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An oceanic vorticity meter

by T. Rossby

ABSTRACT

An acoustical method of determining the relative vorticity of oceanic motions is advanced. By transmitting signals in opposite directions around a closed loop, for example a 3 km triangle, one can obtain directly the circulation, which by virtue of Stoke’s theorem is equivalent to the average vorticity of the “enclosed” fluid times the area. A resolution of $10^{-8} \text{sec}^{-1}$ is regarded as a minimum design objective and appears to be feasible. In the error discussion the major source of uncertainty is found to be the movement of the acoustical transceivers, which due to ray bending and reflection must be buoyed off the ocean bottom. There is some hope that the technique can be applied on oceanwide scales to determine the relative vorticity of entire ocean basins.

The principle of the vorticity meter is demonstrated experimentally with the classical spin-up problem. The relative vorticity injected into the fluid is shown to be proportional to the change in angular velocity of the container and decays exponentially with time in agreement with theory.

1. Introduction

The concept of vorticity and especially its conservation have played a central role in furthering our understanding of the meso- and large scale oceanic and atmospheric circulations. Unlike the meteorologist the oceanographer with his inaccessible environment and limited resources is still seeking to determine many of the elementary characteristics of oceanic variability such as the vertical and horizontal scales of coherence and the lowest order statistics of fluid motion, let alone that of vorticity. It is well known that conventional methods of measuring currents do not allow for an accurate calculation of the vertical component of vorticity on geostrophic time-scales. This is because the error or uncertainty in the local velocity measurement, be it for instrumental reasons or because of local variability, becomes rapidly magnified as one seeks to compute the difference between two partial derivatives, i.e. $(\partial v/\partial x - \partial u/\partial y)$. Nonetheless, as more sophisticated studies of oceanic motion are mounted, it seems likely that increasing stress will be laid on our ability to lay bare the higher order vorticity dynamics.

In this note we propose a new approach to this measurement problem. It is based on an acoustical method for determining the circulation around a closed path.

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Once this is known, we can with Stokes’ theorem obtain the average vorticity of the “enclosed” fluid. Conceptually the problem is simple, we shall try to argue that it is feasible.

2. The vorticity (and current) — Meter — In concept

The central element of the proposed scheme is the assertion that the line integral of fluid velocity along a ray between two points is proportional to the difference in travel time of two acoustic signals, transmitted in opposite directions. Let us examine this by considering two fixed points $O_1$, $O_2$ in a fluid in which the fluid velocity and speed of sound may vary in both space and time, but where we must impose the restriction that they are temporarily steady. (Figure 1).

The distance from $O_1$ to $O_2$ is given by the integral

$$|O_1O_2| = \int_{0}^{\tau_{12}} (c + u_{12}) \, dt$$

(1)

where $u_{12}$ is the component of motion along the ray $O_1O_2 \left( = u \cdot \frac{ds}{|ds|} \right)$ and $\tau_{12}$ is the transit time. If $c$ and $u_{12}$ are temporarily steady, we may average these along the path, i.e.,

$$\frac{1}{|O_1O_2|} \int_{0}^{\tau_{12}} (c + u_{12}) \, dt = (\bar{c} + \bar{u}_{12}).$$

(2)

If this averaging is done inside the integral in equation (1), it then becomes:

$$|O_1O_2| = \int_{0}^{\tau_{12}} (c + u_{12}) \, dt = \tau_{12}(\bar{c} + \bar{u}_{12})$$

(3)

where the ($\bar{\cdot}$) is the path average. For a signal transmitted from $O_2$ to $O_1$ we have

$$|O_2O_1| = \int_{0}^{\tau_{21}} (c - u_{12}) \, dt = \tau_{21}(\bar{c} - \bar{u}_{12}).$$

(4)

On the assumption that the path lengths are the same we obtain from eqs. (3) and (4) the traveltime difference:

$$\Delta \tau_{12} = \tau_{12} - \tau_{21} = \frac{1}{|O_1O_2|} \left( \int_{\tau_{21}}^{\tau_{12}} (c + u_{12}) \, dt - \int_{\tau_{21}}^{\tau_{21}} (c - u_{12}) \, dt \right) = \frac{2\bar{u}_{12}/|O_1O_2|}{\bar{c}^2 - \bar{u}_{12}^2}$$

(5)

$$\approx -\frac{2\bar{u}_{12}/|O_1O_2|}{\bar{c}^2}$$

since \( \frac{u}{c} = O(10^{-5}) \).
The line integral of fluid motion between the same two points can, under the above assumption that the flow is momentarily steady, be written:

$$\int_{O_1}^{O_2} \mathbf{u} \cdot d\mathbf{s} = \bar{u}_{12} / O_1 O_2 = -\frac{\Delta \tau_{12}}{2} \bar{\varepsilon}^2. \quad (6)$$

Thus the line integral around an $N$-cornered polygon becomes

$$\oint \mathbf{u} \cdot d\mathbf{s} = \sum_{i=1}^{N} \mathbf{u}_{i,i+1} / O_i O_{i+1} = -\frac{1}{2} \sum_{i=1}^{N} \Delta \tau_{i,i+1} \bar{\varepsilon}_{i,i+1}^2 \quad (7)$$

where $N + 1 = 1$.

Stokes' theorem tells us that this circulation is equivalent to the surface integral of the normal component of vorticity, but this is the average vorticity, $\bar{\xi}$, times the area:

$$\oint \mathbf{u} \cdot d\mathbf{s} = \iint (\Delta \times \mathbf{u})_z \, dxdy = \bar{\xi} \cdot \text{area}. \quad (8)$$

Our final result is therefore

$$\bar{\xi} = -\frac{1}{\text{area}} \cdot \frac{1}{2} \sum_{i=1}^{N} \Delta \tau_{i,i+1} \bar{\varepsilon}_{i,i+1}^2. \quad (9)$$

We call attention to the fact that for each side of the polygon we also have a line-averaged currentmeter:

$$\bar{u}_{i,i+1} = -\frac{1}{|O_i O_{i+1}|} \cdot \frac{\Delta \tau_{i,i+1} \bar{\varepsilon}_{i,i+1}^2}{2}. \quad (10)$$

Indeed, this method of current measurement is widely used by meteorologists in acoustic anemometers.

3. A few numbers

Let us consider an isosceles triangle and a square both with side length, $a$, and assume a constant speed of sound. Then for the former we have

$$\bar{\xi} = -\frac{2}{\sqrt{3}} \frac{\bar{\varepsilon}^2}{a^2} \sum_{i=1}^{3} \Delta \tau_{i,i+1} \quad (11)$$

and for the square

$$\bar{\xi} = -\frac{\bar{\varepsilon}^2}{2a^2} \sum_{i=1}^{4} \Delta \tau_{i,i+1}. \quad (12)$$

If these arrays or polygons are placed horizontally in the ocean so that only the vertical vorticity is sensed, we can determine from equations (11) and (12) a numerical relationship between $\bar{\xi}$ and the travel time difference around the loop. This is
Figure 2. The ordinate shows the expected travel time difference (in seconds) around a triangle (---) and a square (....) as a function of side length, $a$. The curves (- - -) show the sensitivity of the line averaged current meter (Eqn. 10). The vorticity units are in radians/second.

illustrated in figure 2 where we also show, according to equation (10), the sensitivity of the acoustical current meter.

Although one microsecond is a small quantity, it is possible with very narrow bandpass filters or phase-lock techniques to achieve a timing resolution of this magnitude using a CW frequency of 10 kHz, say. Thus, in principle, a vorticity of $10^{-6}$ sec$^{-1}$ or 10$^0_0$ of the planetary vorticity as well as spatially averaged fluid velocities of very high accuracy can be detected. We must now discuss possible sources of error.

4. Sources of Error

For definiteness we shall have in mind an isosceles triangle or square, 3 km/on the side, place horizontally in the deep ocean. The following discussion can with appropriate adjustments be developed for other geometries.

5. Temporal errors

In the previous conceptual discussion the assumption that $c$ and $u$ are temporarily steady is obviously not valid. However, if the time difference is based on simultaneously transmitted signals, as it certainly should be, then it is only a variability in the time frame between transmit and receive that can affect the measurement (2 seconds for a 3 km path). We assert that variations in $u_{12}$ on this time scale will be negligible. The local speed of sound is especially stable in deep waters and for a given pressure will depend mostly on local temperature variations, which are of order 3–10 centidegrees. That is equivalent to changes in the sound speed of 15–50 cm/sec. Such variations result from the advection of water masses which are large compared to our polygon. The temporal variance in temperature on a “seconds time scale” is one of millidegrees or less. The effect of this on the speed of sound is equivalent to less than 1 cm/sec, and when this is also line averaged, we feel confident in asserting that $\bar{c}$ in deep water should have quite negligible variability within our 2 sec. time frame. Spatial variations in the speed of sound due to temperature and
salinity are by themselves unimportant since the quantity $\Delta \tau$ is a differential measurement along the same path.

6. Movement of Point $O_i, O_{i+1}$

In a practical realization of the polygon, the corners will have to be buoyed off the bottom due to downward ray-bending. These moorings will inevitably have a certain mobility and it would appear that these motions, probably several meters relative displacement corresponding to a millisecond or more change in travel time, would destroy the performance of the instrument. In fact, however, both the velocity and vorticity measurements are differentials which are insensitive to the total travel time. Similar to the discussion above, it is only mooring motions during the measurement time frame that can contribute to an error. We do not know what the variance of mooring motion looks like, but it must be a strong function of scope or stiffness, proximity to vortex shedding elements, mass points, etc. In the design of such a mooring and this will probably be the most critical consideration, primary attention will have to be given to minimizing strumming and high frequency oscillations that can contribute to the displacement variance on the seconds time scale. It is worth remarking that although the time differences are more critical for vorticity measurements than for current measurements the contribution of incoherent mooring motions to the vorticity measurement will be much less, because roughly speaking, the probability of all 3 or 4 corners moving in such a way as to generate an apparent vorticity is much smaller. Moreover, whatever error is introduced, to either the vorticity of the currentmeter, it will have zero average since the moorings have no net displacement and must be limited to the high frequency end of the spectrum. This is readily shown by a worst case argument: Imagine one mooring taut (stationary) and the other free to wander around a circle with 100 meters diameter each day. The maximum relative speed will be 3 mm/sec which will cause a $4 \mu$sec timing error ($a = 3$ km). This is quite small and in any event such relative mooring motion should be easy to avoid. Furthermore, for mooring motions which are incoherent, the errors can at best only be a weak function of separations. The measurement $S/N$ ratio can therefore be improved by increasing the scale of the array.

7. Alignment of the Array

A most significant error will result from vertical shear if the array is not horizontal. If the sides of the polygon are at different depths, then, if this vertical extent is large enough, we will be sensing the vertical rather than the horizontal shear. Accordingly, the slope of the array must be less than the ratio of expected horizontal to maximum vertical shears \(\left(\frac{(\Delta u)_{z}}{(\Delta u)_{H}} = \frac{u_z \Delta z}{u_H \Delta H} = \text{slope} \cdot \frac{u_z}{u_H} \ll 1\right)\). As an example, we may require the resolution of $10^{-6}$ sec$^{-1}$ vorticity in the presence of $10^{-3}$ sec$^{-1}$ vertical shear. Thus the slope must be $< 10^{-3}$, which is stringent indeed. This assumes,
However, that such large shears are coherent for several kilometers. This is doubtful for they have limited vertical extent, \( O \) (10 meters), and probably limited horizontal extent \( O \) (100–1000 meters), although we do not really know. If the lateral scale is less than the array we may depend upon local incoherence and line-averaging to reduce this error (this, n.b. is the strength of the acoustical method). If the scale is larger than the array, we may to some extent depend upon raybending for vertical averaging, a procedure that to a degree is inevitable. In deep waters, where the speed of sound is a nearly linear function of pressure, all rays are almost circular and are convex downwards. The vertical dip of these rays as a function of horizontal separation between transmitter and receiver is easily computed from Snell's law and is shown in Figure 3. Within limits therefore the vertical "insonification" can be used for vertical averaging (albeit nonuniform) of the horizontal velocity field. Unfortunately, rather large separations are necessary, \( O \) (10 km) before it will become effective. This raybending also prescribes the minimum distance the array must buoyed off the bottom in order to avoid grazing. However, reflections off the bottom will require significantly further elevation and additionally some acoustical beam forming may be required.

8. Wave length Ambiguity

We have mentioned that phase-lock techniques allow for the resolution of very small timing differences. We must now, briefly, estimate the largest expected variations for if these are substantially greater than one cycle of the CW signal, we will have to resolve a lane or cycle ambiguity as well. It is unlikely that a vorticity greater than 0.1 f will be observed in deepwater by our conceptual array. According to Figure 2 this corresponds to \( \Delta \tau < 35 \mu \text{sec} \) for the triangle and \( \Delta \tau < 80 \mu \text{sec} \) for the square. Similarly, if we imagine 20 cm/sec as a likely maximum current we see from the same figure that \( \Delta \tau < 500 \mu \text{sec} \). Thus, choosing 10 kHz as our CW tone, we have no lane ambiguity for the vorticity meter and at most 10 cycles for the currentmeter. A standard currentmeter in the array should provide for correct cycle identification. Even this may not be necessary if the record is long enough and the

![Figure 3. Vertical dip of a circular ray in meters (sound velocity increases linearly with depth) between two points horizontally separated.](image-url)
first order velocity statistics are known. This lack of lane ambiguity represents an enormous simplification to the design of such an array. Taking a practical approach one may imagine an array consisting of three corners, each with its transmitter and receiver, connected by cable together to a central unit which contains the functions for transmission, phase comparison and recording. This is a straightforward and symmetrical approach, but suffers in that laying cables between a central unit and buoyed units a certain distance away is awkward, at best. Cables, for synchronization of clocks, are, however, necessary for the line averaged currentmeter. Another possibility is to let two of the three corners be acoustical slaves. That is, one master corner transmits a CW tone to the other two corners. These have phase-locked receivers which regenerate the received signal. These, scaled to another frequency, are retransmitted to the other slave, which in turn regenerates the received signal with a phase-lock loop, scales it to yet another frequency and transmits this back to the master. The purpose of frequency scaling is only a device so that several signals can be transmitted simultaneously without interference. Scaling, as opposed to bandshifting, preserves the phase information. By this procedure the differential timing can be measured around the loop without cable connection between the corners. One disadvantage is that the time frame is larger (6 seconds for the vorticity meter). With either technique the distance between corners can for example be established with acoustic transponders. Multipath interference can be a source of severe timing ambiguity at great distances or when substantial raybending (refraction) takes place. Our own experience, however, and that of Mr. Douglas Webb (private communication) leads us to believe that over a nearly straight 3 km or less path there is very little if any interference, i.e., there is only one dominant ray. This, of course, assumes that surface and bottom reflections can be avoided or discriminated against by beam forming.

9. Discussion

In summary, none of the errors that have been discussed above with the possible exception of that or mooring motion, appear to present major difficulties for the proposed scheme for measuring the vorticity (and fluid velocity). We are greatly encouraged by this result. An extension of this procedure to the study of vorticity fluctuations in the subtropical waters above the main thermocline may be more difficult due to the rather large horizontal and vertical excursion of the moorings. It should be emphasized that we have used 1°/o of the coriolis parameter in the above discussions as a minimum design objective. It should be possible to do better. If there were no hope of achieving this threshold, however, it would be questionable whether the scheme is worth further development, at least as a vorticity meter, since it is doubtful that vorticity fluctuations in excess of 10°/o of the coriolis parameter occur in the open ocean.

2. It may be possible to transmit the synchronization by very low frequency E/M radiation. Although the attenuation would be enormous, there would be no interference. (A. J. Williams, private communication).
Figure 4. Arrangement of vorticity meter in a rotating cylinder. The transmit/receive crystals are positioned at one corner such that the beams cross at the edge of the wall. Reflectors at the two two corners complete the equilateral triangle.

It is tempting to speculate about what one might find with the vorticity meter. The slope of the vorticity spectrum will be $\omega^2 \tau$ that of the kinetic energy density if the flow is non-divergent. This shift in weight to higher frequencies may permit a more rapid convergence towards more stable estimates of the vorticity variance than has been possible for the velocity spectrum where very long data records are required.

The cutoff frequency, according to the frozen field approximation, is determined by the size of the array and the rate of advection of high horizontal wave no. structure past it. For a 3 km array and a 3 cm/sec r.m.s. velocity this is $\sim 1$ cycle/day. This argument does not apply to wavy fields such as internal tides and inertial oscillations. In a stratified ocean, the horizontal wave number is a strong function of frequency (especially if it is near the coriolis frequency), the static stability and the vertical mode number. Given the amplitudes of velocity and vorticity, the horizontal wave number can be obtained directly and hence the vertical mode number, at least if only one (or a band limited set of modes) is present. For inertial oscillations of a given amplitude, the vorticity oscillations will have an amplitude that is proportional to their frequency difference from the coriolis parameter ($\propto \sqrt{\sigma^2 - f^2}$), and will be of the order of $1-10\%$ of $f$.

Extension to oceanwide scales

Perhaps the most remarkable result to emerge is the possibility of measuring the relative vorticity on oceanwide scales. Suppose we wish to resolve an average vorticity of $5 \cdot 10^{-9}$ sec$^{-1}$ corresponding to an average speed of 1 mm/sec around an isosceles triangle, 1500 km to the side. The timing difference will be 4 msec. Whether this can be resolved or not depends on the quality and stability of differential acoustical signalling over such great distance. Multipath interference will be severe, but can be minimized somewhat by signalling in the SOFAR channel where the axial rays clearly dominate. Even so, a different signalling scheme, such as pulsed FM, may be necessary in order to obtain sufficient timing resolution without using high acoustic power level. The overall stability of the average speed of sound for the SOFAR channel is about 1 part in $10^3$, but we also have some evidence that within the time frame of 16 minutes, corresponding to the 1500 km path, the fractional changes will not exceed $\approx 1$ part in $10^5$. This is based on as yet unpublished CW
transmission experiments in the SOFAR channel (R. Watts, private communication). To an unknown degree we may expect changes in the sound speed to manifest themselves simultaneously to both signals traveling in opposite directions. Thus a differential measurement should be less sensitive to these changes. Moreover, if the fluctuations are of high frequency origin (caused by internal waves?), low pass filtering a number of such observations should further reduce the level of uncertainty. 1 mm/sec fluid velocity is $7 \times 10^{-6}$ times that of the speed of sound, which as mentioned has a short term stability of $10^{-5}$. Thus, we are asking that the process of differential measurement and averaging reduce the level of uncertainty by about an order of magnitude. We cannot but help feel that this can be done even though we do not have any evidence to support it. If true, the average vorticity of whole ocean basins may be determined and put in direct relationship to external forcing by wind stresses etc. It is interesting to note that the depth of the SOFAR channel is at the depth of the node of the lowest and most energetic baroclinic mode. Thus an ocean wide vorticity meter would sense primarily the ocean's barotropic response.

A laboratory analogue

The measurement of fluid motion on small scales (<1 meter) using the travel time difference of acoustic pulses (equation 10) has recently been developed to a high level of perfection by Mr. Trygve Gytre at the Chr. Michelsen's Institutt in Bergen, Norway. The technique employs two small piezoelectric crystals, approximately 1 cm in diameter, to transmit to each other sharply beamed acoustic pulses. The travel time difference, although extraordinarily small, can be used to resolve fluid velocities no larger than 1 mm/sec.

In support of the ideas that have been developed in this note, we have adapted this technique to construct a small laboratory model of the vorticity meter. Specifically we have measured the relative vorticity of a homogeneous fluid in the classical spin-up problem (Greenspan and Howard, 1963). Using a lucite cylinder 11.45 cm deep and 13.9 cm wide centered on a rotating table, we beamed two crystals, mounted in the wall at middepth, at two reflectors positioned in such a way that the acoustic pulse travels around an equilateral triangle inscribed in the tank (Figure 4). The electronic circuitry, which provides an analog voltage proportional to the travel time difference, was connected to a strip chart recorder.

Prior to each spin-up/down solid body rotation was established and the recorder output was nulled. A slight change in rotation rate was then introduced. Typical traces of the injected relative vorticity and its exponential decay are shown in Figure 5.
Ideally this vorticity should appear as a step, but the finite time required to change the rotation rate and the need to low pass filter the analog signal to remove the effect of table microphonics gave the step its rounded shape.

As expected the agreement between theory and the experimentally determined spin-up time constant was well within experimental error. Rather than study this further, we ran the experiment at different angular velocities to determine whether the injected vorticity was always proportional to the change in angular velocity. This was done by assuming that the relative vorticity is of the form \( \xi = \xi_0 e^{-t/\tau} \), where \( \xi_0 \) was determined so that the sum \( \sum (\xi_i - \xi_0 e^{-t_i/\tau})^2 \) is a minimum for each run. Since \( \tau = (L^2/\nu Q)^{1/4} \), \( L \) is the half depth, \( \nu \) the kinematic viscosity and \( Q \) is the angular velocity) was given, this calculation became very simple. In Figure 6 we have plotted \( \xi_0 \) against \( \Delta \Omega \). Since the vorticity meter could not be calibrated independently the ordinate is scaled in arbitrary units. Clearly the agreement is very good since the relative vorticity at \( t = 0 \) is linearly proportional to the amplitude and sign of \( \Delta \Omega \) and independent of \( Q \).

Acknowledgement. It is a pleasure for me to acknowledge several fruitful discussions with Mr. Douglas Webb and with Professor Walter Munk, who asked me whether the technique could be applied to oceanwide scales. At first I did not think so, but now I feel the possibility deserves to be given serious consideration. I am grateful to Mr. G. Grimm, who suggested using the acoustical current meter technology to construct the laboratory analogue of the vorticity meter.

REFERENCE