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Default and Punishment in General Equilibrium

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Abstract

We extend the standard model of general equilibrium with incomplete markets to allow for default and punishment. The equilibrating variables include expected delivery rates, along with the usual prices of assets and commodities. By reinterpreting the variables, our model encompasses a broad range of moral hazard, adverse selection, and signalling phenomena (including the Akerlof lemons model and Rothschild–Stiglitz insurance model) in a general equilibrium framework.

We impose a condition on the expected delivery rates for untraded assets that is similar to the trembling hand refinements used in game theory. Despite earlier claims about the nonexistence of equilibrium with adverse selection, we show that equilibrium always exists, even with exclusivity constraints on asset sales, and transactions-liquidity costs or information-evaluation costs for asset trade.

We show that more lenient punishment which encourages default may be Pareto improving because it allows for better risk spreading.

We also show that default opens the door to a theory of endogenous assets.

Keywords: default, incomplete markets, adverse selection, moral hazard, equilibrium refinement, endogenous assets

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1 Introduction

There is a substantial amount of default in the American economy. At first glance this would seem to be a sign of disequilibrium, and to call for economic models that radically depart from the orthodox paradigm of general equilibrium and market clearing.

Indeed, general equilibrium theory has for the most part not made room for default. In the Arrow–Debreu model of general equilibrium with complete contingent markets (GE), and likewise in the general equilibrium model with incomplete markets (GEI), agents keep all their promises by assumption. More specifically, in the GE model, agents never promise to deliver more goods than they personally own. In the GEI model, the definition of equilibrium (that has been developed in a rapidly growing literature) allows agents to promise more of some goods than they themselves have, provided they are sure to get the difference elsewhere. Agents there too must honor their commitments, though no longer exclusively out of their own endowments. Each agent can keep his promises because other agents keep their promises to him.

We build a model that explicitly allows for default, but is broad enough to incorporate conventional general equilibrium theory as a special case. We call the model $GE(R, \lambda, Q)$ because each asset j is defined by its promise R_j, the penalty rate λ_j which determines the punishment for default on the promise, and the quantity restriction Q_j attendant on those who sell it.

Fixing exogenously the set $\mathcal{A}$ of tradeable assets,

$$\mathcal{A} = \{(R_j, \lambda_j, Q_j) : (R_j, \lambda_j, Q_j) \text{ is tradeable}\},$$

we solve for equilibrium $E(\mathcal{A})$. The equilibrating variables include anticipated delivery rates on assets, along with the usual prices of assets and commodities.

One of the central features of our model is that assets are thought of as pools. Different sellers of the same asset will typically default in different events, and in different proportions. The buyers of the asset receive a pro rata share of all the different sellers' deliveries, just as an investor does today in the securitized mortgage market, or in the securitized credit card market. Just as buyers of commodities are assumed in perfect competition to regard prices as fixed, so we assume that buyers of assets regard default rates as fixed. Our general equilibrium model thus stands in contrast to models in which a single lender and a single borrower negotiate with each other. We have avoided a finite-player, game-theoretic treatment of default because, for the massive anonymous financial markets on which we focus attention, perfect competition is a better approximation to reality, and much more analytically tractable.

We have also avoided a (perfectly competitive but) partial equilibrium treatment of our subject because we wanted to evaluate the system-wide consequences of default. In a world in which promises can exceed physical endowments, each default can begin a chain reaction. A creditor in one market where payment does not occur is deprived of the means of delivery in another market where he is the debtor, thereby causing a further default in some other market, etc. The indirect effects of default might be as important as the direct effects, but they are missed in partial equilibrium models. We emphasize that these chain reactions occur exclusively in economies with intermediate levels of financial development, such as the system now in place in the United States. Once the asset markets become complete, the system of interlocking debts will be broken, as in the GE model, and no chain reactions will occur.

Another central feature of our model is that the subset $\mathcal{A}^* \equiv \mathcal{A}^*(E(\mathcal{A})) \subset \mathcal{A}$ of actively traded assets

$$\mathcal{A}^* = \{(R_j, \lambda_j, Q_j) \in \mathcal{A} : (R_j, \lambda_j, Q_j) \text{ is positively traded in } E(\mathcal{A})\}$$

also emerges in equilibrium. The promises, penalties, and sales limitations corresponding to actively traded assets can thus themselves be regarded as endogenous.

A crucial role in the endogenous determination of asset trade is played by the expectations agents have over the deliveries of assets that are not positively traded. (In game theoretic terms, this is analogous to beliefs off the equilibrium path.) We fix these expectations for non-traded assets at reasonable levels by a straightforward equilibrium refinement. The idea is to introduce an external ε-agent who sells ε units of each asset and fully delivers, and to take the limit as $\varepsilon \to 0$. This rules out irrational pessimism on expected deliveries from untraded assets. The simplicity of the refinement is due to our hypothesis of perfect competition consistently applied.[1]

In Sections 2–4 we describe the model and explain who bears the loss from default and how the penalties are administered. We explain that endogenous default necessarily involves adverse selection and moral hazard. Indeed we note that the standard adverse selection and signalling models of Akerlof (1972), Spence (1973), and Rothschild and Stiglitz (1976) are special cases of our model. Section 4 describes and justifies our equilibrium refinement.

Our first goal is to show that if agents have the mental powers to anticipate future rates of default (contingent on future events), just as they are presumed by conventional equilibrium theory to have the mental powers to anticipate future prices (contingent on future events), then default is consistent with the orderly function of markets. In Section 5 we prove the existence of equilibrium with default under exactly the same conditions necessary to prove the existence of equilibrium in the GEI model (where default is ruled out by assumption.) More precisely, we show that our refined equilibrium $E(\mathcal{A})$ exists for every collection $\mathcal{A}$ of assets (R, λ, Q) for which $Q < \infty$, or for which $Q = \infty$ but the promises R are all paid in the same numeraire.

This general existence of equilibrium is somewhat surprising, because default seems linked to disequilibrium, and because we know from the GEI literature that the existence of equilibrium can be compromised when the asset span is endogenous, and because Akerlof and Rothschild and Stiglitz all suggested that equilibrium might not exist. Our general existence proof is also surprising in that it seems to counter the suspicion that asymmetric information creates an obstacle to competitive equilibrium. (See for example, Helpmann and Laffont, 1975). The key to existence is that the asymmetry is one-sided. Each seller has the option to deliver whatever he wants,

[1] To the best of our knowledge, this appears to be the first analogue of the "trembling hand" refinement of game theory in perfectly competitive equilibrium.

while all buyers get the same payoff (per unit purchased). Were the asymmetry two-sided, then indeed equilibrium would be more problematic.[2]

In Section 6 we describe how chain reactions could occur in the model. We also make the obvious but important point that in a very primitive financial world with one or no assets, there cannot be chain reactions of default. Furthermore, at the opposite extreme, in an Arrow–Debreu world, there will also be no chain reactions because no agent need ever promise to deliver more than he himself has on hand. One agent's default will therefore not compromise any other debtor's ability to repay. Chain reactions are thus characteristic of financial economies with intermediate levels of development. They can be shortened by netting promises. Netting, however, creates a nonconvexity in the budget set. But with a continuum of households, we show that equilibrium still exists.

The consequences of default are potentially ruinous, yet many economic systems permit them, at least to a certain extent. (To be sure, some societies are more tolerant of default than others.) Since the imposition of default penalties causes a deadweight loss of utility which could be avoided altogether, either by abolishing the penalties or else by making them so harsh that nobody dares incur them, a rationale for intermediate levels of default penalties is called for. From a historical or legal perspective, many explanations suggest themselves: protection of creditors and debtors, punishment commensurate with the crime, etc.

In Section 7 we give a purely economic explanation for intermediate default penalties by showing that when markets are incomplete, intermediate levels of penalties that encourage a limited amount of default can raise the level of overall economic efficiency, making both creditors and debtors better off, even when the whole chain of indirect effects is accounted for. Sometimes this range of appropriate intermediate levels of default penalties consists of no more than a single point, as in the example presented in Section 7. In such cases we speak of the optimal default penalty. In the Arrow–Debreu world where all contingencies can be foreseen and written into the contract, the first welfare theorem demonstrates that contracts should be strictly enforced, so the optimal default penalty is infinitely harsh. But if some contingencies cannot be written into the contract, as will be the case when markets are incomplete, then it may be advisable not to punish severely those who default, even when the

[2] Consider an outdoor market at which different farmers can put apples up for sale in the same bin. Each farmer may know how many of his own apples are rotten, but suppose all the apples are mixed together in the bin. If buyers canot pick out their favorite apples, but must order by the number of randomly chosen apples, then a single price can clear the market for these (non-homogeneous) apples. But if buyers have asymmetric capacities for detecting rotten apples, and if the buyers were allowed to examine the fruit and to choose their apples, then a homogeneous price per apple might not be enough to clear the apple market.

There is also no problem in clearing the (heterogeneous) apple market with one price if, in addition, buyers can pay with real or counterfeit coins, provided that all sellers get the same distribution of coins. What is crucial is that each market can be separated into two sides α and β such that every trader on the α-side receives the same relative bundle of β-goods, though each β-trader may deliver a different bundle to the market, and similarly each β-trader receives the same proportions of α-goods, though each α-trader may deliver a distinct α-bundle to the market.

We elaborate this general situation in other work.

penalties cannot be varied with the reason for defaults.

We are careful to explain the two reasons why lenient default penalties are advantageous when markets are incomplete. First, default allows agents to tailor-make promises into deliveries that suit them best. In effect they can replace the given assets by more appropriate assets. Second, the span of the asset deliveries can be made much larger than the span of the asset promises, since a single given asset can be made into as many different assets as there are sellers, if different sellers default differently on the same promises.

In Section 8 we postulate transactions costs (that might decrease with increased liquidity), and we postulate that contingent securities cannot be traded until after a fixed "evaluation" cost is paid. This creates a discontinuity in payoffs. We prove nevertheless that, with a continuum of agents, existence of equilibrium with default remains intact.

Both of these market impediments decrease the social efficiency of trading in many asset markets and increase the social benefit of packaging heterogeneous promises into a single security. Thus we should expect to see only a few actively traded assets in $\mathcal{A}^*$, none of which are as delicately state-contingent as Arrow securities. Under these circumstances it is clear that even if it were possible to write any conceivable contingent promise, and to set any degree of harshness for the default penalties, active equilibrium trade will involve fewer promises. From Section 7 we deduce that these promises should be accompanied by lenient penalties.

Our final goal is to show that even in a world without trading costs, assets are endogenous. Whereas in GEI the selection of assets is usually regarded as outside the model, here we can resolve the asset selection problem by focusing on the endogenous determination of positively traded assets $\mathcal{A}^*$. Typically in GEI, every (nonredundant) asset is actively traded, so $\mathcal{A} = \mathcal{A}^*$. However, in equilibrium with default, there will typically be many assets in $\mathcal{A} \backslash \mathcal{A}^*$ which are priced by the market, but neither bought nor sold.[3] The reason is that with default, the sale of an asset is not the negative of its purchase. The buyer receives only what is delivered, but the seller gives up in addition penalties for what is not delivered. The marginal utility of buying may thus be strictly less than the marginal disutility of selling, leaving room for a price in between at which no agent will want to buy or sell.

Recall that each asset (R_j, λ_j, Q_j) is characterized by three dimensions. If the set $\mathcal{A}$ of available assets is comprehensive (i.e., all conceivable levels and combinations of the three asset dimensions are present in $\mathcal{A}$), then we prove in Section 9 that $\mathcal{A}^*$ will in effect select the Arrowian levels: completely spanning promises, with infinite penalties, and nonbinding quantity constraints. On the other hand, if two of the dimensions in $\mathcal{A}$ are *exogenously* restricted away from their Arrowian levels, then the forces of supply and demand will *endogenously* select the levels in the remaining dimensions in $\mathcal{A}^*$ to be far from Arrowian, as we show in Sections 10 and 11.

For example, suppose promises and quantity constraints are fixed exogenously as in Section 7, where we showed that optimal penalties *should* be intermediate. We

[3] In some applications we might choose to limit $\mathcal{A}$ exogenously; the point is that even if $\mathcal{A}$ is inclusive, $\mathcal{A}^*$ will still be limited.

can ask how harsh the penalties *will* be that endogenously emerge in $\mathcal{A}^*$. We find that the forces of supply and demand select the optimal penalties.

Suppose quantity constraints are fixed exogenously at infinity, and default penalties are fixed exogenously to be lenient. We show in Section 9 that there is an equilibrium in which Arrow promises, though available, are not traded, and equilibrium instead selects the non-contingent promise.

In our sequel paper (Dubey–Geanakoplos, 2001a), we show that if promises and penalties are fixed exogenously in a particular way, our model includes the insurance contracts of Akerlof (1972) and Rothschild–Stiglitz (1976). In that case $\mathcal{A}^*$ endogenously selects quantity limits Q_j. This enables us to show how the phenomenon of signalling can be treated in perfect competition, moreover without jeopardizing the existence of equilibrium.

In our final Section 12 we show that equilibrium still exists with non-convex budget sets and payoff functions that need not be concave or continuous. This enables us to include confiscation and trigger penalties into our model of default.

2 Pooling: Adverse Selection and Moral Hazard in Perfect Competition

In keeping with the spirit of perfect competition, which is the hallmark of general equilibrium, we suppose that all trades are mediated by the market at market-given prices. This situation arises in practice when agents trade small quantities with each of many partners via a market. It differs from the standard framework of adverse selection and moral hazard found in so-called "principal–agent" models and "matching" models in which a single buyer confronts a single seller to negotiate a large transaction (see, e.g., Gale, 1992). But it still leaves plenty of room for adverse selection and moral hazard.

In the GEI model agents sell prespecified assets, i.e., promises to deliver commodities and money in the future, contingent on observable states of nature. We extend that model by giving the seller of an asset the *option* of delivering what he promised or of *defaulting* and incurring a penalty. As a result of the option, different agents may pay off differently on the same asset, so that the revenue from purchasing an asset depends not only on the asset's promises, but also on the identities of the sellers.

Adverse selection enters the picture because different sellers may have different proclivities to keep promises, either because they have different disutilities for the penalties incurred by defaulting, or because they have different endowments out of which to pay their debts. Since there is potentially a (negative) correlation between an agent's proclivity to repay and the quantity of promises he is likely to try and sell, buyers must be aware that the default rates they face will be different from those they would get from the median seller. Moral hazard enters the picture twice, first because agents have a choice not to repay, and second because an agent who sells many assets will be less able to fully deliver on any one of them than he would if he had refrained from overextending himself. The degree of adverse selection and moral hazard depends on market prices.

In finite player, game theoretic analyses of the strategic role of asymmetric information, moral hazard and adverse selection play additional roles, that we do not allow here, stemming from the supposition that each agent has a large impact on traders he deals with. For example, those models posit that anyone who lends another agent more money must take into account the moral hazard that the borrower might as a result pursue a larger and riskier project, and hence the probability of repayment might be affected. Similarly, anyone who unilaterally offers a higher price for the same promise (equivalently, a lower interest rate for the same loan) must take into account the adverse (or favorable) selection effect on the kind of people who want that loan. We ignore these complexities and retain the hypothesis of perfect competition.

In our model, agents do not unilaterally set price; the market sets the price. No agent has the power, or perhaps the visibility, to set a price different from the market price.[4] Equivalently, we might say that a buyer of an asset can set any price he wants, but in doing so he makes the cautious assumption that the selection of sellers he will find is no different from what is elicited by the market price, or (in the event of no trade) by a perturbation of the market. Cautious expectations are defendable on their own merits, but it is probably worth pointing out that in markets with a large number of traders, a buyer who credibly and visibly offers a price above the market price will be deluged with more sellers than he can accommodate. From which seller is he likely to buy? The sellers with the greatest incentive to get to him first are the ones who would have already been willing to sell at the low market price and now find an opportunity to make a surplus, not the sellers who just barely prefer to sell at the new high price. With cautious expectations, there is no reason for a buyer to offer more than the market price, since he expects to face the same selection of sellers as at the market price, at which he can already purchase whatever quantity he desires.

In our model, lenders provide money to a pool of heterogeneous borrowers. No lender can observe the personal characteristics of any particular borrower, but he can formulate a judgment about the (state-contingent) rate of repayment for the pool as a whole. Ultimately he will receive a prorata share of the deliveries from the whole pool of borrowers. Furthermore, he supposes that he is so negligible compared to the size of the pool he lends to that nothing he does can affect the general terms of trade. Therefore he should figure that no matter how much money he lends, the rate of repayment in any state of nature will be unaffected. Adverse selection and moral hazard are nevertheless incorporated into a framework of perfect competition by enlarging the traditional set of equilibrating price variables to include rates of repayment.

The large markets on Wall Street conform to our spirit of perfect competition and anonymous trade. For example, in the mortgage backed securities (MBS) market, investors buy shares of a pool of home mortgages. The homeowners have the option to default on their mortgage payments. The investing agent, however, is not matched

[4] We think of each buyer as a point in a continuum of buyers.

with a particular homeowner. On the contrary, he gets a share of the payments of all the homeowners in the pool, so that his risk is diversified. He collects potentially different amounts from different homeowners selling the same asset. MBS payments also differ depending on the identity of the homeowners because the homeowners are given a second option, to prepay the mortgage. In the MBS market there are widely disseminated predictions of the future average rate of default and prepayment, conditional on the realized state of the world (typically specified by interest rates and perhaps one or two other parameters). Similarly we suppose in our model that the state contingent rates of repayment are known as part of the definition of equilibrium.

In the mortgage market, banks act as intermediaries, approving each mortgage after checking the homeowner's credentials, and then selling the mortgage to a Wall Street firm, or first to GNMA or FNMA, which in turn sell them to Wall Street. Many of the finite game theoretic models of default emphasize the bilateral negotiation between the bank and the homeowner. We take the opposite view in this paper, that the bank plays a mostly mechanical role not requiring any judgment about the quality of homeowners beyond checking customer assertions of objective facts that must be passed on to the mortgage market. We therefore concentrate on the decisions made by the homeowner–borrowers who sell the assets and the investor–lenders who purchase the assets, leaving the banks entirely out of the picture. Indeed we find in recent practice that banks are compensated in their mortgage efforts not for their judgment in choosing reliable homeowners, but primarily from servicing fees for collecting payments and other administration.

Mutual funds are another prominent example of securities that aggregate the payments from many parties. For that matter, virtually all companies whose stocks are traded over the New York Stock Exchange can be regarded as conglomerations of different businesses whose profits are summed and distributed to the shareholders. In practice, the purchase of a single asset often brings revenues from many different sources, which can be conveniently approximated by the limiting case of an infinite pool of sources.

In this paper we do not allow for options beyond default, but they could be handled in the same manner. One should note that in many options markets there is a central clearing house. Agents trade the options against the clearing house, not against each other in bilateral negotiations. Often the clearing house guarantees payment (here the option is held by the buyer of the asset instead of by the seller). This guarantee tends to make the payments independent of the identity of the sellers, but if the guarantee should fail then the system would revert to one akin to our model.

3 Default Penalties

Once we allow for default it is evident that society has much to gain from punishing those agents who fail to keep their promises. In a multiperiod world, market forces themselves might provide some incentive to keep promises, since agents who acquired a bad reputation for previous defaults might find it more difficult to obtain new loans. Collateral is also a very important device for guaranteeing at least partial payment (see Geanakoplos, 1997); but here we ignore it. For reasons of simplicity and tractability, we confine attention to a two period model with exogenously specified default penalties which are increasing in the size of the default. These penalties might be interpreted as the sum of third party punishment such as prison terms, pangs of conscience, (unmodeled) reputation losses, and (unmodeled) garnishing of future income. Later, in Section 12, we extend the model to include confiscation of current goods.

Default in our model can either be strategic or due to ill fortune. Penalties are imposed on agents who fail to deliver, whatever the cause. Debtors choose whether to repay or to bear the penalty for defaulting; creditors cannot observe why default occurs. Agents who have no resources to repay will be punished as severely as they would if they had the resources but chose not to repay.[5] The consequences of default penalties are therefore two-fold: they tend to induce agents to keep promises when they are able, and they tend to discourage agents from making promises that they know in advance they will not always be able to keep.

Let d^h_{sj} be the nominal market value of default by agent h in state s on asset j. Although in practice the severity of the penalty (e.g., a felony vs. a misdemeanor) depends on the nominal amount, and that is only adjusted slowly in the face of inflation, we suppose the adjustment is instantaneous, so that the penalties depend on the "real" default. Accordingly, we divide d^h_{sj} by the market price in state s of a fixed basket of goods v_s.

We introduce the parameters λ^h_{sj} to represent the utility penalty on agent h for each real dollar of default in state s on asset j. The payoff to agent h in state s can thus be written

$$w^h_s(x_s, (d^h_{sj})_{j\in J}, p_s) = u^h_s(x_s) - \frac{\sum_{j\in J} \lambda^h_{sj}[d^h_{sj}]^+}{p_s \cdot v_s}$$

where x_s denotes consumption and p_s the prices in state s, and $[y]^+$ denotes the maximum of y and 0. The formula implies that the agent is punished for defaulting but not rewarded for overpaying his promises.

This simple parameterization of extra-economic default penalties was first introduced by Shubik and Wilson (1977). It is meant to capture the idea that as the default gets gradually higher, the penalty gets gradually higher, so utility is continuous and monotonically decreasing in the level of default. Furthermore, by raising the default penalty parameters λ^h_{sj}, we can increase the marginal disutility of default. All other properties implied by our special formulation of the default penalties are irrelevant.

In our formulation, the default penalties do not affect the marginal rate of substitution between goods. If we had wished to allow for the possibility that time in jail affects the relative utility for different kinds of goods, we could have easily dropped the separable form of the utilities we have assumed and inserted both λ^h_{sj} and d^h_{sj}

[5] In our model default penalties do not distinguish fraud from ill fortune. In reality they are hard to separate, but ever since Las siete Partitas of Don Alfonso X "the wise," bankruptcy law has sought to distinguish them.

into the utility function u_s^h. Our main results would remain intact. Similarly, the function w_s^h is concave in the level of default, which is convenient in deriving continuous demand functions. However, since we will be assuming a continuum of agents anyway, there is no difficulty in proving the existence of equilibrium with nonconcave utilities.

In this paper we are especially concerned with penalties, $\lambda_{sj}^h = \lambda$ for all h, s, and j. This is done for simple analytical convenience to reflect the idea that policy makers cannot fine tune the default parameters between people, states, or assets. Since we can always rescale the utilities of different agents differently, this case does not require interpersonal utility comparisons, that is it does not imply that a day in jail is regarded with the same dread by every agent. (It does however suggest that the same agent regards a day in jail with the same horror no matter which state it occurs in.) The extreme version of this case occurs when every λ_{sj}^h is set to infinity, which reduces our model to the standard GEI model.

In our sequel paper we set

$$\lambda_{sj}^h = \begin{cases} \infty & \text{if} \quad s \in \bar{S}^h \subset S \\ 0 & \text{if} \quad s \notin \bar{S}^h \end{cases} .$$

Agents are forgiven completely in some states (perhaps when their endowments are zero) and compelled to repay otherwise. This case will allow us to make insurance a special case of our model. We replace a single insurance contract, that says agent h will receive money if an accident occurs, with two contracts: one which delivers x dollars to agent h in every state, and the other in which agent h promises to deliver x dollars in every state. Since agent h will deliver everything he promised in states with λ_{sj}^h infinite, and default completely without penalty when λ_{sj}^h is 0, on net agent h will receive the same money as he would if he bought insurance paying x dollars in those states $S \setminus \bar{S}^h$ where his λ_{sj}^h is 0.

Finally, an interesting interpretation can be given to the condition

$$\lambda_{sj}^h > \lambda_{sj'}^h$$

namely, that asset j' is junior debt compared to asset j for agent h in state s. The rational agent h will pay off his j debt entirely in state s before redeeming a single dollar of j' debt. This distinction between junior and senior debt will not concern us until we reconsider the Modigliani–Miller principle, which we do in our sequel paper.

4 Default in Equilibrium: The $GE(R, \lambda, Q)$ Model

4.1 The Economy

As in the canonical model of general equilibrium with incomplete markets (GEI), we consider a two-period economy, where agents know the present but face an uncertain future. In period 0 (the present) there is just one state of nature (called state 0), in which H agents trade in L commodities and J assets. Then chance moves and selects one of S states which occur in period 1 (the future). Commodity trades take place again, and assets pay off. The difference from GEI is that in our $GE(R, \lambda, Q)$ model, assets pay off in accordance with what agents opt to deliver. Our notation can be formalized as follows:

$\ell \in L = \{1, ..., L\}$ = set of commodities
$s \in S = \{1, ..., S\}$ = set of states in period 1
$S^* = \{0\} \cup S$ = set of all states
$h \in H = \{1, ..., H\}$ = set of agents
$e^h \in \mathbb{R}_+^{S^* \times L}$ = initial endowment of agent h
$j \in J = \{1, ..., J\}$ = set of assets
$R_j \in \mathbb{R}_+^{S \times L}$ = promises per unit of asset j of each commodity $\ell \in L$ in each state $s \in S$
$u^h : \mathbb{R}_+^{S^* \times L} \to \mathbb{R}$ = utility function of agent h
$\lambda_{sj}^h \in \overline{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$ = real default penalty on agent h for asset j in state s
$Q_j^h \in \mathbb{R}_+$ = bound on sale of asset j by agent h

We assume that no agent has the null endowment, and that all named commodities are present in the aggregate, i.e.,

$$e_s^h = (e_{s1}^h, ..., e_{sL}^h) \neq 0$$

for all $h \in H$ and $s \in S^*$, and

$$e_{s\ell} = \sum_{h \in H} e_{s\ell}^h > 0$$

for all $s\ell \in S^* \times L$. Also each u^h is continuous, concave and strictly increasing in each of its $S^* \times L$ variables. Having assumed strict monotonicity and concavity, there is no further loss of generality in assuming that $u^h(x) \to \infty$ whenever $||x||_\infty \to \infty$.[6]

We can visualize the state space as a simple tree:

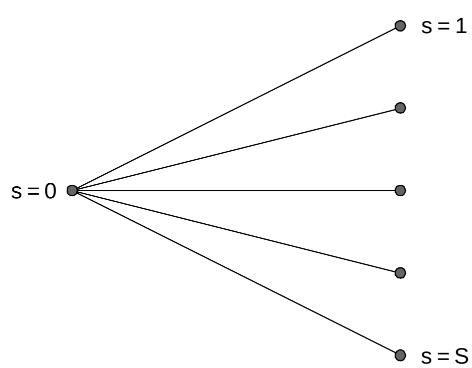

Figure 1

[6]Let $\square = \{x \in \mathbb{R}_+^{S^* L} : ||x||_\infty \leq 2||\sum_h e^h||_\infty\}$. Let $\mathcal{L}$ be the set of affine functions $L : \mathbb{R}_+^{S^* L} \to \mathbb{R}$ such that $L(x) \geq u^h(x)$ for all $x \in \square$. Define $\tilde{u}^h(x) \equiv \inf_{L \in \mathcal{L}} L(x)$. Then equilibrium with u^h and $\tilde{u}^h$ coincide, and $\tilde{u}^h$ has the desired properties.

Agents h have heterogeneous, state-dependent endowments $e_s^h \in \mathbb{R}_+^L$ and disutilities of default λ_{sj}^h.

Adverse selection enters the picture because agents have different endowments out of which to keep their promises, and also different disutilities of default.

Promises must be of a limited kind $j \in J$ fixed a priori. A promise $j \in J$ specifies bundles of goods (or services) to be delivered in each state:

$$\text{Promise } R_j = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \begin{matrix} \} - \text{state 1 goods} \\ \} - \text{state 2 goods} \\ \\ \} - \text{state } S \text{ goods.} \end{matrix}$$

Agents h make promises by selling various quantities φ_j^h of each asset j. An agent's ability to keep a promise depends on how many promises he sells, both of the same kind j, and of other kinds $j' \neq j$. Moral hazard enters the picture, since a buyer of an asset (i.e., lender) does not know which other promises the seller (i.e., borrower) has made, and because borrowers have the option to default.

Each kind of asset prescribes a limit on its sale, $\varphi_j^h \leq Q_j^h$. Limits on sales of promises are necessary to any realistic model of credit.[7] If $Q_j^h = 0$, then agent h is essentially forbidden from selling asset j. If the limits Q_j^h are very large, they may be entirely irrelevant, as they mostly are in the examples of Sections 7 and 8.[8] But if they are small, then they may be used as a signal that the sellers are not making many promises, and hence that the promises are reliable. We explore signalling in our sequel paper.

An economy is defined as a vector

$$\mathcal{E} = \left((u^h, e^h)_{h \in H},\ \left(R_j, ((\lambda_{sj}^h)_{s \in S}, Q_j^h)_{h \in H} \right)_{j \in J} \right).$$

Note again that an asset consists of promises, penalties for default, and limits on sales.

4.2 Equilibrium

In conventional general equilibrium theory, market prices convey all relevant information (trade is anonymous). Furthermore, each agent is very small and unable to affect the prices. In the next section we make this interpretation more tangible by replacing each household h by a continuum of identical households $t \in (h-1, h]$.

The possibility of default forces us to extend the definition of perfect competition. We continue to suppose that agents trade through anonymous markets, that is, they are not able to observe the identity of the agents taking the other side of the trade. In accordance with this anonymity, we suppose that each buyer of asset j ends up

[7] Evidence abounds that finite bounds are always imposed in the extension of credit. Even the best "name" among borrowers has a limited credit line.

[8] In Section 5 we are able to prove the existence of equilibrium even when $Q_j^h = \infty$, provided $\lambda \gg 0$ and the R_j all deliver in the same good.

with sales from *every* seller in proportion to how much they sell. Thus if households of type $h = 1$ and $h = 2$ are the only sellers of asset j, and households of type 2 sell twice as many units as households of type 1, then each buyer of asset j receives 2/3 of his purchases from households of type 2 and 1/3 from households of type 1. Similarly, when a borrower (i.e., a seller of an asset promise) defaults on some asset delivery, the loss is spread out proportionally to all owners of that asset. This is captured in our model by the hypothesis that all buyers of asset j face the same delivery rates. Like prices, buyers regard delivery rates as unaffected by their own actions.

Lenders will naturally try to forecast what fraction of their investments actually deliver. They recognize that their own loans are spread among many borrowers, so that the rates of default will not be affected by how much they loan. Their forecasts of default will of course be conditional forecasts, depending on the state of nature that prevails in the future. In exactly this way the great Wall Street investment banks make forecasts of homeowner prepayment and default rates, conditional on the future level of interest rates and other parts of the "state of nature." In our model we will make the heroic (though standard) "rational expectations" assumption that these conditional forecasts are all correct: the realized rate of default (or delivery) in each state on an asset is exactly the rate anticipated in that state.

If we wish to allow for the possibility that some of the characteristics of the sellers are observed, this is easily accommodated in our model by setting some of the sales limits $Q_j^h = 0$. For example, in the extreme case that the type of the sellers of asset j is perfectly observable, we can think of H different copies of asset j, namely $j_1, ..., j_H$ such that $Q_{j_h}^{h'} = 0$ if $h \neq h'$. In that case a buyer of asset j_h knows that he must be buying from households of type h. Even in this case a buyer makes his purchases from a continuum of sellers, albeit of the same type, so that the buyer never has to worry about the strategic effect of his own loan on an individual seller's actions.

One might suppose that it is a simple matter to describe a $GE(R, \lambda, Q)$ equilibrium with default by respecifying the assets according to what is actually delivered as opposed to what is promised. But what is delivered is determined endogenously and cannot be predicted without solving for the equilibrium. Moreover different agents will make different deliveries on the same asset even though the lenders receive the same aggregated payoffs. Thus our model cannot be fitted into the standard GEI framework.

When an asset market is active, the informational requirements for the $GE(R, \lambda, Q)$ equilibrium are roughly the same as in competitive equilibrium: agents must know the promised delivery of each asset in each state, and the average fraction of delivery of each asset in each state. No trader needs to bother about the identities of those he is trading with.

To define a $GE(R, \lambda, Q)$ equilibrium, first consider the "macrovariables" p, π, K that each agent takes as fixed. Here $p \in \mathbb{R}_{++}^{S^* \times L}$ is the vector of commodity prices; $\pi \in \mathbb{R}_+^J$ is the vector of asset prices; and K is an $S \times J$ matrix with entries K_{sj} between 0 and 1, representing the fraction expected to be delivered of payments promised by asset j in state s.

4.2.1 Household Budget and Payoff

The *budget set* $B^h(p, \pi, K)$ of agent h is given by:

$$B^h(p,\pi,K) = \Big\{(x,\theta,\varphi,D) \in \mathbb{R}_+^{S^*\times L} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{J\times S\times L} :$$
$$p_0 \cdot (x_0 - e_0^h) + \pi \cdot (\theta - \varphi) \leq 0;\ \varphi_j \leq Q_j^h \text{ for } j \in J;\ \text{and, } \forall s \in S,$$
$$p_s \cdot (x_s - e_s^h) + \sum_{j\in J} p_s \cdot D_{sj} \leq \sum_{j \in J} \theta_j K_{sj} p_s \cdot R_{sj} \Big\}$$

Here $x \in \mathbb{R}_+^{S^*\times L}$ is the final consumption of commodities, $\theta \in \mathbb{R}_+^J$ (respectively, $\varphi \in \mathbb{R}_+^J$) gives the purchases (respectively, sales) of the J assets, and $D_{sj} \in \mathbb{R}_+^L$ is the vector of goods delivered by agent h on asset j in state s.

The budget set allows agent h to deliver whatever he pleases. On the other hand, the agent expects to receive a fraction K_{sj} of the promises made to him on asset j in state s. The first constraint says that agent h cannot spend more on purchases of commodities x_0 and assets θ than the revenue he receives from the sale of commodities e_0^h and assets φ. Moreover he can never sell more than Q_j^h of any asset j. The second constraint applies separately in each state $s \in S$. It says that agent h cannot spend more on the purchase of commodities x_s and asset deliveries $\sum_j D_{sj}$ in state s than the revenue he gets in state s from commodity sales e_s^h and asset receipts $\sum_j \theta_j K_{sj} p_s R_{sj}$.

The only reason that agents deliver anything on their promises is that they feel a disutility λ_{sj}^h from defaulting. The payoff of (x, θ, φ, D) given prices p, to agent h is

$$w^h(x,\theta,\varphi,D,p) = u^h(x) - \sum_{j\in J}\sum_{s\in S} \frac{\lambda_{sj}^h[\varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}]^+}{p_s \cdot v_s}.$$

where $v_s \in \mathbb{R}_+^L$ with $v_s \neq 0$. Note that $[\varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}]^+ \equiv \max\{0, \varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}\}$ is exactly the money value of the default of h on his promise to deliver on asset j in state s.

Notice that the budget set is convex, and the payoff function w^h is concave, in the household choice variables (x, θ, φ, D). Had we expressed these choices with other (apparently natural) variables, such as $\delta_{sj}^h \equiv$ delivery per unit promised, the budget set would no longer be convex, nor would w^h be concave.

It is worth noting a *scaling property* of the budget set (which is immediate from its definition and the fact that $e_s^h \neq 0$ and $p_s \gg 0$ for all $s \in S^*$): $(x, \theta, \varphi, D) \in B^h(p, \pi, K)$ and $0 < \alpha < 1 \Rightarrow (\alpha x, \alpha\theta, \alpha\varphi, \alpha D) \in B^h(p', \pi', K')$ for all (p', π', K') sufficiently close to (p, π, K). This property will often be useful to us.[9]

For simplicity (and for the facility of doing comparative statics) we have taken the default penalty to be linear and separable in the amount of default. But as we

[9] An alternative scaling property, also satisfied by the budget set, is obtained if we replace $(\alpha x, \alpha\theta, \alpha\varphi, \alpha D)$ with $(\alpha x, \alpha\theta, \varphi, \alpha D)$. Our entire analysis remains intact with this version of scaling.

noted in Section 3, we can easily accommodate more general payoffs w^h which allow for the marginal rate of substitution between goods to depend on the level of default. All that is needed for Theorem 1 is the continuity and concavity of w^h. For Theorem 2 we need to assume, in addition, that given any x, $w^h(x,\theta,\varphi,D,p) < u^h(e^h)$ if the default in any state, on any asset, is sufficiently large.

We shall also analyze default with netting, transactions costs, and confiscation, which make for nonconvex budget sets, or nonconcave payoffs. Then it becomes necessary to introduce a continuum of households. We defer this discussion to Sections 6, 8, and 12.

4.2.2 Market Clearing

We are now in a position to define a $GE(R,\lambda,Q)$ equilibrium. It is a list $\langle p,\pi,K,(x^h,\theta^h,\varphi^h,D^h)_{h\in H}\rangle$ such that (1) to (4) below hold.

(1) For $h \in H$, $(x^h,\theta^h,\varphi^h,D^h) \in \arg\max w^h(x,\theta,\varphi,D,p)$ over $B^h(p,\pi,K)$

(2) $\sum_{h\in H}(x^h - e^h) = 0$

(3) $\sum_{h\in H}(\theta^h - \varphi^h) = 0$

(4) $K_{sj} = \begin{cases} \sum_{h\in H} p_s \cdot D^h_{sj} / \sum_{h\in H} p_s \cdot R_{sj}\varphi^h_j, & \text{if } \sum_{h\in H} p_s \cdot R_{sj}\varphi^h_j > 0 \\ \text{arbitrary}, & \text{if } \sum_{h\in H} p_s \cdot R_{sj}\varphi^h_j = 0 \end{cases}$

Condition (1) says that all agents optimize; (2) and (3) require commodity and asset markets to clear. Condition (4), together with the definition of the budget set, says that each potential lender (i.e., buyer) of an asset is correct in his expectation about the fraction of promises that do in fact get delivered. Moreover, his expectation $K^h_{sj} = K_{sj}$ of the rate of delivery does not depend on anything he does himself; in particular, it does not depend on the amount θ^h_j he loans (i.e., purchases) of the asset. Every lender gets the same rate of delivery.

Since heterogeneous borrowers may be selling the same asset, the realized rate of delivery K_{sj} is an average of the rates of delivery of each of the borrowers, weighted by the quantity of their sales. It might well happen that those borrowers with the highest rates of default are selling most of the asset, and this is the adverse selection and moral hazard that rational lenders must forecast.

We believe that our definition of $GE(R,\lambda,Q)$ equilibrium embodies the spirit of perfect, anonymous competition, and represents a significant fraction of the mass asset markets of a modern enterprise economy.

In the next sections we investigate the properties of equilibrium.

4.3 Untraded Assets

It is a curious fact that many of the large asset markets that our model seeks to describe have been initiated not by entrepreneurs but by government intervention. The government, for example, began the GNMA mortgage program by guaranteeing delivery on the promises of all borrowers eligible for the program (but not the timing[10] of delivery). It is likely, however, that these mortgage markets would function smoothly even without government guarantees. Private companies indeed do sell insurance on non-GNMA mortgages. A reasonable question to ask is why the pass through mortgage market did not begin on its own?

One possible explanation is provided by our model. When assets are traded, expected deliveries K_{sj} must be equal to actual deliveries. Expectations cannot therefore be unduly pessimistic. But for assets that are not traded, our model makes no assumption about expectations of delivery (see (4)). In the real world, investors with no experience in observing default rates might tend to overestimate their probability. This can create serious problems, in practice as in our model. In the model, so far, there is nothing to stop the expectations from being absurdly pessimistic, which in turn will support trivial equilibria with no trade in the asset. The point is easily seen by a simple example. Consider an equilibrium of an economy in which certain assets are missing. Introduce these new assets j but choose their prices π_j close to zero. Then no agent will be willing to sell them, for he gets very little in exchange, but undertakes a relatively large obligation either to deliver commodities or to pay default penalties. Also choose the K_{sj} to be positive but even smaller. Then in spite of their low price, no agent will be willing to buy the assets since he expects them to deliver virtually nothing. Thus we have obtained trivial equilibria in which there is no trade of the new assets on account of arbitrarily pessimistic expectations regarding their deliveries.

We believe that unreasonable pessimism prevents many real world markets from opening, and provides an important role for government intervention. But it is interesting to study equilibrium in which expectations are always reasonably optimistic. It is of central importance for us to understand which markets are open and which are not, and we do not want our answer to depend on the agents' whimsical pessimism.

4.4 Trembling Hand Equilibrium

Expectations for deliveries by assets that are not traded are analogous to beliefs in game theory "off the equilibrium path." Selten (1975) dealt with the game theory problem by forcing every agent to tremble and play all his strategies with probability at least $\varepsilon > 0$, and then letting $\varepsilon \to 0$. We shall do roughly the same thing, introducing an external ε-agent who sells and buys $\varepsilon = (\varepsilon_j)_{j \in J} \gg 0$ of every asset, and fully delivers on his promises. (One might interpret this agent as a government which guarantees delivery on the first infinitesimal promises.)

An equilibrium $E(\varepsilon)$ obtained with the ε-agent is called an ε-trembling hand equilibrium. Thus any such $E(\varepsilon) = \langle p(\varepsilon), \pi(\varepsilon), K(\varepsilon), (x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon))_{h \in H} \rangle$ must satisfy:

$$(1^*) \quad (x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon)) \in \arg\max w^h(x, \theta, \varphi, D, p(\varepsilon)) \text{ over } B^h(p(\varepsilon), \pi(\varepsilon), K(\varepsilon))$$

[10] A default induces the government to prepay the loan immediately, even if the lender would prefer the scheduled payments.

$$(2^*)\ \sum_{h\in H}(x_s^h(\varepsilon)-e_s^h)=\begin{cases}0 & \text{if } s=0\\ \sum_{j\in J}\varepsilon_j(1-K_{sj}(\varepsilon))R_{sj} & \text{if } s\in S\end{cases}$$

$$(3^*)\ \sum_{h\in H}(\theta^h(\varepsilon)-\varphi^h(\varepsilon))=0$$

$$(4^*)\ K_{sj}(\varepsilon)=\begin{cases}\dfrac{p_s(\varepsilon)\cdot R_{sj}\varepsilon_j+\sum_{h\in H}p_s(\varepsilon)\cdot D_{sj}^h(\varepsilon)}{p_s(\varepsilon)\cdot R_{sj}\varepsilon_j+\sum_{h\in H}p_s(\varepsilon)\cdot R_{sj}\varphi_j^h(\varepsilon)} & \text{if denominator} > 0\ (R_{sj}\neq 0)\\ 1 & \text{otherwise } (R_{sj}=0)\end{cases}$$

Since the ε-agent buys and sells ε_j units of each asset j, asset market clearing (3*) is as before. But since he delivers fully $\varepsilon_j R_{sj}$ on his promises, and gets delivered only $\varepsilon_j K_{sj}(\varepsilon)R_{sj}$, on net he injects the vector of commodities $\sum_{j\in J}\varepsilon_j(1-K_{sj}(\varepsilon))R_{sj}$ into the economy in each state $s\in S$. This explains (2*). Finally, condition (4*) says that delivery rates must recognize the external agent. (The condition when denominator $=0$ is now reduced to the trivial case when promises $p_s(\varepsilon)\cdot R_{sj}=0$, hence when $K_{sj}(\varepsilon)$ is irrelevant.)

An $E=\langle p,\pi,K,(x^h,\theta^h,\varphi,D)_{h\in H}\rangle$ is called a trembling hand equilibrium if there exists a sequence of ε-trembling hand equilibria $E(\varepsilon)$ with $\varepsilon\to 0$ and $E(\varepsilon)\to E$.

Notice that the external agent boosts the delivery rate $K_{sj}(\varepsilon)$ above the level achieved by the real agents $h\in H$ in the ε-economy (unless they too are fully delivering, or not selling). As $\varepsilon\to 0$, this boost disappears for assets that are positively traded in the limit. But if $\varepsilon_j/\sum_{h\in H}\varphi_j^h(\varepsilon)$ does not go to zero and $\sum_{h\in H}\varphi_j^h(\varepsilon)>0$ for all ε, the limiting rates K_{sj} will be boosted (unless there is no default by the real agents).

We could refine equilibrium still further by restricting attention to ε-trembling hand equilibria in which the $\varepsilon=(\varepsilon_j)_{j\in J}$ go to zero in a particular way. We shall have no need for such ultra-refinements in this paper, but we introduce them in Dubey–Geanakoplos (2001b) in connection with seniority of assets.

4.5 Refined Equilibrium

In Section 5 we prove the existence of trembling hand equilibrium. But trembling hand equilibria are not so handy to work with. Hence we introduce a slightly weaker but simpler notion of refined equilibrium, which captures the essential features of trembling hand equilibrium. Furthermore, we gain in generality, because any property of all refined equilibria (such as uniqueness) must automatically hold for the subset of trembling hand equilibria.

To this end we add a condition (5) to conditions (1)–(4) from the definition of equilibrium in Section 4.2. This requires that if a small change in the macro parameters (p,π) could induce some agents to start selling some of an asset j, where none was being sold before, then buyers should expect at least the rate of delivery they would get had the world indeed been so perturbed. (If there are many ways of perturbing (p, π) to induce sales, then we allow the buyers to focus their attention on one of these perturbations.) If prices π_j are so low that no small perturbation will induce any agents to sell asset j, then buyers are required to expect full delivery, $K_{sj} = 1$.

Let $\| \, \|_\infty$ denote the supremum norm, and let $E \equiv \langle p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h \in H} \rangle$, i.e., E is the candidate equilibrium which satisfies conditions (1) to (4). For $s \in S$, let $J(s) = \{j \in J : \sum_{h \in H} p_s \cdot R_{sj} \varphi_j^h = 0\}$. Thus $J(s)$ is the set of assets in state s for which K_{sj} is not determined by market activity in E. We are ready to state

(5) For any $\delta > 0$, there exists $E(\delta) \equiv \langle p(\delta), \pi(\delta), K(\delta), (x^h(\delta), \theta^h(\delta), \varphi^h(\delta), D^h(\delta))_{h \in H} \rangle$ such that

  (i) $(x^h(\delta), \theta^h(\delta), \varphi^h(\delta), D^h(\delta)) \in \arg\max w^h(x, \theta, \varphi, D, p(\delta))$ over $B^h(p(\delta), \pi(\delta), K(\delta))$

  (ii) $\|E - E(\delta)\|_\infty < \varepsilon$

  (iii) $K_{sj}(\delta) \geq \begin{cases} \sum\limits_{h \in H} p_s(\delta) \cdot D_{sj}^h(\delta) / \sum\limits_{h \in H} p_s(\delta) \cdot R_{sj} \varphi_j^h(\delta) & \text{if } \sum\limits_{h \in H} p_s(\delta) \cdot R_{sj} \varphi_j^h(\delta) > 0 \\ 1 & \text{if } \sum\limits_{h \in H} p_s(\delta) \cdot R_{sj} \varphi_j^h(\delta) = 0 \end{cases}$

  for all $s \in S$ and $j \in J(s)$.

Conditions (i), (ii), and (iii) together imply that if asset j is untraded and $K_{sj} < 1$, then there must be arbitrarily small perturbations of the macro variables which induce some agents to sell j, and to deliver (in aggregate) at a rate no more than K_{sj} (that is, to default at rate at least $(1 - K_{sj})$).

Any $E \equiv \langle p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h \in H} \rangle$ satisfying (1)–(5) is called a refined equilibrium.

It is evident that every trembling hand equilibrium is a refined equilibrium. The boost allowed for in the definition of refined equilibrium is made concrete by the ε-external agent.

We could have imagined an external agent who delivers only 70% of his promises, instead of 100%. The corresponding refined equilibrium would require $K_{sj}(\delta)$ to be between .70 and real agent delivery rates with the δ-perturbed macro variables, and to be exactly .70 if there was no real trade. It is clear that any "100% refined equilibrium allocation" is a "70% refined equilibrium allocation," thus explaining why our choice of 100% deliveries gives the sharpest refinement. The converse is certainly not true, as we show by example in Dubey–Geanakoplos (2001b).

In our definition of equilibrium, there is one price for each asset, including those assets that are not traded. There is therefore no possibility for price taking agents to offer different prices with the hope of luring a better selection of sellers.[11] We can interpret this situation in several ways. Suppose that buyers are aware of the

[11] Putting the matter still differently, we regard an asset or contract as setting out the obligations of the seller, including the penalties if he fails to deliver, and the quantity limitations on his other sales. The price of the contract is set by competition between sellers and buyers, that is, by the market. Agents need only think about one prevailing price for each contract. In the Rothschild–Stiglitz view, the price is one of the terms of the contract. In this view, there is no such thing as a single contract;

composition of sales at the market prices, and perhaps of the composition of sales at prices a penny off from market prices. But they lack the knowledge or computing power to infer what the composition would be at prices far from market. They might therefore presume they will get the same selection of sellers no matter what price they quote, giving them no incentive to deviate from market prices. Alternatively, a buyer might understand full well the composition of sales, but he should make the cautious assumption that he alone cannot serve all the potential sellers, and that he is likely to be reached first by the sellers who are most anxious to sell, that is by the sellers who have the lowest reservation price for the asset. Thus again the buyer expects the same selection of sellers as elicited by the market price, and is left with no incentive to deviate from the market price.

In Sections 10 and 11, on the endogeneity of the asset structure, and in our sequel paper, we show that the equilibrium refinement plays a crucial role in determining whether an asset j is positively traded ($j \in \mathcal{A}^*$) or not ($j \in \mathcal{A} \backslash \mathcal{A}^*$).

4.6 A Continuum of Traders

We have mentioned several times that our model is meant to embody the ideal of perfect competition, in which each agent is so small that by himself he cannot influence anyone else. We can make such an interpretation of our model more concrete by replacing each agent h by a continuum of identical agents parameterized by t lying in the interval $(h-1, h]$: each agent $t \in (h-1, h]$ has identical characteristics $(e^h, u^h, \lambda^h, Q^h)$.

For any $(p, \pi, K) \in \mathbb{R}_{++}^{S^* \times L} \times \mathbb{R}_+^J \times [0,1]^{S \times J}$ we can define $B^t(p, \pi, K)$ exactly as before, replacing h by t throughout. Also $GE(R, \lambda, Q)$ can be defined as before, where μ is Lebesgue measure, replacing $\sum_{h \in H}$ by $\int_I d\mu$, "$\forall h \in H$" by "almost all $t \in I$," and the notion of convergence of x, θ, φ, D (which are now integrable functions on I) in condition (5) by almost everywhere pointwise convergence.

The $GE(R, \lambda, Q)$ of the finite agent economy, whose existence we shall prove in Theorems 1 and 2, corresponds to a $GE(R, \lambda, Q)$ of the continuum model with the added feature that $(x^t, \theta^t, \varphi^t, D^t) = (x^{t'}, \theta^{t'}, \varphi^{t'}, D^{t'})$ whenever t and t' are both in $(h-1, h]$, i.e., all agents of the same type behave symmetrically. We shall call such equilibria *type-symmetric* when viewed in the continuum setting.

But we shall shortly consider a variant of our model in which the convexity of budget sets fails to hold. Here the continuum model is necessary for establishing existence of $GE(R, \lambda, Q)$. Even if the economy is finite-type, its equilibria need no longer be type-symmetric, and the consideration of a continuum becomes unavoidable.[12]

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there are as many contracts as there are prices. Notice also that the Rothschild–Stiglitz view must regard market clearing as one of rationing. At most prices, the contract will not be traded, because *either* supply or demand is zero, and the other side of the market is rationed. This point of view has been admirably expressed by Gale. In our view competitive equilibrium should be defined by a single price at which both supply and demand are equal (possibly both zero), as long as expectations at that price are set at rational levels.

[12] Without too much more trouble we could have allowed for an infinity of types. We have made the finite-type assumption only for ease of exposition.

5 The Orderly Function of Markets with Default

Our first goal in this paper is to establish that default is completely consistent with the orderly function of markets. To that end we prove that under fairly general conditions, equilibrium always exists in our model.

The universal existence of equilibrium is somewhat surprising because of the historical tendency to associate default with disequilibrium (or more accurately, to make full delivery part of the definition of equilibrium), as we have already remarked. Furthermore, endogeneity of the asset payoff structure is known to complicate the existence of equilibrium with incomplete markets. But we show that no new existence problems arise from the endogeneity of the asset payoffs due to default.

The universal existence of equilibrium with default is also surprising because the pioneering papers placing adverse selection in a model of competition, by Akerlof (1972) on the market for lemons, and Rothschild and Stiglitz (1976) on insurance markets, purportedly showed that adverse selection is quite commonly inconsistent with equilibrium. Since the Akerlof and Rothschild–Stiglitz models are special cases of our model, a word about them might be illuminating. We discuss Rothschild–Stiglitz in more detail in our sequel papers.

In Akerlof's lemons paper, each seller knows the value of his car, but the buyer only knows the average quality of the cars for sale. This is analogous to our model in which each seller knows his disutility of defaulting (and indeed his intentions to default) but the buyer knows only the overall average default rate. Akerlof's formal analysis consists of a special example with the property that at any price for used automobiles, the sellers will supply automobiles whose average quality is not worth the price. More precisely, suppose that the quality of cars is uniformly distributed between 0 and 1, and that if v is the quality of some car, then the owner (i.e., the potential seller) knows v and values it at v, whereas any potential buyer would value it at $1.1v$ once he got it, but unfortunately does not know what v is. Suppose that there is a large pool of potential buyers with the same preferences. At any price p, the cars with v less than p will be put up for sale, and the average quality to the buyer will be $(0.5)(1.1)p = .55p < p$. As the price falls, so does the average quality of the automobiles put up for sale. Under this extreme hypotheses, there cannot be any trade of automobiles at any price; even though at each price there are cars for sale that are worth more to the buyers than the price, a buyer must count on getting an average quality car, which is worth less than the price.

Needless to say, one could easily imagine less severe conditions under which there would still be some trade for automobiles, though to be sure the quantity would be less than would obtain under complete information about the quality of every car. For example, if the minimum quality level were $m > 0$ instead of 0, then there would be an equilibrium with $p = (11/9)m$. All the cars with $m < v < p$ would be sold, some buyers would be disappointed, others would be pleasantly surprised, but on average the buyers would get value $(1/2)(m + (11/9)m)(1.1) = (11/9)m = p$ equal to what they paid for. Indeed it is accurate to describe the extreme situation in the Akerlof model as one in which there is an equilibrium, with price set at 0. If the equilibrium with no trade seems bad, it is just that: bad in the welfare sense. But it still is an equilibrium. Interpreted properly, Akerlof's paper shows that adverse selection may in extreme situations result in an equilibrium with no trade; it does not provide any reason to suppose that equilibrium and adverse selection are incompatible.

Insurance contracts promise payments conditional on the state of nature, and so can be viewed as assets such as we describe in this paper, as we mentioned earlier. In particular, the Rothschild–Stiglitz model can be expressed as a special case of our general equilibrium model, as we show in our sequel paper. The reason Rothschild and Stiglitz found robust regions with no equilibrium is that they defined equilibrium expectations differently, as we have explained. If buyers had the perfectly competitive expectations that we invoke, namely that each thinks he cannot improve his selection of sellers by unilaterally offering a higher price, then the Rothschild–Stiglitz model would always have an equilibrium, as we show in our sequel paper, even using their "exclusivity" hypothesis.

We are now ready to state our main theorem, which is that $GE(R, \lambda, Q)$ equilibrium always exists, even if we insist on the equilibrium refinement discussed in Section 4.3.

Theorem 1 *For any $\lambda \in \overline{\mathbb{R}}_+^{HSJ}$ and $Q \in \mathbb{R}_+^{HJ}$, a trembling hand $GE(R, \lambda, Q)$ equilibrium exists, where $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \infty$.*

Proof Suppose first that penalties are finite, $\lambda \in \mathbb{R}_+^{HSJ}$. Fix a tremble $\varepsilon = (\varepsilon_j)_{j \in J} \gg 0$. For any small lower bound $b > 0$, define

$$\Delta_b = \left\{ (p, \pi) \in \mathbb{R}_+^{S^* \times L} \times \mathbb{R}_+^J : \sum_{\ell=1}^{L} p_{s\ell} = 1 \ \forall s \in S^*, \right. \\ \left. b \leq p_{s\ell} \ \forall s\ell \in S^* \times L, \text{ and } 0 \leq \pi_j \leq \tfrac{1}{b} \ \forall j \in J \right\}.$$

Choose M large enough to ensure that: $||x||_\infty > M \Rightarrow u^h(x) > u^h(2 \sum_{h' \in H} e^{h'})$ for all $h \in H$. (By assumption, $u^h(x) \to \infty$ as $||x|| \to \infty$, so such an M exists.) Now define, for each $h \in H$, $\Box^h = \{(x, \theta, \varphi, D) \in \mathbb{R}_+^{S^* \times L} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{SLJ} : \|x\|_\infty \leq M,\ \theta_j \leq 2 \sum_{h' \in H} Q_j^{h'},\ \varphi_j^h \leq Q_j^h, \text{ and } ||D||_\infty \leq ||Q||_\infty ||R||_\infty\}$. Let $\Box^H \equiv \mathsf{X}_{h \in H} \Box^h$.

Denote $\eta \equiv (p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h \in H}) \in \Delta_b \times [0,1]^{S \times J} \times \Box^H \equiv \Omega_b$.

Consider the map $\bar{K}_b : \Omega_b \to [0,1]^{S \times J}$ defined by

$$\bar{K}_{bsj}(\eta) = \begin{cases} \min \left\{ \dfrac{p_s \cdot R_{sj}\varepsilon_j + \sum\limits_{h \in H} p_s \cdot D_{sj}^h}{p_s \cdot R_{sj}\varepsilon_j + \sum\limits_{h \in H} p_s \cdot R_{sj}\varphi_j^h}, 1 \right\} & , \text{ if } R_{sj} \neq 0 \\ 1 & , \text{ if } R_{sj} = 0 \end{cases}$$

for each $s \in S$, $j \in J$. Clearly $\bar{K}_{bsj}$ is a continuous function.

Next, consider the correspondence $\psi_b^0 : \Omega_b \Rightarrow \Delta_b$ defined by

$$\begin{aligned}\psi_b^0(\eta) &= \operatorname*{arg\,max}_{(p,\pi)\in\Delta_b} \Bigg\{ p_0 \cdot \sum_{h\in H}(x_0^h - e_0^h) + \pi \cdot \sum_{h\in H}(\theta^h - \varphi^h) \\ &\quad + \sum_{s\in S}\sum_{h\in H} p_s \cdot \left[x_s^h - e_s^h - \sum_{j\in J}(1-\bar{K}_{bsj}(\eta))R_{sj}\varepsilon_j \right] \Bigg\}.\end{aligned}$$

Clearly this map is non-empty and convex-valued, and USC.

Finally for each $h \in H$, define the correspondence $\psi_b^h : \Omega_b \Rightarrow \Box^h$ by

$$\psi_b^h(\eta) = \operatorname*{arg\,max}_{x,\theta,\varphi,D} \{w^h(x,\theta,\varphi,D,p) : (x,\theta,\varphi,D) \in B^h(p,\pi,K) \cap \Box^h\}.$$

Notice that ψ_b^h is non-empty valued and convex-valued, thanks to the continuity and concavity of w^h, for all $h \in H$. To check that $B^h(p,\pi,K) \cap \Box^h$ is LSC, let p^n, π^n, $K^n \xrightarrow{n} \bar{p}$, $\bar{\pi}$, $\bar{K}$ with $\bar{p} \gg 0$. Let $(\bar{x},\bar{\theta},\bar{\varphi},\bar{D}) \in B^h(\bar{p},\bar{\pi},\bar{K})$. Fix $0 < \alpha < 1$. Then $(\alpha\bar{x},\alpha\bar{\theta},\alpha\bar{\varphi},\alpha\bar{D}) \in B^h(p^n,\pi^n,K^n)\cap\Box^h$ for sufficiently large n by the scaling property of the budget set, because $\bar{p}_s \cdot e_s^h > 0 \ \forall s \in S^*$. Since α was arbitrary, this shows that $B^h(p,\pi,K) \cap \Box^h$ is LSC in (p,π,K) whenever $p \gg 0$. Since $B^h(p,\pi,K) \cap \Box^h$ is clearly USC, ψ_b^h is USC by the maximum principle.

Let $\psi_b : \Omega_b \Rightarrow \Omega_b$ be the correspondence defined by

$$\psi_b(\eta) = \psi_b^0(\eta) \times \{\bar{K}_b(\eta)\} \times \bigtimes_{h\in H} \psi_b^h(\eta).$$

By Kakutani's theorem ψ_b has a fixed point $\eta^b \equiv (p^b,\pi^b,K^b,(x^h(b),\theta^h(b),\varphi^h(b),D^h(b))_{h\in H})$. To avoid notational clutter, we suppress the b.

Note that in state 0, $p_0 \cdot (\sum_h (x_0^h - e_0^h)) + \pi \cdot (\sum_h (\theta^h - \varphi^h)) = 0$ (since, given the monotonicity of each u^h, this equality holds for each h individually in his budget-set). It follows that the "price player" could not make the value of excess demand (across commodities and assets) positive in period 0. Suppose for some $j \in J$, $\sum_{h\in H}(\theta_j^h - \varphi_j^h) > 0$. By taking $\tilde{\pi}_j = 1/b$ and $\tilde{\pi}_i = 0$ for $i \neq j$, it follows that

$$\frac{1}{b}\sum_h (\theta_j^h - \varphi_j^h) + \sum_{\ell\in L}\tilde{p}_{0\ell}\sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) \leq 0,$$

for all $\tilde{p} \in \mathbb{P}_b \equiv \{q \in \mathbb{R}_+^L : q_\ell \geq b \ \forall \ell \in L, \sum_{\ell=1}^L q_\ell = 1\}$. Hence

$$\sum_h (\theta_j^h - \varphi_j^h) \leq b||e_0||_\infty.$$

Similarly, if $\sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) > 0$ for some ℓ, then by taking all $\tilde{\pi}_j = 0$ and $\tilde{p}_{0\ell} = 1-(L-1)b$ and $\tilde{p}_{0k} = b$ for all $k \neq \ell$, we get

$$\sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) \leq \frac{(L-1)b||e_0||_\infty}{1-(L-1)b}.$$

From the fact that $\bar{K}_b$ fixed K, and from the fact that agents have optimized so that $p_s \cdot D^h_{sj} \leq p_s \cdot R_{sj}\varphi^h_j$, whenever $R_{sj} \neq 0$ we get

$$K_{sj} = \frac{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot D^h_{sj}}{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot R_{sj}\varphi^h_j} \leq 1.$$

Hence

$$\sum_{h\in H} p_s \cdot D^h_{sj} = \sum_{h\in H} K_{sj} p_s \cdot R_{sj}\varphi^h_j - (1-K_{sj})p_s \cdot R_{sj}\varepsilon_j.$$

From optimization of monotonic utilities in the budget set, we get

$$p_s \cdot (x^h_s - e^h_s) = \sum_{j\in J} K_{sj} p_s \cdot R_{sj}\theta^h_j - \sum_{j\in J} p_s \cdot D^h_{sj}.$$

Adding over agents $h \in H$, and substituting the above expression for $\sum_{h\in H} p_s \cdot D^h_{sj}$ we get

$$\begin{aligned} p_s \cdot \sum_{h\in H}(x^h_s - e^h_s) &= \sum_{j\in J}(1-K_{sj})p_s \cdot R_{sj}\varepsilon_j + \sum_{j\in J}\sum_{h\in H} K_{sj}p_s \cdot R_{sj}(\theta^h_j - \varphi^h_j) \\ &\leq \sum_{j\in J}(1-K_{sj})p_s \cdot R_{sj}\varepsilon_j + J||R||_\infty b||e_0||_\infty. \end{aligned}$$

Suppose $\sum_{h\in H}(x^h_{s\ell} - e^h_{s\ell}) - \sum_{j\in J}(1-K_{sj})R_{s\ell j}\varepsilon_j > 0$ for some $s \in S$. Since we are at a fixed point, the price player cannot increase the value of excess demand in state s by taking $\tilde{p}_{s\ell} = 1-(L-1)b$, and $\tilde{p}_{sk} = b$ for all $k \neq \ell$. Hence

$$\begin{aligned} &\sum_{h\in H}(x^h_{s\ell} - e^h_{s\ell}) - \sum_{j\in J}(1-K_{sj})R_{s\ell j}\varepsilon_j \\ &\leq \frac{1}{1-(L-1)b}\left\{(L-1)b\left[||e_0||_\infty + ||R||_\infty \sum_{j\in J}\varepsilon_j\right] + J||R||_\infty b||e_0||_\infty\right\}. \end{aligned}$$

We now let $b \to 0$. We argue that π_j must remain bounded as $b \to 0$. If $Q^h_j = 0$ $\forall h$, then replace π_j with 1. Otherwise, if $\pi_j \to \infty$, any agent h with $Q^h_j > 0$ could replace his entire action by selling Δ units of j, buying M $(\leq \Delta\pi_j/L)$ units of each period 0 good, and delivering fully. Since $e^h_s \neq 0$ for all s, and commodity price ratios are bounded in each state, agent h can do this without incurring any default. But this gives him utility that exceeds $u^h(2\sum_{h'} e^{h'})$, which is more than he can possibly be getting at the fixed point (with all excess demands close to zero for small b), a contradiction. Thus all asset prices are bounded.

Since all choices and all macrovariables are uniformly bounded for small b, we can pass to convergent subsequences, obtaining $\bar{E} \equiv \langle \bar{p}, \bar{\pi}, \bar{K}, (\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h, \bar{D}^h)_{h\in H}\rangle$ as a limit point. Taking the limit of all inequalities derived above, we conclude that aggregate excess demand for commodities and assets is less than or equal to zero in E. It follows that the limit price ratios $\bar{p}_{s\ell}/\bar{p}_{sk}$ are bounded in each state $s \in S^*$. If $p_{s\ell}/p_{sk}$ became unbounded as $b \to 0$, some agent with $e^h_{s\ell} > 0$ could have consumed M units of commodity sk, obtaining more utility than $u^h(2\sum_{h' \in H} e^{h'})$, for all small b; but since excess demand goes to zero with b, $x^h < 2\sum_{h' \in H} e^{h'}$ for small enough b, contradicting that h has optimized. Thus the limiting $\bar{p} \gg 0$, and all agents have positive income in every state in $\bar{E}$. Thus individuals are optimizing in $\bar{E}$ on their untruncated budget sets. (This uses the concavity of ω^h and the fact that eventually the constraints on x and θ are not binding in his budget set.)

Note finally that if all commodity prices are positive, there cannot be excess supply in any commodity in $\bar{E}$, otherwise the price player would be making negative profits. For the same reason there cannot be excess supply of any asset j in $\bar{E}$, unless $\pi_j = 0$. But then no agent would sell j unless $\lambda^h_{sj} R_{sj} = 0$ for all $s \in S$. Without loss of generality we may in this case take $\theta^h_j = \varphi^h_j = 0$ for all h.

Thus we have shown that $\bar{E}$ is an ε-trembling hand equilibrium. Letting $\varepsilon \to 0$ and taking limits we obtain a trembling hand equilibrium. This proves the theorem for finite penalties λ.

If some penalties are infinite, we take limits of equilibria with increasing penalties. Since all actions must stay bounded along the sequence (because $Q^h_j < \infty$), any cluster point of these equilibria will serve as the desired trembling hand equilibrium.. ■

Our proof has used the fact that $\varphi^h_j \leq Q^h_j$ by assumption. Later the Q^h_j will play an important role as signals, but now the reader may wonder what would happen if they were eliminated, or taken to be enormously large. Recall that there is a pathology that occasionally occurs even when there is no default, for example in the GEI model. Sometimes two assets j and j' that promise different commodities nevertheless become nearly equivalent at some spot prices $(p_s)_{s \in S}$ because they then promise nearly the same money. At these prices the number of independent assets suddenly drops, and demand blows up as agents try to go infinitely long in asset j' and infinitely short in asset j (or vice versa). This destroys the existence of equilibrium. The bounds Q^h_j prevent this, as Radner (1972) long ago pointed out for the GEI model.

In the GEI model without short sale constraints like the Q^h_j, equilibrium can only be guaranteed if all the assets promise payoffs exclusively in the same good (say L) in each state $s \in S$. (See Geanakoplos–Polemarchakis (1986).) The asset matrix R then is effectively reduced to $S \times J$ dimensions.

Default provides another reason why two assets that make different promises might, given certain macro variables (p, π, K), actually deliver the same money in every state. One should therefore wonder if default introduces additional difficulties in proving the existence of equilibrium. We have just seen that in the presence of the bounds Q^h_j it does not. Now we shall show that default also does not complicate the existence picture without the bounds Q^h_j.

Theorem 2 *Let all promises R_j be exclusively in good L for all $s \in S$. Define $GE(R,\lambda) = GE(R,\lambda,Q)$ with $Q^h_j = \infty$, $\forall h \in H$, $j \in J$. Then $GE(R,\lambda)$ exists for any vector $\lambda \in \overline{\mathbb{R}}^{HSJ}_{+}$ with $\lambda \gg 0$.*

Proof Theorem 2 specializes the conditions of Theorem 1. Hence we have a $GE(R,\lambda,Q)$ equilibrium for all finite Q. Consider a sequence of equilibria, $\eta(Q) = (p(Q), \pi(Q), K(Q), (x^h(Q), \theta^h(Q), \varphi^h(Q), D^h(Q))_{h\in H})$, where $Q^h_j = Q \in \mathbb{N}$, for all $h \in H$, $j \in J$.

If there is a single Q with $\varphi^h_j(Q) < Q$, for all $h \in H$, $j \in J$, then by the concavity of each w^h, $\eta(Q)$ is a $GE(R,\lambda)$.

Passing to a convergent subsequence if necessary, we may suppose that for all $h \in H$ and $j \in J$,

$$\frac{\theta^h_j(Q)}{Q} \to \bar{\theta}^h_j, \ \frac{\varphi^h_j(Q)}{Q} \to \bar{\varphi}^h_j.$$

Moreover, we might as well assume that for at least one j and some h and h', $\bar{\theta}^h_j \neq 0$ and $\bar{\varphi}^{h'}_j = 1$.

For notational convenience, we shall write R_{sj} and D_{sj}, instead of the more accurate R_{sLj} and D_{sLj}, and we shall suppose that real default in each state $s \in S$ is measured in terms of the commodity bundle $v_s = 1_L$, which is one in the Lth coordinate, and zero elsewhere. Since all assets are exclusively delivering in the Lth good, no harm results from these simplifications. Finally, w.l.o.g. take $p_{sL} = 1$ for all $s \in S$.

Observe that for any $h \in H$, $s \in S$, $j \in J$, the level of default

$$d^h_{sj}(Q) \equiv [R_{sj}\varphi^h_j(Q) - D^h_{sj}(Q)]^+ \leq \frac{1}{\lambda^h_{sj}}[u^h(e) - u^h(e^h)],$$

for otherwise agent h would have done better not trading at all. (At any $GE(R,\lambda,Q)$, $x^h \leq \sum_{h'} e^{h'} \equiv e$.) Hence if $\varphi^h_j(Q) \to \infty$,

$$\frac{[R_{sj}\varphi^h_j(Q) - D^h_{sj}(Q)]}{\varphi^h_j(Q)} = \frac{[R_{sj}\varphi^h_j(Q) - D^h_{sj}(Q)]^+}{\varphi^h_j(Q)} = \frac{d^h_{sj}(Q)}{\varphi^h_j(Q)} \to 0.$$

It follows that $K_{sj}(Q) \to 1$ for all $s \in S$ with $R_{sj} > 0$, provided that $\sum_{h\in H} \varphi^h_j(Q) = \sum_{h\in H} \theta^h_j(Q) \to \infty$.

Furthermore, since relative prices $p_{s\ell}(Q)/p_{sk}(Q)$ stay bounded,

$$\sum_{j\in J} K_{sj}(Q)R_{sj}\theta^h_j(Q) - \sum_{j\in J} D^h_{sj}(Q)$$

must stay bounded. Otherwise agent h would eventually be consuming a negative quantity in state s, or a quantity exceeding the aggregate endowment e_s, contradicting commodity market clearing.

Putting these last statements together, we must have that

$$\lim_{Q\to\infty} \frac{\sum_{j\in J} K_{sj}(Q)R_{sj}\theta_j^h(Q) - \sum_{j\in J} D_{sj}^h(Q)}{Q} = R_s(\bar{\theta}^h - \bar{\varphi}^h) = 0,$$

for all $h \in H$, $s \in S$.

Consider any h with $\bar{\varphi}^h \neq 0$, and hence $\bar{\theta}^h \neq 0$. For any $Q \geq 1$,

$$\begin{aligned} \hat{\theta}^h &= \theta^h(Q) - \bar{\theta}^h \geq 0 \\ \hat{\varphi}^h &= \varphi^h(Q) - \bar{\varphi}^h \geq 0. \end{aligned}$$

At any large Q, the agent could feasibly have chosen

$$\hat{D}_{sj}^h = D_{sj}(Q) - R_{sj}\bar{\varphi}_j^h \geq 0 \text{ for all } j \in J.$$

With these choices he would pay exactly the same penalty as in the equilibrium $n(Q)$. He would receive exactly the same consumption at time 1 if $K_{sj}(Q) = 1$ for all j with $\bar{\theta}_j^h > 0$, and strictly more consumption otherwise. In order for him not to prefer this deviation, we must therefore have

$$\pi(Q)[\bar{\theta}^h - \bar{\varphi}^h] \leq 0 \text{ for all } h \in H.$$

But since $\bar{\theta}^h$ and $\bar{\varphi}^h$ are limits of $GE(R, \lambda, Q)$ equilibrium portfolios,

$$\sum_{h\in H} \bar{\theta}^h = \sum_{h\in H} \bar{\varphi}^h,$$

hence we must have

$$\pi(Q)[\bar{\theta}^h - \bar{\varphi}^h] = 0 \text{ for all } h \in H.$$

It now follows that household h would still prefer this deviation unless $\forall j \in J, \forall s \in S$,

$$[R_{sj} > 0, \text{ and } \bar{\theta}_j^h > 0 \text{ for any } h \in H] \Rightarrow [K_{sj}(Q) = 1].$$

Note finally that if $\bar{\varphi}_j^h > 0$, there must be some agent i with $\bar{\theta}_j^i > 0$, hence $K_{sj}(Q) = 1$ for all $s \in S$ with $R_{sj} > 0$ and either $\bar{\theta}_j^h > 0$ or $\bar{\varphi}_j^h > 0$.

Replacing $(p(Q), \pi(Q), K(Q), (x^h(Q), \theta^h(Q), \varphi^h(Q), D^h(Q))_{h\in H})$ with $(p(Q), \pi(Q), K(Q), (x^h(Q), \hat{\theta}^h, \hat{\varphi}^h, \hat{D}^h)_{h\in H})$ we get another $GE(R, \lambda, Q)$ with $\hat{\varphi}_j^h(Q) < Q$ for all h and j. (Notice that we are reducing sales and purchases only for assets with $K_{sj} = 1$, which therefore leaves the K unchanged.) ■

6 Chain Reactions, Netting, and Supernetting

6.1 Chain Reactions

In modern financial economies, agents often are long and short in many different assets. They rely on revenues from their loans to keep their own promises. But these revenues are only as reliable as the loans other agents have made to yet different parties, thus opening the possibility of a chain reaction of defaults. If α defaults against β, forcing β to default against γ, forcing γ to default against δ, then in our definition of equilibrium, α, β, and γ will pay default penalties, and the total utility loss from defaults will be large. Curiously this phenomenon is at its most dangerous when the financial system is at an intermediate level of development, with smoothly functioning markets that permit agents to go short, but without some finely tuned assets, forcing agents to hold complicated portfolios to achieve the risk spreading they desire.

Consider a world with four agents and three possible future events, each consisting of many different states of the world. Suppose β wants to consume in the first event, γ in the second event, and δ in the third event. Suppose agents β, γ, and δ have no endowment in the future states. Suppose α wants to consume in the present, but has a considerable endowment of goods in the future, except in one unlikely state ω in the third event.

If there were an advanced financial system of Arrow securities, agent α would in effect sell directly to each of the other three agents. For example, with just three Arrow securities, each one paying off exclusively in a different one of the three events, agent α would sell the first security to β, the second to γ, and the third to δ. Agent α by himself would default in state ω, and he alone would pay a default penalty.

Suppose, however, that in a less advanced financial system there are again three securities available. R_{123} promises 1 dollar in every state, R_{23} promises 1 dollar in (every state in) events 2 and 3, and R_3 promises 1 dollar in (every state in) event 3. Then in equilibrium we could expect α to sell R_{123}, β to buy R_{123} and to sell R_{23}, γ to buy R_{23} and to sell R_3, and δ to buy R_3. In the bad state ω in event three, the chain of defaults indicated above will take place. The penalty that α pays for starting the chain reaction may be very small compared to the total penalty incurred by the rest of the defaulters.

A diagram may make the situation clearer.

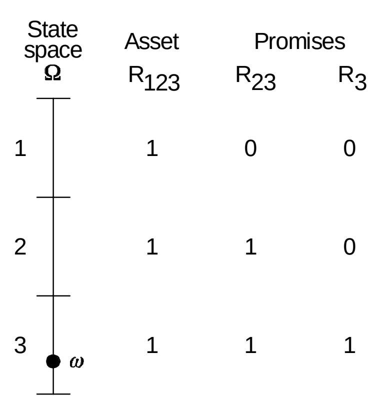

Figure 2

Notice that the asset span is exactly the same as with the three Arrow securities.

What makes the chain of defaults possible is the interlocking asset trade, with investors receiving and delivering in a long chain, in some state. With Arrow securities this chain would never reach more than two links and one default.

One way around these chain reactions is to encourage market intermediation that nets payouts.

6.2 Netting

Consider the variation of our $GE(R, \lambda, Q)$ model in which an agent's purchases and sales of any given asset j are netted, so that he is deemed to have purchased $(\theta_j - \varphi_j)^+$ and sold $(\varphi_j - \theta_j)^+$. In this case, the budget-set and payoff function of an agent $t \in (h-1, h]$ of type h are modified as follows:

$$B^h(p, \pi, K) = \Bigg\{ (x, \theta, \varphi, D) : p_0 \cdot (x_0 - e^h_h) + \pi \cdot (\theta - \varphi) \leq 0; \ \varphi_j \leq Q^h_j \text{ for } j \in J;$$
$$p_s \cdot (x_s - e^h_s) + \sum_{j \in J} p_s \cdot D_{sj} \leq \sum_{j \in J} K_{sj} p_s \cdot R_{sj} (\theta_j - \varphi_j)^+ \text{ for all } s \in S \Bigg\}$$

$$w^h(x, \theta, \varphi, D, p) = u^h(x) - \sum_{s \in S} \sum_{j \in J} \frac{\lambda^h_{sj} \left[p_s \cdot R_{sj}(\varphi_j - \theta_j)^+ - p_s \cdot D_{sj} \right]^+}{p_s \cdot v_s}$$

Moreover, if $(x^t, \theta^t, \varphi^t, D^t)_{t \in (0,H]}$ are the equilibrium choices of the agents, then condition (4) on $GE(R, \lambda, Q)$ becomes:

$$K_{sj} = \int_0^H p_s \cdot D^t_{sj} d\mu(t) \Big/ \int_0^H p_s \cdot R_{sj}(\varphi^t_j - \theta^t_j)^+ d\mu(t)$$

whenever the denominator is positive.

Notice that the budget set is no longer convex, hence an equilibrium may not exist in the finite agents model. However we have

Theorem 3 *A $GE(R, \lambda, Q)$ exists in the finite-type continuum model with netting (though it may not be type symmetric).*

We defer the proof until the next section.

6.3 Supernetting

Here we go a step further and consider netting across different assets that an agent has traded in.[13] Now deliveries are no longer made separately on each asset, but there is one combined payment in every state $s \in S$. This extends the idea in Theorem 1 that agents may deliver differently on the same asset because of default or some other option. Here they may even make different promises. Supernetting combines

[13] Institutionally this may be regarded as a clearinghouse.

this extension with netting both sides of a trade. Thus the delivery choice of an agent is a vector $D \in \mathbb{R}^{SL}_+$, and the constraint in t's budget set must be rewritten (where t is of type h)

$$p_s \cdot (x_s - e^h_s) + p_s \cdot D_s \le K_s[\sum_{j \in J} p_s \cdot R_{sj}(\theta_j - \varphi_j)]^+.$$

His payoff is modified to

$$u^h(x) - \sum_{s \in S} \lambda^h_s \left[\left(\sum_{j \in J} p_s \cdot R_{sj}(\varphi_j - \theta_j) \right)^+ - p_s \cdot D_s \right]^+ .$$

Notice that the K_{sj}, λ^h_{sj} are reduced in this setting to K_s, λ^h_s. Finally, the condition on K_s is

$$K_s = \int_0^H p_s \cdot D^t d\mu(t) / \int_0^H \left(\sum_{j \in J} p_s \cdot R_{sj}(\varphi^t_j - \theta^t_j) \right)^+ d\mu(t)$$

whenever the denominator is positive.

Along the same lines as Theorem 3, we have

Theorem 4 *A $GE(R, \lambda, Q)$ exists in the finite-type continuum model with supernetting (though it may not be type symmetric).*

Remark (1) If the constraint Q is removed, then Theorems 3 and 4 still hold provided asset payoffs are designated in a single commodity.

(2) These theorems also hold when the privilege of netting (supernetting) varies with the type of a player. Indeed in this case $GE(R, \lambda, Q)$ will need to have non-type-symmetric behavior only for those types which are allowed to net (supernet).

(3) For Theorems 3 and 4, we have taken default penalties to be separable and linear for the sake of simplicity. But in fact both theorems hold provided only that w^h is continuous, $w^h(x, \theta, \varphi, D, p) \le u^h(x)$, and $w^h(x, \theta, \varphi, D, p) = u^h(x)$ if h never defaults.

Proofs of Theorems 3 and 4 We prove Theorem 3. The proof of Theorem 4 is similar. We observe first that since $[\alpha x]^+ = \alpha[x]^+$, the budget sets are lower semi continuous exactly as in the proof of Theorem 1: let $(\bar{x}, \bar{\theta}, \bar{\varphi}, \bar{D}) \in B^h(\bar{p}, \bar{\pi}, \bar{K})$, and let $(p(\varepsilon), \pi(\varepsilon), K(\varepsilon)) \xrightarrow{\varepsilon} (\bar{p}, \bar{\pi}, \bar{K})$, where $\bar{p} \gg 0$. Take $\alpha < 1$ and $(x(\varepsilon), \theta(\varepsilon), \varphi(\varepsilon), D(\varepsilon)) = (\alpha\bar{x}, \alpha\bar{\theta}, \bar{\varphi}, \alpha\bar{D})$. For ε near 0, these points are clearly budget feasible, because of the scaling property that holds when $\bar{p}_s \cdot e^h_s > 0$, $\forall s \in S^*$. Since α was taken to be arbitrary, the budget is LSC.

But now we can repeat the rest of the proof of Theorem 1, replacing ψ^h_ε by $\text{conv}(\psi^h_\varepsilon)$. By Kakutani's theorem there is a fixed point $(p(\varepsilon), \pi(\varepsilon), K(\varepsilon), (x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon))_{h \in H})$ of ψ_ε. By Caratheodory's theorem $(x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon)) \in \text{conv}(\psi^h_\varepsilon(p(\varepsilon), \pi(\varepsilon), K(\varepsilon)))$ is in the convex hull of at most $n = S^*L + J + J + SL + 1$

points $(x^{hi}(\varepsilon), \theta^{hi}(\varepsilon), \varphi^{hi}(\varepsilon), D^{hi}(\varepsilon))_{i=1}^{n}$ in $\psi_{\varepsilon}^{h}(p(\varepsilon), \pi(\varepsilon), K(\varepsilon))$. Passing to convergent subsequences as $\varepsilon \to 0$ gives a $^*GE(R, \lambda, Q)$ equilibrium for the continuum economy, in which each type h displays at most n different (but indifferent!) behaviors.

7 The Economic Advantages of Intermediate Default Penalties with Incomplete Markets

There are four fundamental drawbacks to reducing the default penalties λ so far that some agents choose to default in at least some states in equilibrium: (1) creditors, rationally anticipating (on account of direct and indirect reasons) that they might not be repaid, are less likely to lend; (2) borrowers may not repay even in contingencies that have been foreseen, and even though they are able; (3) imposing penalties is a deadweight loss; (4) the default of unreliable agents imposes an externality on reliable agents who, because they cannot distinguish themselves from the unreliable agents, are forced to borrow on less favorable terms.

Akerlof regarded the fourth (externality) cost of default as so important that for this reason alone he suggested it would always be worthwhile to reduce default by imposing penalties on defaulters. By analogy one could ask manufacturers of products to issue guarantees to replace any defective parts, and in addition to pay for all damages caused by defective parts.

Our second goal in this paper is to show that despite myriad reasons why default is socially costly, the benefits from permitting some default often outweigh all of these costs. The benefits from allowing default are basically twofold, and both stem from the fact that markets are incomplete to begin with. First, an agent who defaults on a promise is in effect tailoring the given security and substituting a new security that is closer to his own needs, at a cost of the default penalty. With incomplete markets one set of assets may lead to a socially more desirable outcome than another set. Second, since *each* agent may be tailoring the same given security to his special needs, one asset is in effect replaced by as many assets as there are agents, and so the dimension of the asset span is greatly enlarged. A larger asset span is likely to improve social welfare (although this gain must be weighed against the deadweight loss of the default penalties that are thereby incurred). In short, permitting default allows for a plethora of additional assets that do not have to be specified in advance. Each agent can tailor the simple standard contract to fit his idiosyncratic situation.

A third benefit from allowing default, which is closely related to the first two, is that when there is no netting, agents can go long and short in the same security, thereby doubling their asset span. We make use of this in the following example. (The examples could be presented with netting, or supernetting, but then we would need more assets and a more cumbersome analysis to make the same points.)

Let there be three agents and $S = 3$ states of nature, let there be one good in each state, $L = 1$, and suppose agents have no utility for consumption at $t = 0$. Each agent has the same utility

$$u(x_1, x_2, x_3) = \sum_{s=1}^{3} \log(x_s).$$

The endowments of the agents are

$$e^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; \; e^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \; e^3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We take the collection of asset promises to be

$$R_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \; R_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \equiv 1^1; \; R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv 1^2; \; R_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv 1^3.$$

We take default penalties to be one of three types:

$$\lambda^h_{sj} = \infty, \; \forall h, s, j; \; \lambda^h_{sj} = \lambda > 0, \; \forall h, s, j; \text{ or}$$

$$\lambda^h_{sj} = \begin{cases} \infty & \text{if } e^h_s = 1 \\ 0 & \text{if } e^h_s = 0 \end{cases}.$$

We take

$$Q^h_j = \infty \;\; \forall h, j.$$

Notice that the first two penalties are completely anonymous, since they are the same whatever the name of the defaulter, and whatever his circumstances. The last penalty type is infinite when agents have the resources to pay, and 0 otherwise. They do not depend on the name of the defaulter, but they do depend on his circumstances; they require more information to carry out. The information required is identical to the sort of information an insurance company must obtain to verify that an accident has occurred. Indeed in our sequel paper we shall use these penalties precisely in order to render insurance a special case of default.

Version A0: Arrow Assets: Pure Promises, Infinite Penalties, and Infinite Quality Constraints

The Arrow–Debreu equilibrium in our example can easily be calculated as $p = (1, 1, 1)$ and $x^h = (2/3, 2/3, 2/3)$, $\forall h \in H$. It can be implemented as a $GE(R, \lambda, Q)$ if $\mathcal{A}$ consists of the three Arrow assets $j = 1, 2, 3$. Let $R_j = 1^j$, where 1^j is the jth unit vector, be the pure promise for state j, and let the default penalties and sales constraints be set at infinity, $\lambda^h_{sj} = Q^h_j = \infty$, $\forall h \in H$, $s \in S$, $j \in J$. The equilibrium is given by $(p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h \in H})$ where $p = (1, 1, 1)$, $\pi = (1, 1, 1)$, $K_{sj} = 1$, $\forall sj$, $x^h = (2/3, 2/3, 2/3)$, $\theta^h = 2/3 \cdot 1^h$, $\varphi^h = 1/3 \cdot e^h$, and $D^h_{sj} = 1/3$ if $h \neq s = j$, and 0 otherwise.

In the $GE(R, \lambda, Q)$ equilibrium just described, the volume of trade is $2/3$ in each of the three asset markets. Notice that there is some trivial multiplicity in the equilibria, since agents could engage in wash sales and buy and sell the same asset. We could instead have taken $\theta^h = (2/3, 2/3, 2/3)$, $\varphi^h = e^h$, which has volume of trade equal to 2 in each of three asset markets. However, even with the tiniest of transactions costs, wash sales would be eliminated, and the volume of trade would fall to 2/3.

Version A1: The Optimal Default Penalty with Incomplete Markets

In Version A0 we found that setting $\lambda^h_{sj} = \infty$ gave a Pareto efficient outcome, because it eliminated default. Setting $\lambda^h_{sj} = \lambda < \infty$ would have led to a Pareto worse outcome. Nevertheless, we shall argue in this section that when markets are incomplete, it is often better to set intermediate default penalties. In Version A0, markets for risk sharing were effectively complete.

Consider the economy as in A0, but with only one asset $R_0 = (1, 1, 1)$. Suppose that the reason for default cannot be observed, so $\lambda^h_{sj} = \lambda$, $\forall h$, s, j. Agents who promise delivery but do not have the good will default and suffer the penalty. Anticipating this they will make fewer promises, and risk-sharing will be reduced.

We can calculate the equilibrium and agent utilities for any value of $\lambda \in [0, \infty]$. When $\lambda = 0$ buyers realize that sellers will not deliver anything, so demand will be zero and equilibrium will involve no trade. When $\lambda \to \infty$ buyers will anticipate full delivery, but sellers will realize that with probability 1/3 they will not be able to avoid a crushing penalty, and so again equilibrium trade goes to 0. By setting an intermediate level of default penalties we can make everybody better off. We graph the situation schematically in welfare space:

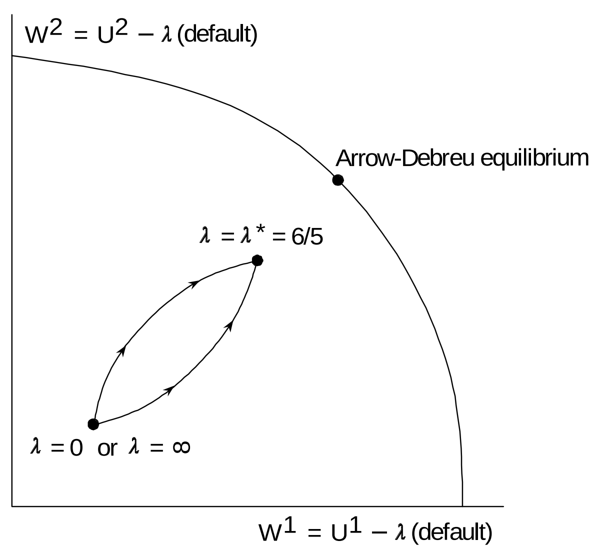

Figure 3

It is worth noting that in our equilibrium different sellers default differently. (Agent h defaults in state $s = h$). The buyers of the asset receive the average deliveries of all the sellers, which works out to 2/3 in every state, when $\lambda = \lambda^* = 6/5$.

Thus our example illustrates the pooling aspect of assets, namely that investors buy shares of a pool of individually sold promises.

A consequence of pooling is that the volume of trade is high. In equilibrium (when $\lambda = 6/5$), each agent sells 1/2 unit of the asset, giving a total volume of trade equal to $3 \cdot 1/2 = 3/2$, much greater than the volume of trade per asset in the Arrow–Debreu equilibrium.

We now proceed to compute equilibrium for all values of λ.

Note first that since the only object traded in period 0 is the asset R_0, we can always take its price $\pi_0 = 1$. Final consumption for agent $h = 1$ will be

$$x^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} D^1_{10} \\ D^1_{20} \\ D^1_{30} \end{pmatrix} + \begin{pmatrix} K_{10}1 \\ K_{20}1 \\ K_{30}1 \end{pmatrix} \theta^1_0.$$

By symmetry we can guess $K_{s0} = K$, and $D^h_{h0} = 0$, $D^h_{s0} = D$, if $h \neq s$, and $\theta^h_0 = \varphi^h_0 = \theta = \varphi$. In any state, two agents will be delivering D, and since all three will be promising $\varphi = \theta$, we must have $K = 2D/3\varphi$. Hence

$$x^1 = \begin{pmatrix} \frac{2}{3}D \\ 1 - \frac{1}{3}D \\ 1 - \frac{1}{3}D \end{pmatrix}.$$

When $0 \leq \lambda < 1$, $K_{s0} = 0$, $\forall s$, and $\theta^h = \varphi^h = 0$. For $\lambda \geq 1$, let us guess that each agent delivers precisely up to the point where the marginal utility of consumption equals λ, defaulting on the rest of his promises. Then $x^1_2 = x^1_3 = 1/\lambda$, and so $D = 3 - 3/\lambda$. Consumption for agent $h = 1$ (the other h are handled symmetrically) is then

$$x^1 = \begin{pmatrix} 2(1 - 1/\lambda) \\ 1/\lambda \\ 1/\lambda \end{pmatrix}.$$

The marginal utility to buying a unit more of the asset is then

$$MU(\text{buyer}) = K\left[\frac{1}{2(1 - 1/\lambda)} + \lambda + \lambda\right]$$

and, since the agent is defaulting on the margin in all three states, the marginal disutility to selling a unit is

$$MDU(\text{selling}) = 3\lambda.$$

These two must be equal in equilibrium, hence

$$K = \frac{3\lambda}{\frac{\lambda}{2(\lambda-1)} + \lambda + \lambda} = \frac{6\lambda - 6}{4\lambda - 3}.$$

Moreover

$$\varphi = \frac{2}{3}\frac{D}{K} = \frac{(2 - \frac{2}{\lambda})\left(\frac{1}{2 - \frac{2}{\lambda}} + \lambda + \lambda\right)}{3\lambda} = \frac{4}{3} - \frac{1}{\lambda}.$$

Notice that φ, D, and K are monotonically increasing in λ. For $1 \leq \lambda < 6/5$, $D < \varphi$, confirming that we have guessed a genuine equilibrium. Note that at $\lambda = 1$, $D = 0$, and the only equilibrium involves no trade. Because marginal utility is infinite at the no trade point, trade jumps immediately for $\lambda > 1$ to $\varphi = 4/3 - 1/\lambda$. As λ rises to $\lambda = 6/5$, φ rises to $1/2$ and D rises to $1/2$, and K rises to $2/3$.

At $\lambda = 6/5$, $x^1 = (1/3, 5/6, 5/6)$, $x^2 = (5/6, 1/3, 5/6)$ and $x^3 = (5/6, 5/6, 1/3)$. By buying and selling $1/2$ unit of the asset R_0, agent h gains $1/3$ when $s = h$ and loses $1/6$ in the two states $s \neq h$. Agent h delivers fully when $s \neq h$ because his marginal utility of consumption after delivery is $1/(5/6) = 6/5 = \lambda^*$. When $s = h$, agent h defaults completely since his marginal utility of consumption $1/(1/3) = 3 > \lambda^*$. Since for any $s \in S$ we have 2 agents with $h \neq s$, $K_{s0} = 2/3$. Thus the asset promise $R_0 = (1, 1, 1)$ actually delivers $(2/3, 2/3, 2/3)$ per unit promise. Agent $h = 1$ delivers $1/2 \cdot (0, 1, 1)$, agent $h = 2$ delivers $1/2 \cdot (1, 0, 1)$, and agent $h = 3$ delivers $1/2 \cdot (1, 1, 0)$. The reason each agent buys and sells only $1/2$ a unit of asset R_0 instead of a full unit to get to the Arrow–Debreu allocation is that the sale of φ units of the asset is accompanied by the loss of $\varphi\lambda$ utiles for the inevitable default in state $s = h$. The marginal utility from buying the asset is $(2/3)(6/5) + (2/3)(6/5) + (2/3) \cdot (2/3) \cdot (3) = 18/5$; the marginal disutility from selling is also $(6/5) + (6/5) + (6/5) = 18/5$. (It is therefore more convenient to take $\pi_0 = 18/5$.)

For $\lambda \geq 6/5$, the agents always deliver if they have the goods on hand. Thus for $\lambda > 6/5$ we can no longer maintain our guess that agents default until marginal utility equals λ. When λ is increased beyond $\lambda^* = 6/5$, marginal utility is less than λ in the good state, and K_{s0} is maintained at $2/3$, but asset trade again begins to drop because the inevitable punishment makes selling less attractive. The formulas are messy and we do not bother to present them here. An increase in the penalty rate beyond $\lambda = 6/5$ does not improve risk bearing (since φ begins to drop), and it also increases the deadweight loss from punishing agents who cannot deliver anyway. It thus strictly lowers welfare.

Furthermore, observe that as λ rises from 1 to $\lambda = 6/5$, the deadweight utility loss from default

$$\lambda\varphi + 2\lambda(\varphi - D) = \frac{4}{3}\lambda - 1 - \frac{10}{3}\lambda + 4 = 3 - 2\lambda$$

actually falls, to $3/5$. Since the allocation is improving, and the default penalty is falling, we deduce that $\lambda^* = 6/5$ leads to the Pareto best outcome among all economies with $\lambda^h_{sj} = \lambda$.

Example A1 illustrates that the optimal default penalty might be low enough to encourage some real default, and the attendant deadweight loss, when markets are incomplete. It also illustrates that the possibility of default makes the asset payoffs endogenous, since we do not know before an equilibrium is calculated what the default rates will turn out to be. If we change the utilities or endowments of the agents, or the default penalties, the equilibrium will change, the default rates will change, and the asset payoffs will be different.

8 Trading Costs and Incomplete Markets

Even if assets and penalties could be chosen simultaneously, there is good reason to suppose that not every Arrow security would be actively traded. In practice, that would be much too costly. In the next two subsections we formalize two kinds of costs.

Agents often differ more in their idiosyncratic selling than in their buying. All risk averse agents, for example, prefer to buy riskless consumption over risky consumption with the same expected payoff. By the same token, agents have idiosyncratic endowments and income streams, so they each have different objects to sell.

If every agent tried to market a personalized asset, tailor made to his needs, buyers would be confronted with a bewildering array of choices. The information processing and evaluation costs would be prohibitive, forcing each buyer to consider only a few of the assets. In addition, transactions costs would also limit the number of assets that could be purchased. And these costs would be all the higher because every market would be thinly traded, with only a few buyers and just one seller.

In the real world promises are standardized, enabling liquid markets, even though deliveries are idiosyncratic. Thus two agents take out the same insurance policy, under the same terms, even though it is perfectly understood that payments on each will come in different states of the world. We present a concrete example to illustrate ideal pooling, and to show how our model of default encompasses insurance. The example shows that in some cases the Arrow–Debreu equilibrium can be achieved with just one asset.

Version A2: The Advantages of Standardized Pooled Assets

Consider the economy described in Section 7 with $H = \{1, 2, 3\}$ and with just one asset $R_0 = (1, 1, 1)$. Suppose that the default penalties are

$$\lambda_{sj}^h = \begin{cases} \infty & \text{if} \quad s \neq h \text{ (i.e., if } e_s^h = 1) \\ 0 & \text{if} \quad s = h \text{ (i.e., if } e_s^h = 0) \end{cases}$$

that is, default penalties are infinite when agents have the resources to pay, and 0 otherwise. We might interpret state s as the state in which a bad accident happens to agent $h = s$.

Let $\pi_0 = 3$, $p = (1, 1, 1)$. Agent h buys *and* sells 1 unit of the asset, delivering fully when his endowment is 1, and defaulting completely when his endowment is 0. The upshot is that on net, agent h has effectively bought an insurance contract. Indeed every agent has formally obtained the same insurance contract (by virtue of making identical asset trades) but each has insured his own idiosyncratic risk.

Since in every state two agent types deliver and the other type defaults, $K_{s0} = 2/3$, $\forall s \in S$. Consumption by h in the state $s = h$ where he has no endowment is thus $K_{s0}\theta_{s0}^h R_{s0} = (2/3)(1)(1) = 2/3$. Consumption in the other states where he delivers is $e_s^h + K_{s0}\theta_{s0}^h R_{s0} - D_{s0}^h = 1 + 2/3 \cdot (1)(1) - 1 = 2/3$. We verify that this is a $GE(R, \lambda, \infty)$ equilibrium by noting that the marginal utility of owning an extra unit of the asset is $\sum_{s=1}^{S}(\partial u/\partial x_s)K_{s0}R_{s0} = \frac{3}{2}\left(\frac{2}{3}\right) + \frac{3}{2}\left(\frac{2}{3}\right) + \frac{3}{2}\left(\frac{2}{3}\right)$, which is equal to the marginal disutility of selling the asset $\sum_{s=1}^{S} R_{s0} \min\left[\frac{\partial u}{\partial x_s}, \lambda_{s0}^h\right] = \frac{3}{2}(1) + \frac{3}{2}(1) + 0$, where $3/2 = [d\log(2/3)]/dx$ is the marginal utility of consumption in each state.

Version A2 seems at first glance like an artificial example,because the penalties themselves are idiosyncratic. But, as we said earlier, they are no more idiosyncratic than insurance contracts.

8.1 Transactions and Liquidity Costs

Trade in any market usually involves some sort of transactions cost. The "competitive market" itself is an abstraction of a complicated set of interrelationships between brokers, middlemen, buyers and sellers, and it should come as no surprise that final buyers do not receive the full value of what sellers give up. As a first approximation, we can represent this wedge by a simple utility loss to transacting, proportional to the quantity of the transaction. We can, however, be a little bit more specific about which asset markets are likely to have higher transactions costs.

When an asset is very finely defined, so as to pay off in exactly those states that a particular small group of people is interested in, then it is not likely to be heavily traded. A seller may have to wait a long time to find a suitable buyer and vice versa. And when such a buyer is found, he will exercise some temporary monopsony power. We say that the market lacks liquidity. As a first approximation we can incorporate liquidity costs simply by assuming that more utility is lost in transactions in less liquid markets.

With liquidity costs and other trading costs, we still need to consider non type-symmetric equilibria, in which actions of two agents t and t' of the same type h may be different. Let

$$x_j^h = \int_{h-1}^{h} x_j^t dt,\ \theta_j^h = \int_{h-1}^{h} \theta_j^t dt,\ \varphi_j^h = \int_{h-1}^{h} \varphi_j^t dt,\ D_{sj}^h = \int_{h-1}^{h} D_{sj}^t dt$$

for all $h \in H$, $j \in T$. Let $\xi^t = (x^t, \theta^t, \varphi^t, D^t)$ be the actions of agent $t \in (0, H]$. For each asset $j \in J$, denote the total volume of purchases and sales by $\theta_j = \sum_{h\in H} \theta_j^h$, and $\varphi_j = \sum_{h\in H} \varphi_j^h$. Then we can define the utility of agent $t \in (h-1, h]$ at $\eta = (p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h\in H})$ by

$${}^*W^t(\xi^t, \eta) = w^t(x^t, \theta^t, \varphi^t, D^t, p) - \sum_{j=1}^{J} \theta_j^t a_j^h(\varphi_j) - \sum_{j=1}^{J} \varphi_j^t b_j^h(\theta_j)$$

where a_j^h and b_j^h are continuous, nonnegative, decreasing functions. When a_j^h and b_j^h are constant functions, we get a simple transactions costs economy. When they are strictly decreasing in the quantity of trade on the other side of the market, they indicate that part of the transactions costs are due to the difficulty of finding an agent with whom to trade.

${}^*GE(R, \lambda, Q)$ equilibrium is defined exactly the same way as $GE(R, \lambda, Q)$ equilibrium, except that ${}^*W^t$ replaces w^t for each agent t. Each agent t regards θ and

φ as fixed when he ponders changing his θ^t and φ^t. Under these circumstances, equilibrium always exists.

Theorem 5.1 *Under the conditions of Theorem* 1*, a* $^*GE(R, \lambda, Q)$ *equilibrium exists, which is type-symmetric.*

Proof The theorem needs no additional proof, since under the hypothesis that each agent regards himself as so small that he cannot affect either θ or φ, his payoff $^*W^t$ is still a concave function of his choice variables. ■

The advantages of conducting trade through large, standardized, liquid markets as opposed to many specialized markets becomes quite clear when we consider transactions costs. It is possible to standardize contracts to some degree because idiosyncratic default on the same standardized contract can sometimes offer almost the same flexibility as completely separate contracts, as we saw in example Version A2.

If we introduced high liquidity costs into the example, efficiency gains of the allocation with default described in Version A2 would be quite striking compared to the complete markets allocation described in Version A0. The same allocation is achieved via one asset, instead of via three assets. And the liquidity of the single asset is 3, instead of 2/3 for each of the three assets in Version A0.[14] Evidently it is socially preferable to have many agents sell the same promise to deliver a dollar unconditionally, and then default in the idiosyncratic states where they cannot pay, rather than to define a separate, idiosyncratic asset for each agent, which only he will sell.

We turn now to an equally important source of trading costs.

8.2 Information and Evaluation Costs

When an agent considers buying a contingent asset he must think carefully about its implications. This computation cost is usually highly nontrivial in practice, and causes most people to shy away from most securities. As a first approximation we can formalize this cost by subtracting a fixed entry cost for buying or selling an asset:

$$^{**}W^t(\eta) = {}^*W^t(\xi^t, \eta) - \sum_{j=1}^{J} \bar{a}_j^h(R_j, (K_{sj})_{s\in S}, \varphi_j)\chi(\theta_j^t) - \sum_{j=1}^{J} \bar{b}_j^h(R_j, (\lambda_{sj}^h)_{s\in S}, \theta_j)\chi(\varphi_j^t)$$

where $\chi(x) = 1$ if $x > 0$, and 0 otherwise, and $\bar{a}_j^h$ and $\bar{b}_j^h$ are continuous, nonnegative functions which are decreasing in θ_j and φ_j, respectively.

[14] We could formally modify our example to incorporate transaction costs $a_j^h(\theta_j)$ and $b_j^h(\varphi_j)$ which are very high for small volumes of trade, but decline (continuously) to near 0 as θ_j and φ_j approach 3. Then the equilibrium described in Version A2 would indeed be very close to a genuine equilibrium with transactions costs (which we do not bother to compute). This equilibrium easily Pareto dominates what could also be accomplished with Arrow securities. Observe that with wash sales, the Arrow securities could be traded in large enough volume to nearly eliminate transactions costs. But as long as the transactions cost is positive, no matter how small, no agent will engage in these wash sales, rendering the Arrow securities prohibitively expensive.

The "evaluation" costs for buyers $\bar{a}_j^h$ and for sellers $\bar{b}_j^h$ of studying a security may depend on the sources of its riskiness. For example, it may be harder to think through contingent defaults than contingent promises. To allow for these possibilities we made the cost of evaluation depend on the rates of payment, as well as on the promises. Furthermore, a large volume of trade in a market may make it very easy to learn about the security, whereas a small volume of trade means an esoteric security for which it is hard to find an expert who can explain it. Thus we also allowed the fixed cost to depend on the liquidity of the market.

Notice that the payoff function $^{**}W^h$ is not continuous at η involving 0 trade in some asset. However, it does have two properties:

(1) $(\xi^t(n), \eta(n)) \to (\xi^t, n) \Rightarrow \limsup {}^{**}W^h(\xi^t(n), \eta(n)) \leq {}^{**}W^h(\xi^t, \eta)$

(2) Let $\eta(n) \equiv (p(n), \pi(n), K(n), (x^h(n), \theta^h(n), \varphi^h(n), D^h(n))_{h\in H})$ converge to $\eta \equiv (p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h\in H})$. Let α_n increase monotonically to 1, and suppose for some agent t, $\xi^t(n) \equiv (x^t(n), \theta^t(n), \varphi^t(n), D^t(n)) = \alpha_n(x^t, \theta^t, \varphi^t, D^t) \equiv \alpha_n \xi^t$. Then $\lim_{n\to\infty} {}^{**}W^h(\xi^t(n), \eta(n)) = {}^{**}W^h(\xi^t, \eta)$.

The first property says that the payoff can only jump up at the limit. (With information and evaluation costs, this happens when trade in some asset becomes 0.) The second property says that when a household's actions converge from below, its payoffs are continuous.

We define $^{**}GE(R, \lambda, Q)$ exactly as $GE(R, \lambda, Q)$ was defined, except that $^{**}W^t$ replaces w^h for all $h \in (h-1, H]$. Again equilibrium must exist.

Theorem 5.2 *Suppose that the payoffs $^{**}W^t$ satisfy conditions* (1) *and* (2) *above. In the finite-type, continuum model, $^{**}GE(R, \lambda, Q)$ equilibrium always exists (though it may not be type-symmetric).*

Sketch of Proof The fixed cost of buying an asset destroys both the continuity of $^{**}W^t$ and its concavity, so at first glance it seems to compromise our existence proof. But in fact on closer inspection one sees that demand is still upper semi continuous, because utility jumps up at zero demand, and zero demand is always feasible. To be slightly more precise, let us abuse notation and use a transparent shorthand for the budget set, demand, and the macro variables. Let x^n be optimal in the budget sets $B^t(p^n) = B^h(p^n)$, where $t \in (h-1, h]$, and let p^n approach p and x^n approach x. If x is not optimal in $B^h(p)$ for $^{**}W^t = {}^{**}W^h$, then there must be some y in $B^h(p)$ with $^{**}W^h(y) > {}^{**}W^h(x)$. But from the proof of lower semicontinuity of the budget set in Theorems 1 and 4, we know that we can approximate y by points $y^n = \alpha_n y$, in $B^h(p^n)$ with $\alpha_n \uparrow 1$. The asset purchases and sales in y^n are less than or equal respectively to the asset purchases and sales in y. Hence if y involves no purchases or no sales of some asset, then so does y^n, and the utilities $^{**}W^h(y^n)$ must approach $^{**}W^h(y)$, using property (2). But for the same reason property (1) holds, and so the utilities $^{**}W^h(x^n)$ approach a number at most equal to $^{**}W^h(x)$. Hence we deduce that x^n is not an optimal demand in $B^h(p^n)$ after all, a contradiction showing that demand is USC.

Since the lack of continuity of $^{**}W^h$ has no effect on the upper semi continuity of demand, the problem is reduced to the lack of concavity of $^{**}W^h$. But that is exactly analogous to the lack of convexity of the budget set, and we dealt with that in Theorems 3 and 4. ■

We now have two reasons (transactions-liquidity costs and information-evaluation costs) why equilibrium cannot support a full set of traded assets. Both reasons give advantages to the standardized contract in Version A2 over the Arrow securities described in A0. In general if we begin with a comprehensive set $\mathcal{A}$ of asset promises, default penalties, and quantity constraints, in equilibrium only a very few of them will be actively traded on account of the trading costs. These will be the endogenously determined assets $\mathcal{A}^*$. The assets in $\mathcal{A}^*$ will be few in number and each one will be far from any Arrow security.

9 Endogenous Asset Structures

In some contexts it has become customary to think of endogenizing the asset structure by allowing atomic agents to invent new assets (often one at a time) to upset a prevailing equilibrium. These asset-creating agents are hypothesized to be motivated by payoffs that might depend on the perceived volume of trade which would take place in their new asset if no other prices changed (or in the new trading equilibrium after all prices equilibrated), or in some other way on their perceived profits from introducing the new asset. When the status quo assets are chosen so that none of these agents has an incentive to introduce a new asset, the asset structure is said to have been endogenously determined. This approach to endogenizing the asset structure almost inevitably involves a combination of comparative price taking behavior and oligopolistic-Nash thinking on the part of the asset creating agents.

By contrast we follow a relentlessly competitive approach to the problem of endogenous assets. Every agent is a price taker. An asset is endogenously missing in our approach if it is not in $\mathcal{A}^*$, i.e., if there is a price at which no agent wants to sell or buy it.

Recall that an asset is specified not just by its vector R_j of promises across states, but also by the associated default penalties λ^h_{sj}, and quantity constraints Q^h_j. If the government could simultaneously and without limitations choose these parameters, it would set them at the Arrowian levels: promises with full span, infinite penalties, and nonbinding quantity constraints. Now we show the market would do the same.

Version A3: Arrow Securities Emerge When Default Penalties Are Infinite

Consider our standard example with $H = \{1,2,3\}$, but with four assets $R_j = 1^j$, $j = 1,2,3$, and $R_0 = (1,1,1)$. Let the penalties be $\lambda^h_{sj} = \infty$ if $j = 1,2,3$, and $\lambda^h_{s0} = \lambda^* = 6/5$ for all h and s. Despite the fact that the default penalty for asset

0 has been chosen "optimally," the unique equilibrium (ignoring redundant trades) is the Arrow–Debreu equilibrium of Version A0, so that asset 0 is not traded at all. The forces of supply and demand determine that the Arrow securities are traded and other assets are not.

We elevate this example to a theorem:

Theorem 6 *Let $\mathcal{E} = ((u^h, e^h)_{h \in H}, (R_j, ((\lambda^h_{sj})_{s \in S}, Q^h_j)_{j \in J}))$ be an economy which includes all the Arrow securities: for each $s \in S$, there is an asset $i = i(s)$ such that $R_{sLi} = 1$ and $R_{s'\ell i} = 0$ otherwise, with $Q^h_i = \infty$ $\forall h$ and $\lambda^h_{si} = \infty$ $\forall h$ and $\forall s$. Then for any $GE(R, \lambda, Q)$ equilibrium $\eta = ((p, \pi, K), (x^h, \theta^h, \varphi^h, D^h)_{h \in H})$, we can find prices $q \in \mathbb{R}^{(1+S)L}_{++}$ such that $(q, (x^h)_{h \in H})$ is an Arrow–Debreu equilibrium. Moreover, if $\lambda \gg 0$, no agent defaults on any actively traded asset in η, even if there are assets $j \in J$ with low λ^h_{sj}. Finally, there is an equilibrium η', possibly η itself,with the same $((p, \pi, K), (x^h)_{h \in H})$ such that the only actively traded assets in η' are Arrow securities.*

Proof Let η be given. Let $q_0 = p_0$ and let $q_s = \pi_{i(s)}(p_s/p_{sL})$, $\forall s \in S$. Let

$$v^h(q) \equiv \max\{u^h(x) : q \cdot x \leq q \cdot e^h,\ x \in \mathbb{R}^{(1+S)L}_{+}\}.$$

Observe that $K_{sj} = 1$ for each asset j with $\lambda^h_{sj} = \infty$ $\forall h, s$, if $R_s \neq 0$, since no agent will default in the refinement. It follows that by never defaulting, each agent h could, by selling and buying the Arrow securities, achieve at least $v^h(q)$, that is,

$$u^h(x^h) \geq u^h(x^h) - \text{default penalty} \geq v^h(q).$$

It follows that $q \cdot x^h \geq q \cdot e^h$ $\forall h \in H$. Since η is an equilibrium $\sum_{h \in H} x^h = \sum_{h \in H} e^h$. Hence $q \cdot x^h = q \cdot e^h$ $\forall h \in H$, and $(q, (x^h)_{h \in H})$ is an Arrow–Debreu equilibrium, and the default penalty actually borne by each agent $h \in H$ is zero.

Clearly each agent is indifferent to achieving x^h via the actively traded assets in η, or via Arrow securities. If every agent trades exclusively via Arrow securities, then supply will equal demand, and η' is a genuine equilibrium. ■

10 Endogenous Default Penalties When Promises Are Incomplete

We have already seen that when all asset promises are available, the market should and will exclusively trade promises with infinite penalties. Let us suppose that the set $\mathcal{A}$ contains only a limited variety of promises, far short of a complete set of Arrow promises. Given these limitations on promises, in Section 7 we were able to ask how severe the default penalties *should* be to promote economic efficiency. Since our model allows for the possibility that different punishment regimes coexist at the same time, we can also ask how harsh the punishment scheme *will* be that endogenously emerges in equilibrium. For example, an agent could indicate his intention to perform a service, he could orally commit to performing the service, he could put in writing that he promised to perform a service, or he could draw up a contract with a lawyer announcing his promise to perform a service. If all four of these promises are treated equally by the courts, then there is no issue of selecting a punishment. But if the punishment in case of default is different for these different manners of making the same promise, then in effect the parties to the agreement are choosing the severity of default penalties attached to the promise. We shall now show that in our example, the forces of supply and demand select the optimal default penalties.

Version A4: Endogenous Default Penalties

Consider the model of version A1 with only one asset promise $R_0 = (1, 1, 1)$ and $\lambda^h_{s0} = \lambda^* = 6/5$, $\forall h \in H$ and $\forall s \in S$. It is natural to regard the penalty λ^* as imposed by a beneficent and knowledgeable government. But we may also regard λ^* as emerging from the equilibrium forces of supply and demand.

Now let there be a finite number of additional assets R_j, all making the same promises $R_j = (1, 1, 1)$, but with default penalties $\lambda_j = \lambda^h_{sj}$ for all $h \in H$, $s \in S$, ranging at intervals of $\lambda^*/100$ from 0 to $100\lambda^*$. We shall now show that despite the myriad of available assets, in every (symmetric) equilibrium, all trade will be conducted in the assets j for which $\lambda^h_{sj} = \lambda^*$. We begin by describing an equilibrium of this type, and then we show it is essentially the only (symmetric) equilibrium.

The equilibrium will involve exactly the same prices, delivery rates, trades, and consumption as described in example A1 for the case $\lambda = \lambda^* = 6/5$. We must now extend that equilibrium to define prices π_j and delivery rates K_{sj} for all the new assets. Set $\pi^* = 18/5$, and set $\pi_j = 6/5 + 6/5 + \min\{\lambda_j, 3\}$ for $\lambda_j \geq \lambda^* = 6/5$, which is the marginal disutility of selling asset j, when $\lambda_j \geq \lambda^*$. At these prices agents are just indifferent between selling j and j^*, so it is optimal to supply zero of j. Recall in example A1, $\pi_0 = 18/5 =$ the marginal utility of buying or selling asset R_0, hence $\pi_j \geq \pi_0$. For $\lambda^* \leq \lambda_j < 3$, set $K_{sj} = 2/3$ for all $s \in S$. The marginal utility of buying such assets j is then 18/5, and this is never higher than the price π_j, hence optimal demand $\theta^h_j = 0$. For $\lambda_j \geq 3$, we let $K_{sj} = 1$, $\forall s \in S$. Here $\pi_j = 12/5 + 3 = 27/5$. But the marginal utility of buying such an asset is also $6/5 + 6/5 + 3 = 27/5$, again optimal demand and supply are zero.

For $\lambda_j < \lambda^*$, define $\pi_j = 3\lambda_j$ and $K_{sj} = 0$, $\forall s \in S$. Since utilities are concave, to check that no agent would sell asset j, we only need to look at the marginal utilities at the allocation achieved in example A1. Clearly on the margin, the disutility of selling asset j is $3\lambda_j$, once again rendering sellers indifferent between j and j^*. Notice that every seller would in fact completely default, hence buyers are rational to anticipate this, and to demand zero.

In every case we set the price equal to the marginal utility of the sellers, and the K_{sj} equal to the rates of payments that would be made with infinitesimally higher π_j. Hence in every case buyer expectations are rational, i.e., they satisfy condition (5) of equilibrium. We have thus displayed an equilibrium in which (almost) any default penalty is available, yet only a single one (namely the Pareto efficient penalty) is used in equilibrium.

We now argue that there can be no other (symmetric) equilibrium. In any (symmetric) equilibrium we have consumption $x^1 = (2x, 1-x, 1-x)$, and similarly $x^2 = (1-x, 2x, 1-x)$, and $x^3 = (1-x, 1-x, 2x)$, with $x \leq 1/3$. If $x = 1/6$, we must be in our original equilibrium. If $x > 1/6$, then agent 1 has delivered up to a point in states 2 and 3 where his marginal utility of consumption $1/(1-x) > 6/5$. He would not have done that unless he was selling an asset with default penalty $\lambda_j \geq 1/(1-x) > 6/5$. If asset j delivers fully in every state, then it is irrelevant, since by symmetry each agent is buying and selling an equal amount of it. But from the argument in the proof of Theorem 2, if the asset did not deliver everywhere, then any agent buying and selling it would default completely in at least one state. Since by symmetry every agent buys and sells it, $K_{sj} \leq 2/3, \forall s \in S$. The marginal utility to purchasing asset j is at most $\frac{2}{3}\left(\frac{1}{2x} + \frac{1}{1-x} + \frac{1}{1-x}\right) = \frac{2}{3}\frac{3x+1}{(1-x)2x} = \frac{1}{1-x} + \frac{1}{(1-x)3x} < \frac{3}{1-x}$ (if $x > 1/6$) of utility in period 1. The marginal disutility of selling asset j is at least $\frac{1}{1-x} + \frac{1}{1-x} + \frac{1}{1-x} = \frac{3}{1-x}$, a contradiction.

If $x < 1/6$, we shall show there can be no equilibrium price π^* for asset $j = j^*$. The marginal disutility of selling asset j^* is $\frac{1}{1-x} + \frac{1}{1-x} + \frac{6}{5}$, since $1/(1-x) < 6/5 = \lambda^*$. Hence, the marginal disutility of selling is less than $18/5$. It also follows that every agent would deliver in each of his two good states if he were selling asset j^*. Hence $K_{s0} \geq 2/3, \forall s \in S$, by our equilibrium refinement. The marginal utility of buying asset j^* is then at least $\frac{2}{3}\frac{1}{1-x} + \frac{2}{3}\frac{1}{1-x} + \frac{2}{3}\frac{1}{2x}$. For $x < 1/6$, the marginal utility of buying is always larger than $18/5$, hence larger than the marginal utility of selling, a contradiction.

11 Endogenous Promises when Default Penalties Are Lenient

Consider a situation in which default penalties are not allowed to be too severe, perhaps because politics do not permit harsh penalties that don't "fit the crime." We shall show that then the "Arrow promises" will often not be actively traded, even if they are available without transactions costs.

Version A5: Adverse Selection with Differential Penalties

Let us reconsider the example with $\#H = 3$ and assets $R_1 = (1,0,0)$, $R_2 = (0,1,0)$ and $R_3 = (0,0,1)$ (either in place of asset $R_0 = (1,1,1)$, or in addition to R_0), where the penalties are $\lambda^h_{sj} = \begin{cases} \infty \text{ if } e^h_s = 1 \\ 0 \text{ if } e^h_s = 0 \end{cases}$. In equilibrium none of the Arrow securities would be traded, since agents of type $h = j$ would be tempted to sell $j \in \{1,2,3\}$ like crazy, thereby reducing K_{sj} to zero, so that π_j would be zero, so that no agents besides those of type $h = j$ would sell asset j even if π_j went up a little.

When default penalties are not uniform, asset promises which pay off in very specific states are not likely to be traded even if they can be written, because there is bound to be some agent who can take advantage of the specificity of the conditions to escape punishment. These agents will debase the value of the asset and prevent others from selling it. In short, when there is a variety of penalties λ_{sj}^h, buyers must beware of an adverse selection of sellers with $\lambda_{sj}^h < \partial u^h/\partial x_s$.

Version A6: Span of Active Assets Shrinks as Default Penalties Fall

Consider the same three agents' utilities and endowments as in examples A0–A4. Suppose now, however, that there are asset promises $R_0 = (1,1,1)$, $R_1 = 1^1$, $R_2 = 1^2$, $R_3 = 1^3$. Fix all the penalties $\lambda_{sj}^h = \lambda$, for all $h \in H$, $s \in S$, $j = 0,1,...,J$, as in Version A1. We wish to illustrate two points. First, we shall see that in equilibrium not all available assets are traded, even though there are no transactions costs. Second, we shall see that as the default penalties λ decline, the span of actively traded asset promises shrinks.

When $3/2 \leq \lambda \leq \infty$, there is (essentially) a unique equilibrium reproducing the Arrow–Debreu equilibrium of Version A0. Each agent h consumes $(2/3, 2/3, 2/3)$, obtained by selling 1/3 units of assets $j \in \{1,2,3\} \setminus h$, and buying 2/3 units of asset h. (Thus agent 1 puts $\varphi_2^1 = \varphi_3^1 = 1/3$ and $\theta_1^1 = 2/3$.) All assets deliver completely, $K_{sj} = 1\ \forall s \in S$, $j \in J$, and $1 = \pi_1 = \pi_2 = \pi_3 = (1/3)\pi_0$. There is no trade in asset 0.

When $3/2 > \lambda > 6/5$, default emerges, but traded asset promises are still "complete." In the unique GEI(R, λ, ∞) equilibrium, $x_s^h = \begin{cases} 1/\lambda & \text{if } h \neq s \\ 2(1-1/\lambda) & \text{if } h = s \end{cases}$. We can guess that each agent h sells φ units of each of the two "Arrow promises" $j \neq h$, and buys 2φ units of asset $j = h$. (The prices of all three assets is the same.) The marginal utility of buying asset $j = h$ is $MU_B = K1/[2(1-1/\lambda)]$ and the marginal disutility of selling either "Arrow promise" $j \neq h$ is $MU_S = \lambda$. Equating the two marginal utilities gives $K = 2(\lambda - 1)$. Final consumption, say for agent $h = 1$, is

$$x' = \begin{pmatrix} 2(1-1/\lambda) \\ 1/\lambda \\ 1/\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ K\varphi \\ K\varphi \end{pmatrix} + \begin{pmatrix} 2K\varphi \\ 0 \\ 0 \end{pmatrix}.$$

This gives $1/\lambda = 1 - K\varphi$. Replacing K with $2(\lambda - 1)$ gives $\varphi = 1/2\lambda$. We can now describe the rest of the equilibrium: $\varphi_j^h = \begin{cases} 1/2\lambda & \text{if } j \in \{1,2,3\} \setminus h \\ 0 & \text{if } j \in \{0,h\} \end{cases}$; $\theta_j^h = \begin{cases} 0 & \text{if } j \in \{0,1,2,3\} \setminus h \\ \frac{1}{\lambda} & \text{if } j = h \end{cases}$; $K_{sj} = 2(\lambda-1)\ \forall s \in S$, $j \in \{1,2,3\}$; $K_{s0} = 2/3\ \forall s \in S$; $\lambda = \pi_1 = \pi_2 = \pi_3 = (1/3)\pi_0$. For example, if $\lambda = 4/3$, then $x^1 = (1/2, 3/4, 3/4)$; $\varphi_0^1 = 0$, $\varphi_1^1 = 0$, $\varphi_2^1 = 3/8$, $\varphi_3^1 = 3/8$, $\theta_0^1 = 0$, $\theta_1^1 = 3/4$, $\theta_2^1 = 0$, $\theta_3^1 = 0$; $K_{sj} = 2/3$, $j = 1,2,3$, $\forall s \in S$; $K_{s0} = 2/3$, $\forall s \in S$; $\pi_1 = \pi_2 = \pi_3 = 4/3$; $\pi_0 = 4$. Note that agent 1 sells 3/8 units of asset 2, delivers $2/3 \cdot (3.8) = 2/8$, and thus consumes $x_2^1 = 1 - 2/8 = 3/4$. Notice that the marginal disutility of selling one unit of asset 0 is to default in each state, which costs $3(4/3)$ utiles, or 4, which is equal to the price π_0. The marginal utility of buying one unit of asset 0 is $2/3 \cdot (4/3) + 2/3 \cdot (4/3) + 2/3 \cdot (2) =$

$28/9 < 4$, so no agent wants to buy asset 0. For $6/5 \geq \lambda > 1$, the same equilibrium persists. But there is another.

When $6/5 \geq \lambda > 1$, there is an equilibrium in which asset trades shrink to one dimension. We can guess from example Version A1 and from the above calculations that $x_s^h = \begin{cases} 1/\lambda & \text{if } h \neq s \\ 2(1-1/\lambda) & \text{if } h = s \end{cases}$; $\varphi_0^h = \theta_0^h = 4/3 - 1/\lambda$, $\varphi_j^h = \theta_j^h = 0$, $\forall j \in \{1,2,3\}$, $K_{s0} = (6\lambda - 6)/(4\lambda - 3)$, $\forall s \in S$, $K_{sj} = 2(\lambda - 1)$, $\forall s \in S$, $j \in \{1,2,3\}$; $\lambda = \pi_1 = \pi_2 = \pi_3 = 1/3 \cdot \pi_0$. To verify equilibrium condition (5) for the given K_{sj}, for asset $j = 1,2,3$, note that each agent h could equally well have achieved exactly the same consumption and default penalties by trading via the assets $j \in \{1,2,3\}$, exactly as described in the last paragraph. Had all three agents done so, delivery rates on these assets really would have been $K_{sj} = 2(\lambda - 1)$. For example, if $\lambda = 8/7$, then $x^1 = (1/4, 7/8, 7/8)$, $\varphi_0^1 = \theta_0^1 = 11/24$, $K_{s0} = 6/11$; $\pi_1 = \pi_2 = \pi_3 = 8/7$, $\pi_0 = 24/7$. Observe that by buying $\theta_0^1 = 11/24$ units of asset 0, agent 1 consumes $x_1^1 = K_{10}\theta_0^1 1 = 6/11 \cdot (11/24) = 1/4$, as claimed. Note that agent 1 delivers $D_0^1 = (0, 9/24, 9/24)$. Note also that the marginal disutility of selling asset 0 is $24/7$, while the marginal utility of buying another unit of asset 0 is $6/11 \cdot (4) + 6/11 \cdot (8/7) + 6/11 \cdot (8/7) = 264/77 = 24/7$. Similarly the marginal utility to agent 1 from buying or selling asset 1 is $2/7 \cdot (4) = 8/7$. The marginal disutility to selling assets 2 or 3 is also $8/7$, while the marginal utility of buying them is 0.

When $1 > \lambda \geq 0$, the actively traded asset span shrinks to zero dimensions, since there is no trade in equilibrium.

12 Confiscation and Trigger Penalties

In practice there may be a legal system that confiscates resources from defaulters. The detailed rules of confiscation can take many forms; we shall describe the simplest. The legal system may also impose penalties that are discontinuous in the size of the default, for example trigger penalties that jump to a minimum level at the first infinitesimal default. Our existence theorems have not explicitly allowed for these possibilities. But in fact, with a continuum of households, existence of equilibrium remains intact by Theorem 7 below.

Trigger penalties can be modeled by discontinuous payoffs $W^t(\xi^t, \eta)$, such as we saw in Section 8.2.

An extreme version of confiscation prohibits a household from consuming any goods until he has redeemed all his debts. Formally, let

$$C^h(p, \pi, K) = \Big\{ (x, \theta, \varphi, D) \in B^h(p, \pi, K) : \text{ for any } s \in S, \\ \text{if for some } j \in J,\ p_s \cdot D_{sj} < \varphi_j p_s \cdot R_{sj},\ \text{then } x_s = 0 \Big\}.$$

Here an outside agency like the court enforces delivery when debtors have resources to make good on their promises. This draconian confiscation may so hinder asset sales as to completely eliminate default at equilibrium. In Section 7 we saw that lenient penalties which encourage default can be welfare improving. Thus this extreme confiscation is not socially advisable. Bankruptcy law implicitly recognizes this, putting limits on how much can be confiscated.

The correspondence C^h describing extreme confiscation satisfies three properties which would also hold for a variety of milder forms of confiscation:

- (0*) $(e^h, 0, 0) \in C^h(p, \pi, K) \subset B^h(p, \pi, K)$
- (1*) C^h is upper semi-continuous
- (2*) C^h has the scaling property: $(x, \theta, \varphi, D) \in C^h(p, \pi, K)$ and $0 \leq \alpha < 1 \Rightarrow (\alpha x, \alpha\theta, \alpha\varphi, \alpha D) \in C^h(\hat{p}, \hat{\pi}, \hat{K})$ for $(\hat{p}, \hat{\pi}, \hat{K})$ sufficiently close to (p, π, K)

Theorem 7 *Define equilibrium with budget sets $C^h(p, \pi, K)$ for $t \in (h-1, h]$, and payoffs $W^t(\xi^t, \eta^t)$. Suppose C^h satisfies properties (0*), (1*), (2*) and W^t satisfies (1) and (2) of Section 8.2, and the quantity constraints Q^h_j are all finite. Then in the finite-type continuum model, equilibrium exists (though it may not be type symmetric).*

Proof Same as the proof of Theorem 5.2. ■

Confiscation and discontinuous default penalties are thus subsumed by Theorem 7. Theorem 7 also paves the way for dealing with exclusivity restrictions on asset sales. But we leave this for our sequel paper.

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