frac{j-1}{m})^{2d}\log^i\left(\frac{j-1}{m}\right)\right|\\
\le\ &\sup_{\theta\in\Theta_n}\left|(\frac{j}{m})^{2d}\log^i\left(\frac{j}{m}\right)-(\frac{j-1}{m})^{2d}\log^i\left(\frac{j}{m}\right)\right|\\
&+\left|(\frac{j-1}{m})^{2d}\log^i\left(\frac{j}{m}\right)-(\frac{j-1}{m})^{2d}\log^i\left(\frac{j-1}{m}\right)\right|\\
\le\ &\sup_{\theta\in\Theta_n}\left|\log^i\left(\frac{j}{m}\right)(\frac{j}{m})^{2d}\right|\left|1-(1-\frac{1}{j})^{2d}\right|\\
&+\sup_{\theta\in\Theta_n}\left((\frac{j-1}{m})^{2d}i\left|\log^{i-1}\left(\frac{j-1}{m}\right)\frac{1}{j-1}\right|\right)\\
=\ &O(j^{-1}\log^i m) \text{ for all } j.
\end{aligned}
\tag{A.41}
$$

Therefore

$$
L_1 = O_p(\frac{\log^2\lambda_m}{\sqrt{m}}\lambda_m^{2d_1}+\frac{|\log\lambda_m|\log m}{\sqrt{m}}\lambda_m^{2d_1}+\frac{\log^2 m}{\sqrt{m}}\lambda_m^{2d_1}) = o(1). \tag{A.42}
$$

We then show $L_2 = o_p(1)$. For $i = 0, 1, 2$, define $L_{2i}$ as $L_{1i}$ is defined, but with $U_j$ replaced by $\varepsilon_j$. Since $\sup_{\theta\in\Theta_n} \frac{1}{m}\sum_{j=1}^m \left|(j/m)^{2d}\log^i(j/m) - 1/m\sum_{k=1}^m (k/m)^{2d}\log^i(k/m)\right| = O(1)$, we have

$$L_2 = O_p\left(\lambda_m^{6d_0}(\log^2\lambda_m + 2\log|\lambda_m| + 1)\right) = o_p(1). \tag{A.43}$$

We next show that $L_3 = o_p(1)$. Following a similar procedure, we bound $L_3$ by $\sup_{\theta\in\Theta_n} |8\beta(d-d_0)\lambda_m^{2d}|(\log^2\lambda_m L_{31} + 2|\log\lambda_m|L_{32} + L_{33})$, where

$$L_{3i+1} = \sup_{\theta\in\Theta_n}\left|\left(\frac{1}{m}\sum_{j=1}^m(\frac{j}{m})^{2d}\log^{i+1}(\frac{j}{m})\right) - \left(\frac{1}{m}\sum_{j=1}^m(\frac{j}{m})^{2d}\log^{i}(\frac{j}{m})\right)\left(\frac{1}{m}\sum_{j=1}^m\log(\frac{j}{m})\right)\right|. \tag{A.44}$$

In view of $1/m\sum_{j=1}^m (j/m)^k \log(j/m) = -(k+1)^{-2} + o(1)$, $k \geq 0$, it is easy to show that $L_{3i+1}, i = 0, 1, 2$ are bounded. Hence

$$L_3 = O_p\left(\lambda_m^{2d_1}(\log^2\lambda_m + 2|\log\lambda_m| + 1)\right) = o_p(1). \tag{A.45}$$

Continuing, we show that $L_4 = o_p(1)$. Since $\sup_{\theta\in\Theta_n}\left|\beta_0\lambda_j^{2d_0} - \beta\lambda_j^{2d}\right| = O(1)$, it is easy to see that

$$L_4 = O_p\left(\lambda_m^{2d_1}(\log^2\lambda_m + 2|\log\lambda_m| + 1)\right) = o_p(1). \tag{A.46}$$

Therefore $\sup_{\theta\in\Theta_n} |\beta/m\sum_{j=1}^m e_j(\log\lambda_j^2)^2\lambda_j^{2d}| = o_p(1)$.

Following the same procedure, we can show that

$$\sup_{\theta\in\Theta_n} |\lambda_m^{-2d_0} m^{-1}\sum_{j=1}^m e_j\left(\log\lambda_j^2\right)\lambda_j^{2d}| = o_p(1). \tag{A.47}$$

The details are omitted.

**Part (b)** We consider the individual elements of $\sup_{(d,\beta)'\in\Theta_n} ||D_n^{-1}[J_n(d,\beta) - J_n(d_0,\beta_0)]D_n^{-1}||$ in turn.

Since $x_{1j} = -2\log\lambda_j(1+o(1))$, the (1,1) element can be readily shown to be $o(1)$. Similarly, the (1,2) element can be written as $\sup_{(d,\beta)'\in\Theta_n} 2|L_5 - L_6|(1+o(1))$ where

$$\begin{aligned} L_5 &= -\frac{1}{m}\sum_{k=1}^m\left((\frac{j}{m})^{2d} - \frac{1}{m}\sum_{k=1}^m(\frac{k}{m})^{2d}\right)\left(\log(\frac{j}{m}) - \frac{1}{m}\sum_{k=1}^m\log(\frac{k}{m})\right) \text{ and} \\ L_6 &= -\frac{1}{m}\sum_{j=1}^m\left[\left((\frac{j}{m})^{2d_0} - \frac{1}{m}\sum_{k=1}^m(\frac{k}{m})^{2d_0}\right)\left(\log(\frac{j}{m}) - \frac{1}{m}\sum_{k=1}^m\log(\frac{k}{m})\right)\right]. \end{aligned}$$

Approximating sums by integrals yields

$$L_5 = -\frac{4d}{(2d+1)^2}(1+o(1)), \text{ and } L_6 = -\frac{4d_0}{(2d_0+1)^2}(1+o(1)). \tag{A.48}$$

Therefore, the (1,2) element is

$$\sup_{(d,\beta)'\in\Theta_n} 2|\frac{4d_0}{(2d_0+1)^2} - \frac{4d}{(2d+1)^2}|(1+o(1)) = o(1).$$

Finally, the (2,2) element is

$$\sup_{(d,\beta)'\in\Theta_n}\left|\frac{1}{m}\sum_{j=1}^{m}\left((\frac{j}{m})^{2d}-\frac{1}{m}\sum_{k=1}^{m}(\frac{k}{m})^{2d}\right)^2-\frac{1}{m}\sum_{j=1}^{m}\left((\frac{j}{m})^{2d_0}-\frac{1}{m}\sum_{k=1}^{m}(\frac{k}{m})^{2d_0}\right)^2\right|$$

$$= \sup_{(d,\beta)'\in\Theta_n}\left|\frac{4d^2}{(4d+1)(2d+1)}-\frac{4d_0^2}{(4d_0+1)(2d_0+1)}\right| = o_p(1). \qquad (A.49)$$

**Part (c)** Part (c) holds by using $x_{1j}=-2\log\lambda_j(1+o(1))$ and $x_{2j}=\lambda_j^{2d_0}$ and approximating sums by integrals.

**Part (d)** Let $\xi_j=(\xi_{1j},\xi_{2j})'$, where

$$\xi_{1j}=-2\log\frac{j}{m}+\frac{2}{m}\sum_{k=1}^{m}\log\frac{k}{m},\;\; \xi_{2j}=(\frac{j}{m})^{2d_0}-\frac{1}{m}\sum_{k=1}^{m}(\frac{k}{m})^{2d_0}. \qquad (A.50)$$

Then, we can rewrite $D_n^{-1}S_n(d_0,\beta_0)$ as

$$D_n^{-1}S_n(d_0,\beta_0)=-\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\xi_j(U_j+\varepsilon_j)(1+o(1)). \qquad (A.51)$$

Note that $\sum_{j=1}^m\xi_{1j}\varepsilon_j$ equals

$$\begin{aligned}
&\{\sigma_u>0\}\lambda_m^{4d_0}\textstyle\sum_{j=1}^m\left(-2\log\frac{j}{m}+\frac{2}{m}\sum_{k=1}^m\log\frac{k}{m}\right)\left(-\frac{f_w^2(0)}{2f_u^2(0)}(\frac{j}{m})^{4d_0}\right)(1+o(1))\\
&+\{\sigma_u=0\}\lambda_m^2\textstyle\sum_{j=1}^m\left(-2\log\frac{j}{m}+\frac{2}{m}\sum_{k=1}^m\log\frac{k}{m}\right)\left((\frac{j}{m})^2(\frac{f_w''(0)}{2f_w(0)}+\frac{d_0}{12})\right)(1+o(1))\\
=\;&\{\sigma_u>0\}m\lambda_m^{4d_0}\frac{f_w^2(0)}{2f_u^2(0)}\frac{8d_0}{(4d_0+1)^2}(1+o(1))-\{\sigma_u=0\}m\lambda_m^2\left(\frac{f_w''(0)}{f_w(0)}+\frac{d_0}{6}\right)\frac{2}{9}(1+o(1))
\end{aligned}$$

$$(A.52)$$

and $\sum_{j=1}^m\xi_{2j}\varepsilon_j$ equals

$$\begin{aligned}
=\;&\{\sigma_u>0\}\lambda_m^{4d_0}\textstyle\sum_{j=1}^m\left((\frac{j}{m})^{2d_0}-\frac{1}{m}\sum_{k=1}^m(\frac{k}{m})^{2d_0}\right)\left(-\frac{f_u^2(0)}{2f_w^2(0)}(\frac{j}{m})^{4d_0}\right)(1+o(1))\\
&+\{\sigma_u=0\}\lambda_m^2\textstyle\sum_{j=1}^m\left((\frac{j}{m})^{2d_0}-\frac{1}{m}\sum_{k=1}^m(\frac{k}{m})^{2d_0}\right)\left((\frac{j}{m})^2\left(\frac{f_w''(0)}{2f_w(0)}+\frac{d_0}{12}\right)\right)(1+o(1))\\
=\;&-\{\sigma_u>0\}m\lambda_m^{4d_0}\frac{f_w^2(0)}{2f_u^2(0)}\frac{8d_0^2}{(2d_0+1)(4d_0+1)(6d_0+1)}(1+o(1))\\
&+\{\sigma_u=0\}m\lambda_m^2\left(\frac{f_w''(0)}{f_w(0)}+\frac{d_0}{6}\right)\frac{2d_0}{3(2d_0+3)(2d_0+1)}(1+o(1)).
\end{aligned}$$

$$(A.53)$$

Therefore

$$D_n^{-1}S_n(d_0,\beta_0)+b=\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\xi_jU_j+o(1). \qquad (A.54)$$

We now prove that for any vector $v = (v_1, v_2)'$, $\frac{1}{\sqrt{m}}\sum_{j=1}^m v'\xi_j U_j \Rightarrow N(0, \frac{\pi^2}{6}v'\Omega v)$. Write

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m} v'\xi_j U_j = T_1 + T_2 + T_3, \tag{A.55}$$

where

$$\begin{aligned} T_1 &= \frac{1}{\sqrt{m}}\sum_{j=1}^{[\log^8 m]} a_j U_j, \quad T_2 = \frac{1}{\sqrt{m}}\sum_{j=[\log^8 m]+1}^{[m^{0.5+\delta}]} a_j U_j \\ T_3 &= \frac{1}{\sqrt{m}}\sum_{j=[m^{0.5+\delta}]}^{m} a_j U_j, \quad a_j = v'\xi_j, \end{aligned} \tag{A.56}$$

for some $0 < \delta < 0.5$.

Since $\max_{1\le j\le m}|\xi_{1j}| = O(\log m)$ and $\max_{1\le j\le m}|\xi_{2j}| = O(\log m)$, we have $\max_{1\le j\le m}|a_j| = O(\log m)$. Therefore the proofs in HDB that $T_1 = o_p(1)$ and $T_2 = o_p(1)$ are also valid in the present case. We now show that $T_3 \to N(0, \frac{\pi^2}{6}v'\Omega v)$ by verifying that the sequence $\{a_j\}$ satisfies (25) with $\rho = v'\Omega v$. The first condition of (25) holds as $\max_{1\le j\le m}|a_j| = O(\log m) = o(m)$. The second condition holds because

$$\begin{aligned} \sum_{j=[m^{0.5+\delta}+1]}^{m} a_j^2 &= \sum_{j=1}^{m} a_j^2 - \sum_{j=1}^{[m^{0.5+\delta}]} a_j^2 = \sum_{j=1}^{m} a_j^2 + o(m) \\ &= mv'(\frac{1}{m}\sum_{j=1}^{m}\xi_j'\xi_j)v + o(m) \sim mv'\Omega v. \end{aligned} \tag{A.57}$$

The last equality follows because we can show that $\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^m \xi_j'\xi_j = \Omega$ by approximating the sums by integrals. The third condition holds because

$$\begin{aligned} \sum_{j=[m^{0.5+\delta}]+1}^{m} |a_j|^p &\le 2^p|v_1|\sum_{j=[m^{0.5+\delta}]+1}^{m}|\xi_{1j}|^p + 2^p|v_2|\sum_{j=[m^{0.5+\delta}]+1}^{m}|\xi_{2j}|^p \\ &= O(m) + 2^p|v_1|\sum_{j=[m^{0.5+\delta}]+1}^{m}|\xi_{2j}|^p \\ &= O(\sum_{j=[m^{0.5+\delta}]+1}^{m}|(\frac{2\pi j}{m})^{2d_0}|^p) + O\left\{\sum_{j=[m^{0.5+\delta}]+1}^{m}\left[\frac{1}{m}\sum_{j=1}^{m}(\frac{2\pi j}{m})^{2d_0}\right]^p\right\} + O(m) \\ &= O(m) + O(m) + O(m) = O(m). \end{aligned} \tag{A.58}$$

Here we have employed $\sum_{j=[m^{0.5+\delta}]+1}^m |\xi_{1j}|^p = O(m)$. See $(A18)$ in HDB (1998).

The above results combine to establish part (d). □

**Proof of Theorem 3**

Scaling the first order conditions, we have

$$\begin{aligned}-D_n^{-1}S_n(d_0,\beta_0) &= D_n^{-1}H_n(d_0,\beta_0)D_n^{-1}D_n(\widehat{d}-d_0,\widehat{\beta}-\beta_0)' \\ &\quad +D_n^{-1}[H_n(d^*,\beta^*)-H_n(d_0,\beta_0)]D_n^{-1}D_n(\widehat{d}-d_0,\widehat{\beta}-\beta_0)'. \end{aligned} \tag{A.59}$$

Thus

$$\begin{aligned}&D_n(\widehat{d}-d_0,\widehat{\beta}-\beta_0)' \\ =\ & -\left\{D_n^{-1}H_n(d_0,\beta_0)D_n^{-1}+D_n^{-1}[H_n(d^*,\beta^*)-H_n(d_0,\beta_0)]D_n^{-1}\right\}^{-1}D_n^{-1}S_n(d_0,\beta_0).\end{aligned} \tag{A.60}$$

But since $\widehat{d}-d_0 = O_p((m/n)^{2d_0})$, we know that $(\widehat{d},\widehat{\beta})$ and $(d^*,\beta^*)$ belong to $\Theta_n$ with probability approaching one. Therefore,

$$\begin{aligned}&||D_n^{-1}[H_n(d^*,\beta^*)-H_n(d_0,\beta_0)]D_n^{-1}|| \\ \le\ & \sup_{(d,\beta)'\in\Theta_n}\left(||D_n^{-1}[H_n(d,\beta)-J_n(d,\beta)]D_n^{-1}||+||D_n^{-1}[H_n(d_0,\beta_0)-J_n(d_0,\beta_0)]D_n^{-1}||\right) \\ & + \sup_{(d,\beta)'\in\Theta_n}||D_n^{-1}[J_n(d,\beta)-J_n(d_0,\beta_0)]D_n^{-1}|| \\ =\ & o_p(1),\end{aligned} \tag{A.61}$$

by Lemma 5. Furthermore,

$$\begin{aligned}D_n^{-1}H_n(d_0,\beta_0)D_n^{-1} &= D_n^{-1}[H_n(d_0,\beta_0)-J_n(d_0,\beta_0)]D_n^{-1}+D_n^{-1}J_n(d_0,\beta_0)D_n^{-1} \\ &= \Omega+o(1).\end{aligned} \tag{A.62}$$

Consequently,

$$\begin{aligned}D_n(\widehat{d}-d_0,\widehat{\beta}-\beta_0)'-\Omega^{-1}b_n &= -\Omega^{-1}\left(D_n^{-1}S_n(d_0,\beta_0)+b_n\right)+o_p(1) \\ &\Rightarrow -\Omega^{-1}N(0,\frac{\pi^2}{6}\Omega) =_d N(0,\frac{\pi^2}{6}\Omega^{-1}).\end{aligned} \tag{A.63}$$

□

**Proof of Theorem 4**

Let $\delta_n=(\delta_{n1},\delta_{n2})'=-H_n^{-1}(d^*,\beta^*)S_n(d_0,\beta_0)$. It is easy to show that
(a) when $\delta_{n2}\ge 0$,

$$D_n(\widehat{d}-d_0,\widehat{\beta}-\beta_0)'=-D_nH_n^{-1}(d^*,\beta^*)S_n(d_0,\beta_0)=-\Omega^{-1}D_n^{-1}S_n(d_0,\beta_0)(1+o_p(1));$$

(b) when $\delta_{n2}<0$, $\widehat{\beta}-\beta_0=0$ and

$$\sqrt{m}(\widehat{d}-d_0)=-\sqrt{m}H_{n,11}^{-1}(d^*,\beta^*)S_{n,1}(d_0,\beta_0)=-\Omega_{11}^{-1}m^{-1/2}S_{n,1}(d_0,\beta_0)(1+o_p(1)).$$

Let $\eta_n=(\eta_{n,1},\eta_{n,2})'=D_n^{-1}S_n(d_0,\beta_0)$, then

$$\begin{aligned}\sqrt{m}(\widehat{d}-d_0) &= -\left(\widetilde{\Omega}_{11}\eta_{n1}+\widetilde{\Omega}_{12}\eta_{n2}\right)\left\{\widetilde{\Omega}_{12}\eta_{n1}+\widetilde{\Omega}_{22}\eta_{n2}\le 0\right\}(1+o_p(1)) \\ &\quad -\Omega_{11}^{-1}\eta_{n1}\left\{\widetilde{\Omega}_{12}\eta_{n1}+\widetilde{\Omega}_{22}\eta_{n2}>0\right\}(1+o_p(1))\end{aligned} \tag{A.64}$$

$$\sqrt{m}\lambda_m^{2d_0}\left(\widehat{\beta}-\beta_0\right)=-\left(\widetilde{\Omega}_{12}\eta_{n1}+\widetilde{\Omega}_{22}\eta_{n2}\right)\left\{\widetilde{\Omega}_{12}\eta_{n1}+\widetilde{\Omega}_{22}\eta_{n2}\le 0\right\}(1+o_p(1)). \tag{A.65}$$

The proof is completed by invoking the continuous mapping theorem. □

Figure 1: Performances of the NLP estimator and the LP estimator with $d_0 = 0.45$ and $\sigma_u^2 = 0$ for sample size 512

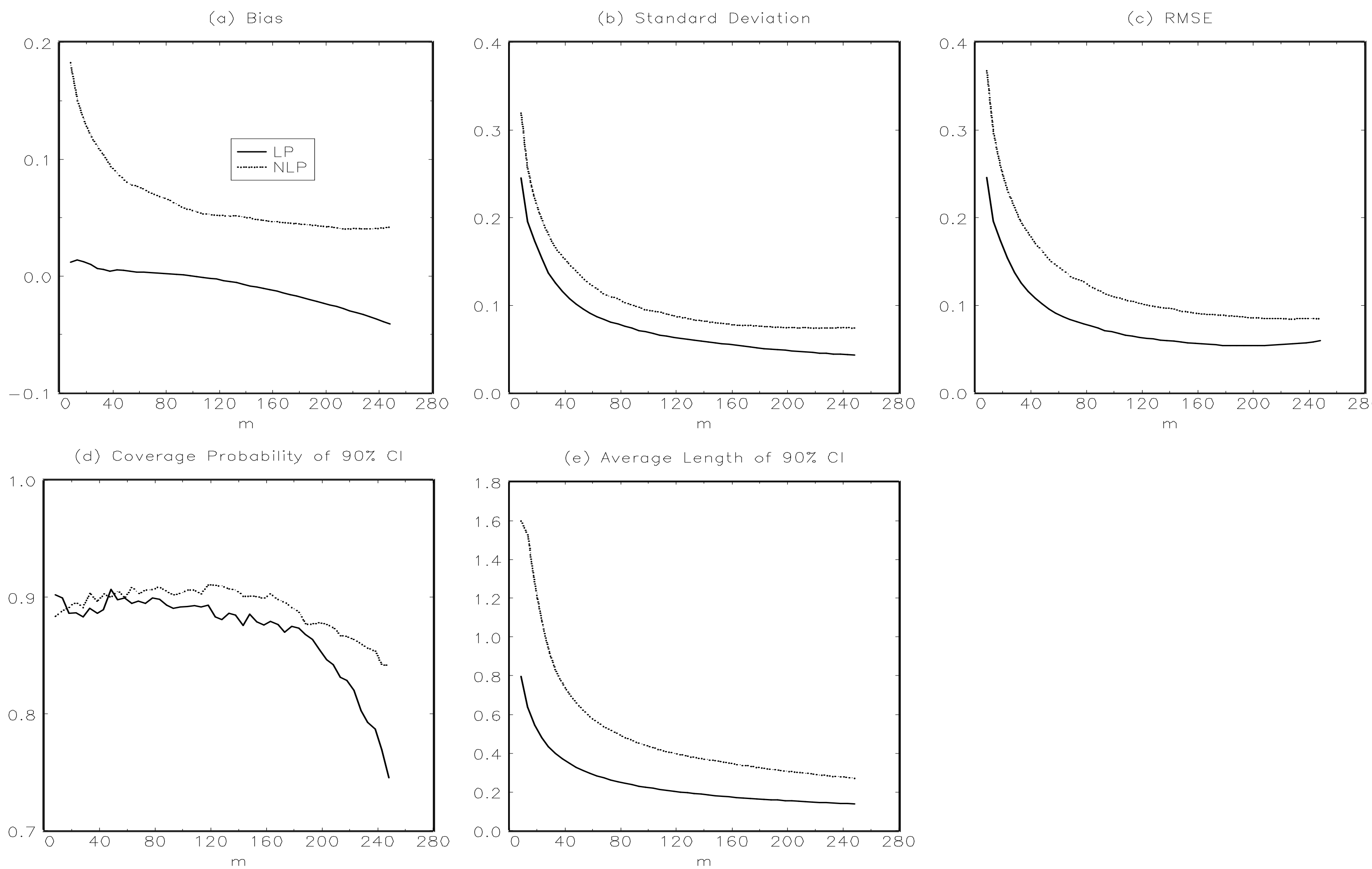


Figure 2: Performances of the NLP estimator and the LP estimator with $d_0 = 0.45$ and $\sigma_u^2 = 8$ for sample size 512

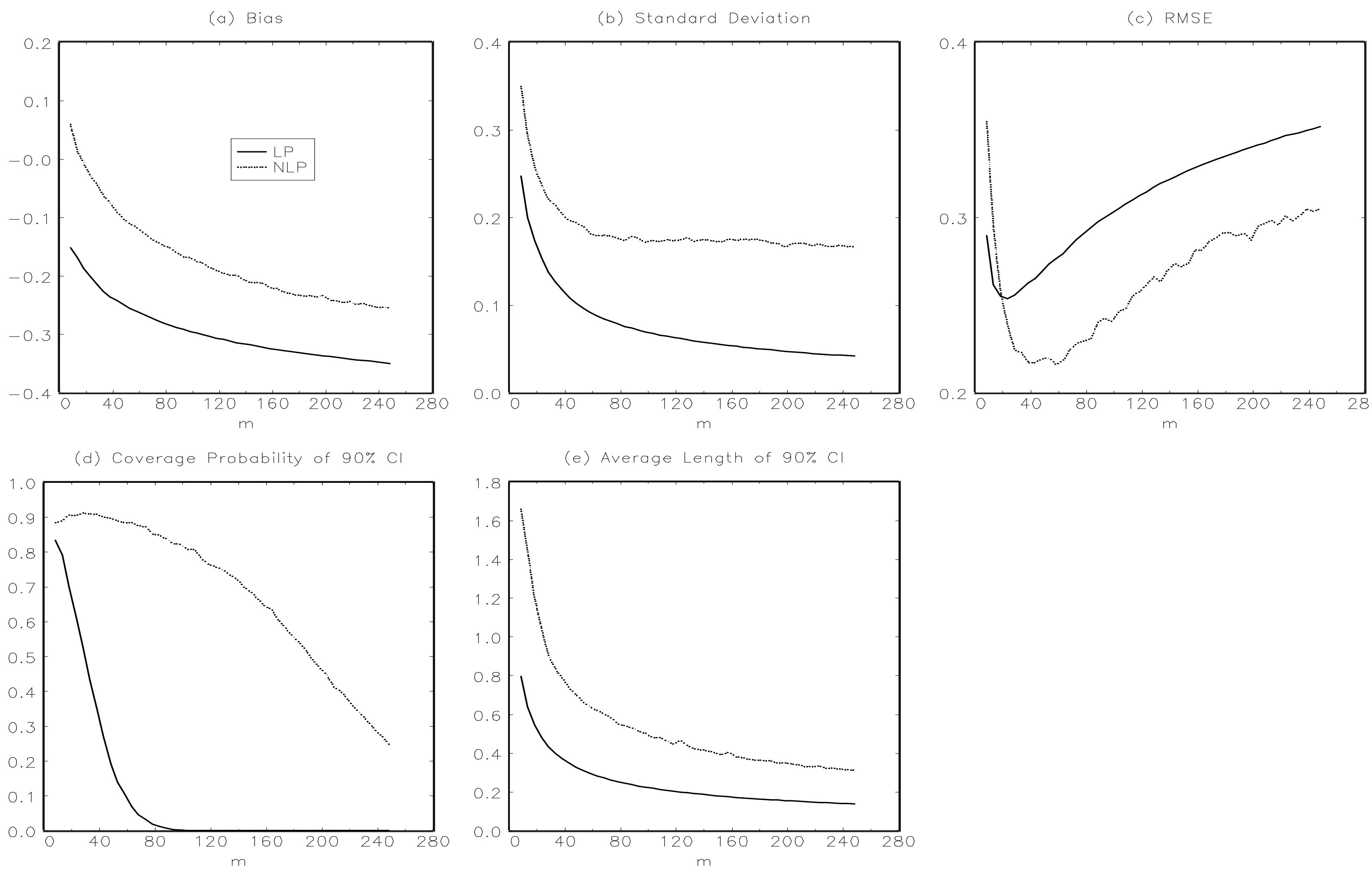


Figure 3: Performances of the NLP estimator and the LP estimator with $d_0 = 0.85$ and $\sigma_u^2 = 0$ for sample size 512

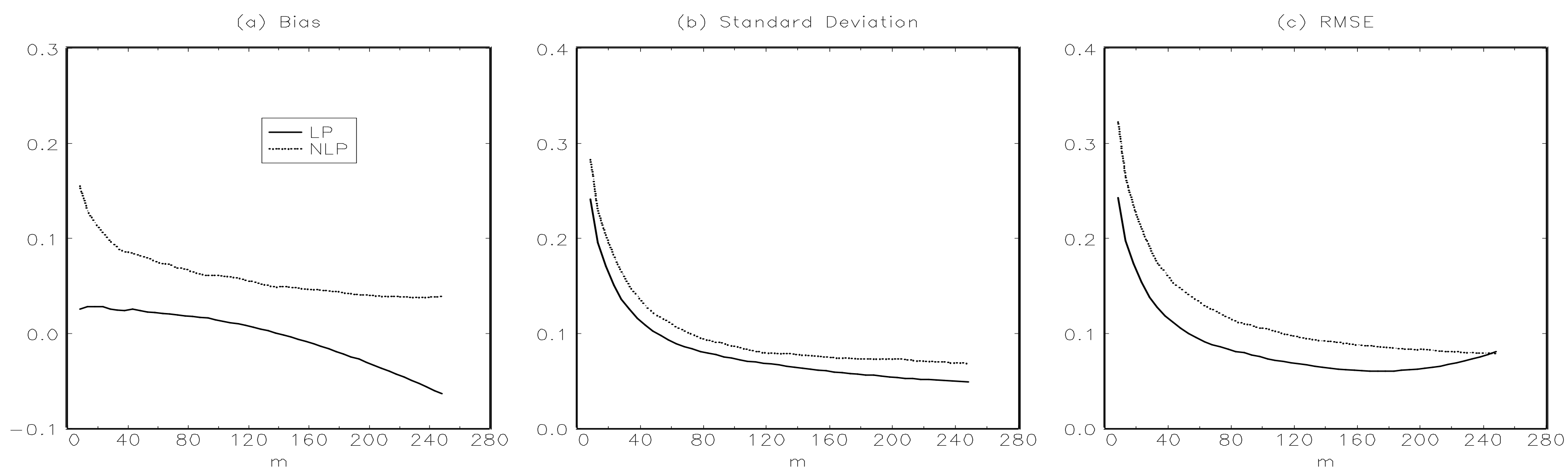



Figure 4: Performances of the new estimator and the GPH estimator with $d_0 = 0.85$ and $\sigma_u^2 = 8$ for sample size 512

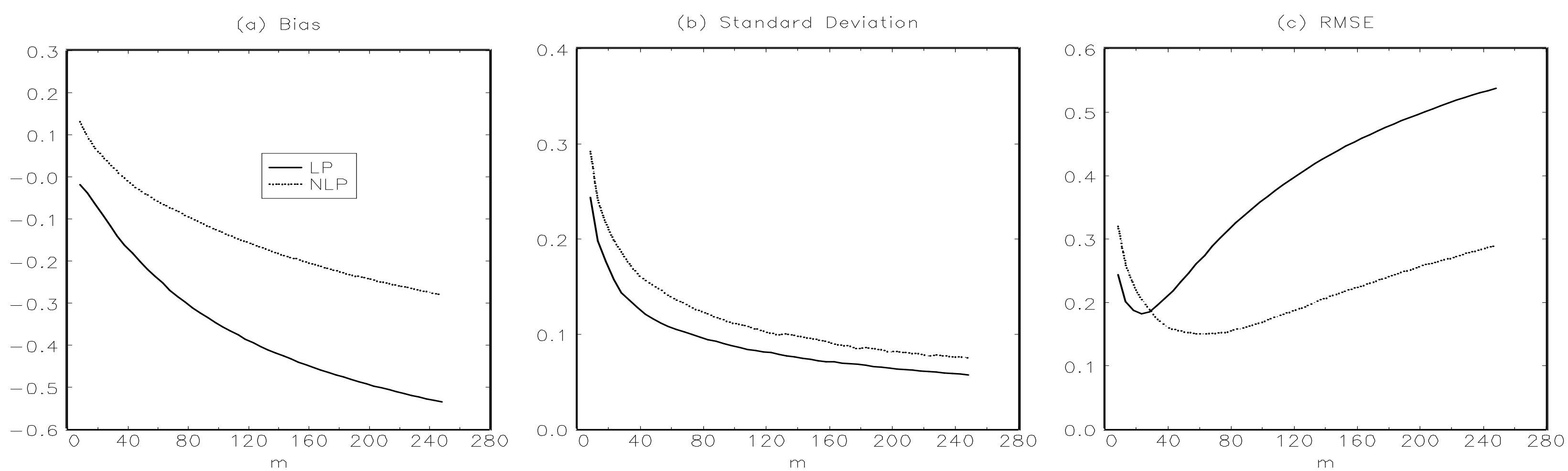


Figure 5: Performances of the NLP estimator and the LP estimator with $d_0 = 0.45$ and $\sigma_u^2 = 8$ for sample size 128

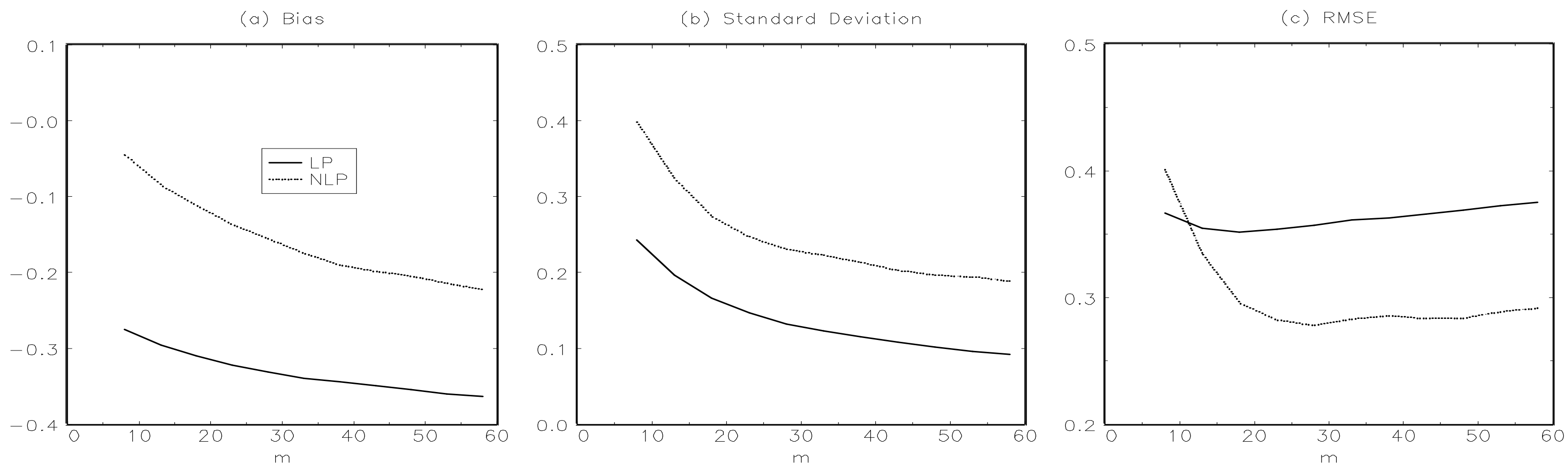
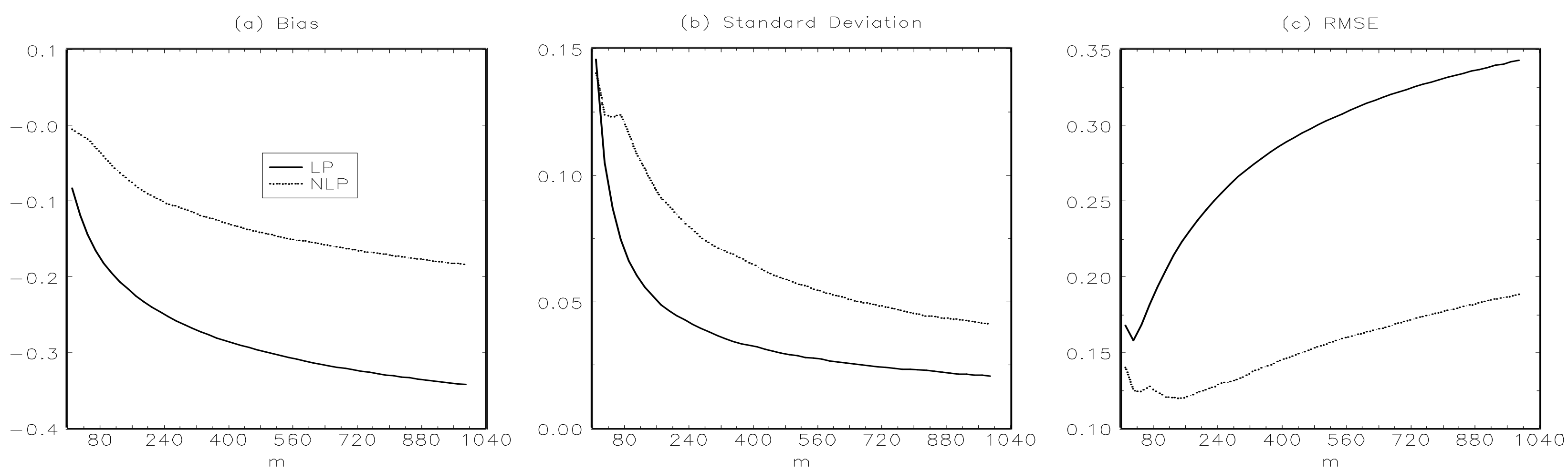



Figure 6: Performances of the NLP estimator and the LP estimator with $d_0 = 0.45$ and $\sigma_u^2 = 8$ for sample size 2048



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