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On the Number of Nash Equilibria in a Bimatrix Game

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A MODEL OF MIGRATION

Thomas Quint and Martin Shubik

December 1994

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ABSTRACT

A simple game-theoretic model of migration is proposed, in which the players are animals, the strategies are territories in a landscape to which they may migrate, and the payoffs for each animal are determined by its ultimate location and the number of other animals there. If the payoff to an animal is a decreasing function of the number of other animals sharing its territory, we show the resultant game has a pure strategy Nash equilibrium (PSNE). Furthermore, this PSNE is generated via “natural” myopic behavior on the part of the animals.

Finally, we compare this type of game with congestion games and potential games.

1 Introduction

In this paper we consider a special class of noncooperative games, in which space — the ability of individuals to see and to move distances — influences players’ strategies, and crowding on the same piece of land influences their payoffs. The class of games we propose is somewhat related to, but different from, the congestion games/potential games ([1], [2]), which have already been studied. We first present a heuristic sketch and then the formal model.

We begin with a landscape consisting of a set of territories, together with a population of animals. Each animal’s movement is limited to some subset of the territories. The payoff obtained by any animal is a nonincreasing function of the number of animals in the territory to which it finally migrates. Intuitively, it would seem possible that the animals could keep moving without ever achieving an equilibrium state. However, we are able to show that even under these mild assumptions a pure strategy Nash equilibrium (PSNE) exists, and that it can be achieved by a best response sequence of moves by the animals.

As we regard many of the more interesting questions concerning the spread of population as involving dynamics, we are concerned with the possibility of isolating classes of games for which one can show that there is a reasonably natural disequilibrium dynamic which converges to an equilibrium. In Remark 2 of Section 4, we argue

that our model is such a class. So far as we know, the only other such class is the aforementioned Monderer–Shapley potential games. Hence we devote a significant portion of the paper (Section 5) toward comparing the two classes of games. We find there is overlap, but that neither class contains the other.

Finally, we remark that at the level of generality utilized here, we are essentially treating space as a network of nodes connected by arcs. However, *we do not yet impose a distance metric*. On the other hand, our main result is concerned with situations in which more crowding is always less desirable than less crowding, and there is only one species. Certainly, models of interspecies competition, especially those involving predator–prey relationships, would violate these assumptions. Perhaps results concerning such models could be obtained if one limited oneself to games where some distance metric is present.

2 Background: Potential games and congestion games

We first review the theory of potential games as presented in Monderer and Shapley [1]. Let $\Gamma = (N, \{\Sigma^i\}, H)$ denote a game in strategic form with the finite player set $N = \{1, \dots, n\}$, pure strategy sets Σ^i , $i \in N$, and payoff functions $H^i : \Sigma \rightarrow \mathbb{R}$, $i = 1, \dots, n$. Let $\Sigma = \Sigma^1 \times \dots \times \Sigma^n$, σ denote a generic element of Σ , and define the notation $\sigma|_i d$ to mean the element of Σ in which the players play according to σ , except i plays d . A *pure strategy Nash Equilibrium* (PSNE) is an element $\sigma \in \Sigma$ for which $H^i(\sigma) \geq H^i(\sigma|_i d)$ for all $i \in N$ and $d \in \Sigma^i$.

An *exact potential* for Γ is a function $P : \Sigma \rightarrow \mathbb{R}$ in which

$$P(\sigma|_i d) - P(\sigma) = H^i(\sigma|_i d) - H^i(\sigma) \quad (1)$$

holds for all $\sigma \in \Sigma$, $i \in N$, and $d \in \Sigma^i$. An *ordinal potential* for Γ is a function P in which

$$\text{sgn}[P(\sigma|_i d) - P(\sigma)] = \text{sgn}[H^i(\sigma|_i d) - H^i(\sigma)]$$

holds for all $\sigma \in \Sigma$, $i \in N$, and $d \in \Sigma^i$, where $\text{sgn}(x)$ denotes the sign of x , namely, $+1$, -1 , or 0 .

A game admitting an exact (resp., ordinal) potential function is called an *exact* (resp., *ordinal*) *potential game*. It is clear that the class of exact potential games is a proper subset of the class of ordinal potential games.

Given game Γ , an *improvement path* is a sequence $\sigma_1, \dots, \sigma_n$ of elements of Σ , in which one player at a time (in any order) changes to a better strategy. Thus, if σ_v and σ_{v+1} are consecutive in the sequence, then, for some $i \in N$ and $d \in \Sigma^i$,

$$\sigma_{v+1} = (\sigma_v|_i d), \text{ and } H^i(\sigma_{v+1}) > H^i(\sigma_v).$$

A game in which all improvement paths are finite in length is said to have the *finite improvement path* (FIP) property.

Theorem 1 (Monderer–Shapley) *Every ordinal (and hence every exact) potential game a) has the FIP property, and thus b) has a PSNE.*

Finally, a *congestion game* [2] is defined as follows: There are given a finite player set N and a finite set F of *facilities*. Associated with each facility $f \in F$ is a real-valued function c_f , depending only on the number u_f of users of f , and interpreted as the benefit per user of the facility. No requirements are placed on the functions c_f .

The strategy set for each player is a given set of elements of 2^F , i.e., each player i chooses a set $F^i (\subseteq F)$ of facilities he will use. Given F^1, \dots, F^n , the number of users of facility f is given by

$$u_f(F^1, \dots, F^n) = |\{i \in N : f \in F^i\}|.$$

Hence, the payoff for player i , given strategies F^1, \dots, F^n are chosen, is

$$H^i(F^1, \dots, F^n) = \sum_{f \in F^i} c_f(u_f(F^1, \dots, F^n)).$$

Theorem 2 (Monderer–Shapley) *The class of congestion games coincides with the class of exact potential games.*

3 The migration model

Consider a landscape on which live a set $N = \{1, \dots, n\}$ of *animals*. The landscape itself is partitioned into a group of *territories* $T = \{T_1, \dots, T_\ell\}$. Due perhaps to its initial location (or other physical factors), animal i has a set $m(i) \subseteq T$ of feasible territories to which it may migrate. These will be its strategies in a game theoretic sense. Define $\mathcal{T} = m(1) \times \dots \times m(n)$, and let τ be a generic element of \mathcal{T} .

The object for each animal is to try to migrate to a feasible territory it thinks “best.” Two factors determine the desirability of a territory: physical features of the territory and the number of other animals present there. To this end, define the payoff function h by: $h(i, t, k)$ is the payoff to animal i if it migrates to territory t , and there is a population of k animals there (including itself).

Given $\tau = (t^1, \dots, t^n)$ (for which $t^i \in m(i)$ for all i), we define the integer-valued function $u(\tau)$ by $u_t(\tau) = |\{i \in N : t^i = t\}|$, $t \in T$. Hence, the t -th component of u gives the number of animals that migrate to territory t , given that the animals use strategy vector τ . It follows that if strategy vector τ is chosen, the payoff to animal i is then $H^i(\tau) = h(i, t^i, u_{t^i}(\tau))$.

The quantities (N, T, m, h) are sufficient to define a *migration game*. This is a noncooperative game in which the player set is N , the strategy set for each animal is its set of feasible territories, and the payoffs are given by H .

A *pure strategy Nash equilibrium* (PSNE) is a strategy vector $\tau = (t^1, \dots, t^n)$ for which $H^i(\tau) \geq H^i(\tau|_i t)$ for all $i \in N$ and $t \in m(i)$, where the strategy vector $\tau|_i t$ is defined as τ , except that i moves to t instead of t^i . This corresponds to the usual definition of PSNE for a noncooperative game, with the usual justification in terms of stability.

4 The theorem

Theorem 3 *Suppose that in a migration game the function $h(i, t, k)$ is nonincreasing in k for all i, t . Then the game has a PSNE.*

Remark 1 The assumption concerning h is natural if the animals are competitors — the more competitors there are on a particular piece of land, the worse it is for each particular one of them. For this reason, we call a migration game which satisfies this assumption a *competitive migration game* (CMG).

Remark 2 We feel that the proof below gives a process by which animals might naturally arrange themselves so as to form a PSNE: The animals arrive into the landscape one by one. Each time an animal is introduced, it immediately heads for its most desired territory, given the distribution of “older” animals. This perhaps produces a “ripple effect,” by which the older animals adjust to the new arrival, until a new PSNE is reached. The situation is then stable until the next animal arrives, etc.

Remark 3 We remark that since the result is concerned with *pure strategy* Nash equilibria, the function h need only be defined *ordinally* in order for the PSNE existence result to hold. In this case, the “competitive” assumption is simply that, if $k_1 < k_2$, any animal i prefers going to territory t with k_1 animals there over going to territory t with k_2 animals there.

Proof of Theorem 3 We consider an induction on the number of animals n (for any ℓ). If $n = 1$, the conclusion is obvious. For arbitrary n , the induction hypothesis tells us that we may place animals $2, \dots, n$ so as to form a PSNE. Do this. Suppose this PSNE places k_t animals in territory t , $t = T1, \dots, T\ell$. Now place animal #1 so as to maximize its payoff, i.e., into the territory which maximizes $h(1, t, k_t + 1)$ over $t \in m(1)$. Call this territory t^1 . Now, for animals not in t^1 , the placement of animal #1 in this way has only the effect of making t^1 seem no more desirable to them (because h is nonincreasing in k). Hence, if the n -animal arrangement is not a PSNE, it must be that some animal (not #1) in t^1 has incentive to move. Pick any such animal, and label it as #2.

Since #2 has incentive to move, it must be that the argmax of $h(2, t, k_t + 1)$ over $t \in m(2)$ is not t^1 . WLOG suppose that it is t^2 , and now move #2 from t^1 into t^2 , creating a new arrangement of the n animals. Is this new arrangement a PSNE? First, it is clear that any animal not in t^1 or t^2 will have no reason to move. Second, no animal left behind in t^1 will have reason to move — #1 because it chose t^1 and #2’s departure only strengthens that choice, and all other animals because the original $n - 1$ -player game PSNE has essentially only been altered by adding an extra player to t^2 . Hence, if the arrangement is not a PSNE, it must be that some animal (not #2) in t^2 now has incentive to move. Pick any such animal, and label it as #3.

Continuing this process, we see that in each iteration there is one animal that moves, and that in each case the “mover” i moves so as to maximize $h(i, t, k_t + 1)$

over $m(i)$. Hence it is impossible for any player to be the “mover” more than once. Because N is finite, this in turn implies that the process must terminate. Since the process can terminate only at a PSNE, the Theorem is proven. \square

5 Comparison with potential games

It seems apparent that the class of CMGs should be intimately related to that of ordinal/exact potential games. Our first result reflects the fact that the only sense in which CMGs are more general than congestion games is that players are permitted to have different utility functions:

Lemma 1 *Suppose in a CMG that h is invariant under changes in i . Then the game is a congestion game.*

Proof Follows from the definitions.

A different class of CMGs are those in which h is invariant as a function of t ; this is the situation in which the territories themselves are, for all intents and purposes, identical. In this case, it is easily seen that the set of PSNEs will be a symmetric set, consisting of all strategies which send to each territory either $\lfloor \frac{n}{\ell} \rfloor$ or $\lfloor \frac{n}{\ell} \rfloor + 1$ animals. In particular, there will be a unique PSNE payoff¹ in the case where n is divisible by ℓ .

Next, we move back to the study of general CMGs. We show via three examples that there is no obvious relation between CMGs and potential games.

Example 1 *An example of a congestion game which is not a CMG.* Let $N = \{1, 2\}$, and let $F = \{a, b, c, d\}$. Player 1 has two strategies, namely to use facilities a and b , or to use facilities c and d . Player 2 also has two strategies, using facilities a and d , or using b and c . The benefits are given by:

	Facility a	Facility b	Facility c	Facility d
1 User	6	5	4	6
2 Users	2	0	2	4

This gives rise to the strategic form game

	Strategy $a-d$	Strategy $b-c$
Strategy $a-b$	(7,8)	(6,4)
Strategy $c-d$	(8,10)	(8,7)

¹There will be $n!/[(n/\ell)!]^\ell$ such PSNEs, each with the same payoffs.

Is this a CMG? Well, we have 2 players and 2 strategies for each player. Suppose WLOG that animal #1's strategies (in an equivalent CMG) are to go to Territory 1 or Territory 2. Then it must be that #2's strategies must also be to go to Territory 1 or 2 — otherwise, if #2 could go to some third territory (say Territory 3), its payoff would be the same no matter what #1 did. But this is impossible because all 4 of its (= #2's) payoffs are different.

Hence, if this is to be a CMG, it must be that there are two territories, call them $T1$ and $T2$, and both animals may migrate to either territory.

In addition, since “10” is the highest payoff for #2, it must be that this occurs when #1 goes to one territory (say $T1$) and #2 goes to the other ($T2$). Hence $h(1, T1, 1) = 8$ and $h(2, T2, 1) = 10$. This partial knowledge concerning h is displayed diagrammatically as:

	$T1 (k = 1)$	$T1 (k = 2)$	$T2 (k = 1)$	$T2 (k = 2)$
Animal #1	8			
Animal #2				

Now, when #2 switches strategies, the payoff is (8, 7). Hence we have

	$T1 (k = 1)$	$T1 (k = 2)$	$T2 (k = 1)$	$T2 (k = 2)$
Animal #1	8	8	10	
Animal #2	7			

And, if instead #1 switches strategies, the payoff is (7, 8). So we have

	$T1 (k = 1)$	$T1 (k = 2)$	$T2 (k = 1)$	$T2 (k = 2)$
Animal #1	8	8	10	
Animal #2	7			

Finally, if #1 goes to $T2$ and #2 to $T1$, the payoff is (6, 4), giving

	$T1 (k = 1)$	$T1 (k = 2)$	$T2 (k = 1)$	$T2 (k = 2)$
Animal #1	8	8	6	7
Animal #2	4	7	10	8

This is a contradiction because the nonincreasingness of h as a function of k is violated.

Example 2 *An example of a CMG which violates the FIP property.* Let $n = \ell = 3$, with $m(i) = \{T1, T2, T3\}$ for $i = 1, 2, 3$. The function h is given by:

	$T1$			$T2$			$T3$		
	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
Animal #1	1	1	0	4	1	0	3	2	1
Animal #2	4	2	1	3	3	0	2	2	2
Animal #3	5	4	0	3	2	0	6	3	2

This gives rise to the game in strategic form below:

		$t^1 = T1$	$t^2 = T2$	$t^3 = T3$
$t^3 = T1$	$t^1 = T1$	(0,1,0)	(1,3,4)	(1,2,4)
	$t^1 = T2$	(4,2,4)	(1,3,5)	(4,2,5)
	$t^1 = T3$	(3,2,4)	(3,3,5)	(2,2,5)

		$t^1 = T1$	$t^2 = T2$	$t^3 = T3$
$t^2 = T2$	$t^1 = T1$	(1,2,3)	(1,3,2)	(1,2,3)
	$t^1 = T2$	(1,4,2)	(0,0,0)	(1,2,2)
	$t^1 = T3$	(3,4,3)	(3,3,2)	(2,2,3)

		$t^1 = T1$	$t^2 = T2$	$t^3 = T3$
$t^3 = T3$	$t^1 = T1$	(1,2,6)	(1,3,6)	(1,2,3)
	$t^1 = T2$	(4,4,6)	(1,3,6)	(4,2,3)
	$t^1 = T3$	(2,4,3)	(2,3,3)	(1,2,2)

Note that there are two PSNEs here, which pay off (4, 4, 6) and (3, 3, 5). But suppose we start at the outcome in which Animals #2 and #3 choose to migrate to T1, while Animal #1 goes to T2. This gives a payoff vector of (4, 2, 4). Now consider the following improvements in order:

- a) Animal #2 switches to T2 (from T1), yielding payoffs (1, 3, 5).
- b) Animal #3 switches to T3 (from T1), yielding payoffs (1, 3, 6).
- c) Animal #1 switches to T3 (from T2), yielding payoffs (2, 3, 3).
- d) Animal #2 switches to T1 (from T2), yielding payoffs (2, 4, 3).
- e) Animal #3 switches to T1 (from T3), yielding payoffs (3, 2, 4).
- f) Animal #1 switches to T2 (from T3), yielding payoffs (4, 2, 4).

It is easily seen that this cycle repeats endlessly, and so we have violated the FIP property. Furthermore, we note that at each stage of the cycle, the switching animal is playing a best response against what the others are doing — so in fact this example violates a weaker version of the FIP property, which we would call the “Finite Best-Response Improvement Path” (FBRIP) property.

Example 3 *An example of a CMG which is an ordinal potential game but is not an exact potential game.* Let $n = \ell = 2$, with $m(i) = \{T1, T2\}$ for $i = 1, 2$. The function h is given by:

	$T1 (k = 1)$	$T1 (k = 1)$	$T2 (k = 1)$	$T2 (k = 2)$
Animal #1	5	2	7	2
Animal #2	4	2	5	2

This gives the strategic form game

		$t^2 = T1$	$t^2 = T2$
$t^1 = T1$	$t^2 = T1$	(2,2)	(5,5)
	$t^2 = T2$	(7,4)	(2,2)

If we define P by: $P(T1, T1) = 0$, $P(T1, T2) = 1$, $P(T2, T1) = 1$, and $P(T2, T2) = 0$, we find that P is an ordinal potential, and so the game is an ordinal potential game. However, suppose there is an exact potential P for this game, given by $P(T1, T1) = w$, $P(T1, T2) = x$, $P(T2, T1) = y$, and $P(T2, T2) = z$. Using (2.1) four times, we have $x - w = 3$, $y - w = 5$, $y - z = 2$, and $x - z = 3$. These four equations are inconsistent.

6 A question of order

We might wonder whether or not *all* PSNEs of a CMG can be found in the manner described in Remark 2 of Section 4, possibly by introducing the animals in different orders. The answer turns out to be negative. For instance, in Example 2 we may verify that the procedure produces the PSNE paying off (4,4,6) no matter which of the six orderings of animals is used. [The equilibrium paying off (3,3,5) is never reached.] On the other hand, lest we believe that the same PSNE is always reached using any ordering, consider Example 3. Here, if the ordering is (Animal #1, Animal #2) we get the PSNE paying off (7,4); while, if the ordering is (Animal #2, Animal #1), the other PSNE (paying off (5,5)) is attained.

We further note in this last example that Animal #1 gets a payoff of 7 from the PSNE when it is first in the ordering, and a payoff of only 5 if it is second. Similarly, #2 earns more in the case where it is first. This may cause us to wonder whether there is a theorem stating that an animal's payoff from a resultant PSNE is a nonincreasing function of its "lateness" in the ordering.² This is in fact true in the case where h is invariant as a function of t , if we assume that animals randomly choose among territories that appear equally attractive to them. In expectation, the first $l * \text{int}(n/\ell)$ animals all do equally well, and better than the last $n \pmod{\ell}$ animals in the ordering. In general, as the following example shows, the answer is again negative:

Example 3 $n = \ell = 3$, $m(i) = \{T1, T2, T3\}$ for $i = 1, 2, 3$, and h is given by:

	T1			T2			T3		
	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
Animal #1	3	1	0	0	0	0	2	0	0
Animal #2	3	1	0	2	0	0	0	0	0
Animal #3	3	1	0	2	0	0	0	0	0

If the animals are introduced in the order (Animal #1, Animal #2, Animal #3), we can see that the PSNE generated is the one which places Animal #1 in T3, Animal #2 in T2, and Animal #3 in T1. On the other hand, if the ordering is (Animal #1, Animal #3, Animal #2), the procedure generates the PSNE where Animal #1 goes to T3, Animal #2 to T1, and Animal #3 to T2. Hence, as far as Animals #2 and #3

²This is in fact true in the case where h is invariant as a function of t , if we assume that animals randomly choose among territories that appear equally attractive to them. In expectation, the first $l * \text{int}(n/\ell)$ animals all do equally well, and better than the last $n \pmod{\ell}$ animals in the ordering.

are concerned, (given Animal #1 is to be first), it is better to be last in the ordering rather than second.

7 Some thoughts on generalizing the model

One might envision a dynamic generalization of the model as follows. Again, the set of animals is $N = \{1, \dots, n\}$, and the set of territories is $T = \{T1, \dots, T\ell\}$. Initially, animal i is located in territory $I(i)$, $i \in 1, \dots, n$.

Given it is in Territory t , the set of territories to which animal i is able to move, *in one time period*, is given by $m(i, t)$. We assume $t \in m(i, t)$ for all i and t , as it is always feasible for an animal to stay put. In addition, we would define the function $s : N \times T \rightarrow T$, where $s(i, t)$ gives the set of territories that animal i can “see,” given it is in territory t . The idea is that an animal might see a particularly desirable place, to which it can’t migrate in one period, but to which it can plan to arrive after a longer spell. In general, we might expect $m(i, t) \subseteq s(i, t)$ for all i and t .

The payoffs for animal i are given by the sum $\sum_x h(i, t_x^i, k_{xt_x^i})$, where t_x^i is the territory i is in during time period x , $k_{xt_x^i}$ is the number of animals there at that time, and the summation is taken over all x .

In general, we anticipate the analysis of such dynamic (competitive) migration games to be difficult, but in one case an easy result is obtained:

Lemma 2 *Suppose $m(i, t) = m(i, I(i))$ for all i and $t \in m(i, I(i))$. Then the dynamic CMG has a Nash equilibrium in pure strategies.*

Proof We may consider the animals to be playing a CMG in each period. Taking a PSNE from each period in this case gives a Nash equilibrium for the entire game.

References

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- [2] Rosenthal, Robert: A Class of Games Possessing Pure-Strategy Nash Equilibria. International Journal of Game Theory **2**, 65–67 (1973).