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Vertical Integration, Networks, and Markets

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COMPETITION FOR GOODS IN BUYER-SELLER NETWORKS

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Abstract: This paper studies competition in a network and how network structure determines agents' individual payoffs. It constructs a general model of competition that can serve as a reduced form for specific models. The paper shows how agents' outside options, and hence their shares of surplus, derive from "opportunity paths" connecting them to direct and indirect alternative exchanges. Analyzing these paths, results show how third parties' links affect an agent's bargaining power. Even distant links may have large effects on an agent's earnings. These payoff results, and the identification of the paths themselves, should prove useful to further analysis of network structure.

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1. Introduction

Networks of buyers and sellers are a common exchange environment. Networks are distinguished from markets by specific assets, or "links," between particular buyers and sellers that enhance the value of exchange. In many industries, for example, manufacturers train particular suppliers or otherwise "qualify" suppliers to meet certain criteria. The asset may also be less formal, as when a supplier's understanding of a manufacturer's idiosyncratic needs develops through repeated dealings.[1]

This paper studies competition for goods in a network. The theory we develop explains how third parties may affect the terms of a bilateral exchange. Bargaining theory focuses primarily on bilateral negotations. Yet strictly bilateral settings seem to be the exception rather than the rule. The network model we develop allows for arbitrarily complex multilateral settings. New links introduce new potential exchange partners. We show how such changes in opportunities affect matchings and divisions of surplus. In particular, we evaluate how third parties can affect an agent's "bargaining power" by changing, perhaps indirectly, its outside options.

The networks we consider consist of buyers, sellers, and the pattern of links that connect them. Each buyer demands a single unit of an indivisible good, and each seller can produce one unit. A buyer can only purchase from a seller to whom it is linked. Competition for goods will then depend on the link pattern. If a linked buyer and seller negotiate terms of trade, each agent's links to other agents determine their respective "outside options." Since alternative sellers or buyers may be linked to yet other agents, the entire pattern of links affects the value of these outside options and each agent's "bargaining power." For example, in Figure 1 below, the price buyer 3 would pay to either seller 1 or seller 3 depends on their links to buyers 1 and 2 or buyers 4 and 5 respectively, and further depends on these buyers' links to other sellers. The heart of this paper is identifying precisely how such indirect links affect competitive prices and each agent's ability to extract surplus from an exchange.

The paper first develops a theory of competition in a network. There are potentially many ways to model negotiations or competition for goods. This paper takes a general approach: we characterize competitive prices and allocations as those that satisfy a "supply equals demand"

[1] Kranton and Minehart (1998) introduces a model of buyer-seller networks, and Kranton and Minehart (1999) explores the role of networks in industrial organization and discusses industry examples.

condition for the network setting. We show that these prices yield payoffs that are individually rational and pairwise stable. These are the minimal conditions that payoffs resulting from any negotiation process or competition should satisfy. We show that there is range of such competitive prices, as in Demange and Gale (1985). Basic results also show the equivalence between these payoffs and the core of an assignment game [Shapley and Shubik (1972)]. Competitive prices, then, distribute the surplus generated by an efficient allocation of goods.

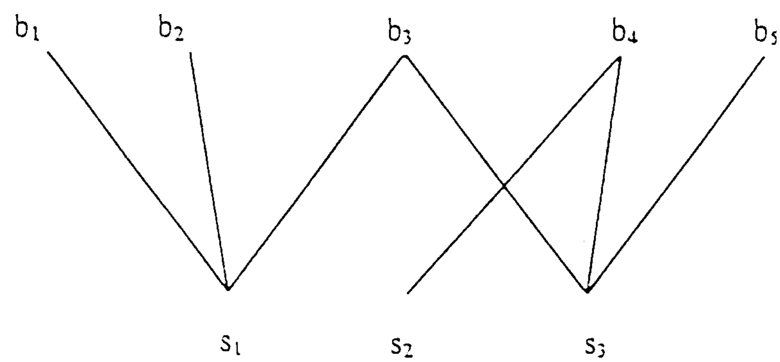

Figure 1

Armed with our general results, we turn to our central objective: studying the relationship between network structure and agents' competitive gains from trade.

We first characterize the range of competitive prices in terms of network structure. To do so, we define the notion of an *opportunity path.* An opportunity path is a path of links from a buyer to a seller to another buyer to a seller, and so on. These paths capture direct and indirect competition for a set of sellers' goods. The influence of network structure, expressed in these paths, is quite intuitive. For example, for the lower bound of a competitive price, we show that a buyer i that obtains a good must pay at least the valuation of a particular buyer. This buyer is not obtaining a good and is connected by an opportunity path to buyer i. It is therefore buyer i's (perhaps indirect) competitor and can replace buyer i in an allocation of goods. To prevent this replacement, buyer i must pay a sufficiently high price: it must pay at least what this buyer would be willing pay to obtain a good.[2]

The paper then asks how new links affect the prices paid by third parties. Consider a manu-

[2]This value is also the social opportunity cost of buyer i obtaining a good.

facturer that "qualifies" a particular supplier. This investment increases the value of an exchange between the two. It also affects the payoffs of all other manufacturers and suppliers. Since the supplier now has an additional sales option, it could extract greater rents from its other buyers. We call this *a supply stealing effect*. On the other hand, since the manufacturer has access to another source of supply, there is also a *supply freeing effect*. We evaluate these effects by identifying two types of paths in a network, what we call *buyer paths* and *seller paths*. A path between a buyer i and the seller with the new link, a *seller path*, is detrimental to buyer i's payoffs. The link confers the *supply stealing effect*. A path between a buyer i and the buyer with the new link, a *buyer path*, is beneficial to buyer i's payoffs. It confers a *supply freeing effect*.

For example, consider the network in Figure 1 and add a link between b_2 and s_2. The link, for example, frees supply for b_1 which can more often obtain goods from s_1. Buyers with buyer paths benefit from the supply freeing effect. In contrast, b_4 suffers from a supply stealing effect. This effect will extend to b_5, which now must sometimes compete with b_4 for s_3's output. As for sellers, the supply freeing effect hurts s_1 and the supply stealing effect helps s_2 and s_3.

These results provide a general framework to understand competition in a network setting. With this general model, we can place specific models of network competition in context. For example, an ascending-bid auction for a network [Kranton and Minehart (1998)] yields the lowest competitive prices. Other extensive form models would yield the same or different splits of surplus, or introduce trade frictions that drive the allocation away from efficiency.

The theory of buyer and seller paths explains how third parties can affect an agent's bargaining power. Previous theories of stable matchings in marriage problems and other such settings (e.g. Roth and Sotomayer (1990), Demange and Gale (1985)) view preferences (i.e., links) as exogenous. Hence, such comparative statics are not an issue. In our setting, links are specific investments over which agents ultimately make choices.[3] The comparative statics provide a methodology for studying how one agent's investments in specific assets impact others' returns. Ultimately, then, these results can inform the study of strategic incentives to invest in specific assets.[4]

[3]In Kranton and Minehart (1998), we develop a model of network formation in which agents invest in links. All the results in this paper apply to that model.

[4]Incentives to invest in specific assets is a major theme in industrial organization and theory of the firm literature. Classic contributions include Grossman and Hart (1990), Hart and Moore (1990), and Williamson (1975). Most studies to date consider specific asset investment in bilateral settings; the "outside option" is assumed but not modeled.

Section 2 builds a model of buyer-seller networks and develops a general notion of competition for goods. Section 3 characterizes the range of competitive prices in terms of the network structure. Section 4 considers how changes in the link pattern impact agents' competitive payoffs. Section 5 concludes.

2. Competition in a Network

2.1. Model of Buyer-Seller Networks[5]

There is a finite set of sellers $\mathbf{S}$ that number $\bar{S} \equiv |\mathbf{S}|$ who each have the capacity to produce one indivisible unit of a good at zero marginal cost. There is a finite set of buyers $\mathbf{B}$ that number $\bar{B} \equiv |\mathbf{B}|$ who each demand one indivisible unit of a good. Each buyer i, or b_i, has valuation v_i for a good, where $\mathbf{v} = (v_1, ..., v_{\bar{B}})$ is the vector of buyers' valuations. We restrict attention to generic valuations where $v_i > 0$ for all buyers i and $v_i \neq v_k$ for all buyers $i \neq k$.[6]

A buyer and seller can engage in exchange only if they are "linked." A *link pattern*, or graph, $\mathcal{G}$ is a $\bar{B}\times\ \bar{S}$ matrix, $[g_{ij}] \in \{0,1\}$, which indicates linked pairs of buyers and sellers. For buyer i and seller j, $[g_{ij}] = 1$ when b_i and s_j are linked, and $[g_{ij}] = 0$ when the pair is not linked. For a given link pattern and a set of buyers $\mathcal{B} \subseteq \mathbf{B}$, let $L(\mathcal{B}) \subseteq \mathbf{S}$ denote the set of sellers linked to any buyer in $\mathcal{B}$. We call $L(\mathcal{B})$ the buyers' *linked set* of sellers. Similarly, for a set of sellers $\mathcal{S} \subseteq \mathbf{S}$, let $L(\mathcal{S}) \subseteq \mathbf{B}$ denote the sellers' *linked set* of buyers.

Allocations of goods are feasible only when they respect the links between buyers and sellers. An *allocation of goods*, A, is a $\bar{B}\times\ \bar{S}$ matrix, $[a_{ij}] \in \{0,1\}$, where $[a_{ij}] = 1$ indicates that buyer i obtains a good from seller j. For a given link pattern $\mathcal{G}$, an allocation of goods is *feasible* if and only if $[a_{ij}] \leq [g_{ij}]$ for all i,j and for each buyer i, if there is a seller j such that $[a_{ij}] = 1$ then $[a_{ik}] = 0$ for all $k \neq j$ and $[a_{lj}] = 0$ for all $l \neq i$. The social surplus associated with an allocation A is the sum of the valuations of the buyers that secure goods in A. We denote the surplus as $w(A;\mathbf{v})$.[7]

[5] The following model of buyer-seller networks is from Kranton and Minehart (1998).

[6] This assumption is without loss of generality when buyers' valuations are independently and identically distributed with a continuous distribution. In this case, we would be concerned with expected valuations and non-generic valuations arise with probability zero.

[7] We can write $w(\mathbf{v},A) = \mathbf{v} \cdot A \cdot \mathbf{1}$, where $\mathbf{1}$ is an $\overline{S}\ \times 1$ matrix where each element is 1.

2.2. Competitive Prices and Allocations

We next consider competition for goods in this setting. Consider a price vector $\mathbf{p} = (p_1, ..., p_{\overline{S}})$ which assigns a price p_j to each seller s_j. Let u_i^b and u_j^s denote payoffs for each buyer i and seller j, respectively. Let $\mathbf{u}^b = (u_1^b, ...u_{\overline{B}}^b)$ and $\mathbf{u}^s = (u_1^s, ...u_{\overline{S}}^s)$ denote payoff vectors. For a price vector and allocation $(\mathbf{p}, A)$, payoffs are as follows: For seller j, $u_j^s = p_j$. For buyer i, $u_i^b = v_i - p_j$ if it obtains a good from seller j in A. Otherwise, $u_i^b = 0$.

We say a price vector and allocation ($\mathbf{p}$,A) is competitive when it satisfies the following "supply equals demand" conditions for the network setting:

Definition 1. *For a graph $\mathcal{G}$ and valuation $\mathbf{v}$, a price vector and allocation $(\mathbf{p},A)$ is competitive if and only if (1) if a buyer i and a seller j exchange a good, then $v_i \geq p_j \geq 0$ and $p_j = \min\{p_k | s_k \in L(b_i)\}$, (2) if a buyer i does not buy a good then $v_i \leq \min\{p_k | s_k \in L(b_i)\}$ and (3) if a seller j does not sell a good then $p_j = 0$.*[8]

The first two requirements are that there is no excess demand: given prices $\mathbf{p}$, a buyer would want to buy a seller's good if and only if it is assigned the good in A. That is, no more than one buyer demands any seller's good. The last requirement is that there is no excess supply: given $\mathbf{p}$, a seller would want to supply a good if and only if it provides a good in A. That is, no seller that does not have a buyer would wish to sell a good.

Our first set of results characterizes these competitive price vectors and allocations.

We show that each competitive price vector and allocation ($\mathbf{p}$,A) yields payoffs $(\mathbf{u}^b, \mathbf{u}^s)$ that are both *individually rational* and *pairwise stable*. Individual rationality simply requires that no agent earn negative payoffs. Pairwise stablity requires no linked buyer and seller can generate more surplus together than they earn in their joint payoffs. Formally,

[8] While these conditions may appear asymmetric with respect to buyers and sellers, they are not. More complicated notation would allow us to define competitive prices in an obviously symmetric manner. We have chosen to use the simpler notation in the text. The following definition is payoff equivalent to the one above with appropriate translation of notation: Consider a price vector (p_j^i) for $i = 1, ..., \overline{B}$ and $j = 1, ..., \overline{S}$. A price vector and allocation A are then competitive if and only if (1) if a buyer i and a seller j exchange a good, then $v_i \geq p_j^i \geq 0$ and $p_j^i = \min\{p_k^i | s_k \in L(b_i)\}$ and $p_j^i = \min\{p_j^k | b_k \in L(s_j)\}$; (2) if a buyer i does not buy a good then $v_i \leq \min\{p_k^i | s_k \in L(b_i)\}$ and (3) if a seller j does not sell a good then $0 = \min\{p_j^k | b_k \in L(s_j)\}$.

Definition 2. *A feasible*[9] *payoff vector* $(\mathbf{u}^b, \mathbf{u}^s)$ *is stable if and only if (i) (individual rationality)* $u_i^b \geq 0$; $u_j^s \geq 0$ *for all* i, j; *and (ii) (pairwise stability)* $u_i^b + u_j^s \geq v_i$ *for all linked pairs* b_i *and* s_j.[10]

We should expect any model of competition or negotiations in networks to yield stable payoffs, absent undue frictions in the negotiation process. It is straightforward to show that our "supply equals demand" conditions for prices and allocations are equivalent to these stability conditions. For example, if at given prices, only one buyer demands any seller's good, then there is no buyer that could offer a seller a different price that would make them both better off.

Proposition 1. *If* $(\mathbf{p},A)$ *is competitive, then* $\mathbf{u}^s = \mathbf{p}$ *and the associated payoffs for buyers* $\mathbf{u}^b$ *are stable. If* $(\mathbf{u}^b, \mathbf{u}^s)$ *are stable payoffs, then there is an allocation* A *and the price vector* $\mathbf{p} = \mathbf{u}^s$ *such that* $(\mathbf{p},A)$ *is competitive.*

Proof. Proofs are provided in the Appendix.

Our next result shows that a competitive price and allocation $(\mathbf{p},A)$ always involves an efficient allocation of goods.[11] For a graph $\mathcal{G}$ and valuation $\mathbf{v}$, an *efficient allocation* of goods yields the greatest possible social surplus and is defined as follows:

Definition 3. *For a given* $\mathbf{v}$, *a feasible allocation* A *is efficient if and only if, given* $\mathcal{G}$, *there does not exist any other feasible allocation* A' *such that* $w(A'; \mathbf{v}) > w(A; \mathbf{v})$.

It is easy to understand why competitive prices are always associated with efficient allocations. If it were not the case, then there would be excess demand for some seller's good. A buyer that is not purchasing but has a higher valuation than a purchasing buyer would also be willing to pay the sales price.

[9] Feasiblility requires that payoffs can derive from a feasible allocation of goods. The payoffs $(\mathbf{u}^b, \mathbf{u}^s)$ are *feasible* if there is a feasible allocation A such that (i) $u_i^b = 0$ for any buyer i who does not obtain a good, (ii) $u_j^s = 0$ for any seller j who does not sell a good, and (iii) $\sum_i u_i^b + \sum_j u_j^s = w(A; \mathbf{v})$.

[10] We do not write the stability condition for buyers and sellers that are not linked because it is always trivially satisfied.

[11] This and the remainder of the results in this section derive from basic results on assignment games. Assignment games consider stable pairwise matching of agents in settings such as marriage "markets." In our setting, the value of matches would be given by $\mathbf{v}$ and the graph $\mathcal{G}$. Shapley and Shubik (1972) develop the basic results we use in this section. Roth and Sotomayor (1990, Chapter 8) provide an excellent exposition. We refer the reader to their work and our (1998) working paper for proofs and details.

Proposition 2. *For a graph $\mathcal{G}$ and valuation $\mathbf{v}$, if a price vector and allocation $(\mathbf{p},A)$ is competitive, then A is an efficient allocation.*

We next present a result that greatly simplifies the analysis of competitive prices and allocations. The first part of the proposition shows the "equivalence" of efficient allocations: in any efficient allocation, the same set of buyers obtains goods.[12] The second part of the proposition shows that the set of competitive price vectors is the same for all efficient allocations. With this result we can ignore the particular efficient allocation and refer simply to the set of competitive price vectors for a graph $\mathcal{G}$ and valuation $\mathbf{v}$. The result implies that the set of agents' competitive payoffs is uniquely defined; it is the same for all efficient allocations of goods.

Proposition 3. *For a network $\mathcal{G}$ and valuation $\mathbf{v}$:*

(a) If A and A' are both efficient allocations, then a buyer obtains a good in A if and only if it obtains a good in A'.

(b) If for some efficient allocation A, $(\mathbf{p},A)$ is competitive, then for any efficient allocation A', $(\mathbf{p},A')$ is also competitive.

Our final result of this section shows that the set of competitive price vectors for a graph $\mathcal{G}$ and valuation $\mathbf{v}$ has a well-defined structure. Competitive price vectors exist, and convex combinations of competitive price vectors are also competitive. There is a maximal and a minimal competitive price vector. The maximal price vector gives the best outcome for sellers, and the minimal price vector gives the best outcome to buyers. We will later examine how changes in the network structures affect these bounds.

Proposition 4. *The set of competitive price vectors is nonempty and convex. It has the structure of a lattice. In particular, there exist extremal competitive prices $\mathbf{p}^{\max}$ and $\mathbf{p}^{\min}$ such that*

[12] Proposition 3 below requires generic valuations. Otherwise, efficient allocations could involve different sets of buyers. For example, in a network with one seller and two linked buyers, if the two buyers have the same valuation v, then either one could obtain the good. The assumption of generic valuations simplifies our proofs, but the results obtain for all valuations. If two efficient allocations A and A' involve different sets of buyers, and if $(\mathbf{p},A)$ is competitive then there is a $(\mathbf{p}',A')$ that gives the same payoffs and is also competitive. That is, each allocation is associated with the same set of stable payoffs. In the two buyer example, for instance, the buyer obtaining the good always pays $p = v$ and both buyers earn $u^b = 0$.

$\mathbf{p}^{\min} \leq \mathbf{p} \leq \mathbf{p}^{\max}$ *for all competitive prices* $\mathbf{p}$. *The price* $\mathbf{p}^{\max}$ *gives the worst possible outcome for each seller and the best possible outcome for each buyer.* $\mathbf{p}^{\min}$ *gives the opposite outcomes.*

2.3. Competitive Prices, Opportunity Cost, and Network Structure

In this section, we determine the relationship between network structure and the set of competitive prices. To do so, we use the notion of "outside options" to characterize the extremal competitive prices; that is, we relate $\mathbf{p}^{\max}$ and $\mathbf{p}^{\min}$ to agents' next-best exchange opportunities. We will see that the private value of these opportunities can be determined by quite distant indirect links. The relationships we derive below are a basis for our comparative static results on changes in the link pattern.

We first formalize the physical connection between a buyer, its exchange opportunities, and its direct and indirect competitors. A buyer's exchange opportunities and competitors in a network are determined by its links to sellers, these sellers' links to other buyers, and so on. In a graph $\mathcal{G}$, we denote a *path* between two agents as follows: a path between a buyer i and a buyer m is written as $b_i - s_j - b_k - s_l - b_m$, meaning that b_i and s_j are linked, s_j and b_k are linked, b_k and s_l are linked, and finally s_l and b_m are linked. For a given feasible allocation A, we use an arrow to indicate that a seller j's good is allocated to a buyer $k : s_j \rightarrow b_k$.

For a feasible allocation A, we define a particular kind of path, an *opportunity path*, that connects an agent to its alternative opportunities and the competitors for those exchanges. Consider some buyer which we label b_1. We write an opportunity path connecting buyer 1 to another buyer n as follows:

$$b_1 - s_2 \rightarrow b_2 - s_3 \rightarrow ...b_{n-1} - s_n \rightarrow b_n.$$

That is, buyer 1 is linked to seller 2 but not purchasing from seller 2. Seller 2 is selling to buyer 2, buyer 2 is linked to seller 3, and so on until we reach b_n. An opportunity path begins with an "inactive" link, which gives buyer 1's alternative exchange. The path then alternates between "active" links and "inactive" links, which connect the direct and indirect competitors for that exchange. Since the path must be consistent both with the graph and the allocation, we refer to a path as being "in $(A, \mathcal{G})$." We say a buyer has a "trivial" opportunity path to itself.

Opportunity paths determine the set of competitive prices. We next show that $\mathbf{p}^{\max}$ and $\mathbf{p}^{\min}$ derive from opportunity paths in $(A^*, \mathcal{G})$, where A^* is an efficient allocation of goods for a given valuation $\mathbf{v}$. The results show how prices relate to third party exchanges along an opportunity path and build on the following reasoning. Suppose for given competitive prices, some buyer 1 obtains a good from a seller 1 at price p_1. Suppose further that buyer 1 is also linked to a seller 2, through which it has an opportunity path to a buyer n, as specified above. Because buyer 1 does not buy from seller 2 and prices are competitive, it must be that $p_2 \geq p_1$. That is, seller 2's price is an upper bound for p_1. Furthermore, since buyer 2 buys from seller 2 but not seller 3, it must be that $p_3 \geq p_2$. That is, seller 3's price provides an upper bound on p_2 and hence on p_1. Repeating this argument tells us that buyer 1 's price is bounded by the prices of all the buyers on the path. That is, *if a buyer buys a good, the price it pays can be no higher than the prices paid by buyers along its opportunity paths.*[13]

Building on this argument, let us characterize $\mathbf{p}^{\max}$. No price paid by any buyer is higher than its valuation. Therefore, $p_1^{\max}$ is no higher than the lowest valuation of any buyer linked to buyer 1 by an opportunity path. We label this valuation $v^L(b_1)$.[14] Our next result shows that when $p_1^{\max} \neq 0$, it exactly equals $v^L(b_1)$. To prove this, we argue that we can raise $p_1^{\max}$ up to $v^L(b_1)$ without violating any stability conditions. When the price of exchange between a buyer and a seller changes, the stability conditions of all linked sellers and buyers change as well. The proof shows that we can raise the price simultaneously for a particular group of buyers in such a way as to maintain stability for all buyer-seller pairs. For $p_1^{\max} = 0$, we show that buyer 1 has an opportunity path to a buyer that is linked to a seller that does not sell its good. This buyer obtains a price of 0, which then forms an upper bound for buyer 1 's price. We have:

Proposition 5. *Suppose that in* $(A^*, \mathcal{G})$*, a buyer* 1 *obtains a good from a seller* 1*. If* $p_1^{\max} > 0$*, then* $p_1^{\max} = v^L(b_1)$ *where* $v^L(b_1)$ *is the lowest valuation of any buyer linked to buyer* 1 *by an opportunity path. If* $p_1^{\max} = 0$*, then buyer* 1 *has an opportunity path to a buyer that is linked to a seller that does not sell its good.*

We can understand the value $v^L(b_1)$ as buyer 1's "outside option" when purchasing from seller 1. If buyer 1 does not purchase from seller 1, the worst it can possibly do is pay a price of $v^L(b_1)$ to obtain a good. This is the valuation of the buyer that buyer 1 would displace by changing

[13]This observation is central to the proofs of most of our subsequent results. We present is as a formal lemma in the appendix.

[14]Since buyer 1 has an opportunity path to itself, $p_1^{\max} \leq v_1$.

sellers. This displaced buyer could be arbitrarily distant from buyer 1. Buyer 1 can purchase from a new seller and, in the process, displace a buyer n on an opportunity path pictured above as follows: buyer 1 obtains a good from seller 2 whose former buyer 2 now purchases from seller 3 whose former buyer 3 now purchases from seller 4 and so on, until we reach buyer n who no longer obtains a good. In order to accomplish this displacement, buyer 1 must pay its new seller a price of at least v_n. This price becomes a lower bound for the prices paid along the opportunity path, and is just high enough so that buyer n is no longer interested in purchasing a good. As indicated by the above Proposition, the easiest such buyer to displace is the one with the lowest valuation on opportunity paths from buyer 1.

We next characterize the minimum competitive price $\mathbf{p}^{\min}$ in terms of opportunity paths. The opportunity paths from a seller also determine a seller's "outside option." Consider a seller 1 that is selling to buyer 1. We write an opportunity path connecting seller 1 to another buyer n as follows:

$$s_1 - b_2 \rightarrow s_2 - b_3 \rightarrow ...s_{n-1} - b_n.$$

The path begins with an "inactive" link, then alternates between active and inactive links and ends with a buyer. If s_1 has opportunity path(s) to buyers that do not obtain goods, s_1 will receive $p_1 > 0$. The non-purchasing buyers at the end of the paths set the lower bound of p_1. If p_1 were lower than these buyers' valuations, there would be excess demand for goods. Therefore, $p_1^{\min}$ must be no lower than the highest valuation of these non-purchasing buyers. We label this valuation $v^H(s_1)$; it is the highest valuation of any buyer that does not obtain a good and is linked to seller 1 by an opportunity path. The proof of the next result shows that if $p_1^{\min} > 0$, then $p_1^{\min}$ is exactly equal to $v^H(s_1)$. As in the previous proposition, we show this by supposing $p_1^{\min} > v^H(s_1)$ and showing it is possible to decrease the price in such a way as to maintain all stability conditions. If and only if $p_1^{\min} = 0$, then s_1 has no opportunity paths to buyers that do not obtain goods. We have

Proposition 6. *Suppose that in $(A^*, \mathcal{G})$, a buyer 1 obtains a good from a seller 1. If $p_1^{\min} > 0$, then $p_1^{\min} = v^H(s_1)$, where $v^H(s_1)$ is the highest valuation of any buyer that does not obtain a good and is linked to seller 1 by an opportunity path. If and only if $p_1^{\min} = 0$, then all buyers linked to seller 1 by an opportunity path obtain a good in A.*

We can understand the value $v^H(s_1)$ as seller 1's "outside option" when selling to buyer 1. The worst seller 1 can do if it does not sell to buyer 1 is earn a price $v^H(s_1)$ from another buyer. This price is the valuation of the buyer that would replace buyer 1 in the allocation of goods. The replacement occurs along an opportunity path from seller 1 to a buyer n as follows: seller 1 no longer sells to buyer 1, but sells instead to buyer 2, whose former seller 2 now sells to buyer 3, and so on until seller $n-1$ now sells to buyer n. To accomplish this replacement, seller 1 can charge its new buyer a price no more than v_n. This price forms a new lower bound on the opportunity path, and is just low so that the new buyer b_n is willing to buy. Out of all the buyers n that could replace buyer 1 in this way, the best for seller 1 is the buyer with highest valuation.[15]

We conclude the section with a summary of our results on the set of competitive prices and opportunity paths.

Proposition 7. *A price vector* $\mathbf{p}$ *is a competitive price vector if and only if for an efficient allocation* A, $\mathbf{p}$ *satisfies the following conditions: (i) if a buyer* i *and a seller* i *exchange a good, then* $v^L(b_i) \geq p_i \geq v^H(s_i)$ *and* $p_i = \min\{p_k | s_k \in L(b_i)\}$, *(ii) if a seller* i *does not sell a good then* $p_i = 0$.

We illustrate these results in the example below. We show the efficient allocation of goods and derive the buyer-optimal and the seller-optimal competitive prices, $\mathbf{p}^{\min}$ and $\mathbf{p}^{\max}$, from opportunity paths.

Example 1. *For the network in Figure 2 below, suppose buyers' valuations have the following order:* $v_2 > v_3 > v_4 > v_5 > v_6 > v_1$. *The efficient allocation of goods involves* b_2 *purchasing from* s_1, b_3 *from* s_2, b_4 *from* s_3, *and* b_6 *from* s_4, *as indicated by the arrows. In a competitive price vector* $\mathbf{p}$, p_1 *is in the range* $v_3 \geq p_1 \geq v_1$*: To find* $p_1^{\min}$, *we look for opportunity paths from* s_1. *Seller 1 has only one opportunity path to a buyer that does not obtain a good - to* b_1. *Therefore,* $v^H(s_1) = v_1$. *For* $p_1^{\max}$, *we look for opportunity paths from* b_2. *Buyer 2 has only one opportunity path - to* b_3.[16] *Therefore,* $v^L(b_2) = v_3$. *The price* p_2 *for seller 2 is in the range*

[15] Note that $v^H(s_1)$ is exactly the social opportunity cost of allocating the good to buyer 1. If buyer 1 did not purchase, the buyer that would replace it in an efficient allocation of goods, the "next-best" buyer, has valuation of $v^H(s_1)$.

[16] The path from b_2 to b_4, for example, is not an opportunity path, because it does not alternative between inactive and active links.

$v_3 \geq p_2 \geq v_5$: Seller 2 has two opportunity paths to buyers who do not obtain a good - to b_1 and b_5. Since $v_5 > v_1$, $v^H(s_2) = v_5$. Buyer 3 has only a "trivial" opportunity path to itself. Therefore, $v^L(b_3) = v_3$. We can, similarly, identify the maximum and minimum prices for s_3 and s_4, giving us $\mathrm{p}^{\min} = (v_1, v_5, v_5, 0)$ and $\mathrm{p}^{\max} = (v_3, v_3, v_4, v_6)$. Any convex combination of these upper and lower bounds, $(\beta v_1 + (1-\beta)v_3, \beta v_5 + (1-\beta)v_3, \beta v_5 + (1-\beta)v_4, (1-\beta)v_6)$ where $\beta \in [0,1]$, are also competitive prices.

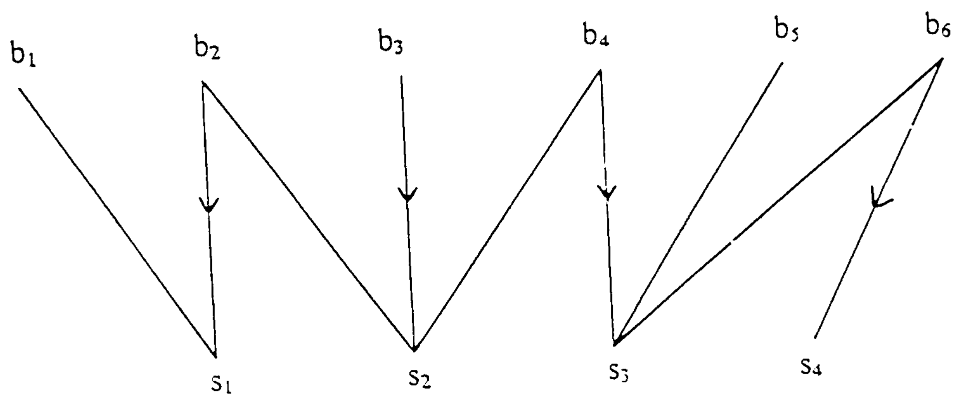

Figure 2

3. Network Comparative Statics

In this section we explore how changes in a network impact agents' competitive payoffs.

3.1. Payoffs as Functions of the Graph

To compare payoffs between graphs, we first make a unique selection from set of competitive payoffs for each graph. For a graph $\mathcal{G}$ and valuation v, we define the price vector $\mathrm{p}(\mathcal{G}) \equiv q\mathrm{p}^{\min}(\mathcal{G}) + (1-q)\mathrm{p}^{\max}(\mathcal{G})$, where $q \in [0,1]$ and $\mathrm{p}^{\min}(\mathcal{G})$ and $\mathrm{p}^{\max}(\mathcal{G})$ are the lowest and highest competitive prices for $\mathcal{G}$ given v. We assume that q is the same for all graphs and valuations. By Proposition 4, the set of competitive prices is convex, so the price vector $\mathrm{p}(\mathcal{G})$ is competitive. For a given q and given valuation v, let $u_i^b(\mathcal{G})$ and $u_j^s(\mathcal{G})$ denote the competitive payoffs of buyer i and seller j as a function of $\mathcal{G}$. Taking an efficient allocation for $(\mathcal{G}, \mathrm{v})$, for a buyer i that purchases from seller j, we have $u_j^s(\mathcal{G}) = p_j(\mathcal{G})$ and $u_i^b(\mathcal{G}) = v_i - p_j(\mathcal{G})$. Buyers who do not obtain a good receive a payoff of zero, as do sellers who do not sell a good.

This parameterization allows us to focus on how changes in a network affects an agent's "bargaining power." With q fixed across graphs, the difference in an agent's ability to extract surplus depends on the changes in the outside options, as determined by the graphs. We can see this as follows: The total surplus of an exchange between a buyer i and a seller j is v_i. Of this surplus, in graph $\mathcal{G}$ a buyer i earns at least its outside option $v_i - p_j^{\max}(\mathcal{G}) = v_i - v^L(b_i)$, where $v^L(b_i)$ is derived from the opportunity paths in $\mathcal{G}$. Similarly, seller j earns at least its outside option $p_j^{\max}(\mathcal{G}) = v^H(s_i)$. The buyer then earns a proportion q of the remaining surplus, and the seller earns a proportion $(1-q)$. We have

$$\begin{aligned} u_i^b(\mathcal{G}) &= v_i - v^L(b_i) + q\left[v_i - \left(v_i - v^L(b_i) + v^H(s_j)\right)\right] = v_i - p_j(\mathcal{G}), \\ u_j^s(\mathcal{G}) &= v^H(s_j) + (1-q)\left[v_i - \left(v_i - v^L(b_i) + v^H(s_j)\right)\right] = p_j(\mathcal{G}). \end{aligned}$$

A change in the graph would impact $v^L(b_i)$ and $v^H(s_j)$ through a change in an agent's set of opportunity paths, and thereby affect agents' shares of the total surplus from exchange.[17]

The proportion q could depend on some (unmodeled) features of the environment, such as agents' discount rates.[18] An assumption of a "Nash bargaining solution" would set $q = \frac{1}{2}$. Specific price formation processes may also yield a particular value of q. An ascending-bid auction for the network setting, for example, gives $q = 1$ (see Kranton and Minehart (1998)). In this sense, our parameterization provides a framework within which to place specific models of network competition and bargaining. As long as q does not depend on the graph, our payoffs are a reduced form for any model that yields individually rational and pairwise stable payoffs.

3.2. Comparative Statics on an Agent's Network: Population and Link Pattern

We now study how changes in a network affect agent's competitive payoffs. We show changes in payoffs for any valuation $\mathbf{v}$. We first consider the payoff implications of adding a link, holding fixed the number of buyers and sellers. We then consider adding new sets of buyers or sellers to a network. A priori, the impact of these changes is not obvious. As mentioned in the introduction, there are possibly many externalities from changing the link pattern.

[17] This approach to "bargaining power" is often used in the literature on specific assets. For instance, in a bilateral settting Grossman and Hart (1986) fix a 50/50 split ($q = 1/2$) of the surplus net agents' outside options. They then analyze how different property rights change agents' outside options.

[18] In a bilateral bargaining with alternating offers, Rubinstein (1982) and others derives q from agents' relative rates of discount.

Adding Links We begin with preliminary results to help identify the source of price changes when a link is added to a network. Consider a link pattern $\mathcal{G}$ and add a link between a buyer and a seller that are not already linked. Denote the buyer b_a, the seller s_a, and the augmented graph $\mathcal{G}'$. The first result shows that an efficient allocation A' for $\mathcal{G}'$ involves at most one new buyer with respect to an efficient allocation A for $\mathcal{G}$. We can trace all price changes to this buyer. This buyer either replaces a buyer that purchased in A or is simply added to this set of buyers. It is also possible that no new buyer obtains a good. In this case, the second result says we can simply restrict attention to an allocation that is efficient in both graphs.

If a new buyer does obtain a good, the efficient allocation changes along what we call a *replacement path*, a form of opportunity path. The new buyer n, which we call the *replacement buyer*, obtains a good from a seller, whose previous buyer obtains a good from a new seller, and so on, along an opportunity path from buyer n to some buyer 1. Buyer 1 either no longer obtains a good or obtains a good from a previously inactive seller. Critically, we show that this replacement involves the new link, and no other changes can strictly improve economic welfare. (If any such improvement were possible, it could not involve the new link and so would have been possible in the original graph $\mathcal{G}$, and, hence, the original allocation A could not have been efficient.)

Lemma 1. *For a given* $\mathbf{v}$, *if an efficient allocation* A *for* $\mathcal{G}$ *and an efficient allocation* $A^{'}$ *for* $\mathcal{G}'$ *involve different sets of buyers, then* A *and* A' *are identical except on a set of* n *or* $n+1$ *distinct agents* $\mathcal{H} = \{(s_1), b_1,\ s_2,\ b_2,\ ...,\ s_n,\ b_n\}$ *that may or may not include the seller* s_1. *These agents are connected by an opportunity path* $(s_1) \rightarrow b_1 - s_2 \rightarrow b_2 - s_2 \rightarrow ..b_{n-1} - s_n \rightarrow b_n$ *in* $(A', \mathcal{G}')$. *The path includes (relabeled) the agents with the additional link* $s_a \rightarrow b_a$. *In* $(A, \mathcal{G})$, *this path is in two pieces* $b_n - s_n \rightarrow b_{n-1} - ...s_{a+1} \rightarrow b_a$ *and* $s_a \rightarrow b_{a-1} - ...s_2 \rightarrow b_1 - (s_1)$, *with the new link between* s_a *and* b_a *the "missing" link. Buyer* n *obtains a good in* A' *but not in* A. *Buyer* 1 *obtains a good in* A. *Buyer* 1 *obtains a good in* A' *if and only if* $s_1 \in \mathcal{H}$.

Lemma 2. *For a given* $\mathbf{v}$, *if an efficient allocation* A *for* $\mathcal{G}$ *and an efficient allocation* $A^{'}$ *for* $\mathcal{G}'$ *involve the same set of buyers, then* A *is efficient for both graphs.*

With these preliminary results, we can evaluate the impact of an additional link on payoffs for different buyers and sellers in a network. Our first result considers the direct effects of the link. We show that the buyer and seller with the additional link (b_a and s_a) enjoy an increase in their competitive payoffs. Intuitively, the buyer (seller) is better off with more direct sources of supply (demand).

Proposition 8. *For the buyer and seller with the additional link (b_a and s_a), $u_a^b(\mathcal{G}') \geq u_a^b(\mathcal{G})$ and $u_a^s(\mathcal{G}') \geq u_a^s(\mathcal{G})$.*

The result is proved by examining opportunity paths. Suppose that when the link is added, a new buyer (the replacement buyer) obtains a good. By Lemma 1, this buyer, b_n, has an opportunity path to b_a in $(A, \mathcal{G})$. Because b_n does not obtain a good in A, it must be facing a prohibitive "best" price of at least v_n. This can only happen if b_a's price, which is an upper bound of the prices of sellers along the opportunity path, is at least v_n. In $(A', \mathcal{G}')$, the direction of the opportunity path is reversed. That is, b_a now has an opportunity path to b_n. The price that b_n pays is now an upper bound on b_a's price. Since b_n pays at most its valuation, b_a's price is at most v_n. We have, thus, shown that b_a pays a (weakly) lower price and receives a higher payoff in $(A', \mathcal{G}')$.

Our next results consider the indirect effects of a link. One effect, as mentioned in the introduction, is the *supply stealing effect.* When a link is added between b_a and s_a, b_a can now directly compete for s_a's good. Buyers with direct or indirect links to s_a, then, should be hurt by the additional competition. Sellers should be helped. On the other hand, there is a *supply freeing effect.* When a link is added between b_a and s_a, b_a depends less on its other sellers for supply. Some buyer n that is not obtaining a good may now obtain a good from a seller $s_k \in L(b_a)$. With less competition for these sellers' goods, sellers should be hurt and buyers helped.

We identify the two types of paths in a network that confer these payoff externalities. If in $\mathcal{G}$, there is a path connecting an agent and b_a, we say the agent has a *buyer path* in $\mathcal{G}$. If in $\mathcal{G}$ there is a path connecting an agent and s_a, we say the agent has a *seller path* in $\mathcal{G}$. Buyer paths confer the supply freeing effect: A buyer i with a buyer path is indirectly linked to buyer a and is, thus, in competition for some of the same sellers' output. When b_a establishes a link with another seller, it frees supply for b_i. Sellers along the buyer path face lower demand and receive weakly lower prices. Seller paths confer the supply stealing effect: If b_i has a seller path, b_i faces more competition for s_a's good; that is, b_a steals supply from b_i. Competition for goods increases, hurting b_i and helping sellers along the seller path.

We use these paths to show how new links affect the payoffs of third parties. It might seem natural that the size of the externality depends on the length of the path. The more distant the new link, the weaker the effect. The next example shows that this is not the case. It is not the length of a path that matters, but how it is used in the allocation of goods.

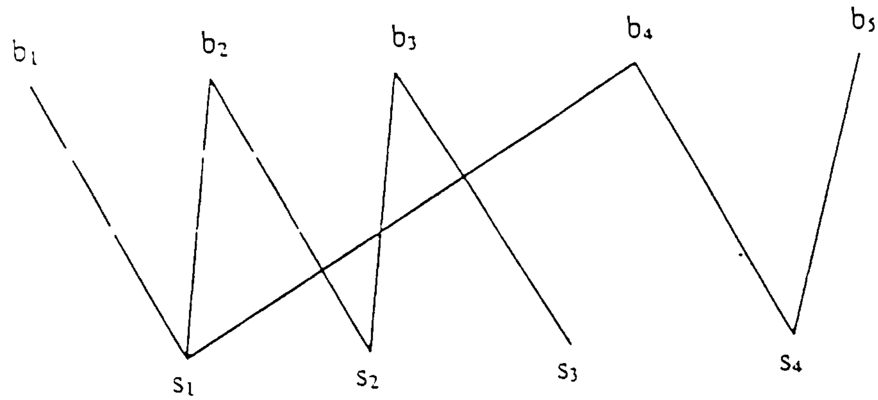

Figure 3

Example 2. *In Figure 3 above, consider the impact on b_2 of a link between b_4 and s_3. b_2 has a short buyer path and a long seller path. However, the supply stealing effect (through the seller path) dominates. Without the link between b_4 and s_3, b_2 always obtains a good (b_2 always buys from s_2, and b_3 always buys from s_3). With the link, b_2 is sometimes replaced by b_4 and no longer obtains a good. This occurs for particular valuations* v. *For other* v, *b_2 is not replaced, but the price it pays is weakly higher. Therefore, b_2's competitive payoffs fall for any* v.

We next show how the impact of buyer and seller paths depend on the network structure. Our first result demonstrates the payoff effects when an agent only has one type of path. Following results indicate payoff effects when agents have both buyer and seller paths

If an agent has only a buyer path or only a seller path in $\mathcal{G}$, the effect of the new link on its payoffs is clear. A buyer that has only a buyer path (seller path) is helped (hurt) by the additional link. A seller that has only a buyer path (seller path) is hurt (helped) by the additional link. We have the following proposition, which we illustrate below.

Proposition 9. *For a buyer i that has only buyer paths in $\mathcal{G}$, $u_i^b(\mathcal{G}') \geq u_i^b(\mathcal{G})$. For a buyer i that has only seller paths in $\mathcal{G}$, $u_i^b(\mathcal{G}') \leq u_i^b(\mathcal{G})$. For a seller j that has only buyer paths in $\mathcal{G}$, $u_j^s(\mathcal{G}') \leq u_j^s(\mathcal{G})$. For a seller j that has only seller paths in $\mathcal{G}$, $u_j^s(\mathcal{G}') \geq u_j^s(\mathcal{G})$.*

Example 3. *In the following graph, consider adding a link between buyer 4 and seller 3. Sellers 1 and 2 have only seller paths and are better off. Seller 4 with only a buyer path is worse off. Buyer 5 is better off because it has only a buyer path. Buyers 1, 2, and 3 with only seller paths are worse off.*

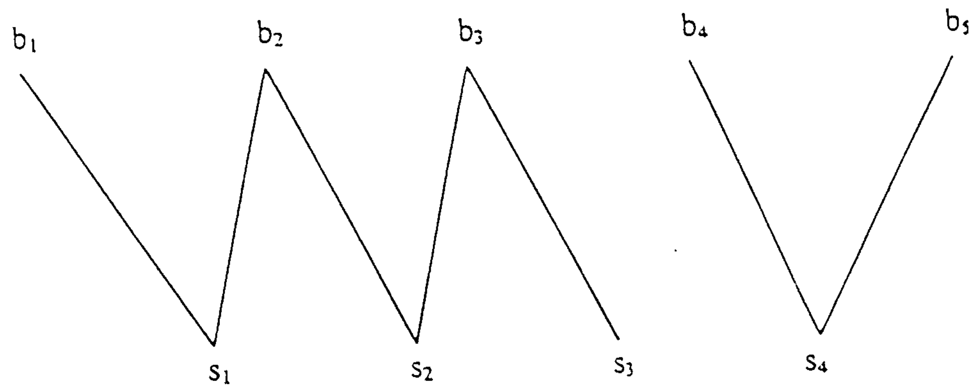

Figure 4

When agents have both buyer and seller paths, the overall impact on payoffs is less straightfoward. Supply freeing and supply stealing effects go in opposite directions. In many cases, however, we can determine the overall impact of a new link. We begin with the agents that have links to b_a or s_a. We show that the buyers (sellers) linked to the seller (buyer) with the additional link are always weakly worse off. For buyers (sellers), the supply stealing (freeing) effect dominates.

Proposition 10. *For every* $b_i \in L(s_a)$ *in* $\mathcal{G}, u_i^b(\mathcal{G}') \leq u_i^b(\mathcal{G})$. *For* $s_j \in L(b_a)$ *in* $\mathcal{G}$, $u_j^s(\mathcal{G}') \leq u_j^s(\mathcal{G})$.

The proof argues that for these buyers (sellers), there is in fact no new supply freeing (stealing) effect associated with the new link. To see this, consider a $b_i \in L(s_a)$. Potentially, b_i could benefit from the fact that b_a's new link frees the supply of b_a's *other* sellers. We show that this hypothesis contradicts the efficiency of the allocation A in $\mathcal{G}$. Suppose, for example, that b_i benefits from the new link because it is the replacement buyer. That is, b_i obtains a good in $(A', \mathcal{G}')$, but not in $(A, \mathcal{G})$. By Lemma 1, b_i replaces a buyer 1 along an opportunity path such as $b_1 - s_a \rightarrow b_a - s_i \rightarrow b_i$ in $(A', \mathcal{G}')$, as pictured below in Figure 5 where the new link is dashed. That is, in this example,

b_1 obtained a good directly from s_a in A and does not obtain a good at all in A'. Then, since b_i is linked to s_a by hypothesis, b_i could have replaced buyer 1 along the path $b_1 - s_a \rightarrow b_i$ in $\mathcal{G}$. If the replacement is efficient in $\mathcal{G}'$ then it is also efficient in $\mathcal{G}$. Hence, A could not have been an efficient allocation.[19]

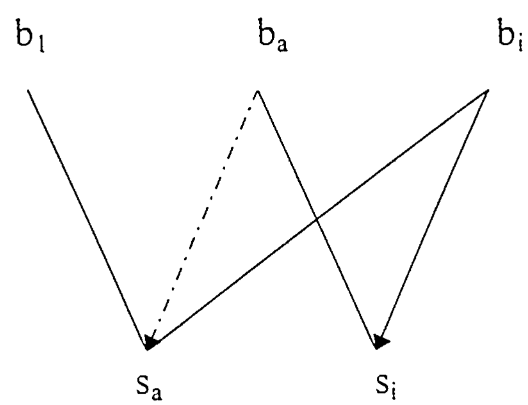

Figure 5

We can show further that any buyer that is only linked to sellers that are, in turn, linked to b_a is always better off with the additional link. For such a buyer, the supply-freeing effect dominates any supply stealing effect. We provide the proposition, then illustrate below. The intuition here is simple. If the buyer obtains a good, it must be from a seller linked to b_a. By our previous Proposition 10, this seller is worse off in $\mathcal{G}'$. So any of its possible buyers must be better off.

Proposition 11. *For every b_i such that $L(b_i) \subseteq L(b_a)$ in $\mathcal{G}$, $u_i^b(\mathcal{G}') \geq u_i^b(\mathcal{G})$.*

The next example shows how to apply this and previous results to evaluate the impact of a link in a given graph.

Example 4. *In Figure 6 below, consider the impact of a link between b_3 and s_2. By Proposition 8, b_3 and s_2 both enjoy an increase in their competitive payoffs. By Proposition 10, b_2, s_1, b_4, and s_3 all have lower payoffs. By Proposition 11, b_1 and b_5 have higher payoffs. We can further show that b_6 has higher payoffs and s_4 has lower payoffs, since their only paths to the agents with the additional link is through b_5.*

[19] The proof of Proposition 10 involves some subtlety. For example, the result does not generalize to buyers linked to sellers linked to buyers linked to s_a.

We illustrate the example below.

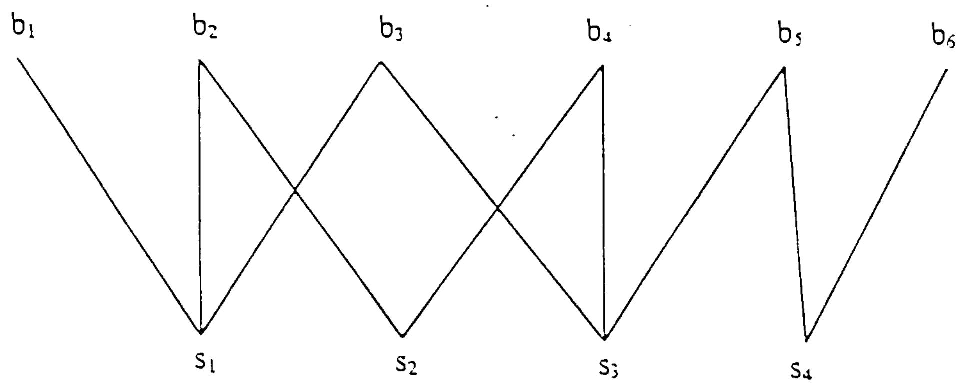

Figure 6

Changing Network by Adding Buyers or Sellers We conclude our analysis by placing the above propositions in the context of earlier results on assignment games. The literature on assignment games considered adding agents on one side of a matching "market." In our framework, this would be equivalent to adding new buyers, or sellers, to a network. In this case, what we call the supply stealing/freeing effects are easier to analyze because the new buyers or sellers do not have any existing links; buyers and sellers are added along with all their links. Intuitively, adding a new seller can only free supply and adding a new buyer can only steal it.

The results below show that indeed adding a seller (buyer) along with all its links must cause a net supply freeing (stealing) effect. The above intuition aside, these results are interesting because, given the necessity of links for exchange in a network, adding buyers (sellers) does not necessarily increase (decrease) the effective buyer/seller ratio of an agent. For example, in the network in Figure 1, suppose s_1 is subtracted from the network. Because s_1 provides the links to the rest of the network for buyers 1 and 2, these buyers are also effectively removed, and the buyer-seller ratio would decrease for the remaining agents. At first glance, it would seem that a lower buyer-seller ratio should help some buyers and hurt some sellers. The next result, however, shows that this is not the case.

A buyer is always better off when sellers are added to its network, regardless of the number of new buyers that compete for the albeit increased supply. Sellers are always worse off. The proposition is proved by an application of an earlier result due to Demange and Gale (1985, Corollary 3).

Proposition 12. *Consider a graph $\mathcal{G}$ linking a collection of buyers and sellers. Add any set of sellers S together with their links to the graph, and let $\mathcal{G}'$ denote the new graph. For all buyers i we have $u_i^b(\mathcal{G}) \leq u_i^b(\mathcal{G}')$. For every seller $j \notin S$, we have $u_j^s(\mathcal{G}) \geq u_j^s(\mathcal{G}')$.*

We have an analogous result for adding buyers to a network. A seller is always better off, regardless of the number of competing sellers that are effectively added to the seller's network. A buyer is always worse off.

Proposition 13. *Consider a graph $\mathcal{G}$ linking a collection of buyers and sellers. Add any set of buyers B together with arbitrary links to the graph, and let $\mathcal{G}'$ denote the new graph. For all sellers j, we have $u_j^s(\mathcal{G}) \leq u_j^s(\mathcal{G}')$. For every buyer $i \notin B$, we have $u_i^b(\mathcal{G}) \geq u_i^b(\mathcal{G}')$.*

4. Conclusion

This paper studies competition in buyer-seller networks, with particular attention to the role of network structure. When prior relationships are necessary for exchange, we show that agents' "outside options" depend on the entire web of direct and indirect links. Even distant links may have large effects on an agent's earnings. In contrast, many models of bargaining and exchange simply assume a fixed reservation value as an outside option. Some consider a limited number of alternative trading partners, as in Bolton and Whinston (1993) where a buyer may deal with two sellers. The present paper, to the best of our knowledge, is the first to analyze outside opportunities when agents on both sides of an exchange can have multiple alternative partners.[20]

We first develop a general model of network competition. This model characterizes prices that satisfy a natural "supply equals demand" condition for the network setting. Resulting payoffs are both individually rational and pairwise stable. No individual agent or pair of agents can do better. Any specific model of competition that yields individually rational, pairwise stable payoffs can be represented by our payoff functions. A parameter $q \in [0,1]$ allows for different

[20] A series of papers considers price formation in a market where anonymous buyers and sellers meet pairwise (e.g. Gale (1987), Rubinstein and Wolinsky (1985)). This effort differs from ours, because we require buyers and sellers to be linked in order to engage in exchange.

splits of the surplus of exchange net agents' "outside options." It is these outside options, then, that determine an agent's bargaining power.

We show how these outside options derive from a given network structure. We define a particular type of path in a network called an *opportunity path*. *Opportunity paths* connect agents to their alternative exchange opportunities and to their direct and indirect competitors. These paths determine agents' outside options. For example, the (perhaps indirect) demand from a (perhaps distant) buyer along a path ensures that a seller receive at least a certain price elsewhere, if it does not sell to its current buyer. This distant demand gives the seller its outside option, which guarantees at least a certain share of the surplus from exchange.

Finally, we consider how changes in third parties' links impact agents' payoffs. That is, we conduct comparative statics on the link pattern. Again, we parse a network into paths. *Seller (buyer) paths* connect an agent to a particular seller (buyer) and tell us whether it will be helped or hurt by that seller's (buyer's) additional links. Seller paths generate what we call a *supply stealing effect*, since the new link establishes an additional source of demand. Buyer paths generate a *supply freeing effect*, since the new link establishes another source of supply. Using these paths, we prove several results about differently connected buyers and sellers.

In conclusion, the paper provides a general model of "outside options," and hence bargaining power, when exchange is limited by pre-existing relationships. The model of network competition can serve as a reduced form for specific models of competition and bargaining. The network structure, through opportunity paths, buyer paths, and seller paths should affect the outcomes of these games in the same way as they affect our competitive payoffs. These payoff results, and the identification of the paths themselves, should also prove useful to further analysis of network structure.

5. APPENDIX

Proof of Proposition 1

Part 1: We show that if a price vector and allocation $(\mathbf{p}, A)$ satisfies Conditions (1), (2), and (3), then the associated payoff vector is stable.

Individual Rationality: For each buyer i and seller j that exchange a good, individual rationality is satisfied since $0 \leq p_j \leq v_i$. Buyers and sellers that do not exchange goods all earn a payoff of 0 which is also individually rational.

Pairwise Stability. First consider linked buyers and sellers that exchange goods in A: For a buyer i and seller j, $u_i^b + u_j^s = v_i - p_j + p_j = v_i$ satisfying pairwise stability. Next consider linked buyers and sellers that do not exchange goods in A. For each buyer i linked to seller k but obtains a good from seller j, the joint payoffs of buyer i and seller k are $u_i^b + u_k^s = v_i - p_j + p_k$. By condition (1), $p_j \leq p_k$, which implies that $u_i^b + u_k^s \geq v_i$,satisfying pairwise stability. For each buyer i linked to seller k but that does not obtain a good from any seller, the joint payoffs of buyer i and seller k are $u_i^b + u_k^s = p_k$. Condition (2) implies that $p_k \geq v_i$, so $u_i^b + u_k^s \geq v_i$, satisfying pairwise stability.

Part 2: A stable payoff vector $(\mathbf{u}^b, \mathbf{u}^s)$ is defined for a feasible allocation of goods A. We show that $(\mathbf{p}, A)$ is competitive, where $\mathbf{p} = \mathbf{u}^s$. That is, we show that $(\mathbf{p}, A)$ derived from $(\mathbf{u}^b, \mathbf{u}^s)$ satisfies Conditions (1), (2), and (3).

Condition (1): For a buyer i purchasing from a seller j: Individual rationality implies that $0 \leq p_j \leq v_i$. Pairwise stability implies that $u_i^b + u_k^s = v_i - p_j + p_k \geq v_i$ for all $s_k \in L(b_i)$. This implies $p_k \geq p_j$ for all $s_k \in L(b_i)$, or, in other words, $p_j = \min\{p_k | s_k \in L(b_i)\}$.

Condition (2): For a buyer i that is not purchasing a good, by the definition of feasible stable payoffs, $u_i^b = 0$. Pairwise stability then implies that $0 + p_k \geq v_i$ for all $s_k \in L(b_i)$. That is, buyer i's valuation is lower than the price charged by any of its linked sellers: $v_i \leq \min\{p_k | s_k \in L(b_i)\}$.

Condition (3): For a seller j that is not selling a good, by the definition of feasible stable payoffs, $u_j^s = 0$, which implies $p_j = 0$. ■

Lemma A1 *Suppose that in $(A, \mathcal{G})$, a buyer 1 has an opportunity path to a buyer n. Let $\mathbf{p}$ be a competitive price vector. If buyer 1 obtains a good from a seller 1, then $p_1 \leq p_n$. If buyer 1 does not obtain a good, then $v_1 \leq p_n$.*

Proof. Since $b_{n-1} \in L(s_n)$ but $s_{n-1} \to b_{n-1,}$ we have $p_{n-1} \leq p_n$. Since $b_{n-2} \in L(s_{n-1})$, but $s_{n-2} \to b_{n-2}$, we have $p_{n-2} \leq p_{n-1}$. Repeating this reasoning, we obtain $p_2 \leq ... \leq p_n$. If buyer 1 obtains a good from a seller 1, then since $b_1 \in L(s_2)$, we have $p_1 \leq ... \leq p_n$ as desired. If buyer 1 does not obtain a good from seller 1, then since $b_1 \in L(s_2)$, we must have $v_1 \leq p_2 \leq .. \leq p_n$.

Lemma A2 *Suppose that in $(A, \mathcal{G})$, a buyer 1 has an opportunity path to a buyer n. Let $\mathbf{p}$ be a competitive price vector. If buyer 1 obtains a good from a seller 1, then $p_1 \leq v_n$. If buyer 1 does not obtain a good, then $v_1 \leq v_n$.*

Proof. By individual rationality, a buyer never pays a price higher than its valuation. Therefore $p_n \leq v_n$. If buyer 1 obtains a good from a seller 1, then Lemma A1 implies that $p_1 \leq v_n$. If buyer 1 does not obtain a good, then Lemma A1 implies that $v_1 \leq p_n$. So we have that $v_1 \leq p_n \leq v_n$ as desired.

Proof of Proposition 5

We show that $p_1^{\max}$ is exactly equal to $v^L(b_1)$, the lowest valuation of any buyer on an opportunity path from buyer 1. The logic is that we can raise p_1 to $v^L(b_1)$ without any violation of pairwise stability, but any higher price would violate pairwise stability.

Let $\widehat{p}^{\max}(b_1) = \min\{p_k^{\max} | s_k \in L(b_1), \, k \neq 1\}$. By individual rationality and pairwise stability for buyer 1, we must have $p_1^{\max} \leq \min\{v_1, \widehat{p}^{\max}(b_1)\}$. If $p_1^{\max} < \min\{v_1, \widehat{p}^{\max}(b_1)\}$, then we can raise $p_1^{\max}$ up to $\min\{v_1, \widehat{p}^{\max}(b_1)\}$ without violating pairwise stability for any buyer-seller pair containing b_1. Raising $p_1^{\max}$ also does not violate pairwise stability for other buyer-seller pairs, since other buyers in $L(s_1)$ already find seller 1's price to be prohibitively high. This contradicts the maximality of $p_1^{\max}$. So we must have $p_1^{\max} = \min\{v_1, \widehat{p}^{\max}(b_1)\}$.

Let $B^{\widehat{p}^{\max}(b_1)} = \{b_k | b_1$ has an o.p.("opportunity path") to b_k and b_k pays a price $p_k^{\max} = \widehat{p}^{\max}(b_1)\}$. Fix any $b_k \in B^{\widehat{p}^{\max}(b_1)}$. By pairwise stability, we have that for all $s_m \in L(b_k)$, $p_m^{\max} \geq \widehat{p}^{\max}(b_1)$. The inequality is strict ($p_m^{\max} > \widehat{p}^{\max}(b_1)$) if and only if s_m sells its good to a buyer $b_m \notin B^{\widehat{p}^{\max}(b_1)}$.

If $b_k \in B^{\widehat{p}^{\max}(b_1)}$ then $v_k \geq \widehat{p}^{\max}(b_1)$.

Case I. $p_1^{\max} > 0$.

We argue that if $p_1^{\max} > 0$, then there is a $b_k \in B^{\widehat{p}^{\max}(b_1)}$ with $v_k = \widehat{p}^{\max}(b_1)$. If $b_k \in B^{\widehat{p}^{\max}(b_1)}$ then $v_k \geq \widehat{p}^{\max}(b_1)$. Suppose that for all $b_k \in B^{\widehat{p}^{\max}(b_1)}$, we have $v_k > \widehat{p}^{\max}(b_1)$. If there is a $b_k \in$

$B^{\widehat{p}^{\max}(b_1)}$ linked to an inactive seller, then it must be that $\widehat{p}^{\max}(b_1) = 0$ and hence that $p_1^{\max} = 0$ contradicting our assumption.

Otherwise, it is possible to raise the price $\widehat{p}^{\max}(b_1)$ paid by all the buyers in $B^{\widehat{p}^{\max}(b_1)}$ without violating stability: that is, individual rationality for any agent or pairwise stability for any pair of agents. The pairs affected are all those (b_i, s_j) for which either u_i^b or u_j^s changes. These are of two types: (i) (b_i, s_j) where $s_j \in L(b_i)$ and $b_i \in B^{\widehat{p}^{\max}(b_1)}$ and (ii) (b_i, s_j) where $s_j \in L(b_i)$, $b_i \notin B^{\widehat{p}^{\max}(b_1)}$, and s_j sells a good to some $b_k \in B^{\widehat{p}^{\max}(b_1)}$.

To preserve stability, we raise $\widehat{p}^{\max}(b_1)$ by a small enough amount that (1) for each $b_k \in B^{\widehat{p}^{\max}(b_1)}$, the inequality $v_k > \widehat{p}^{\max}(b_1)$ is still satisfied; and (2) for each $s_j \in L(b_k)$ selling to a buyer $b_j \notin B^{\widehat{p}^{\max}(b_1)}$, the inequality $p_j^{\max} > \widehat{p}^{\max}(b_1)$ is still satisfied.

Requirement (1) insures that individual rationality still obtains for all buyers whose payoffs have changed. Since sellers who sell goods to these buyers get higher prices, their payoffs are also individually rational.

Consider pairwise stability. First consider pairs (b_i, s_j) of type (i) above. We have argued that s_j must sell a good to some b_j. If $b_j \in B^{\widehat{p}^{\max}(b_1)}$, then $p_j^{\max} = \widehat{p}^{\max}(b_1)$. So s_j receives the same price that b_i pays and pairwise stability for (b_i, s_j) is trivial. If $b_j \notin B^{\widehat{p}^{\max}(b_1)}$, then Requirement (2) insures pairwise stability for (b_i, s_j). Next consider pairs (b_i, s_j) of type (ii) above. The seller j receives a higher payoff $\widehat{p}^{\max}(b_1)$ than before. Buyer i 's payoff is unchanged, so pairwise stability still holds.

We have shown that we can raise $\widehat{p}^{\max}(b_1)$ without violating the stability conditions for any agents. This is a contradiction to the assumption that the prices were maximal. Therefore, we must have $v_k = \widehat{p}^{\max}(b_1)$ for some $b_k \in B^{\widehat{p}^{\max}(1)}$. We can write $p_1^{\max} = \min\{v_1, v_k\}$.

We next argue that $p_1^{\max} = \min\{v_1, v_k\}$ is the lowest valuation out of all buyers linked to buyer 1 by an o.p . (including itself). If b_1 has an o.p. to any other buyer n then by Lemma A2, $p_1^{\max} \leq v_n$ so that $\min\{v_1, v_k\} \leq v_n$ as desired.

We have argued that if $p_1^{\max} > 0$, then $p_1^{\max} = v^L(b_1)$ where $v^L(b_1)$ is the lowest valuation out of all buyers linked to buyer 1 by an o.p (including buyer 1 itself).

Case II. $p_1^{\max} = 0$.

Finally, suppose that $p_1^{\max} = 0$. By our genericity assumption, $v_k \neq 0$ for all buyers k. So if $p_1^{\max} = 0$, it must be that $\widehat{p}^{\max}(b_1) = 0$. It follows from the proof above that there is a $b_k \in$

$B^{\widehat{p}^{\max}(b_1)}$ linked to an inactive seller. (Otherwise, we could raise $\widehat{p}^{\max}(b_1)$ to be above 0 without violating pairwise stability.) We have thus shown that b_1 has an opportunity path to a buyer who is linked to a seller who does not sell its good. ■

Proof of Proposition 6

The proof is similar to the proof of Proposition 5 and is available from the authors on request.

Proof of Proposition 7

We will show the equivalence of conditions (i) and (ii) to the definition of a competitive price vector.

Neccessity: If $(\mathbf{p},A)$ is a competitive price vector, then conditions (i) and (ii) are an immediate implication of Propositions 5 and 6.

Sufficiency: We show that a price vector satisfying conditions (i) and (ii) satisfies Conditions (1), (2), and (3) in the definition of a competitive price vector.

We first show that if a buyer i and seller j exchange a good, then $0 \leq p_j \leq v_i$. Since, by condition (i), $p_j \geq v^H(s_i) > 0$ (for generic $\mathbf{v}$), we have $p_j > 0$. Since buyer i has a trivial opportunity path to itself, $v^L(b_i) \leq v_i$. Condition (i) that $p_j \leq v^L(b_i)$ then implies $p_j \leq v_i$.

We next show that if a buyer i does not obtain a good, then $v_i \leq \min\{p_k | s_k \in L(b_i)\}$. Consider an $s_k \in L(b_i)$. If s_k is selling its good to some other buyer l, then $p_k \geq v^H(s_k)$. Since buyer i is on an opportunity path to seller k, it must be that $v^H(s_k) \geq v_i$. So $p_k \geq v_i$ as desired. If s_k is not selling its good, then since buyer i is not obtaining a good, there is a violation of efficiency (since $v_i > 0$ by our genericity assumption). ■

Proof of Lemma 1

We restate the lemma, because the notation is important.

Lemma 1 *If efficient allocations in $\mathcal{G}$ and $\mathcal{G}'$ involve different sets of buyers, then there are efficient allocations A for $\mathcal{G}$ and A' for $\mathcal{G}'$ such that A and A' are identical except on a set of firms $\mathcal{H} = \{(s_1), b_1,\ s_2,\ b_2,\ ...,\ s_n,\ b_n\}$ that may or may not include the seller s_1. These firms are connected by a path $(s_1) \to b_1 - s_2 \to b_2 - s_2 \to ..b_{n-1} - s_n \to b_n$ in $(A',\ \mathcal{G}')$. The path includes (relabeled) $s_a \to b_a$. In $(A, \mathcal{G})$, the same firms are connected by two paths $b_n - s_n \to .b_{n-1} - ...s_{a+1} \to b_a$ and $s_a \to b_{a-1} - ...s_2 \to b_1 - (s_1)$. Buyer n obtains a good in A' but not in*

A. Buyer 1 obtains a good in A. Buyer 1 obtains a good in A' if and only if $s_1 \in \mathcal{H}$.

Proof. For each new buyer and any efficient allocations, we first construct paths that have the structure of the paths in the Lemma. We then argue that there can be at most one new buyer in A'. We then argue that we can choose the efficient allocations to have the desired structure.

Choose any efficient allocations A and A'. Let buyer n be a buyer that obtains a good in A' but not in A. Buyer n buys a good in A', say from seller n. If s_n did not sell a good in A then b_n should have obtained s_n's good in $(A, \mathcal{G})$ unless b_n and s_n were not linked. That is, unless $b_n = b_a$ and $s_n = s_a$, we have contradicted the efficiency of A. If $b_n = b_a$ and $s_n = s_a$, the hypothesis in the Lemma about opportunity paths is trivially satisfied.

Otherwise, it must be that s_n did sell a good in A, say to b_{n-1} where $b_{n-1} \neq b_n$. If b_{n-1} does not obtain a good in A', then the efficiency of A' implies that $v_{n-1} \leq v_n$. If $v_{n-1} < v_n$, this contradicts the efficiency of A because b_n could have replaced b_{n-1} in A. We rule out the case $v_{n-1} = v_n$ as non-generic.

So it must be that b_{n-1} does obtain a good in A', say from s_{n-1} where $s_{n-1} \neq s_n$. Repeating the above argument shows that s_{n-1} must have sold its good in A to a b_{n-2} who also obtains a good in A', and so on. Eventually, this process ends with $b_{n-k} = b_a$ and $s_{n-k} = s_a$ and $s_a \to b_a$ in $(A', \mathcal{G}')$. (By construction, the process always picks out firms not already in the path. Also by construction, the process does not end unless we reach $b_{n-k} = b_a$ and $s_{n-k} = s_a$, but it must end because the population of buyers is finite.)

We have constructed two paths. In A', we have constructed an opportunity path from b_a to b_n. In A, we have constructed an o.p (opportunity path) from b_n to b_a .

If s_a is inactive in A, we now have paths that have the structure of the paths in the lemma. Otherwise, s_a sells its good to a buyer, say b_{n-k-1} in A. If b_{n-k-1} does not obtain a good in A' then we again now have paths that have the structure of the paths in the lemma. Otherwise b_{n-k-1} obtains a good in A' from a seller, say s_{n-k-1}. If this seller is inactive in A, we now have paths that have the structure of the paths in the lemma. And so on. Eventually, this process must end because it always picks out new firms from the finite population of firms. This constructs the paths in the lemma.

We next argue that there can be at most one new buyer in A'. Suppose there are two new buyers n and n'. For each one, we can construct a path to b_a and s_a as above. But this is a contradiction: since each seller has only one unit of capacity, it is impossible for the two paths from buyers n and n' to overlap.

Finally, we show that we can choose A and A' as in the hypothesis of the Lemma. Fix any efficient allocations A and A' and construct the paths as above. Suppose that the path construction process above ends with an inactive seller s_1. In $\mathcal{G}'$ at the allocation A, buyer n has a path to $s_1 : b_n - s_n \rightarrow b_{n-1} - ...s_{a+1} \rightarrow b_a - s_a \rightarrow b_{a-1} - ...s_2 \rightarrow b_1 - s_1$. We replace this with the path: $s_1 \rightarrow b_1 - s_2 \rightarrow b_2 - s_2 \rightarrow ...b_{n-1} - s_n \rightarrow b_n$. This gives us an allocation $\widetilde{A}'$ that is necessarily efficient in $\mathcal{G}'$ (the efficient set of buyers obtains goods) and is related to A as in the hypothesis of the lemma.

Suppose that the path construction process above ends with a buyer b_1 who does not obtain a good in $(A', \mathcal{G}')$. In $\mathcal{G}'$ at the allocation A, buyer n has an opportunity path to $b_1 : b_n - s_n \rightarrow .b_{n-1} - ...s_{a+1} \rightarrow b_a - s_a \rightarrow b_{a-1} - ...s_2 \rightarrow b_1$. We replace this with the path: $b_1 - s_2 \rightarrow b_2 - s_2 \rightarrow ....b_{n-1} - s_n \rightarrow b_n$. This gives us an allocation $\widetilde{A}'$ that is necessarily efficient in $\mathcal{G}'$ (the efficient set of buyers obtains goods) and is related to A as in the hypothesis of the lemma. ■

Proof of Lemma 2

Since by hypothesis the same set of buyers obtains goods in A as in A', A yields the same welfare as any efficient allocation in $\mathcal{G}'$. Since $\mathcal{G} \subset \mathcal{G}'$, A is also feasible in $\mathcal{G}'$ and hence efficient. ■

We call the set $\mathcal{H}$ from Lemma 1 the *replacement set.* We also refer to the paths in the lemma as the *replacement paths.* Buyer n is the *replacement buyer*, and we say that buyer 1 is replaced by buyer n.

The next four lemmas will be used in proofs below. They use the notation and set up of Lemma 1. The first two characterize the maximal prices for buyers in the replacement set. There are corresponding results for the minimal price. These second two results (which we state without proof) pin down the minimal prices quite strongly.

Lemma A3 *Let A and A' be efficient allocations in $\mathcal{G}$ and $\mathcal{G}'$ involving different sets of buyers as in Lemma 1. We assume the notation from Lemma 1. In $\mathcal{G}$, $v_n \leq p_n^{\max} \leq p_{n-2}^{\max} ... \leq p_{a+1}^{\max}$ and $p_a^{\max} = ... = p_2^{\max}$. If buyer 1 is replaced, then $p_2^{\max} = v_1$. If buyer 1 is not replaced, then $p_2^{\max} = 0$.*

Proof. The inequalities $p_n^{\max} \leq p_{n-2}^{\max} ... \leq p_{a+1}^{\max}$ follow from the fact that there is an opportunity path from buyer n to buyer a in $(A, \mathcal{G})$. Since b_n is linked to s_n but does not obtain a good in A, it must be that $v_n \leq p_n^{\max}$.

First suppose that buyer 1 is not replaced by buyer n in A'. Then in A, buyer 1 is linked to an inactive seller and so pays a price $p_2^{\max} = 0$ to seller 2. There is an opportunity path from any buyer in $\{b_2, ...b_{a-1}\}$ to b_1, so by Proposition 5 we have $0 = p_2^{\max} = p_3^{\max} = ...p_a^{\max}$.

Now suppose that buyer 1 is replaced by buyer n in $(A', \mathcal{G}')$. Let b_i be one of the set $\{b_1, b_2, ...b_{a-1}\}$. If b_i pays a price of $p_{i+1}^{\max} = 0$ in $\mathcal{G}$, then by Proposition 5, b_i has an opportunity path to a buyer l who is linked to an inactive seller. In $\mathcal{G}'$ *with the allocation* A, buyer n has an opportunity path to b_i and hence to b_l. But then buyer n could be added to the set of buyers who obtain a good without replacing buyer 1. This contradicts the efficiency of A'.

So b_i pays a positive price $p_{i+1}^{\max}$. Let buyer L be the "price setting" buyer—that is, the.buyer with valuation $v^L(b_i) = p_{i+1}^{\max}$. (We will say $v^L(b_i) = v_L$ for short.) By Proposition 5, b_i has an opportunity path to buyer L . In $\mathcal{G}'$ *with the allocation* A, buyer n has an opportunity path to b_i and hence to b_L. If $v_L < v_1$, then it is more efficient for buyer n to replace b_L than to replace b_1. This contradicts the efficiency of A'. So it must be that $p_{i+1}^{\max} \geq v_1$. Buyer i also has an opportunity path to buyer 1. This implies that $p_{i+1}^{\max} \leq v_1$.So it must be that $p_{i+1}^{\max} = v_1$ as desired. ■

Lemma A4 *Let A and A' be efficient allocations in $\mathcal{G}$ and $\mathcal{G}'$ involving different sets of buyers as in Lemma 1. We assume the notation from Lemma 1. In $\mathcal{G}'$, $v_n = p_n^{\max\prime} = ... = p_{a+1}^{\max\prime} = p_a^{\max\prime} \geq ... \geq p_2^{\max\prime}$. If buyer 1 is replaced, then $p_2^{\max\prime} \geq v_1$. If buyer 1 is not replaced, then it buys from s_1 and $p_2^{\max\prime} \geq p_1^{\max\prime}$.*

Proof. There is an opportunity path in $(A', \mathcal{G}')$ from b_2 to b_n. This implies that $p_2^{\max\prime} \leq p_3^{\max\prime} \leq ... \leq p_n^{\max\prime}$. Since b_n buys a good from s_n, $p_n^{\max\prime} \leq v_n$.

We argue that $p_a^{\max\prime} = v_n$. This implies that $v_n = p_n^{\max\prime} = ... = p_{a+1}^{\max\prime} = p_a^{\max\prime}$.

By Proposition 5, the price $p_a^{\max\prime}$ is determined by buyer $a's$ opportunity path's in $(A', \mathcal{G}')$. All of these opportunity paths are also opportunity path's in $(A, \mathcal{G})$ except the one from buyer a to buyer n : $b_a - s_{a+1} \rightarrow b_{a+1} - ....s_n \rightarrow b_n$. By Lemma A3, buyer a pays a strictly positive price $p_{a+1}^{\max}$ in $(A, \mathcal{G})$ and so has a price setting buyer L . Buyer L has the lowest valuation $v^L(b_a)$ (or v_L for short) of all buyers to which buyer a has an opportunity path in $(A, \mathcal{G})$. There is an opportunity path from buyer n to buyer L in $(A, \mathcal{G})$. (Join the o.p. from buyer n to buyer a $[b_n - s_n \rightarrow b_{n-1} - ...s_{a+1} \rightarrow b_a]$ to the o.p from buyer a to buyer L.) Therefore $v_L \leq v_n$. If $v_L < v_n$, we have a contradiction to the efficiency of A in $\mathcal{G}$ because we could have replaced buyer L with buyer n in $(A, \mathcal{G})$. Therefore $v_L = v_n$ or equivalently $p_a^{\max\prime} = v_n$.

To finish the proof, suppose that buyer 1 is replaced, so that it does not obtain a good in A'. Since $b_1 \in L(s_2)$, it must be that $v_1 \leq p_2^{\max\prime}$. If buyer 1 is not replaced, then it buys from s_1. Since $b_1 \in L(s_2)$ it must be that $p_2^{\max\prime} \geq p_1^{\max\prime}$. ■

Lemma A5 *Let A and A' be efficient allocations in $\mathcal{G}$ and $\mathcal{G}'$ involving different sets of buyers as in Lemma 1. We assume the notation from Lemma 1. In $\mathcal{G}$, $v_n \leq p_n^{\min} \leq p_{n-2}^{\min} ... \leq p_{a+1}^{\min}$ and $p_a^{\min} = ... = p_2^{\min}$. If buyer 1 is replaced, then $p_a^{\min} = v^H(s_a) \leq \min\{v_1, ..., v_{a-1}\}$. If buyer 1 is not replaced, then $p_a^{\min} = 0$.*

Proof. The proof is available on request from the authors. It is similar to the proofs of Lemmas A3 and A4.

Lemma A6 *Let A and A' be efficient allocations in $\mathcal{G}$ and $\mathcal{G}'$ involving different sets of buyers as in Lemma 1. We assume the notation from Lemma 1. In $\mathcal{G}'$, $p_n^{\min\prime} = ... = p_{a+1}^{\min\prime} = p_a^{\min\prime} = ... = p_2^{\min\prime}$. If buyer 1 is replaced, then $p_2^{\min\prime} = v_1$. If buyer 1 is not replaced, then $p_2^{\min\prime} = p_1^{\min\prime} = 0$.*

Proof. The proof is available on request from the authors. It is similar to the proofs of Lemmas A3 and A4.

Proof of Proposition 8

We will prove the result for $q = 0$ ($\mathbf{p} = \mathbf{p}^{\max}$). For $q = 1(\mathbf{p} = \mathbf{p}^{\min})$, we proved this result in Kranton and Minehart (1998) using the fact that revenues are realized by an ascending bid auction. For other q the revenue functions are given by a convex combination of the extremal revenue functions.

Fix $q = 0$. For a valuation $\mathbf{v}$, we will choose efficient allocations A in $\mathcal{G}$ and A' in $\mathcal{G}'$ as in Lemmas 2 and 1. That is either $A = A'$ or A and A' differ only on the replacement set of firms.

I. Buyers: $u_a^b(\mathcal{G}') \geq u_a^b(\mathcal{G})$.

Consider a valuation $\mathbf{v}$. If $A = A'$, then every opportunity path for buyer a in $\mathcal{G}$ is also an opportunity path for buyer a in $\mathcal{G}'$. If buyer a does not obtain a good in A, then its payoff is 0 in both graphs. Otherwise, let buyer a obtain a good from seller j. By Proposition 5, buyer a's price is the lowest valuation of any buyer along an opportunity path. Since buyer a has a larger set of opportunity paths in $\mathcal{G}'$ we have $p_j^{\max} \geq p_j^{\max\prime}$. That is, buyer a earns a higher maximal payoff in $\mathcal{G}'$ than in $\mathcal{G}$.

If $A \neq A'$, we use the notation from Lemma 1. The replacement buyer n has an opportunity path to buyer a in $(A, \mathcal{G})$:

$$b_n - s_n \rightarrow b_{n-1} - ...s_{a+1} \rightarrow b_a.$$

Because $b_n \in L(s_n)$ and b_n does not obtain a good, it must be that $p_n^{\max} \geq v_n$. Therefore buyer a's price satisfies $p_{a+1}^{\max} \geq v_n$.

In $(A'$, $\mathcal{G}')$, buyer a has an opportunity path to buyer n :

$$b_a - s_{a+1} \rightarrow b_{a+1} - ....s_n \rightarrow b_n.$$

Buyer a obtains a good from seller a. By Lemma A2, $p_a^{\max\prime} \leq v_n$.

We have shown that buyer a pays a lower price in $(A'$, $\mathcal{G}')$ than in $(A, \mathcal{G})$. Therefore, buyer a earns a higher maximal payoff in $\mathcal{G}'$ than in $\mathcal{G}$ for all generic $\mathbf{v}$.

II. Sellers: $u_a^s(\mathcal{G}') \geq u_a^s(\mathcal{G})$.

Consider a valuation $\mathbf{v}$. If $A = A'$, then every opportunity path for buyer a in $\mathcal{G}$ is also an opportunity path for buyer a in $\mathcal{G}'$. If seller a does not sell a good under A, then its payoff is 0 in both graphs. Otherwise, let some buyer b_1 obtain a good from seller a in $(A, \mathcal{G})$. (This buyer can not be b_a.) Consider an opportunity path for buyer 1 in $\mathcal{G}'$. The path has the form

$$b_1 - s_2 \rightarrow b_2 - s_3 \rightarrow ...b_{n-1} - s_n \rightarrow b_n.$$

If the path is not in opportunity path in $\mathcal{G}$, then it must contain the link $b_a - s_a$. But then the path has the form

$$b_1 - s_2 \rightarrow b_2 - s_3 \rightarrow ...b_a - s_a \rightarrow b_1 - ...$$

That is, the path takes us from s_a back to buyer 1. The path therefore does not link buyer 1 to any buyers that it was not already linked to by an opportunity path in $\mathcal{G}$. By Proposition 5, seller a has the same price and hence the same payoff in both graphs.

If $A \neq A'$, we use the notation from the Lemma 1. There is a replacement buyer n and a buyer 1 that buyer n may or may not replace. If buyer 1 is not replaced, then by Lemma A3, $p_a^{\max} = 0$. That is, seller a earns a maximal payoff of 0 in $(A, \mathcal{G})$ and so is weakly better off in $(A', \mathcal{G}')$. If buyer 1 is replaced, then by Lemma A3, $p_a^{\max} = v_1$. Efficiency of A' in $\mathcal{G}'$implies that $v_n \geq v_1$. So $p_a^{\max} \leq v_n$.

By Lemma A4, $p_a^{\max\prime} = v_n$. Since seller a earns a weakly higher price in $\mathcal{G}'$, it is weakly better off in $\mathcal{G}'$ than in $\mathcal{G}$.

We have shown that seller a earns a weakly higher maximal payoff in $\mathcal{G}'$ than in $\mathcal{G}$ for all generic $\mathbf{v}$. ■

Proof of Proposition 9

We omit a proof of this as it is very similar to the proofs of Propositions 10 and 11**.** It is available from the authors on request.

Proof of Proposition 10

We will prove these results for $q = 0$ $(\mathbf{p} = \mathbf{p}^{\max})$. For $q = 1(\mathbf{p} = \mathbf{p}^{\min})$, we proved the result in Kranton and Minehart (1998). For other q, the results follow from the fact that the payoffs are a convex combination of the payoffs for $q = 0$ and $q = 1$.

For a valuation $\mathbf{v}$, we will choose efficient allocations A in $\mathcal{G}$ and A' in $\mathcal{G}'$ as in Lemmas 2 and 1. That is either $A = A'$ or A and A' differ only on the replacement set of firms.

I. For every $b_i \in L(s_a)$, $u_i^b(\mathcal{G}') \leq u_i^b(\mathcal{G})$.

Fix a valuation $\mathbf{v}$. Suppose $A = A'$. If b_i does not obtain a good, its payoff is 0 in both graphs, so we are done. Let $\mathcal{O}_i$ denote the set of buyers connected to b_i by an opportunity path in $(A, \mathcal{G})$ and let $\mathcal{O}_i'$ denote the set of buyers connected to b_i by an opportunity path in $(A', \mathcal{G}')$. We argue that these two sets are the same. Clearly $\mathcal{O}_i' \supseteq \mathcal{O}_i$. Suppose there is some $b_k \in \mathcal{O}_i'$ that is not in $\mathcal{O}_i$. The o.p. from b_i to b_k in $(A', \mathcal{G}')$ must contain the link $b_a - s_a$, and since no good is exchanged b_a must precede s_a in the o.p as follows:

$$b_i - s_{i+1} \rightarrow b_{i+1} - \ldots \rightarrow b_a - s_a \rightarrow b_{a+1} - \ldots \rightarrow b_k.$$

But then since $b_i \in L(s_a)$ in $\mathcal{G}$, there is a more direct o.p. from b_i to b_k that does not contain the link $b_a - s_a$ given by:

$$b_i - s_a \rightarrow b_{a+1} - ... \rightarrow b_k.$$

Since this o.p. does not contain $b_a - s_a$, it is also an o.p.in $(A\ ,\mathcal{G})$, so $b_k \in \mathcal{O}_i$ which contradicts to our assumption.

We have shown that $\mathcal{O}_i = \mathcal{O}_i'$. By Proposition 5, b_i pays the same price in both graphs.

If $A \neq A'$, we use the notation from Lemma 1. Suppose that b_i is the replacement buyer b_n. There is an o.p. $b_i - s_a \rightarrow b_{a-1} - ...s_2 \rightarrow b_1$ in $(A, \mathcal{G})$. So b_i could have replaced b_1 in $\mathcal{G}$. This contradicts the efficiency of A.

Suppose that b_i is replaced by b_n then it is worse off in $\mathcal{G}'$, and we are done.

Otherwise b_i obtains a good in A and A'. Suppose that $b_i \in \mathcal{H}$. If $b_i \in \{b_1, b_2, ..., b_{a-1}\}$ then by Lemmas A3 and A4, we are done because b_i pays a higher price in $\mathcal{G}'$ and so is worse off. Because b_i obtains a good in both A and A', we have $b_i \neq b_n$. If $b_i \in \{b_a, ..., b_{n-1}\}$, then in $(A, \mathcal{G})$, b_i obtains its good from a seller $s_* \in \{s_{a+1}, s_{a+2}, ..., s_n\}$ and there is an o.p. from b_n to b_1 in $(A, \mathcal{G})$ using the link $b_i - s_a$ as follows:

$$b_n - s_n \rightarrow b_{n-1} - s_{n-1} \rightarrow ...s_* \rightarrow b_i - s_a \rightarrow b_{a-1} - s_{a-1} \rightarrow ... \rightarrow b_{1.}$$

Therefore, b_n could have replaced b_1 (or bumped b_1 to an inactive seller) in $\mathcal{G}$ contradicting the efficiency of A.

Suppose that $b_i \notin \mathcal{H}$. First note that b_i does not obtain a good from s_a in either graph. Instead b_i has an o.p. to b_1 in $(A, \mathcal{G})$. So by Lemma A2, b_i has $p_i^{\max} \leq v_1$.

In $\mathcal{G}'$, suppose that b_i pays a positive price. Then by Proposition 5, there is an o.p.from b_i to a price setting buyer l, (that is, $v_l = v^L(b_i)$) so that $p_i^{\max\prime} = v_l$. If this path is also an o.p. in $(A, \mathcal{G})$ we are done because by Lemma A2, $p_i^{\max} \leq v_l$ and so b_i is worse off in $(A', \mathcal{G}')$. Otherwise the o.p.intersects $\mathcal{H}$. The path must come in to a seller and leave at a buyer. Let b_k be the last buyer in the intersection. The portion of the o.p. from b_k to b_l is also an o.p. in $(A, \mathcal{G})$. If $k \leq a-1$, then b_i has an o.p. to b_k in $(A, \mathcal{G})$ (namely: $b_i - s_a \rightarrow b_{a-1} - s_{a-1}... \rightarrow b_k$) and hence to b_l in $(A, \mathcal{G})$. (Remark: This last step could not be generalized to $b_i \in L(L(L(s_a)))$.) Then we are done because by Lemma A2, $p_i^{\max} \leq v_l$ and b_i is worse off in $(A', \mathcal{G}')$. If $k \geq a$, then b_n has an o.p. to b_k and hence to b_l in $(A, \mathcal{G})$. This implies that $p_i^{\max\prime} = v_l \geq v_n$, because otherwise b_n should replace b_l contradicting the efficiency of A. Since $v_n \geq v_1$, we then have that $p_i^{\max\prime} \geq v_1$.We have already argued that $p_i^{\max} \leq v_1$ so b_i is weakly better off in $\mathcal{G}$.

In $\mathcal{G}'$, suppose that b_i pays a price of $p_i^{\max\prime} = 0$. Then b_i has an o.p to a buyer that is linked to an inactive seller. By Lemma 1, this seller was also inactive in A. If the o.p. is also an o.p. in $(A, \mathcal{G})$, we are done because b_i also pays a price of $p_i^{\max} = 0$ in $\mathcal{G}$ and so is weakly better off in $\mathcal{G}$. Otherwise the o.p intersects $\mathcal{H}$. The path must come in to a seller and leave at a buyer. Let b_k be the last buyer in the intersection. The portion of the o.p. from b_k to the buyer linked to the inactive seller is also an o.p. in $(A, \mathcal{G})$. If $k \geq a$, then there is an o.p. in $(A, \mathcal{G})$ from b_n to b_k. Joining the two paths gives an o.p. from b_n to the inactive seller. But this contradicts the efficiency of A because b_n could obtain a good without replacing any buyer. If $k \leq a-1$, there is an o.p. in $(A, \mathcal{G})$ from b_i to b_k and hence from b_i to the buyer linked to the inactive seller. (Remark: This last step could not be generalized to $b_i \in L(L(L(s_a)))$.) We are done because b_i also pays a price of $p_i^{\max} = 0$ in $\mathcal{G}$ and so is weakly better off in $\mathcal{G}$.

We have shown that b_i is weakly better off in $\mathcal{G}$ for generic $\mathbf{v}$.

II. For $s_j \in L(b_a), u_j^s(\mathcal{G}') \leq u_j^s(\mathcal{G})$.

Fix a valuation $\mathbf{v}$. Suppose $A = A'$. If s_j is inactive, it gets 0 in both graphs and we are done. If s_j sells a good to b_j, then by Proposition 5, $p_j^{\max}$ is the lowest valuation of a buyer linked to b_j by an o.p. Since every o.p. in $(A, \mathcal{G})$ is also an o.p. in $(A', \mathcal{G}')$, s_j's price must be weakly lower in $\mathcal{G}'$ and we are done.

If $A \neq A'$, we use the notation from Lemma 1. There is an o.p. from b_n to b_a in $(A, \mathcal{G})$. If s_j were inactive in A, then b_n could have been added to the set of buyers who obtain goods by using this o.p. as a replacement path: $s_j \rightarrow b_a - s_{a+1} \rightarrow b_{a+1}$ ... $\rightarrow b_n$. This contradicts the efficiency of A.

Suppose that s_j is active in A. If $s_j \in \mathcal{H}$, and $j \geq a+1$, then by lemmas A1 and A2, $p_j^{\max} \geq v_n$ and $p_j^{\max\prime} = v_n$. So s_j is weakly worse off in $\mathcal{G}'$ and we are done. If $s_j \in \mathcal{H}$, and $j \leq a$, then because s_j is active, $j \neq 1$. In A, s_j sells its good to b_{j-1}. There is an o.p. from b_n to b_a in $(A, \mathcal{G})$ and also one from b_{j-1} to b_1. These paths can be joined by the linkage $b_a - s_j \rightarrow b_{j-1}$ to give an o.p. from b_n to b_1 in $(A, \mathcal{G})$. But this contradicts the efficiency of A, because b_n could be brought in to the set of buyers who obtain goods in $\mathcal{G}$ using this o.p. as the replacement path. (Remark: This last step could not be generalized to $s_j \in L(L(L(b_a)))$ because the buyer that s_j is linked to in $L(L(b_a))$ need not be in the replacement path.)

If $s_j \notin \mathcal{H}$, then s_j sells its good to the same b_j in both graphs where $b_j \notin \mathcal{H}$. If $p_j^{\max} = 0$, then the fact that $b_a \in L(s_j)$ together with Lemma A1 implies that $p_{a+1}^{\max} = p_{a+2}^{\max} = ... = p_n^{\max} = 0$. (Buyer n has an o.p. to every buyer in $\{b_a, ..., b_{n-1}\}$.) But this contradicts pairwise stability for these prices in $(A, \mathcal{G})$, because b_n would want to buy the good from s_n.

If $p_j^{\max} > 0$, then $p_j^{\max} = v_l$ for the price setting buyer l (that is, $v^L(b_j) = v_l$). There is an o.p. from b_j to b_l and from b_n to b_a. These paths can be joined by the linkage $b_a - s_j \to b_j$ to give an o.p. from b_n to b_l. Since b_n could replace b_l along this o.p., but does not, it must be that $p_j^{\max} = v_l \geq v_n$.

In the o.p. from b_j to b_l is still an o.p. in $(A', \mathcal{G}')$, we are done because $p_j^{\max\prime} \leq v_l$ and so s_j is weakly worse off in $\mathcal{G}'$. Otherwise the o.p. intersects $\mathcal{H}$. The o.p. enters $\mathcal{H}$ for the first time at a seller s_k. Up to that point the path (from b_j to s_k) is the same in $(A', \mathcal{G}')$ as in $(A, \mathcal{G})$. There is an o.p. in $(A', \mathcal{G}')$ from b_k to b_n. So we can join these by the link $s_k \to b_k$ to form an o.p. in $(A', \mathcal{G}')$ from b_j to b_n. By Lemma A2, we have $p_j^{\max\prime} \leq v_n$. Therefore we have $p_j^{\max\prime} \leq v_n \leq p_j^{\max}$ and s_j is weakly worse off in $\mathcal{G}'$.

We have shown that s_j is weakly better off in $\mathcal{G}$ for generic $\mathbf{v}$. ■

Proof of Proposition 11

We prove this result for $q = 0$ $(\mathbf{p} = \mathbf{p}^{\max})$. Similar techniques prove the result for $q = 1$, and hence for other q between 0 and 1.

For a valuation $\mathbf{v}$, we will choose efficient allocations A in $\mathcal{G}$ and A' in $\mathcal{G}'$ as in Lemmas 2 and 1. That is either $A = A'$ or A and A' differ only on the replacement set of firms.

Fix a valuation $\mathbf{v}$. Suppose $A = A'$. If b_i does not obtain a good, its payoff is 0 in both graphs. Otherwise, it obtains a good from $s_j \in L(b_a)$. By the proof of Proposition 10, s_j receives a weakly lower payoff in $\mathcal{G}'$ than in $\mathcal{G}$. So its price must be weakly lower, which means that b_i receives a weakly higher payoff in $\mathcal{G}'$ and we are done.

If $A \neq A'$, we use the notation from Lemma 1. If b_i does not obtain a good in either graph, its payoff is 0 in both graphs and we are done. If b_i receives a good only in $\mathcal{G}'$ (b_i is the replacement buyer), then it must be weakly better off in $\mathcal{G}'$ and we are done. If b_i receives a good only in $\mathcal{G}$ ($i = 1$ where b_1 is the replaced buyer), then by Lemma A1, b_i paid a price in $\mathcal{G}$ exactly equal to its valuation. So it earns a payoff of 0 in both graphs and we are done.

Suppose that b_i obtains a good in both graphs. Let s_i denote the seller that sells its good to b_i in A. If b_i pays a strictly positive price to s_i, let b_l be the price setting buyer (that is, $v^L(b_i) = v_l$). Then b_i has an o.p. to b_l. If this path is also an o.p. to b_l in $(A', \mathcal{G}')$ then b_i pays a price in $(A', \mathcal{G}')$ that is weakly lower than v_l. So b_i is weakly better off in $\mathcal{G}'$ and we are done.

Otherwise, the path intersects the replacement set $\mathcal{H}$. Suppose that neither b_i nor b_l is in the replacement set. The intersection must begin with a seller and end with a buyer. Let b_k be the last buyer in the intersection. The portion of the o.p. from b_k to b_l is also an o.p. in $(A', \mathcal{G}')$. If $k \leq a-1$, consider the part of the o.p. from b_i to b_k. Join the o.p. $b_n - s_n \rightarrow ...b_{a+1} - s_{a+1} \rightarrow b_a - s_i \rightarrow b_i$ to the beginning,(this uses the assumption that $s_i \in L(b_a)$) and the o.p. from b_k to b_1 to the end. This forms an o.p. from b_n to b_1 in $(A, \mathcal{G})$. But this implies that the allocation A' is feasible in $\mathcal{G}$ which contradicts efficiency. If $k \geq a$, then joining the o.p. from b_n to b_k to the o.p. from b_k to b_l forms an o.p. from b_n to b_l in $(A, \mathcal{G})$. This means that b_n could replace b_l in $(A, \mathcal{G})$. Since it does not, it must be that $v_l \geq v_n$. That is, b_i pays $p_i^{\max} \geq v_n$. In $(A', \mathcal{G}')$, there is an o.p. from b_i to b_n (Because $k \geq a$, the o.p. from b_i to b_l in $(A, \mathcal{G})$ must intersect the replacement set for the first time at a seller m with $m \geq a+1$. The part of the o. p. from b_i to s_m is an o.p. in $(A', \mathcal{G}')$.) Join this to the path $s_m \rightarrow b_m - s_{m+1} \rightarrow ..b_n$ to form an o.p. from b_i to b_n in $(A', \mathcal{G}')$. But then by Lemma A2 it must be that $p_i^{\max\prime} \leq v_n$. That is, b_i pays a weakly lower price in $\mathcal{G}'$ and so is weakly better off. The cases that b_i or b_l are in the replacement set is similar (these are essentially special cases of what we have just proved.)

If b_i pays a price of 0 in $(A, \mathcal{G})$, then it has an o.p. to a buyer who is linked to an inactive seller. A similar argument to the previous paragraph (see also the proof of this case in Proposition 10, I.) implies that b_i pays a price of 0 in $\mathcal{G}'$ and so is equally well off in both graphs.

Therefore, b_i's payoff is at least as high in $(A', \mathcal{G}')$ as in $(A, \mathcal{G})$ for all valuations $\mathbf{v}$. ■

Proof of Proposition 12

This result follows from Demange and Gale (1985) Corollary 3 (and the proof of Property 3 of which the Corollary is a special case). Demange and Gale identify two sides of the market, P and Q. We may identify P with buyers and Q with sellers. (This identification could also be reversed.) The way Demange and Gale add agents is to assume that they are already in the initial population, but they have prohibitively high reservation values so that they don't engage in exchange. "Adding" an agent is accomplished by lowering its reservation value. (In our framework, we reduce the reservation value from a very large number to zero.) They show that the minimum payoff for each seller is increasing in the reservation value of *any* other seller (including itself). That is, when sellers are added, the minimum payoff of each original seller weakly decreases. They also show that the maximum payoff for each buyer is decreasing in the reservation value of any seller. That is, when sellers are added, the maximum payoff of each buyer weakly increases.

To complete the proof, we interchange the role of buyers and sellers. (In Demange and Gale (1985), the game is presented in terms of payoffs, and there is no interpretational issue involved in switching the roles of buyers and sellers. In our framework, there is an interpretational issue in switching the roles, but, technically, in terms of payoffs there is no issue.) Corollary 3 states that when buyers are added, the maximum payoff for buyers is weakly decreasing and the minimum payoff for sellers is weakly increasing. Interchanging the roles of buyers and sellers gives us that when sellers are added to a population, the maximum payoff for each original seller weakly decreases and the minimum payoff for each buyer weakly increases.

We have shown that when sellers are added to a network, both the minimum and maximum payoff for each original seller weakly decreases. And both the minimum and maximum payoff for each buyer weakly increases. Convex combinations of these payoffs therefore share the same property. That is, the buyers are better off and the original sellers are worse off. ■

Proof of Proposition 13

The proof is analogous to the one above and is available from the authors on request. ■

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