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Information and the Existence of Stationary Markovian Equilibrium

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Abstract

We describe conditions for the existence of a stationary Markovian equilibrium when total production or total endowment is a random variable. Apart from regularity assumptions, there are two crucial conditions: (i) *low information* - agents are ignorant of both total endowment and their own endowments when they make decisions in a given period, and (ii) *proportional endowments* - the endowment of each agent is in proportion, possibly a random proportion, to the total endowment. When these conditions hold, there is a stationary equilibrium. When they do not hold, such equilibrium need not exist.

1 Introduction

This paper is one of a series of investigations of a mass-market economy with stochastic elements in which the optimization problems faced by each of a continuum of agents are modelled as parallel dynamic programming problems. The model used is a strategic market game at the highest level of aggregation, in order to concentrate on the monetary aspects of a stochastic environment. Although there are several previous papers which provide economic motivation and modelling details (Karatzas, Shubik & Sudderth (1994) and (1997), and Geanakoplos, Karatzas, Shubik & Sudderth (1998), referred to below as [KSS1], [KSS2], and [GKSS], respectively), we have attempted to make this paper as self-contained as possible. However, we will make use of several theorems established in these earlier works.

We consider an economy with a stochastic supply of goods where: (i) the endowment of each agent is in proportion (possibly a random proportion) to the total amount of goods available; and (ii) the agents must bid for goods in each period before knowing both the total supply of goods available and the realization of their own random endowments.

For such an economy, a stationary equilibrium will be shown to exist, where the optimal bid of an agent in each period depends only on the current wealth of the agent. In equilibrium there will be a stationary distribution of wealth among agents, although prices and the wealth of individual agents will fluctuate as random processes. This will be true whether or not the opportunity is available for agents to borrow from, or deposit in, an outside (government) bank.

When either endowments are not proportional, or the agents have additional information in the form of advance knowledge of the total supply of goods, then there need not exist such an equilibrium. This will be illustrated by two examples. One interpretation of these results is that *better short-term forecasting can be destabilizing*. We plan further investigation of these "high information" phenomena in a subsequent paper.

The next section has some preliminary discussion of our model. Sections 3 and 4 treat the the model without lending, Sections 5 and 6 are on the model with lending and possible bankruptcy, whereas the final Section 7 treats five simple examples that illustrate the existence and non-existence of stationary equilibrium.

2 Preliminaries

For simplicity we omit production from consideration. Instead, we consider an economy where all consumption goods are bought for cash (*fiat money*) in a competitive market. Each individual agent begins with an initial endowment of money and a claim to the proceeds from the consumption goods which are sold in the market. The goods enter the economy in each period as if they were "manna" from an undescribed production process, and are owned by the individual agents. However, the agents are required to sell the goods in the market, and do not receive the proceeds until the start of the subsequent period. The assumption that all goods go through the market is probably a better approximation of the realities in a modern economy than the reverse, where each agent can consume everything directly without the interface of markets and prices.

Our model has a continuum of agents indexed by the unit interval $I = [0,1]$ and runs in discrete time units $n = 0, 1, \cdots$. At the beginning of each time-period n, every agent $\alpha \in I$ receives an endowment $Y_n^\alpha(\omega)$ in units of a non-durable commodity. The random variables Y_n^α and all other random variables in this paper are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We shall consider the *no-lending model* of [KSS1], and also the *lending with possible bankruptcy model* of [GKSS]. Unlike these earlier papers, it will no longer be assumed that total production Q is constant from period to period, but instead that production

$$Q_n(\omega) = \int Y_n^\alpha(\omega)\phi(d\alpha)$$

in period n is a random variable, for all $n = 0, 1, \cdots$.

The following assumption will be in force until the examples of the final section.

Assumption 1. *1. The total-production variables $Q_1, Q_2, \cdots$ are IID (independent and identically distributed) with common distribution ζ. It will also be assumed that the Q_n's are strictly positive with finite mean.*

2. The individual endowment variables $Y_n^\alpha(\omega)$ are proportional to the $Q_n(\omega)$, in the sense that

$$Y_n^\alpha(\omega) = Z_n^\alpha(\omega) Q_n(\omega) \qquad \text{for all } \alpha \in I,\ n \in \mathbb{N},\ \omega \in \Omega, \tag{2.1}$$

where the sequences $\{Z_1^\alpha, Z_2^\alpha, \ldots\}$ and $\{Q_1, Q_2, \ldots\}$ are independent, $Z_n^\alpha \geq 0$, $EZ_n^\alpha = 1$ and $Z_1^\alpha, Z_2^\alpha, \cdots$ are IID with common distribution λ^α, for each $\alpha \in I$.

As a consequence of these assumptions, we have

$$EY_n^\alpha = (EZ_n^\alpha)(EQ_n) = EQ_n. \tag{2.2}$$

3 The Model without Lending

Let φ be a non-atomic probability measure on the Borel subsets of $I = [0,1]$ that corresponds to the "spatial" distribution of agents. For $\alpha \in I$ and $n \geq 1$, let $S_{n-1}^\alpha(\omega)$ and $\mathcal{F}_{n-1}^\alpha$ denote respectively the wealth and information $\sigma-$algebra available to agent α at the beginning of period n. As in [KSS1], agent α bids an amount $b_n^\alpha(\omega) \in [0, S_{n-1}^\alpha(\omega)]$ of money for the consumption good *before* knowing the value of $Q_n(\omega)$ or $Y_n^\alpha(\omega)$. We call this the *low-information* condition. (In other words, the information $\mathcal{F}_{n-1}^\alpha$ available to the agent at the beginning of period n, measures the values of past quantities including $S_0^\alpha,\ S_k^\alpha,\ Q_k,\ Z_k^\alpha,\ b_k^\alpha$ for $k = 1, \cdots, n-1$, but *not* of $Q_n,\ Y_n^\alpha$.)

Once all agents have placed their bids, the total amount of fiat money bid for the consumption good is given by

$$B_n(\omega) = \int b_n^\alpha(\omega) \varphi(d\alpha),$$

and a new price is formed as

$$p_n(\omega) = \frac{B_n(\omega)}{Q_n(\omega)}.$$

Each agent α receives an amount

$$x_n^\alpha(\omega) = \frac{b_n^\alpha(\omega)}{p_n(\omega)} = \frac{b_n^\alpha(\omega)}{B_n(\omega)} \cdot Q_n(\omega)$$

in goods, as well as

$$p_n(\omega) Y_n^\alpha(\omega) = \frac{B_n(\omega)}{Q_n(\omega)} \cdot Z_n^\alpha(\omega) Q_n(\omega) = B_n(\omega) Z_n^\alpha(\omega) \tag{3.1}$$

in cash income, and then enters the next period with wealth

$$S_n^\alpha(\omega) = S_{n-1}^\alpha(\omega) - b_n^\alpha + B_n(\omega) Z_n^\alpha(\omega) \tag{3.2}$$

in fiat money.

Each agent α is assumed to have a *utility function* $u^\alpha : [0,\infty) \to [0,\infty)$ for consumption of goods; this function is continuous and continuously differentiable, strictly concave, strictly increasing, and satisfies $u^\alpha(0) = 0,\ (u^\alpha)'_+(0) \in (0,\infty)$. The utility earned by agent α in period n is $u^\alpha(x_n^\alpha)$, and the agent seeks to maximize the expected value of his total discounted utility

$$\sum_{n=0}^{\infty} \beta^n u^\alpha(x_{n+1}^\alpha).$$

A *strategy* π^α for agent α specifies the sequence of bids $\{b_n^\alpha\}_{n=1}^\infty$. The strategy π^α is called *stationary*, if it specifies the bids in terms of a single function $c^\alpha(\cdot)$ of wealth, in the form

$$b_n^\alpha = c^\alpha(S_{n-1}^\alpha), \quad n \geq 1, \tag{3.3}$$

where $c^\alpha(s) \in [0, s],\ s \geq 0$. We call $c^\alpha(\cdot)$ the *consumption function for* the strategy π^α.

The *wealth distribution on day* n is the random measure $\nu_n(\cdot,\omega)$ given by

$$\nu_n(A,\omega) = \varphi\left(\{\alpha \in I \,:\, S_n^\alpha(\omega) \in A\}\right), \qquad A \in \mathcal{B}([0,\infty)). \tag{3.4}$$

We are now ready to define the type of equilibrium that we want to study in this note.

Definition 3.1. A collection of stationary strategies $\{\pi^\alpha, \alpha \in I\}$ and a probability distribution μ on $\mathcal{B}((0,\infty))$ form a *stationary equilibrium*, if
(a) given that $\nu_0 = \mu$ and that every agent α plays strategy π^α, we have $\nu_n = \mu$ for all $n \geq 1$, and
(b) given that $\nu_0 = \mu$, every strategy π^α is optimal for agent α, when every other agent β plays π^β $(\beta \in I, \beta \neq \alpha)$.

Unlike [KSS1], there is no mention of price in Definition 3.1. This is, in part, because the sequence $\{p_n\}$ will not be constant - even in stationary equilibrium - for the model studied here. Indeed, if the consumption function $c^\alpha(\cdot)$ for π^α is the same across all agents $\alpha \in I$, and equal to $c(\cdot)$, then

$$p_n(\omega) = \frac{\int b_n^\alpha(\omega)\,\varphi(d\alpha)}{Q_n(\omega)} = \frac{\int c(s)\,\mu(ds)}{Q_n(\omega)},$$

where the sequence of total bids

$$B_n(\omega) = B = \int c(s)\,\mu(ds)$$

is constant in equilibrium; see Theorem 4.2 below. Thus, the prices $\{p_n\}$ form then a sequence of IID random variables, because the $\{Q_n\}$ do so by assumption. The constant B will play the same mathematical role that was played by the price p in [KSS1] and [GKSS], but the interpretation will, of course, be different.

4 Existence of Stationary Equilibrium for the Model without Lending

The methods of [KSS1] can be adapted to construct a stationary equilibrium for the present model. As in [KSS1], we first consider the one-person game faced by an agent α, assuming that the economy is in stationary equilibrium. For ease of notation we suppress the superscript α while discussing the one-person game. Furthermore, we shall also assume that the agents are *homogeneous*, in the sense that they all have the same utility function $u(\cdot)$ and the same distribution λ for their income variables. This assumption makes the existence proof more transparent, but is not necessary; the proof in [KSS1] works for many types and can be adapted to the present context.

We introduce a new utility function defined, for each $b \geq 0$, by

$$\tilde{u}(b) := E\left[u(bQ(\omega))\right] = \int u(bq)\,\zeta(dq). \tag{4.1}$$

Observe that the expected utility earned by an agent who bids b when faced by a random price $p(\omega) = B/Q(\omega)$, can be written

$$E\left[u\left(\frac{b}{p(\omega)}\right)\right] = E\left[u\left(\frac{b}{B}Q(\omega)\right)\right] = \tilde{u}\left(\frac{b}{B}\right). \tag{4.2}$$

It is straightforward to verify that $\tilde{u}(\cdot)$ has all the properties, such as strict concavity, that were assumed for $u(\cdot)$.

Let $V(\cdot)$ be the value function for an agent playing in equilibrium. In essence, the agent faces a discounted dynamic programming problem and, as in [KSS1], $V(\cdot)$ satisfies the Bellman equation

$$V(s) = \sup_{0 \leq b \leq s}\left[\tilde{u}\left(\frac{b}{B}\right) + \beta \cdot EV(s - b + BZ)\right]. \tag{4.3}$$

This dynamic programming problem is of the type studied in [KSS1], and Theorem 4.1 of that paper has information about it. In particular, there is a unique optimal stationary plan $\pi = \pi(B)$ corresponding to a consumption function $c : [0, \infty) \to [0, \infty)$. We sometimes write $c(s) = c(s; B)$ to show its dependence on the quantity B.

Consider now the Markov chain $\{S_n\}$ of successive fortunes for an agent who plays the optimal strategy π given by $c(\cdot)$. Then we have

$$S_{n+1} = S_n - c(S_n; B) + BZ_{n+1}, \tag{4.4}$$

where $Z_1, Z_2, \dots$ are IID with common distribution λ. By Theorem 5.1 of [KSS1], this chain has a unique stationary distribution $\mu(\cdot) = \mu(\cdot\,; B)$ defined on $\mathcal{B}([0, \infty))$. Now assume that Z has a finite second moment. Then, by Theorem 5.7 of [KSS1], the stationary distribution μ has a finite mean. The following lemma expresses the fact that the total amount bid by all agents is B, when the wealth distribution is $\mu(\cdot\,; B)$.

Lemma 4.1. $\int c(s;B)\mu(ds;B) = B$.

Proof. Assume that S_0 has the stationary distribution μ. Then take expectations in (4.4) to obtain

$$ES_{n+1} = ES_n - \int c(s;B)\,\mu(ds;B) + B \cdot EZ$$

and the desired formula follows, since $ES_{n+1} = ES_n$ and $EZ = 1$. □

Theorem 4.2. *For each $B > 0$, there is a stationary equilibrium for the no-lending model with wealth distribution $\mu(\cdot) = \mu(\cdot\,;B)$ and stationary strategies $\pi^\alpha = \pi(B)$, for all agents $\alpha \in I$.*

Proof. Construct the variables $Z_n^\alpha(\omega) = Z_n(\alpha,\omega)$ using the technique of Feldman & Gilles (1985), so that

$Z_1(\alpha,\cdot), Z_2(\alpha,\cdot),\ldots$ are IID with distribution λ, for every $\alpha \in I$, and

$Z_1(\cdot,\omega), Z_2(\cdot,\omega),\ldots$ are IID with distribution λ, for every $\omega \in \Omega$.

Then the chain $\{S_n(\alpha,\omega)\}$ has the same dynamics for each fixed $\omega \in \Omega$ as it does for each fixed $\alpha \in I$. The distribution μ is stationary for the chain when α is fixed, and will therefore be a stationary wealth distribution for the many-person game if the total bids $B_1(\omega), B_2(\omega),\ldots$ remain equal to B. Now, if $S_0(\cdot,\omega)$ has distribution μ, then

$$B_1(\omega) = \int c(S_0(\alpha,\omega))\,\varphi(d\alpha) = \int c(s)\,\mu(ds) = B,$$

by Lemma 4.1. By induction, $B_n(\omega) = B$ for all n and ω. Hence, the wealth distributions ν_n are all equal to μ.

The optimality of $\pi^\alpha = \pi(B)$ follows from its optimality in the one-person game together with the fact that a single player cannot affect the value of the total bid. □

5 The Model with Lending and Possible Bankruptcy

We now assume that there is a Central Bank which makes loans and accepts deposits. The bank sets two interest rates in each time period n, namely $r_{1n}(\omega) = 1 + \rho_{1n}(\omega)$ to be paid by borrowers and $r_{2n}(\omega) = 1 + \rho_{2n}(\omega)$ to be paid to depositors. These rates are assumed to satisfy

$$1 \le r_{2n}(\omega) \le r_{1n}(\omega), \quad r_{2n}(\omega) \le \frac{1}{\beta} \tag{5.1}$$

for all $n \in \mathbf{N}, \boldsymbol{\omega} \in \boldsymbol{\Omega}$.

Agents are required to pay their debts back at the beginning of the next period, when they have sufficient funds to do so. However, it can happen that they are unable to pay back their debts in full, and are thus forced to pay a *bankruptcy penalty* in units of utility, before they are allowed to continue play. For this reason, we assume now that each agent α has a utility function $u^\alpha : \mathbf{R} \to \mathbf{R}$ defined on the entire real line, and satisfies all the other assumptions made above. For $x < 0$, the quantity $u^\alpha(x)$ is negative and measures the "disutility" for agent α of going bankrupt by an amount x; for $x > 0$, the quantity $u^\alpha(x)$ is positive and measures the utility derived by α from consuming x units of the commodity, just as before.

Suppose that an agent α begins in period n with wealth $S^\alpha_{n-1}(\omega)$. If $S^\alpha_{n-1}(\omega) < 0$, then agent α has an unpaid debt from the previous period and is assessed a penalty of $u(S^\alpha_{n-1}(\omega)/p_{n-1}(\omega))$. The debt is then forgiven, and the agent continues play from wealth-position 0. If $S^\alpha_{n-1}(\omega) \geq 0$, then agent α is not in debt and plays from position $S^\alpha_{n-1}(\omega)$. In both cases, an agent α, possibly after being punished, plays from the wealth-position $(S^\alpha_{n-1}(\omega))^+ = \max\{S^\alpha_{n-1}(\omega), 0\}$.

Based on knowledge of past quantities $S^\alpha_0,\ S^\alpha_k,\ Z^\alpha_k,\ Q_k, r_{1k},\ r_{2k}$ for $1 \leq k \leq n-1$, agent α chooses a bid

$$b^\alpha_n(\omega) \in [0, (S^\alpha_{n-1}(\omega))^+ + k^\alpha],$$

where $k^\alpha \geq 0$ is an upper bound on loans to α. As before, α must bid in ignorance of both the total endowment $Q_n(\omega)$ and his personal endowment $Y^\alpha_n(\omega)$ for period n.

The total bid B_n, the price p_n, and agent α's quantities of goods x^α_n and fiat money $p_n Y^\alpha_n = B_n Z^\alpha_n$ are formed exactly as in the no-lending model of Section 3. Formula (3.2) for the dynamics now takes the form

$$S^\alpha_n = \begin{cases} -r_{1n}(b^\alpha_n - (S^\alpha_{n-1})^+) + B_n Z^\alpha_n, & (S^\alpha_{n-1})^+ \leq b^\alpha_n, \\ r_{2n}((S^\alpha_{n-1})^+ - b^\alpha_n) + B_n Z^\alpha_n, & (S^\alpha_{n-1})^+ > b^\alpha_n. \end{cases} \tag{5.2}$$

The wealth-distribution ν_n on day n is defined by formula (3.3) as before, but with the understanding that A now ranges over Borel subsets of the whole real line, since some agents may have negative wealth. An agent α's utility in period n is now given by

$$\xi^\alpha_n(\omega) = \begin{cases} u^\alpha(x^\alpha_n(\omega)), & S^\alpha_{n-1}(\omega) \geq 0, \\ u^\alpha(x^\alpha_n(\omega)) + u^\alpha(S^\alpha_{n-1}(\omega)/p_{n-1}(\omega)), & S^\alpha_{n-1}(\omega) < 0. \end{cases}$$

As before, agent α seeks to maximize the expected value of his total discounted utility

$$\sum_{n=0}^{\infty} \beta^n \xi^\alpha_n(\omega).$$

We extend now the definition of stationary equilibrium to the model with lending.

Definition 5.1. A *stationary equilibrium* for the model with lending, consists of a wealth distribution μ (i.e. a probability distribution) on the Borel subsets of the real line, of interest rates $r_1,\ r_2$ with $1 \le r_2 \le r_1,\ r_2 \le 1/\beta,$ and of a collection of stationary strategies $\{\pi^\alpha,\ \alpha \in I\}$ such that, if the bank sets interest rates r_1 and r_2 in every period, and if the initial wealth distribution is $\nu_0 = \mu$, then
(a) $\nu_n = \mu$ for all $n \ge 1$ when every agent α plays strategy π^α, and
(b) every strategy π^α is optimal for agent α when every other agent β plays π^β $(\beta \in I, \beta \neq \alpha)$.

Suppose that the model is in stationary equilibrium, and that each stationary strategy π^α specifies its bids b_n^α by the same consumption function $c^\alpha \equiv c(\cdot)$ with $0 \le c^\alpha(s) \le s + k$ for all $s \ge 0$, and the same upper-bound $k^\alpha \equiv k$ on loans, for all $\alpha \in I$. Then the total bid

$$B = B_n(\omega) = \int c(s^+)\, \mu(ds)$$

remains constant, while the prices $\{p_n\}$ form an IID sequence just as in the no-lending model of Section 3.

6 Existence of Stationary Equilibrium for the Model with Lending and Possible Bankruptcy

The methods and results of [GKSS] can be used here, as those of [KSS1] were used in Section 4. We consider the one-person game faced by an agent when the economy is in stationary equilibrium. We suppress the superscript α and assume that agents are homogeneous, with common utility function $u(\cdot)$, income distribution λ, and loan limit k. We define the utility function $\tilde{u}(\cdot)$ as in (4.1) and observe that (4.2) remains correct. Formula (5.2) for the dynamics can be written in the simpler form

$$S_n = g\left((S_{n-1})^+ - b_n\right) + BZ_n \tag{6.1}$$

where

$$g(x) = \begin{cases} r_1 x, & x < 0, \\ r_2 x, & x \ge 0. \end{cases}$$

The Bellman equation becomes

$$V(s) = \begin{cases} \sup_{0 \le b \le s+k} \left[\tilde{u}(b/B) + \beta \cdot EV(g(s-b) + BZ\right], & s \ge 0 \\ \tilde{u}(s/B) + V(0), \quad s < 0. \end{cases} \tag{6.2}$$

This equation is of the type studied in [GKSS], and all of the major results of that paper have counterparts here. For example, Theorem 4.2 of [GKSS] applies, to tell us that there is a unique stationary optimal strategy $\pi = \pi(B)$

corresponding to a consumption function $c(\cdot) = c(\cdot\,; B)$. The Markov chain $\{S_n\}$ for the fortunes of an agent who plays $\pi(B)$ evolves according to the dynamics

$$S_{n+1} = g((S_n)^+ - c((S_n)^+; B)) + BZ_{n+1}. \tag{6.3}$$

Conditions for this chain to have a stationary distribution μ with finite mean are available in Theorem 4.3 of [GKSS]. For μ to be the wealth-distribution of a stationary equilibrium, we must also assume that the bank balances its books under μ.

Assumption 2. *(i) The Markov chain $\{S_n\}$ of (6.3) has an invariant distribution μ with finite mean.*

(ii) Under the wealth distribution μ, the total amount of money paid back to the bank by borrowers in a given period, is equal to the sum of the total borrowed plus the amount of interest paid by the bank to lenders. This condition can be written as

$$\int\int [Bz \wedge \gamma_1 d(s^+)]\,\mu(ds)\lambda(dz) = \int d(s^+)\,\mu(ds) + \rho_2 \int l(s^+)\,\mu(ds),$$

where $d(s) = (c(s) - s)^+$ and $l(s) = (s - c(s))^+$ are the amounts borrowed and deposited, respectively, under the stationary strategy $c(\cdot)$ by an agent with wealth $s \geq 0$.

Theorem 6.1. *If Assumption 2 holds, then there is a stationary equilibrium with wealth distribution μ, and interest rates r_1, r_2 in which every agent plays the plan π.*

The proof of this theorem is the same as that of Theorem 4.2 once the following lemma is established.

Lemma 6.2. $\int c(s^+, B)\,\mu(ds) = B.$

The proof of the lemma is similar to that of Lemma 5.1 in [GKSS].

Theorem 6.1 is intuitively appealing, and useful for verifying examples of stationary equilibria. However, it is inadequate as an existence theorem because condition (ii) of Assumption 1 is delicate and difficult to check. There are two existence theorems in [GKSS], Theorems 7.1 and 7.2, that do not rely on such an assumption. Here we present the analogue of the second of them.

Theorem 6.3. *Suppose that the variables $\{Z_n^\alpha\}$ are uniformly bounded, and that the derivative of the utility function $u(\cdot)$ is bounded away from zero. Then a stationary equilibrium exists.*

The proof is similar to that of Theorem 7.2 in [GKSS], with the constant B again playing the mathematical role played by the price p in [GKSS]. The utility function $\tilde{u}(\cdot)$ replaces $u(\cdot)$ in the argument, and the hypothesis that $\inf u'(\cdot) > 0$ implies that the same is true for $\tilde{u}(\cdot)$.

7 Examples

Here we present five examples. The first two illustrate the existence theorem (4.2) for the model without lending.

Example 7.1. Suppose that the utility function is linear, namely $u(x) = x$. The endowment variables $\{Y_n^\alpha\}$ satisfy the proportionality assumption (2.1) but are otherwise arbitrary. The function $\tilde{u}(\cdot)$ of (4.1) is also linear, since

$$\tilde{u}(b) = \int bq\,\zeta(dq) = b \cdot E(Q).$$

We shall show that the optimal policy π of Theorem 4.2 is given by the "spend all" consumption function $c(s) = s$. To see this, let $I(\cdot)$ be the return function for π. Then

$$I(s) = \tilde{u}(s/B) + \beta \cdot E[I(s - s + BZ)] = \frac{s}{B}E(Q) + I^*,$$

where $I^* := \beta \cdot E[I(BZ)]$. To prove optimality, we have to check that $I(\cdot)$ satisfies the Bellman equation:

$$I(s) = \max_{0 \le b \le s} \Big[\tilde{u}(b/B) + \beta \cdot E[I(s - b + BZ)]\Big].$$

Now

$$E[I(s - b + BZ)] = E\left[\frac{s - b + BZ}{B} \cdot E(Q) + I^*\right] = \frac{s - b + B}{B} \cdot EQ + I^*$$

so that the function

$$\tilde{u}(b/B) + \beta \cdot E[I(s - b + BZ)] = b(1 - \beta)\frac{E(Q)}{B} + \beta\left[\left(\frac{s}{B} + 1\right)E(Q) + I^*\right]$$

attains its maximum on $[0, s]$ at $b = c(s) = s$. Hence, the Bellman equation holds and π is optimal. Notice that under π,

$$S_{n+1} = S_n - S_n + BZ_{n+1} = BZ_{n+1}$$

and the stationary distribution μ is that of BZ_1.

Example 7.2. Assume that the utility function is

$$u(b) = \begin{cases} b\,, & 0 \le b \le 1, \\ 1\,, & b > 1, \end{cases}$$

that the distribution ζ of the endowment variables $\{Q_n\}$ is the two-point distribution

$$\zeta(\{1/2\}) = \zeta(\{1\}) = 1/2,$$

and that the distribution λ of the proportions $\{Z_n\}$ of the total endowment is the two-point distribution

$$\lambda(\{0\}) = 3/4, \quad \lambda(\{4\}) = 1/4.$$

Suppose also that the total bid B is 1. Then the price $p = B/Q$ fluctuates between $p_1 = 1$ (when $Q = 1$) and $p_2 = 2$ (when $Q = 1/2$). The modified utility function of (4.1) is given by

$$\tilde{u}(b) = \frac{1}{2}u(b) + \frac{1}{2}u(b/2) = \begin{cases} \frac{3b}{4} , & 0 \le b \le 1, \\ \frac{2+b}{4} , & 1 \le b \le 2, \\ 1 , & b \ge 2. \end{cases} \tag{7.1}$$

Clearly, an agent with this utility function should never bid more than 2. However, for small values of β, it is optimal to bid all up to a maximum of 2. In fact, we will show that, for $0 < \beta < 3/7$, the optimal policy π is of the form

$$c(s) = \begin{cases} s & , \quad 0 \le s \le 2, \\ 2 & , \quad s \ge 2. \end{cases}$$

To establish the optimality of π, it suffices to show that the return function $I(\cdot)$ satisfies the Bellman equation (4.3). First observe that $I(\cdot)$ satisfies the functional equation

$$I(s) = \begin{cases} \frac{3s}{4} + \beta \cdot EI(Z) , & 0 \le s \le 1, \\ \frac{2+s}{4} + \beta \cdot EI(Z) , & 1 \le s \le 2, \\ 1 + \frac{\beta}{4} I(s+2) + \frac{3\beta}{4} I(s-2) , & s \ge 2. \end{cases} \tag{7.2}$$

In particular,

$$I'(s) = \begin{cases} 3/4 , & 0 < s < 1, \\ 1/4 , & 1 < s < 2, \\ (\beta/4)\, I'(s+2) + (3\beta/4)\, I'(s-2) , & \text{for nonintegers } s > 2. \end{cases} \tag{7.3}$$

As a step toward the verification of the Bellman equation, we shall see that the function $I(\cdot)$ satisfies the concavity condition:

$$1 \ge \beta I'_+(4) + 3\beta I'_+(0). \tag{7.4}$$

Note that, if we can compute $I(0) = \beta \cdot EI(Z)$, then, by (7.2), we know the function $I(\cdot)$ on the interval [0,2].

Now write $a_k := I(2k)$, $k = 0, 1, \ldots$ and,by (7.2), we have the recursion

$$a_k = 1 + \frac{\beta}{4} a_{k+1} + \frac{3\beta}{4} a_{k-1}, \qquad k \ge 1. \tag{7.5}$$

A particular solution of (7.5) is $a_k \equiv \frac{1}{1-\beta}$, so the general solution is given by

$$I(2k) = a_k = \frac{1}{1-\beta} + A\theta^k, \qquad k = 1, 2, \ldots \tag{7.6}$$

for a suitable real constant A. Here θ is the root of the equation

$$f(\xi) := \xi^2 - \frac{4}{\beta}\xi + 3 = 0$$

in the interval $(0,\beta)$. Note that $f(0) = 3 > 0$, $f(\beta) = \beta^2 - 1 < 0$, and $\theta = \frac{2-\sqrt{4-3\beta^2}}{\beta}$.

Using (7.2) and (7.6), we see that $I(2) = 1 + I(1)$ and hence $I(0) = \frac{1}{1-\beta} + A\theta - 1 = \frac{\beta}{1-\beta} + A\theta$. Also $I(0) = \frac{\beta}{4}I(4) + \frac{3\beta}{4}I(0)$. Thus

$$(1 - \frac{3\beta}{4})I(0) = \frac{\beta}{4}I(4)$$

or

$$(1 - \frac{3\beta}{4})(\frac{\beta}{1-\beta} + A\theta) = \frac{\beta}{4}(A\theta^2 + \frac{1}{1-\beta}).$$

Hence, $A = -\frac{1}{1-\theta}$, $I(0) = \frac{\beta}{1-\beta} - \frac{\theta}{1-\theta} > 0$, and $I(1) = \frac{3}{4} + I(0) = \frac{1}{1-\beta} - \frac{\theta}{1-\theta} - \frac{1}{4}$.

More generally, with $d_k := I(2k+1)$, $k = 0, 1, \ldots$, we have the recursion

$$d_k = 1 + \frac{\beta}{4}d_{k+1} + \frac{3\beta}{4}d_{k-1}, \qquad k = 1, 2, \ldots$$

with general solution

$$d_k = \frac{1}{1-\beta} + D\theta^k, \qquad k = 1, 2, \ldots.$$

Plugging this last expression into the equality $I(3) = 1 + \frac{\beta}{4}I(5) + \frac{3\beta}{4}I(1)$, and substituting the value of $I(1)$ from above, we obtain $D = A\theta - \frac{1}{4} = -\frac{\theta}{1-\theta} - \frac{1}{4}$.

With these computations in place, we are now in a position to check the concavity condition (7.5). Indeed, $I'_+(4) = I(5) - I(4) = d_2 - a_2 = (D-A)\theta^2 = [A(\theta-1) - \frac{1}{4}]\theta^2 = \frac{3}{4}\theta^2 = \frac{3}{4}(\frac{4}{\beta}\theta - 3) = \frac{3\theta}{\beta} - \frac{9}{4}$. Thus

$$\beta I'_+(4) + 3\beta I'_+(0) = 3\theta - \frac{9}{4}\beta + 3\beta \cdot \frac{3}{4} = 3\theta < 1.$$

It follows that $f(\frac{1}{3}) < 0$ or $\frac{1}{9} - \frac{4}{3\beta} + 3 < 0$, and this last condition is equivalent to our assumption that $\beta < 3/7$.

We are now prepared to complete the proof that π is optimal by showing that its return function $I(\cdot)$ satisfies the Bellman equation (4.3). Equivalently, we have to check that the function

$$\psi_s(b) := \tilde{u}(b) + \beta EI(s-b+Z) = \tilde{u}(b) + \frac{\beta}{4}I(s-b+4) + \frac{3\beta}{4}I(s-b)$$

attains its maximum over $b \in [0,s]$ at $b^* = c(s)$. We consider three cases.

Case I: $0 \le s \le 1$. *In this case, for* $0 < b < s$*:*

$$\psi_s(b) = \frac{3}{4}b + \frac{\beta}{4}I(s-b+4) + \frac{3\beta}{4}I(s-b)$$

and

$$\psi_s'(b) = \frac{3}{4} - [\frac{\beta}{4}I_+'(4) + \frac{3\beta}{4}I_+'(0) > 0.$$

Thus $\psi_s'(s-) > 0$, *and* $b^* = s \equiv c(s)$ *is the location of the maximum.*

Case II: $1 < s \le 2$. *Here we use (7.1) and (7.3) to obtain*

$$\psi_s'(b) = \begin{cases} \frac{3}{4} - [\frac{\beta}{4}I_+'(5) + \frac{3\beta}{4}I_+'(1)], & 0 < b < s-1, \\ \frac{3}{4} - [\frac{\beta}{4}I_+'(4) + \frac{3\beta}{4}I_+'(0)], & s-1 < b < 1, \\ \frac{1}{4} - [\frac{\beta}{4}I_+'(4) + \frac{3\beta}{4}I_+'(0)], & 1 < b < s. \end{cases}$$

In particular, $\psi_s'(\cdot) > 0$ *on* $[0, s]$, *thus* $b^* = s \equiv c(s)$, *as follows from Lemma 7.1 below.*

Case III: $s > 2$. *In this case, we have*

$$\psi_s'(b) = \begin{cases} \frac{3}{4} - [\frac{\beta}{4}I_+'(s+3) + \frac{3\beta}{4}I_+'(s-1)], & 0 < b < 1, \\ \frac{1}{4} - [\frac{\beta}{4}I_+'(s+2) + \frac{3\beta}{4}I_+'(s-2)], & 1 < b < 2, \\ -\frac{\beta}{4}I_+'(\lfloor s-b \rfloor + 4) - \frac{3\beta}{4}I_+'(\lfloor s-b \rfloor), & b > 2. \end{cases}$$

The function $\psi(\cdot)$ *now attains its maximum at* $b* = 2 \equiv c(s)$, *since* $\psi'(\cdot) > 0$ *on* $(0, 2)$ *and* $\psi'(\cdot) < 0$ *on* $(2, s)$ *as follows from Lemma 7.1 below.*

Lemma 7.1. *The function* $I(\cdot)$ *satisfies:*

$$I_+'(0) > I_+'(1) > I_+'(2) > \cdots > 0.$$

Proof. The first three of these inequalities amount to

$$3/4 > 1/4 > \frac{\beta}{4}I_+'(4) + \frac{3\beta}{4}I_+'(0) = I_+'(0),$$

and have been established already. So we have to prove

$$I_+'(2k) > I_+'(2k+1) > I_+'(2(k+1)) > 0, \qquad \text{for } k \ge 2. \tag{7.7}$$

Now

$$I_+'(2k) = I(2k+1) - I(2k) = d_k - a_k = (D-A)\theta^k,$$

and

$$I_+'(2k+1) = I(2(k+1)) - I(2k+1) = a_{k+1} - d_k = (A\theta - D)\theta^k.$$

So the inequalities of (7.7) amount to

$$D - A > A\theta - D > \theta(D-A) > 0.$$

But $D - A = A(\theta - 1) - 1/4 = 3/4$, $A\theta - D = 1/4$, and these inequalities reduce to $0 < \theta < 1/3$, which has already been proved. □

The optimality of the strategy π for an agent playing in equilibrium with $B = 1$ has now been established. The stationary distribution μ for the corresponding Markov chain as in (4.4) is supported by the set of even integers $\{0, 2, \cdots\}$ and given by

$$\mu(\{0\}) = 1/2, \quad \mu(\{2\}) = 1/6, \quad \mu(\{2k\}) = (2/3)(1/3)^{k-1} \quad \textit{for } k \geq 2.$$

(The calculation of μ is explained in some detail in Example 2.5 of [KSS1].) The family of stationary strategies $\pi^\alpha \equiv \pi$ and the wealth distribution μ form a stationary equilibrium as in Theorem 4.2 for $0 < \beta < 3/7$. There will also exist equilibria for other values of β, but we shall not calculate them here.

The next example provides a simple illustration of Theorem 6.1.

Example 7.3. Let the utility function be

$$u(b) = \begin{cases} b, & b \geq 0, \\ 2b, & b < 0. \end{cases}$$

Suppose that the common distribution ζ of the variables $\{Q_n\}$ is $\zeta(\{1\}) = \zeta(\{3\}) = 1/2$ and that the distribution λ of the $\{Z_n\}$ is $\lambda(\{0\}) = \lambda(\{2\}) = 1/2$. The modified utility $\tilde{u}$ is then

$$\tilde{u}(b) = \frac{1}{2}u(b) + \frac{1}{2}u(3b) = \begin{cases} 2b, & b \geq 0, \\ 4b, & b < 0. \end{cases}$$

Take the interest rates to be $r_1 = r_2 = 2$ and the bound on lending to be $k = 1$. Finally assume that the total bid B is 1.

Although the penalty for default is heavy, as reflected by the larger value of $u'(b)$ for $b < 0$, it is to be expected that an agent will choose to make large bids for β sufficiently small. Indeed we shall show that the optimal strategy π for $0 < \beta < 1/3$ is to borrow up to the limit and spend everything, corresponding to $c(s) = s+1$ for all $s \geq 0$. (Recall that an agent with wealth $s < 0$ is punished in amount $u(s)$ and then plays from position 0. Thus a strategy need only specify bids for nonnegative values of s.)

Let $I(\cdot)$ be the return function for π. Then, for $s \geq 0$

$$I(s) = \tilde{u}(s+1) + \beta E[I(2(s-(s+1)) + Z)] = 2s + 2 + \beta[I(-2) + I(0)],$$

and, for $s < 0$,

$$I(s) = \tilde{u}(s) + I(0) = 4s + I(0).$$

Thus

$$I'(s) = \begin{cases} 2, & s > 0, \\ 4, & s < 0. \end{cases}$$

To verify that $I(\cdot)$ satisfies the Bellman equation (6.2), consider the function

$$\begin{aligned}\psi_s(b) &= \tilde{u}(b) + \beta E I(2(s-b)+Z) \\ &= 2b + \frac{\beta}{2}[I(2(s-b)) + I(2(s-b)+Z)] \\ &= \begin{cases} 2b - 4\beta b + c_1 & , \ 0 \le b < s, \\ 2b - 6\beta b + c_2 & , \ s < b < s+1, \end{cases}\end{aligned}$$

where c_1 and c_2 are constants. Thus

$$\psi_s'(b) = \begin{cases} 1 - 2\beta, & 0 < \beta < s, \\ 1 - 3\beta, & s < b < s+1, \end{cases}$$

and we see that $\psi_s(\cdot)$ attains its maximum on $[0, s+1]$ at $s+1$ because of our assumption that $\beta < 1/3$. It follows that $I(\cdot)$ satisfies the Bellman equation and π is optimal. The Markov chain $\{S_n\}$ of (6.3) becomes

$$S_{n+1} = 2((S_n)^+ - ((S_n)^+ + 1)) + Z = Z - 2.$$

The stationary wealth-distribution, namely, the distribution of $Z-2$, has mass $1/2$ at -2 and mass $1/2$ at 0. Obviously, clause (i) of Assumption 2 is satisfied. Clause (ii) is also satisfied, because every agent borrows one unit of money and spends it; one-half of the agents receive no income and pay back back nothing, whereas the other half receive an income of 2 units of money, all of which they pay back to the bank since the interest rate is $r_1 = 2$. As there are no lenders, the books balance. Theorem 6.1 now says that we have a stationary equilibrium in which half of the agents are in debt 2 units of money and the other half hold no money at the beginning of each period. All of the money is held by the bank.

Suppose now that the discount factor is larger so that agents will be more concerned about the penalties for default. In particular, assume that $1/3 < \beta < 1/2$. Then an argument similar to that above shows that an optimal strategy is for an agent to borrow nothing and spend what he has; that is, the optimal strategy π corresponds to $c(s) = s$ for every $s \ge 0$. This induces the Markov chain

$$S_{n+1} = 2((S_n)^+ - (S_n)^+) + Z = Z$$

with stationary distribution equal to the distribution of Z assigning mass $1/2$ each to 0 and $+2$. This time the books obviously balance, since no one borrows and no one pays back. In fact, the bank has no role to play.

For the next example we drop the assumption that individual endowments are proportional to total production (Assumption 1, part 2) and show that a stationary equilibrium need not exist.

Example 7.4. For simplicity, we return to a no-lending model for this example. Assume that the utility function is $u(b) = b$, and let the distribution ζ of the variables $\{Q_n\}$ be the two-point distribution: $\zeta(\{1\}) = \zeta(\{3\}) = 1/2$. Suppose that when $Q_n = 1$, the variables $\{Z_n^\alpha, \alpha \in I\}$ are equal to 0 or 2 with probability 1/2 each, but that when $Q_n = 3$, each of the Z_n^α is also equal to 3. Thus the $\{Q_n\}$ and the $\{Z_n^\alpha\}$ are not independent as we had assumed in Assumption 1.

Now suppose, by way of contradiction, that a stationary equilibrium exists with wealth distribution μ and optimal stationary strategies $\{\pi^\alpha,\ \alpha \in I\}$ corresponding to consumption functions $c^\alpha(\cdot), \alpha \in I$. The total bid in each period is then $B = \int c^\alpha(s)\, \mu(ds)$ and the prices $p_n = B/Q_n$ are independent, and equal to B and $B/3$ with probability 1/2 each.

Consider next the spend-all strategy π' with consumption function $c(s) = s$. We will sketch the proof that π' is the unique optimal strategy. First we calculate the return function $I(\cdot)$ for π':

$$I(s) = E\left[\frac{s}{p} + \beta \cdot I(BZ)\right] = \frac{2s}{B} + \beta \cdot E[I(BZ)].$$

It is easy to check that $I(\cdot)$ satisfies the Bellman equation

$$I(s) = \sup_{0 \le b \le s} E\left[\frac{b}{p} + \beta \cdot I(s - b + BZ)\right],$$

and that the supremum above is uniquely achieved at $b = s$. It follows that π' is the unique optimal strategy. Thus we must have $\pi^\alpha = \pi'$ for all α. So every agent α spends his entire wealth at every stage n and enters the next stage with wealth

$$S_{n+1}^\alpha(\omega) = BZ_{n+1}^\alpha(\omega).$$

But the distribution of Z_{n+1}^α depends on the value of Q_n. Thus the distribution of wealth varies with the value of Q_n and cannot be identically equal to the equilibrium distribution μ as we had assumed.

In our final example, we assume that agents know the value of the production variable Q_n in each period n before they make their bids. It is not surprising that agents playing optimally take advantage of this additional information, and therefore that a stationary equilibrium need not exist. What sort of equilibrium is appropriate for this “high information" model is a question that we plan to treat in a subsequent paper.

Example 7.5. As in the previous example, we consider a no-lending model with the linear utility $u(x) = x$ and with the distribution ζ of the variables Q_n given by $\zeta(\{1\}) = \zeta(\{3\}) = 1/2$. However, we assume that the individual endowments are proportional so that, as in Assumption 1, the variables Z_n^α are independent of the Q_n.

Suppose, by way of contradiction, that a stationary equilibrium does exist with wealth distribution μ and optimal stationary strategies $\{\pi^\alpha,\ \alpha \in I\}$ corresponding to consumption functions $\{c^\alpha(\cdot),\ \alpha \in I\}$. Let $B = \int c^\alpha(s)\, \mu(ds)$ be the total bid in each period so that the price in period n is $B/3$ if $Q_n = 3$ and is B if $Q_n = 1$. It is not difficult to show that, in a period when the price is low (i.e. when $Q_n = 3$), the optimal bid for an agent is $c(s) = s$. Thus we must have $c^\alpha(s) = s$ for all α and s. However, in a period when the price is high (i.e. when $Q_n = 1$), an agent who spends one unit of money receives in utility $\frac{1}{B}$, whereas an agent who saves the money and spends it in the next period expects to receive $\beta[\frac{1}{2}\frac{1}{B} + \frac{1}{2}\frac{3}{B}]$. Thus, for $\beta \in (\frac{1}{2}, 1)$, it is optimal for an agent to spend nothing in a period when the price is high. But then $c^\alpha(s) = 0$, a contradiction.

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