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A Note on the Averaged Momentum Balance in Two-dimensional Water Waves

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ABSTRACT

For time-periodic two-dimensional water waves of small amplitude, it is shown that a static balance exists between the mean dynamic pressure force at the bottom, the gradient of the mean sea level, and the divergence of the radiation-stress tensor. The balance holds for quite arbitrary wave pattern and bottom variation so long as dissipation and steady currents are insignificant. This is the further generalization of a result obtained by Longuet-Higgins (1972) for the gradual refraction of a progressive wave where the bottom pressure is unimportant.

1. Introduction. The concept of the radiation stresses (Longuet-Higgins and Stewart 1960, 1961, 1962, 1964) has advanced the understanding of a number of nonlinear phenomena in water waves. These stresses, arising from averaging over the wave fluctuations and integrating over the water depth,
represent physically the excess mean momentum fluxes due to waves. Among many applications, they have been used to show for certain special cases that the variation in the radiation stresses is in static equilibrium with the gradient of the mean sea level. This balance is valid up to the second order in wave slope, i.e., $O(ka)^2$, with the conditions that extraneous currents do not exceed $O(ka)^2$ and that dissipation is negligible. These special cases are: (i) a progressive wave refracted by a variable bottom of gentle slope [Longuet-Higgins and Stewart 1962, for one-dimensional topography, $h = h(x)$; Longuet-Higgins 1972, for two-dimensional topography, $h = h(x,y)$], and (ii) a standing wave on a horizontal bottom (Longuet-Higgins and Stewart 1964).

In this note a similar result is deduced for a quite general periodic wave system and bottom topography. In particular we show that in the mean the sea is in static equilibrium between the gradient of the mean sea level, the divergence of the radiation-stress tensor, and the horizontal force due to the mean dynamic pressure at the bottom. Applied to nearshore problems, this result implies that the wave fluctuations do not produce any current-driving force outside the surf zone.

An example in which the bottom-pressure force is important is given. Conditions under which this force can be neglected are then discussed for slowly varying bottoms. Finally, certain more explicit formulae for the mean quantities are given for the case of constant depth but arbitrary monochromatic waves.

2. The Conservation of Averaged Momentum. The approach taken here follows the general derivation by Phillips (1966: § 3.6) who, however, neglected in the averaged momentum equation a term that is important for the present discussion. It is therefore worthwhile to retrace some of his steps in order to bring out this omission. We assume that the fluid is inviscid and incompressible so that the instantaneous equations of motion are

$$
\rho \left( \frac{\partial \mathbf{\hat{q}}}{\partial t} + \mathbf{\hat{q}} \cdot \mathbf{\nabla} \mathbf{\hat{q}} \right) = -\mathbf{\nabla} p - \rho g \mathbf{\hat{e}}_z,
$$

(2.1)

$$\nabla \cdot \mathbf{\hat{q}} = 0
$$

(2.2)

for $-h(x,y) \leq z \leq \zeta(x,y,t)$, where $\zeta$ denotes the free-surface displacement from $z = 0$. Denoting by $u_\alpha, \alpha = 1,2$, the horizontal and $w$ the vertical components of $\mathbf{\hat{q}}$, the boundary conditions are

$$p = 0,
$$

(2.3)

$$\frac{\partial \zeta}{\partial t} + u_\beta \frac{\partial \zeta}{\partial x_\beta} = w
$$

(2.4)
on $z = \zeta$, and
\[ u \beta \frac{\partial h}{\partial x_\beta} = -w \]  
\[ \text{(2.5)} \]
on \( z = -h \).

We consider the case where the fluctuations are strictly periodic in time and are of the first-order significance, \( O(ka) \), and where the mean currents, if any (as the result of nonlinearity or else), are of the second order only. The horizontal length scales of the wave fluctuations, of the mean quantities, and of the bottom variation are assumed to be the same so that diffraction may be included in general. Define \( \overline{\dot{q}} \) to be the time average of \( \dot{q} \) at a point and \( \dot{q}' \) the fluctuating part at the same point, i.e.,

\[ \dot{q} = \overline{\dot{q}} (x,y,z) + \dot{q}' (x,y,z,t), \]
\[ \text{(2.6)} \]

where \( \overline{\dot{q}} = O(ka)^2 \), \( \dot{q}' = O(ka) \). Integrating a horizontal component of (2.1) vertically from \( z = -h \) to \( z = \zeta \), using the boundary conditions (2.3–2.5), and taking the time averages, it follows that, exactly,

\[ \frac{\partial}{\partial x_\beta} \int_{-h}^{\zeta} (q u_\alpha u_\beta + p \delta_\alpha \beta) \, dz = \frac{\partial h}{\partial x_\alpha} \overline{p}_h \quad \alpha, \beta = 1, 2, \]
\[ \text{(2.7)} \]

where the overhead bar denotes the time mean and the subscript \((\quad)_h\) denotes the value of \((\quad)\) on the bottom \( z = -h \). This is eq. (3.6.7) in Phillips (1966). At this stage, Phillips took \( \overline{p}_h \) to be just the hydrostatic pressure on the bottom, hence \( \overline{p}_h = \rho g (\zeta + h) \). This is not the case in general and the difference, being the mean dynamic pressure on the bottom,

\[ \overline{P}_h \equiv \overline{p}_h - \rho g (\zeta + h), \]
\[ \text{(2.8)} \]
is in general of \( O(ka)^3 \), as will be shown hereafter. Inserting (2.6) and (2.8) into (2.7) and anticipating that \( \zeta = O(ka)^2 \), it follows that

\[ \overline{P}_h \frac{\partial h}{\partial x_\alpha} - \rho g h \frac{\partial \zeta}{\partial x_\alpha} - \frac{\partial S_{\alpha \beta}}{\partial x_\beta} = O(ka)^3, \]
\[ \text{(2.9)} \]

where

\[ S_{\alpha \beta} = \rho \int_{-h}^{0} u'_\alpha u'_\beta \, dz + \delta_{\alpha \beta} \left[ \int_{-h}^{\zeta} \overline{p} \, dz - \frac{\rho g}{2} (\zeta + h)^2 + \frac{\rho g}{2} (\zeta')^2 \right] \]
\[ \text{(2.10)} \]

are the components of the radiation-stress tensor. The term \( \overline{P}_h \partial h/\partial x_\alpha \) is absent in Phillip's result [1966: eq. (3.6.11)] and in the special cases cited before. It is just the horizontal force due to the mean dynamic pressure acting on the fluid by the sea bottom; it may be calculated by first integrating the vertical component of (2.1):
\[ \rho(x, y, z, t) = \rho g(\zeta - z) + \rho \frac{\partial}{\partial t} \int_{-z}^{\zeta} w dz + \rho \frac{\partial}{\partial x} \int_{z}^{\zeta} u \beta w dz - \rho w^2. \]  
(2.11)

Taking the time averages and letting \( z = -h \), we have

\[ \bar{P}_h = \rho g (\bar{\zeta} + h) = \rho \int_{-h}^{0} \frac{\partial}{\partial x} \bar{u} \beta w' dz - \rho (w'h)^2 + O(ka)^3, \]  
(2.12)

which is \( O(ka)^3 \) and nonzero in general. It is pertinent for experiments to note that, to measure the mean sea level, \( \bar{\zeta} \), we cannot in general just take the reading of the mean bottom pressure without further adjustment. The shoaling progressive wave on a mild beach (Bowen et al. 1968) is an exception (Longuet-Higgins and Stewart 1962; see also §2b below).

2a. An Example—Stokes Standing Edge Wave. As an illustration, we take the Stokes standing edge wave on a sloping beach as the first-order solution. Let the bottom be given by \( h = -x \tan \theta \), \( 0 < \theta < \pi/2 \), the frequency by \( \omega \), and the amplitude at the shore by \( a \). The first-order velocity potential and the free-surface height are (Lamb 1932)

\[ \varphi = -\frac{iga}{\omega} a e^{k(z \sin \theta - x \cos \theta)} \cos ky, \]  
(2.13)

\[ \zeta' = a e^{-kz \cos \theta} \cos ky, \]  
(2.14)

with the dispersion relationship \( \omega^2 = gk \sin \theta \). Calculating the velocity components by means of

\[ (u', v', w') = \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z} \right) \varphi \]  
(2.15)

and substituting into (2.10) and (2.12), it follows after some algebra that

\[ \begin{align*}
S_{xx} &= T \left\{ \cos^2 \theta \cos^2 ky - 1/2 \right\} + \frac{\rho k}{2} \left( \frac{g a}{\omega} \right)^2 e^{-2kx \cos \theta} F, \\
S_{yy} &= S_{yx} = \frac{1}{2} T \cos \theta \sin 2ky,
\end{align*} \]  
(2.16a)

\[ S_{xy} = S_{yx} = \frac{1}{2} T \cos \theta \sin 2ky, \]  
(2.16b)

\[ \bar{P}_h = \frac{\rho}{2} \left( \frac{gka}{\omega} \right)^2 e^{-2kx \cos \theta} \left( \cos^2 \theta \cos^2 ky - \frac{1}{2} \cos 2ky - \frac{1}{2} e^{-2kx \sec \theta} \right), \]  
(2.17)

where

\[ T = \frac{\rho}{2} \left( \frac{gka}{\omega} \right)^2 \frac{e^{-2kx \cos \theta} - e^{-2kx \sec \theta}}{2k \sin \theta}, \]  
(2.18a)
\[ F = k h \left( \cos^2 \theta \cos^2 ky - \frac{1}{2} \cos 2 ky \right) + \frac{1}{2} \sin \theta \cos^2 ky. \]  

(2.18b)

The mean sea level is most simply calculated from the Bernoulli equation

\[ -g \zeta = \left\{ \zeta' \frac{\partial u'}{\partial t} + \frac{1}{2} \left[ (u')^2 + (v')^2 + (w')^2 \right] \right\}_{z=0} \]

\[ = \frac{1}{4} \left( \frac{g k a}{\omega} \right)^2 e^{-2 kx \cos \theta} (\cos 2 \theta \cos^2 ky + \sin^2 ky). \]

(2.19)

All the above quantities, being the result of quadratic nonlinearity, vary in \( y \) with the period half that of the edge-wave length in the longshore direction.

We have checked after straightforward but tedious algebra that these quantities indeed satisfy (2.9).

2b. Slowly Varying Depth. The dynamic-pressure term in (2.9) can sometimes be relatively insignificant when compared with the remaining two terms. Consider (2.12) for a slowly varying bottom so that \( (1/kh)(\partial h/\partial x_\alpha) \ll 1 \), and the higher derivatives of \( h \) are small. The term \( (w'h')^2 \) is \( O[(\partial h/\partial x_\alpha) ka]^2 \) by virtue of (2.5) and hence is negligible. In general the length scale of the mean quantities can still be comparable to the wavelength depending on the complexity of the waves (e.g., standing waves); hence \( P_h(\partial h/\partial x_\alpha) \) is only \( \partial h/\partial x_\alpha \) times smaller than the remaining terms. Further if the wave is progressive (simple refraction, Longuet-Higgins and Stewart 1962, Longuet-Higgins 1972), then for constant depth, \( \zeta, S_{\alpha\beta} = O(ka)^2 \), \( u'w' = o(ka)^2 \), and for gently varying depth, the orders of magnitude of the three terms in (2.9) are, respectively,

\[ (ka)^2 |\nabla h|^2, \quad \nabla^2 h |\nabla h|, \quad (ka)^2 |\nabla h|, \quad (ka)^2 |\nabla h|, \]

where \( \nabla = \partial/\partial x, \partial/\partial y \). Hence, the dynamic pressure on the bottom is insignificant and the static balance is merely between the gradient of the mean sea level and the divergence of the radiation-stress tensor.

2c. Constant Depth. For \( h = \text{constant} \), more explicit information can be obtained from the general result (2.9). First of all, the dynamic pressure does not give rise to a horizontal force and (2.9) becomes

\[ -gh \frac{\partial \zeta}{\partial x_\alpha} - \frac{\partial S_{\alpha\beta}}{\partial x_\beta} = O(ka)^3, \]

(2.20)

implying that the second term must be the gradient of a scalar. We verify this aspect by evaluating \( S_{\alpha\beta} \) in terms of the general complex amplitude function, \( \eta(x,y) \), which is related to the potential and the free-surface height by
\[ \varphi = -\frac{ig}{\omega} \eta \frac{ch k (z + h)}{ch k h} e^{-i \omega t}, \]  
(2.21)

and is governed further by
\[ \eta + k^2 \eta = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta + k^2 \eta = 0 \]
(2.23)

where \( \omega = gh \). Straightforward calculation in accordance with (2.10) yields
\[ S_{\alpha \beta} = \frac{\rho g}{4} \left\{ \text{Re} \left( \frac{\partial \eta}{\partial x_\alpha} \frac{\partial \eta^*}{\partial x_\beta} \right) \frac{1}{k^2} \left( 1 + \frac{2kh}{sh 2kh} \right) + \right. \]
\[ \left. + \delta_{\alpha \beta} \left[ |\eta|^2 - k^2 |\eta|^2 \right] \frac{1}{2k^2} \left( 2kh \cosh 2kh - 1 \right) \right\}. \]
(2.25)

Now, the divergence of the first term may be manipulated as follows:
\[ \frac{\partial}{\partial x_\beta} \text{Re} \frac{\partial \eta}{\partial x_\alpha} \frac{\partial \eta^*}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \frac{1}{2} \left( \frac{\partial \eta}{\partial x_\alpha} \frac{\partial \eta^*}{\partial x_\beta} + \frac{\partial \eta^*}{\partial x_\alpha} \frac{\partial \eta}{\partial x_\beta} \right) \]
\[ = \frac{1}{2} \frac{\partial}{\partial x_\alpha} \left( \frac{\partial \eta}{\partial x_\beta} \frac{\partial \eta^*}{\partial x_\beta} \right) - \frac{\partial \eta^*}{\partial x_\alpha} \right] = \frac{1}{2} \left[ (|\nabla \eta|^2 - k^2 |\eta|^2 ) \right], \]
(2.26)

which is a gradient vector. Use has been made of (2.24). Thus \( S_{\alpha \beta} \) may be written as
\[ S_{\alpha \beta}(x, y) = C_{\alpha \beta} + \delta_{\alpha \beta} S(x, y), \]
(2.27)
\[ S(x, y) = \frac{\rho g g h}{4 \omega^2} \left[ |\nabla \eta|^2 - \frac{\omega^4}{g^2} |\eta| \right], \]
(2.28)

where \( C_{\alpha \beta} \) are constants. It follows that only the normal-stress components can be space dependent. Combining (2.27) with (2.20), we have
\[ -g h \frac{\partial \zeta}{\partial x_\alpha} = \frac{\partial S}{\partial x_\alpha}, \]
(2.29)

which implies physically that the mean sea level change is balanced by the variation in the radiation tensile or compressive stresses. By integrating (2.29), we have
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\[
\bar{\zeta} = -\frac{1}{4} \frac{g}{\omega^2} \left( |\nabla \eta|^2 - \frac{\omega^4}{g^2} |\eta|^2 \right),
\]

which can also follow from the averaged Bernoulli equation if the Bernoulli constant is taken to be zero. Since the one-term potential (2.21) describes the propagating mode of all free monochromatic waves that are a few wavelengths away from localized obstacles or disturbances, the results obtained here are quite general. If the obstacles are vertical cylinders extending throughout the entire water depth, they apply everywhere outside the wall boundary layers.

The following limiting cases are recorded below. Their derivations from the general formulae are straightforward and are therefore omitted.

(i) Deep Water: \( kh \gg 1, \omega^2 = gk \)

\[
\begin{align*}
S_{xx}/h &= S_{yy}/h = \frac{\rho g}{k} \left( |\nabla \eta|^2 - k^2 |\eta|^2 \right) \\
S_{xy}/h &= S_{yx}/h = 0.
\end{align*}
\]

(ii) Shallow Water: \( kh \ll 1, \omega = (gh)^{1/2} k \)

\[
\begin{align*}
S_{xx} &= \frac{\rho g}{2k^2} \left( \frac{1}{k^2} \left| \frac{\partial \eta}{\partial x} \right|^2 + |\eta|^2 \right) \\
S_{yy} &= \frac{\rho g}{2k^2} \left( \frac{1}{k^2} \left| \frac{\partial \eta}{\partial y} \right|^2 + |\eta|^2 \right) \\
S_{xy} &= S_{yx} = \frac{\rho g}{2k^2} \mathcal{R} \frac{\partial \eta}{\partial x} \frac{\partial \eta^*}{\partial y}
\end{align*}
\]

The mean sea level, \( \bar{\zeta} \), for either case follows simply from (2.30) by inserting the proper dispersion relationship.

(iii) Waves Incident Toward and Reflected from a Straight Wall: Let the \( x \) axis be the solid wall so that

\[
\eta = a e^{-i \mu x} \cos \nu y, \quad \mu^2 + \nu^2 = k^2;
\]

\[
\begin{align*}
S_{xx} &= \frac{\rho ga^2}{4} \left[ \cos 2 \nu y \left( \frac{1}{2} + \frac{kh}{sh 2kh} + \frac{\mu^2}{k^2} \frac{kh}{sh 2kh} - \frac{\nu^2}{k^2} kh \cosh 2kh \right) + \right. \\
&\quad + \left. \frac{\mu^2}{k^2} \left( \frac{1}{2} + \frac{kh}{sh 2kh} \right) + \frac{kh}{sh 2kh} \right], \\
S_{yy} &= \frac{\rho ga^2}{4} \left[ \cos 2 \nu y \left( \frac{\mu^2}{k^2} - \frac{\nu^2}{k^2} \cosh 2kh \right) \frac{kh}{sh 2kh} + \right. \\
&\quad + \left. \frac{\nu^2}{k^2} \left( \frac{1}{2} + \frac{kh}{sh 2kh} \right) + \frac{kh}{sh 2kh} \right]
\end{align*}
\]
These results have been worked out independently by P. L-F Liu in a related study. The two subcases of a pure progressive wave ($\mu = k$, $\nu = 0$) and a one-dimensional standing wave ($\mu = 0$, $\nu = k$) have been given by Longuet-Higgins and Stewart (1964). For standing waves there are some apparent differences between their formulae and ours; these differences can be reconciled by noting that (i) they defined $S_{xx}$ by including the hydrostatic pressure, $-\rho g h \bar{\zeta}$, and (ii) they imposed a further condition that the horizontal spatial mean of $\bar{\zeta}$ be zero, which implies a nonzero Bernoulli constant.

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