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A BOUND ON THE NUMBER OF NASH EQUILIBRIA
IN A COORDINATION GAME

Thomas Quint and Martin Shubik

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A BOUND ON THE NUMBER OF NASH EQUILIBRIA
IN A COORDINATION GAME

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Abstract

We prove that a “nondegenerate” $m \times n$ coordination game can have at most $2^M - 1$ Nash equilibria, where $M = \min(m, n)$.

1. Introduction.

In Quint-Shubik (1994), we conjectured that a “nondegenerate” $n \times n$ bimatrix game could have at most $2^n - 1$ Nash equilibria. In this paper we prove a generalization of this result for a special class of bimatrix games. In particular, we show that a “nondegenerate” $m \times n$ coordination game (i.e., a bimatrix game in which the payoff matrices for the two players are identical) can have at most $2^M - 1$ Nash equilibria, where $M = \min(m, n)$.²

2. Background.

Let there be two players in a game, denoted by I and II. Player I has m pure strategies at his disposal, denoted by $I = \{1, \dots, m\}$, while II has pure strategy set $J = \{1, \dots, n\}$. A mixed strategy for player I is a row-vector p of the $m - 1$ -dimensional simplex P , in which p_i is interpreted to be the probability that he plays pure strategy i . Similarly, the set of mixed strategies for II are the column-vectors q of the $n - 1$ -dimensional simplex Q . Given $p \in P$, define the support of p , or $supp(p)$, to be the set $\{i \in I : p_i > 0\}$, and define $supp(q)$ for $q \in Q$ similarly. Finally, denote by e^i the mixed strategy in which I plays i with probability 1, and by e^j that in which II plays j with probability 1.

We are also given two $m \times n$ matrices A and B , where a_{ij} and b_{ij} represent the payoffs

¹ The authors wish to thank Dicky Yan for supplying a missing step in one of the proofs.

² When we consider the class of all bimatrix games, there are two extreme cases. They are games of coordination (where payoffs for the two players are identical) and zero-sum games (where they are diametrically opposed). It is well known that any “nondegenerate” zero-sum game has exactly 1 Nash equilibrium.

for players I and II respectively, if I plays mixed strategy e^i and II plays e^j . Hence, if I chooses mixed strategy $p \in P$ and II chooses $q \in Q$, the expected payoff for I is pAq , while that for II is pBq . Since the two payoff matrices are sufficient to define a bimatrix game, we shall use the terminology “bimatrix game (A, B) ”.

Given $q \in Q$, p^* is a best response for I against q if $p^*Aq \geq pAq \forall p \in P$. Similarly, q^* is a best response for II against p if $pBq^* \geq pBq \forall q \in Q$. Denote by $R_I(q)$ the set of all best responses for I against q , and by $R_{II}(p)$ the set of all best responses for II against p . A Nash Equilibrium (NE) (Nash, 1950, 1953) is a pair $(p^*, q^*) \in P \times Q$ where $p^* \in R_I(q)$ and $q^* \in R_{II}(p^*)$.

In order to aid us in finding NEs, let us define the sets $\mathcal{R}_I(q)$ and $\mathcal{R}_{II}(p)$ as follows: $\mathcal{R}_I(q) = \{i \in I : e^iAq \geq e^kAq \forall k \in I\}$ and $\mathcal{R}_{II}(p) = \{j \in J : pBe^j \geq pBe^k \forall k \in J\}$. In words, $\mathcal{R}_I(q)$ is the set of best pure strategy responses for I against q , while a similar interpretation holds for $\mathcal{R}_{II}(p)$. The following Lemma is then readily apparent (see, e.g., Shapley (1974) or Jansen (1981)):

Lemma 1: A mixed strategy pair (p, q) is a NE of bimatrix game (A, B) iff $\text{supp}(p) \subseteq \mathcal{R}_I(q)$ and $\text{supp}(q) \subseteq \mathcal{R}_{II}(p)$.

In our upcoming analysis we will want to avoid having to consider the “degenerate” class of games in which there is an infinitude of NEs. To this end, we mention a version of the nondegeneracy assumption from Quint-Shubik (1994):

Nondegeneracy Assumption (NA): If $p \in P$ satisfies $|\text{supp}(p)| = z$ (the $|\cdot|$ notation denotes the cardinality of a set), then there are no more than z pure strategy best responses for II against p . Similarly, if $|\text{supp}(q)| = z$, we have $|\mathcal{R}_I(q)| \leq z$.

Not only does the NA assure the existence of only a finite number of NEs, but we also have the following:

Lemma 2: Suppose the NA holds, that (p, q) is a NE, and that $|\text{supp}(p)| = z$. Then

a) $|\text{supp}(q)| = z$

- b) $\text{supp}(p) = \mathcal{R}_I(q)$
- c) $\text{supp}(q) = \mathcal{R}_{II}(p)$
- d) For any other NE (p^2, q^2) , either $\text{supp}(p^2) \neq \text{supp}(p)$ OR $\text{supp}(q^2) \neq \text{supp}(q)$.

3. Coordination Games and the Theorem.

A coordination game is a bimatrix game in which $A = B$. Since in this case only one matrix is needed to define the game, we use the terminology “coordination game A ”.

Theorem: Suppose a coordination game satisfies the NA. If (p^1, q^1) and (p^2, q^2) are distinct NEs of the game, then a) $\text{supp}(p^1) \neq \text{supp}(p^2)$ AND b) $\text{supp}(q^1) \neq \text{supp}(q^2)$.

Remark 1: We remark that the Theorem is not necessarily true for bimatrix games which are not coordination games. For instance, in the game

$$\left(\begin{array}{cccc} (4, 4) & (0, 3) & (2, 2) & (0, 1) \\ (0, 0) & (2, 1) & (0, \frac{3}{2}) & (4, \frac{11}{6}) \end{array} \right),$$

there are three NEs in which Player I uses both pure strategies with positive probability:

$$p^1 = (\frac{1}{2}, \frac{1}{2}), q^1 = (\frac{1}{3}, \frac{2}{3}, 0, 0).$$

$$p^2 = (\frac{1}{3}, \frac{2}{3}), q^2 = (0, \frac{1}{2}, \frac{1}{2}, 0).$$

$$p^3 = (\frac{1}{4}, \frac{3}{4}), q^3 = (0, 0, \frac{2}{3}, \frac{1}{3}).$$

Next, since there are only $2^m - 1$ possible “supports” for a mixed strategy p , and $2^n - 1$ for q , we have the following:

Corollary: Suppose an $m \times n$ coordination game satisfies the NA. Let $M = \min(m, n)$. Then the game has no more than $2^M - 1$ NEs.

Remark 2: It is easy to construct examples of coordination games A which achieve the bound expressed in the Corollary. Indeed, if $m \leq n$ (so $M = m$), define A by letting its first m columns define an identity matrix, and then judiciously add dominated strategies³ to fill in the last $n - m$ columns. Likewise, if $m \geq n$, again place an $M \times M$ identity matrix in the upper left, but now add $m - n$ dominated rows.

³ This must be done so as not to violate the NA.

Remark 3: In Eaves (1971), it was shown there is a one-to-one correspondence between NEs of coordination game A and solutions to the linear complementarity problem (LCP)

$$Ix - \begin{pmatrix} 0 & A + k_1 E \\ A^T - k_2 E & 0 \end{pmatrix} y = \begin{pmatrix} -1_m \\ 1_n \end{pmatrix}, \quad xy = 0, \quad x, y \geq 0.$$

[I is the identity matrix, E is the matrix of all 1's, k_1 and k_2 are constants so that $A + k_1 E > 0$ and $A^T - k_2 E < 0$, and 1_m (1_n) is the m -vector (n -vector) of all 1's.] Hence our Theorem places an upper limit of $2^M - 1$ on the number of solutions to LCPs of a certain class in which the “M-matrix” is of dimension $(m + n) \times (m + n)$.

Proof of Theorem: We prove conclusion a) of the Theorem (the proof of part b) is similar). So suppose a) were false for some coordination game A . Since raising all coefficients of a bimatrix by the same amount does not change the set of NEs, we may assume $A > 0$. Let A_j denote the j th column of A , and A^i the i th row. Since we are assuming the Theorem false, there exist two NEs, (p^1, q^1) and (p^2, q^2) , for which $\text{supp}(p^1) = \text{supp}(p^2)$. WLOG assume $|\text{supp}(p^1)| = |\text{supp}(p^2)| = z$. By relabeling if necessary, assume $\text{supp}(p^1) = \text{supp}(p^2) = \{1, \dots, z\}$, i.e., both NEs “use” the first z rows.

From the NA, the fact that (p^1, q^1) and (p^2, q^2) are NEs, and the fact that $A > 0$, we know that there exist positive constants s, t, u , and v satisfying

$$p^1 A_j = s \text{ if } j \in \text{supp}(q^1) \tag{3.1}$$

$$p^1 A_j < s \text{ if } j \notin \text{supp}(q^1) \tag{3.2}$$

$$p^2 A_j = t \text{ if } j \in \text{supp}(q^2) \tag{3.3}$$

$$p^2 A_j < t \text{ if } j \notin \text{supp}(q^2) \tag{3.4}$$

$$A^i q^1 = u \text{ for } i \in 1, \dots, z \tag{3.5}$$

$$A^i q^2 = v \text{ for } i \in 1, \dots, z \tag{3.6}$$

Next, we note that the NA implies that $|\text{supp}(q^1)| = |\text{supp}(q^2)| = z$. By relabeling columns if necessary, assume $\text{supp}(q^1) = \{1, \dots, z\}$ and $\text{supp}(q^2) = \{w + 1, \dots, w + z\}$. [The

index w can take on any value from 1 to $n - z$, but cannot take on the value 0 because of the NA.] Define the $z \times z$ matrix C as the submatrix of A defined by rows $\{1, \dots, z\}$ and columns $\{1, \dots, z\}$, i.e, the submatrix defined by the rows in $\text{supp}(p^1)$ and the columns of $\text{supp}(q^1)$. Similarly, define the $z \times z$ matrix D as the submatrix of A defined by rows $\{1, \dots, z\}$ and the columns of $\text{supp}(q^2)$. Note that C and D will share exactly $z - w$ columns if $w < z$, and none otherwise.

We denote by C_j the j 'th column of C , i.e., the first z elements of A_j . Similarly, D_j denotes the j 'th column of D , i.e., the first z elements of A_{w+j} . Hence, if $w < z$, we have $C_{w+j} = D_j$ for $j = 1, \dots, z - w$.

Claim: Matrices C and D are nonsingular (hence, C^{-1} and D^{-1} exist).

Proof of Claim: We prove the Claim for C ; the proof for D is similar. Suppose C were singular. Then there exist constants $\alpha_1, \dots, \alpha_z$, not all zero, such that $\alpha_1 C_1 + \dots + \alpha_z C_z = 0$. Furthermore, since $C > 0$, at least one of the α_j 's is positive and at least one is negative.

Given NE (p^1, q^1) , define a new mixed strategy q^{1*} by

$$q_j^{1*} = \begin{cases} \frac{q_j^1 + \frac{\alpha_j}{N}}{Z} & \text{if } j \in \text{supp}(q^1) = \{1, \dots, z\}; \\ 0 & \text{otherwise,} \end{cases}$$

where N is a large finite number, and $Z = \sum_{j \in \text{supp}(q^1)} (q_j^1 + \frac{\alpha_j}{N}) = 1 + \frac{\sum_{j \in \text{supp}(q^1)} \alpha_j}{N}$ is a normalizing constant. Since at least one α_j is positive and at least one α_j is negative, we note that $(\alpha_1, \dots, \alpha_z)$ is not a multiple of (q_1^1, \dots, q_z^1) , and so q^{1*} is distinct from q^1 .

Now consider the pair (p^1, q^{1*}) . The support of q^{1*} is the same as that for q^1 , so, since $\text{supp}(q^1) \subseteq \mathcal{R}_{II}(p^1)$, we have $\text{supp}(q^{1*}) \subseteq \mathcal{R}_{II}(p^1)$. Furthermore, by the construction, all pure strategies in $\text{supp}(p^1)$ pay off the same for Player I against q^{1*} , so, if N is sufficiently large, they all will be elements of $\mathcal{R}_I(q^{1*})$. [This holds because they all were elements of $\mathcal{R}_I(q^1)$, and, if N is large, q^{1*} is very close to q^1 .] Hence, (p^1, q^{1*}) is also a NE.

However, the fact that (p^1, q^1) and (p^1, q^{1*}) are both NEs is a contradiction of the NA, because of Lemma 2, part d).

Define \hat{p}^1 as the z -vector consisting of the z (nonzero) components of p^1 , i.e., $\hat{p}^1 =$

(p_1^1, \dots, p_z^1) . Define \hat{p}^2 similarly. Finally, define \hat{q}^1 and \hat{q}^2 as the z -vectors consisting of the z nonzero components of q^1 and q^2 respectively.

Using the notation described above, we may rewrite conditions (3.1)-(3.6) as follows

$$\hat{p}^1 C = (s, \dots, s) \implies \hat{p}^1 = (s, \dots, s) C^{-1} \quad (3.7)$$

$$\hat{p}^1 D_j \begin{cases} = s & \text{if } j \in 1, \dots, z - w \text{ (and } w < z); \\ < s & \text{otherwise.} \end{cases} \quad (3.8)$$

$$\hat{p}^2 D = (t, \dots, t) \implies \hat{p}^2 = (t, \dots, t) D^{-1} \quad (3.9)$$

$$\hat{p}^2 C_j \begin{cases} = t & \text{if } j \in w + 1, \dots, z \text{ (and } w < z); \\ < t & \text{otherwise.} \end{cases} \quad (3.10)$$

$$C \hat{q}^1 = (u, \dots, u)^T \implies \hat{q}^1 = C^{-1} (u, \dots, u)^T \quad (3.11)$$

$$D \hat{q}^2 = (v, \dots, v)^T \implies \hat{q}^2 = D^{-1} (v, \dots, v)^T \quad (3.12)$$

Now, since \hat{q}^1 is a positive probability vector, we have $\hat{q}_j^1 \in (0, 1]$ for $j \in 1, \dots, z$. Substituting using (3.11), we have that $C_i^{-1} (u, \dots, u)^T \in (0, 1]$ for $i = 1, \dots, z$. Since $u > 0$, this implies that the row sums of C^{-1} are all positive. A similar argument using \hat{q}^2 tells us the same thing about D^{-1} ; hence we have shown

Proposition: The row sums of C^{-1} and D^{-1} are all positive.

Next, substituting in (3.8) using the expression for \hat{p}^1 found in (3.7) gives

$$(s, \dots, s) C^{-1} D_j \begin{cases} = s & \text{if } j \in 1, \dots, z - w \text{ (and } w < z); \\ < s & \text{otherwise.} \end{cases}$$

This in turn implies (I represents the identity matrix)

$$[(s, \dots, s)(C^{-1} D - I)]_j \begin{cases} = 0 & \text{if } j \in 1, \dots, z - w \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases}$$

Finally, since $s > 0$, this gives

$$[(1, \dots, 1)(C^{-1} D - I)]_j \begin{cases} = 0 & \text{if } j \in 1, \dots, z - w \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases} \quad (3.13)$$

Similarly, substituting (3.9) into (3.10) gives

$$(t, \dots, t)D^{-1}C_j \begin{cases} = t & \text{if } j \in w+1, \dots, z \text{ (and } w < z); \\ < t & \text{otherwise.} \end{cases}$$

which implies

$$[(1, \dots, 1)(D^{-1}C - I)]_j \begin{cases} = 0 & \text{if } j \in w+1, \dots, z \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases} \quad (3.14)$$

Note that in both (3.13) and (3.14), the strict inequality holds for at least one j , because $w \neq 0$.

The Theorem will now be proven if we can show that (3.13) and (3.14) are inconsistent.

To this end, we note that (3.13) implies that

$$1 - [(1, \dots, 1)C^{-1}D]_j \begin{cases} = 0 & \text{if } j \in 1, \dots, z-w \text{ (and } w < z); \\ > 0 & \text{otherwise.} \end{cases}$$

Next, by the Proposition we know that the row sums of D^{-1} are positive; hence, the vector $D^{-1}(1, \dots, 1)^T$ has all positive components. Hence

$$[(1, \dots, 1) - (1, \dots, 1)C^{-1}D] \times D^{-1}(1, \dots, 1)^T > 0,$$

which gives

$$(1, \dots, 1)D^{-1}(1, \dots, 1)^T - (1, \dots, 1)C^{-1}(1, \dots, 1)^T > 0. \quad (3.15)$$

Similarly, starting with (3.14), we have

$$1 - [(1, \dots, 1)D^{-1}C]_j \begin{cases} = 0 & \text{if } j \in w+1, \dots, z \text{ (and } w < z); \\ > 0 & \text{otherwise.} \end{cases}$$

Again, by the Proposition we know that the row sums of C^{-1} are positive; hence, the vector $C^{-1}(1, \dots, 1)^T$ has all positive components. Hence

$$[(1, \dots, 1) - (1, \dots, 1)D^{-1}C] \times C^{-1}(1, \dots, 1)^T > 0,$$

which gives

$$(1, \dots, 1)C^{-1}(1, \dots, 1)^T - (1, \dots, 1)D^{-1}(1, \dots, 1)^T > 0. \quad (3.16)$$

Indeed, inequalities (3.15) and (3.16) are inconsistent.

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