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Wave-Wave Interactions of Surface and Internal Waves

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ABSTRACT

A resonant energy-transfer theory is applied to the weak wave-wave interactions of random surface gravity waves and internal gravity waves in a constant Väisälä-frequency (exponential-density) ocean. Oceanographic measurements have been used to evaluate energy-transfer rates for two interaction processes: (i) the generation of internal waves by higher-frequency surface (wind) waves, and (ii) the mutual interactions of internal waves for frequencies that are low compared with the mean Väisälä frequency. The computations indicate that the transfer rates for the second process may be generally much larger than those for the first process and that both interactions may be the strongest in shallow depths over the continental shelves.

INTRODUCTION.

1.1. Motivation. A preliminary attempt is made to evaluate the importance of weak resonant interactions on the overall energy balance of internal gravity waves in the ocean. Internal wave-like motions have been observed almost everywhere in the ocean since the beginning of this century, but it is still not definitely known what causes them. Therefore, it is important to investigate all physical processes that may affect the internal wave-energy balance. One physical process that has not been fully investigated is the resonant interaction of internal waves with internal waves and with surface waves.

The resonance theory was first applied to surface gravity waves by Phillips (1960), but similar theories have been known to physicists, beginning with Peierls (1929). Ball (1964) applied the resonance theory to a two-layer density model and showed that the conditions for resonance could be satisfied by triplets of interacting waves. Thorpe (1966) extended Ball's results to a continuously stratified density model and proposed some laboratory experiments that would exhibit the energy transfer between a discrete number of waves in a wave

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tank. Hasselmann (1966) has given a general energy equation for a continuous spectrum of resonantly interacting random-wave components and has worked out the coupling coefficients for several different wave fields, including the coefficient for the interactions of surface waves and internal waves. Hasselmann (1967) has extended the theory to include interactions between wave fields and nonwave fields.

The resonance theory is well understood, but the relative importance for the ocean of the three basic interactions that involve surface waves and internal waves has not been determined. In this paper the relative importance of two interactions is estimated: (i) the generation of internal waves due to interactions of surface waves, and (ii) the mutual interactions of internal waves. Energy-transfer rates for several examples of these two interactions are computed from Hasselmann’s (1966) energy equation based on a constant Väisälä-frequency (exponential-density) model and on ocean-temperature measurements. The computations together with a dimensional analysis of the energy equation indicate that the energy-transfer rates for the second interaction may be generally larger than those for the first interaction. In other words, the time scale for energy input from the surface mode to the internal modes is generally much larger than the time scale for the energy redistribution within the internal modes. It is not expected that this result is strongly affected by the constant Väisälä-frequency assumption.

THE THEORY.

2.1. Energy Equation. Hasselmann (1966) has derived an equation that gives the resonant rate of energy transfer for weak wave-wave interactions in a statistically homogeneous random-wave field. The form of the equation that is appropriate for a vertically stratified fluid is

$$\frac{\partial F_3}{\partial t} = \sum_{v_1, v_2 > 0} \left\{ \sigma_{v_1 v_2 v_3}^{v_3} \omega_3 \left[ \omega_3 F_1 F_2 - \omega_2 F_1 F_3 - \omega_1 F_2 F_3 \right] \right\} + 2 \sigma_{v_1 v_2 v_3}^{v_3} \omega_3 \left[ \omega_3 F_1 F_2 + \omega_2 F_1 F_3 - \omega_1 F_2 F_3 \right] dk_1 dk_2$$

where

$$\sigma_{v_1 v_2 v_3}^{v_3} = \pi 2 ! 2^2 \left| 3 D_{k_1 k_2 k_3}^{v_1 v_2 v_3} \right|^2 \delta (k_1 + k_2 + k_3) \delta (\omega_1 + \omega_2 + \omega_3)$$

and the notation is \((j = 1, 2, 3)\):

\(k_j = \) horizontal vector wave numbers, \(k_j = |k_j|\);
\(v_j = \) vertical mode numbers, \(v_j = 0\) for surface waves, \(v_j = 1, 2, 3, \ldots\) for internal waves;
\(\omega_j = \) angular frequencies, \(\omega_j = \omega_j (k_j, v_j) = -\omega_j (k_j, -v_j)\);
\(F_j = \) energy spectra, \(F_j = F_j (k_j, v_j)\);
\(D_{k_1 k_2 k_3}^{v_1 v_2 v_3} = \) the coupling coefficient.
Eq. (1) represents the time rate of change of an arbitrary initial-energy spectrum due to all weak interactions of wave triplets that satisfy the resonance conditions expressed by the delta functions in (2). Given the initial-energy spectrum and the coupling coefficient in (2), the energy-transfer rate can be computed in a straight-forward manner by performing the indicated summation and integration in (1) on a digital computer. In § 4, transfer rates are computed from initial spectra based on oceanographic measurements. The coupling coefficient has been worked out for an arbitrary stratification by Hasselmann (1966) and Kenyon (1966), who used different methods: the two coefficients agree numerically for the constant Väisälä-frequency model used in the computations of § 4. The basic structure of the coupling coefficient consists of algebraic combinations of the three interacting wave numbers, mode numbers, and frequencies multiplied by integrals over the depth of triple products of eigenfunctions and their derivatives.

Some general properties of the energy transfer are: (i) the conservation of total energy, \[ \sum \int \frac{d\mathbf{k}}{\partial t} (\frac{\partial F}{\partial t}) = 0, \]
(ii) the vanishing of energy transfer for a given interacting triplet unless at least two of the components have finite energy, (iii) the involvement of at least two different mode numbers in all interactions, and (iv) generally an energy flow in two directions: a flow toward high wave numbers and low mode numbers (high frequencies) and a flow toward low wave numbers and high mode numbers (low frequencies); i.e., an energy transfer to smaller horizontal and vertical length scales, respectively. Properties (i) and (ii) follow from the form of (1) (Hasselmann 1962), and properties (iii) and (iv) follow from the resonance conditions in (2) and from the convex shape of the dispersion curves (Phillips 1960, Yanowitch 1962).

Additional properties of the interaction that can generate internal waves in the absence of an initial internal-wave field are: (i) two surface waves and one internal wave in the interacting triplets, (ii) nearly the same frequency (large compared with the internal wave [difference] frequency) in the surface waves, (iii) nearly the same wave length (generally of the same order of magnitude as that of the internal wave) in the surface waves, and (iv) the propagation of the generated internal waves at right angles to the mean direction of the surface waves. These properties follow from the resonance condition (2) and the assumed small-density differences within the fluid.

The energy transfers can be represented conveniently in two complementary ways—by Hasselmann's (1966) transfer diagrams and by characteristic time scales. The transfer diagrams are given along with the computations in § 4; the characteristic time scales are given in § 3.

2.2. Ocean Model. The ocean is assumed to be incompressible, stably stratified, nonrotating, and of constant mean depth, \( h \). For algebraic simplicity, a constant Väisälä-frequency stratification is used; i.e., \( N = \left( \frac{g}{\rho} \frac{d\rho}{dz} \right)^{1/2} \).
= constant, where \( N \) is the Väisälä frequency, \( g \) is the acceleration of gravity, and \( \rho = \rho(z) \) is the undisturbed density, which is a function of the vertical coordinate, \( z \), measured positive upward. The ocean bottom is at \( z = -h \), and the surface is at \( z = 0 \); \( \mathbf{x} \) is the horizontal coordinate vector. Also, characteristic density differences in the ocean are assumed to be small; i.e., \( \Delta \rho / \rho \ll 1 \) (typically \( \Delta \rho / \rho \) is the order of \( 10^{-3} \)).

In the model used for the mutual internal-wave interactions, the ocean surface is assumed to be rigid and parallel to the bottom. This is a good approximation (Cox and Sandstrom 1962). The linear vertical displacement, \( \xi_i(\mathbf{x}, z, t) \), for the \( i^{th} \) mode is then proportional to

\[
\xi_i(\mathbf{x}, z, t) \propto \sin \left[ \pi \frac{z + h}{h} \right] e^{i(k_i \cdot \mathbf{x} \pm \omega_i t)},
\]

and the dispersion relation is given by

\[
\omega_i = \frac{N}{\left[ 1 + \left( \frac{\pi \frac{z + h}{h}}{k_i h} \right)^2 \right]^{1/2}},
\]

where \( i \) stands for internal waves.

One consequence of the assumptions of a constant Väisälä frequency and a rigid surface is an additional resonance condition on the mode numbers. The resonance conditions become

\[
\begin{align*}
\omega_1 \pm \omega_2 &= \omega_3, \\
k_1 \pm k_2 &= k_3, \\
v_1 \mp v_2 &= v_3,
\end{align*}
\]

where the top signs correspond and lead to an energy transfer to higher frequencies and lower mode numbers while the lower signs correspond and lead to a transfer to lower frequencies and higher mode numbers. The third equation in (5) follows from the general form of the coupling coefficient (see § 2.1), since the eigenfunctions are trigonometric functions of the depth as indicated by (3). The third equation in (5) greatly reduces the total number of interaction possibilities that satisfy the first two equations in (5).

The model used for interactions of surface waves of high frequency and internal waves of much lower frequency is the same except that the ocean surface (\( z = 0 \)) is assumed to be free (constant pressure). The surface waves are virtually unaffected by the stratification for frequencies that are large compared with the Väisälä frequency, \( N \) (Eckart 1960). The linear vertical displacement of the surface waves is proportional to

\[
\xi_s(\mathbf{x}, z, t) \propto \sinh \left[ k_s(z + h) \right] e^{i(k_s \cdot \mathbf{x} \pm \omega_s t)},
\]

and the dispersion relation is given by
\[ \omega_s = [g k_s \tanh k_s h]^{1/2} \text{ for } [N/\omega]^2 \ll 1, \]

where \( s \) stands for surface waves. Neglecting \( \Delta \theta/\theta \) compared with 1, the internal waves are well represented by (3) and (4), except that the vertical displacement at the free surface is of order \( \Delta \theta/\theta \) (Cox and Sandstrom 1962) and cannot be regarded as zero.

**Time Scales.**

3.1. *Mutual Internal-wave Interactions.* A nondimensional time scale of energy transfer, \( Q_{iii} \), is defined by

\[
Q_{iii} = \frac{\sum_v \int F_i d\mathbf{k}_i}{T_i \frac{1}{2} \sum_v \left| \frac{\partial F_i}{\partial t} \right| d\mathbf{k}_i},
\]

where \( T_i \) is a characteristic internal-wave period. By analogy with electric-circuit theory, \( Q_{iii} \) is a \( Q \) for the weak resonant mutual interactions of internal waves, since \( Q_{iii} \) is defined as the total internal-wave energy divided by the decrease in this energy in a characteristic internal-wave period, assuming that the energy-transfer rate is constant over a wave period. The factor of \( 1/2 \) is included in (8) because the total rate of energy decrease equals the total rate of increase, with the total energy conserved.

By inserting into (1) and (8) a typical internal-wave period \( (T_i) \), wave length \( (\lambda_i) \), mode number \( (v) \), and vertical displacement \( (\zeta_i) \), the orders of magnitude of \( Q_{iii} \) are estimated for the model ocean in the two limiting cases of shallow water, \( [v \lambda_i/h]^2 \gg 1 \), and deep water, \( [v \lambda_i/h]^2 \ll 1 \), respectively

\[
Q_{iii} \propto [v \zeta_i/h]^{-2} \text{ for } [v \lambda_i/h]^2 \gg 1, \quad \text{(9)}
\]

\[
Q_{iii} \propto [\zeta_i/\lambda_i]^{-2} \text{ for } [v \lambda_i/h]^2 \ll 1. \quad \text{(10)}
\]

A large \( Q_{iii} \) represents a weak interaction. According to (9) and (10), the interactions are weak if the vertical displacement is small compared with both wave length and ocean depth.

3.2. *Interactions of Surface Waves and Internal Waves.* A nondimensional time scale, \( Q_{ssi} \), for this interaction is defined by

\[
Q_{ssi} = \frac{\int F_s d\mathbf{k}_s}{T_s \left| \frac{\partial F_s}{\partial t} \right| d\mathbf{k}_s},
\]

where \( F_s = F_s (\mathbf{k}_s, \phi) \) is the energy density of the surface mode. \( Q_{ssi} \) represents the total surface-wave energy divided by the decrease in this energy.
in a typical surface-wave period that results from energy lost to internal waves, assuming that the transfer rate is constant over a surface-wave period. A dimensional analysis of (11) for the ocean model for the shallow-water and deep-water limits yields, respectively,

\[ Q_{ssi} \propto \left[ \frac{Ti}{Ts} \right]^4 \left( \frac{\zeta s}{\lambda s} \right)^{-2} \frac{[\nu \lambda i / h]^2}{\nu} \left[ \frac{\nu \lambda i / h}{\lambda s / h} \right] \geq 1, \lambda s / h \gg 1, \]  
\[ \text{(12)} \]

\[ Q_{ssi} \propto \left[ \frac{Ti}{Ts} \right]^4 \left( \frac{\zeta s}{\lambda s} \right)^{-2} \nu \left[ \frac{[\nu \lambda i / h]^2}{\nu} \right] \ll 1, \lambda s / h \ll 1, \]  
\[ \text{(13)} \]

where \( Ts \) is a typical surface-wave period and \( \lambda s \) is a surface-wave length.

It may be anticipated that the time scale of the interaction of surface waves with internal waves is larger than the time scale of the mutual interactions of surface waves. If the nondimensional time scale, \( Q_{ssss} \), is defined as the total surface-wave energy divided by the decrease in this energy in a surface-wave period due to mutual surface-wave interactions, then a dimensional analysis of \( Q_{ssss} \) from Hasselmann (1962), which is valid in the range \( 0 \leq \lambda s / h \leq 1 \), gives

\[ Q_{ssss} \propto \left[ \frac{\zeta s}{\lambda s} \right]^{-4} \quad \text{for} \quad 0 \leq \lambda s / h \leq 1. \]  
\[ \text{(14)} \]

Also, a dimensional analysis of (11) for \( \lambda s / h \) and \( [\nu \lambda i / h]^2 \) of order unity gives

\[ Q_{ssi} \propto \left[ \frac{Ti}{Ts} \right]^4 \left( \frac{\zeta s}{\lambda s} \right)^{-2} \quad \text{for} \quad [\nu \lambda i / h]^2 \approx \lambda s / h \approx 1. \]  
\[ \text{(15)} \]

Comparison of (14) and (15) shows that \( Q_{ssi} \approx Q_{ssss} \) except for very weak interactions (low wave energies). For example, for \( Ti = 10 \) min (about the minimum for the ocean) and for \( Ts = 10 \) sec, the two \( Q \)'s are approximately equal when \( Q_{ssi} \approx Q_{ssss} \approx \left[ \frac{Ti}{Ts} \right]^8 \) = order \( 10^{-4} \), which occurs when the wave slope is approximately \( \zeta s / \lambda s \approx \left[ \frac{Ts}{Ti} \right]^2 \) = order \( 10^{-4} \). For slopes that are larger than this the mutual surface-wave interactions dominate. Comparison of (14) and (13) shows that the interaction of surface waves and internal waves is still less important in the deep-water limit. Therefore, except for very weak interactions, for which both transfer rates are negligible, the time scale for the interaction of surface waves and internal waves is much larger than the time scale for the mutual surface-wave interaction.

For comparison of the interaction of surface and internal waves with the mutual interaction of internal waves, it is convenient to define a nondimensional time scale,

\[ \bar{Q}_{ssi} = \frac{\sum_f \int F_i dki}{Ti \sum_f \frac{\partial F_i}{\partial t} dki} \]  
\[ \text{(16)} \]

which is the total internal-wave energy divided by the increase in this energy in a typical internal-wave period due to interactions with surface waves, as-
suming a constant transfer rate over an internal-wave period. Then comparison of \( Q_{\text{SSI}} \) and \( Q_{\text{III}} \) by dimensional analysis gives, for the shallow-water and deep-water limits, respectively,

\[
\frac{Q_{\text{SSI}}}{Q_{\text{III}}} \propto \left[ \frac{T_i}{T_s} \right] \left[ \frac{\zeta_i}{\zeta_s} \right]^4 v^4 \quad \text{for} \quad [v\lambda_i/h]^2 \gg 1, \quad \lambda_s/h \gg 1 \quad (17)
\]

\[
\frac{Q_{\text{SSI}}}{Q_{\text{III}}} \propto \left[ \frac{T_i}{T_s} \right] \left[ \frac{\zeta_i}{\zeta_s} \right]^4 [v\lambda_i/h]^{-4} v^2 \quad \text{for} \quad [v\lambda_i/h]^2 \ll 1, \quad \lambda_s/h \ll 1. \quad (18)
\]

Eq. (17) shows that, in the shallow-water limit, the time scale for the interaction of surface waves and internal waves will generally be larger than the time scale for the mutual internal-wave interaction; i.e., the \( Q \) ratio is greater than 1. This is due mainly to the period ratio (which comes from frequency ratios in the coupling coefficient) of internal and surface waves in (17). Typical vertical displacements of internal waves are larger than those of surface waves; this also makes the \( Q \) ratio large. The \( Q \) ratio in (17) is a minimum and could be of order 1 for large surface waves interacting in highly stratified water. Eq. (18) shows that the \( Q \) ratio will always be large in the deep-water limit; this is due to the poor matching of the eigenfunctions [see (3) and (6)] over the depth in the coupling coefficient.

Therefore, assuming constant transfer rates, the time required for the generation, from zero, of a typical internal wave spectrum by interactions of a typical surface-wave spectrum would normally be much longer than the time required for the complete redistribution of the internal wave energy by mutual internal wave interactions. This conclusion is a result of the large mismatch in the frequencies and eigenfunctions of the surface and internal waves; therefore, it probably does not depend strongly on the constant Väisälä-frequency assumption.

**Commutations.**

4.1. **Mutual Internal-wave Interactions in the Deep Ocean.** In this section are given some numerical computations of energy-transfer rates for the mutual internal-wave interactions. The computations are based on eq. (1), on the constant Väisälä-frequency and rigid-surface model, and on ocean-temperature measurements in water about 4 km deep.

At about 370 km southwest of San Diego, California, Cox et al. (1965) measured temperatures every five minutes for 14 days at 15 depths in water 3836 m deep. Fig. 1, taken from Lee and Cox (1966), shows the temperature spectra, \( T(n, \omega) \), in °C²/c/h as a function of frequency, \( n \), at 15 depths, \( z \). The feature that stands out most prominently above the background in most of the spectra is the peak at the frequency of the \( M_2 \) tidal component (about 0.08 c/h). Computations are presented of energy-transfer rates based on this spectral peak. A possible explanation of the observed spectra is that the \( M_2 \) internal tide is generated by the interaction between the surface \( M_2 \) tide and
the rough sea floor (Lee and Cox 1966, Cox and Sandstrom 1962) and that
the energy of the internal tide is then spread to higher and lower frequencies
by the mutual interactions of the internal waves, thereby contributing energy
to the background.

For the computations, eq. (1) was transformed to polar coordinates $kj, \alpha j$.
Following elimination of $k_2, \alpha_2$, and $\alpha_1$ by means of the delta functions in (2),
eq (1) involves a single integration over $k_1$ and a single summation over $v_1$
[because of (5)].

The spectra of temperature fluctuations, $T(n, z)$, in Fig. 1 were converted
to spectra of vertical particle displacements, $A(n, z)$, by using the local average temperature gradients, $\partial \bar{\theta} / \partial z$, given by Lee and Cox (1966), where

$$A(n, z) = T(n, z) \left| \frac{\partial \bar{\theta}}{\partial z} \right|^{-2}. \tag{19}$$

Fig. 2 shows a plot of $A(n, z)$ obtained in this way for $n = 0.08 \text{ c/h}$. The numbers of the points on the curve in Fig. 2 correspond to the numbers of the temperature spectra in Fig. 1; the numbered points show the mean depths at which the temperature records were taken.
The relationship between the energy spectrum, \( f(k, \alpha, \nu) \), in polar coordinates and the amplitude spectrum, \( A(n, z) \), was taken as follows. First

\[
f(k, \alpha, \nu) = \bar{\rho} \left| \frac{dn}{dk} \right| E_2(n, \alpha, \nu),
\]

(20)

where \( n = \omega/2 \pi \), \( \omega \) is given by (4), and \( \bar{\rho} \) is the average density. Then \( E_2(n, \alpha, \nu) \) was assumed to have the form

\[
E_2(n, \alpha, \nu) = E_1(n) S(\alpha) I(\nu),
\]

\[
\int_{-\pi}^{\pi} S(\alpha) d\alpha = 1, \quad \sum_{\nu} I(\nu) = 1.
\]

(21)

For a constant Väisälä frequency and rigid upper surface it can be shown (Kenyon 1966) that

\[
E_1(n) = \left[ \frac{N^2 h}{g} \right] \left[ \frac{1}{h} \int_{-h}^{h} A(n, z) dz \right].
\]

(22)

It has been assumed in particular that

\[
A(n, z) = G(n) A(0.08 c/h, z),
\]

(23)

where \( G(n) \) is sharply peaked about \( n = 0.08 \) c/h in accord with the temperature spectra in Fig. 1, \( G(0.08 \) c/h) = 1, and \( A(0.08 \) c/h, z) is given in Fig. 2. For convenience, \( G(n) \) was taken to be

\[
G(n) = c_1 e^{-(n/c_2 + c_3/n)^2},
\]

(24)

with \( c_1 = e^{-16}, c_2 = 1.11 \times 10^{-5} \) c/s, and \( c_3 = 4c_2 \).

The horizontal directional properties cannot be determined from Cox's data; therefore, in the computations the energy spectra were assumed to be horizontally isotropic, so that \( S(\alpha) = 1/2 \pi \) from (21). Some aspects of anisotropy will be considered in the shallow-ocean case. The depth was taken to be constant at 3836 m and the Väisälä frequency was assumed to be constant at \( 4.76 \times 10^{-3} \) rad/sec (22-min. period).

The partition of energy with respect to mode number, \( \nu \), was difficult to determine. Fig. 2 suggests that the first two or three modes, and possibly some higher ones, may be present. Energy-transfer rates have been computed for three different examples; the energy was assumed to be initially in the first mode only (example 1), equally divided between the first two modes (example 2), and equally divided among the first three modes (example 3); thus the total initial energy was the same in all three examples.

The initial energy spectra, \( E_2(n, \alpha, \nu) \), and the transfer rates, \( \partial E_2(n, \alpha, \nu)/\partial t \),
Table I. Interaction $Q$'s for deep-ocean mutual internal-wave interactions.

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$I(v)$</th>
<th>$10^{-3} Q_{iii}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v = 1$</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$v &gt; 1$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1/2$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$v &gt; 2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1/3$</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>$v &gt; 3$</td>
<td></td>
</tr>
</tbody>
</table>

for examples 1 and 2 are shown in Figs. 3 and 4. From (8), the $Q_{iii}$ have been computed for the three examples by using the computed transfer rates and $T_i = (0.08 \text{ c/h})^{-1}$; the results are given in Table I together with the function $I(v)$ in (21). Table I shows that $Q_{iii}$ decreases with an increase in the number of interacting modes, as expected from the limiting expression (9) for low frequencies [see also (4)].

Fig. 3 shows that, if only the first mode contains energy initially, the energy transfer takes place only among the first two modes; Fig. 4 shows that, if only the first two modes contain energy initially, the energy transfer takes place among the first four modes. The results for example 3 were not plotted, but, if the first three modes contain energy initially, the energy transfer takes place among the first six modes. This limitation on the number of interaction possibilities is due to the third equation in (5).

The main features of Figs. 3 and 4 can be summarized by using Hassel-
mann's (1966) transfer diagrams for conservative wave-wave interactions. The energy transfer in Fig. 3 is represented by the single transfer diagram (a) in Fig. 5, and the energy transfer in Fig. 4 is represented by all five transfer diagrams in Fig. 5. The energy transfer for example 3 is represented by the five diagrams in Fig. 5 plus seven other diagrams (not shown). All of the diagrams in Fig. 5 have the same form, two ingoing arrows and one outgoing arrow. Each arrow of a diagram represents an interacting wave component (v). The length and direction of an arrow correspond to the magnitude and direction of a wave-number vector. For low frequencies, the arrow length is also proportional to the wave frequency as determined from (4). An arrow with a stroke represents an antiwave component (v), with negative wave-number vector and frequency. An arrow points in the direction of wave propagation for both wave and antiwave components. The energy transfer of each component is proportional to the frequency of that component, and the outgoing component always receives a net transfer of energy from the ingoing components. Each diagram is constructed to satisfy the resonance conditions and the condition on the mode numbers (5). The energy transfer for a diagram vanishes if the spectral energy density associated with either ingoing component is initially zero. The diagrams in Fig. 5, therefore, can be drawn directly from (5) and (21).
Figure 5. Transfer diagrams for internal waves in an ocean with constant Väisälä frequency and rigid surface. Diagram (a) corresponds to the energy transfer in Fig. 3; all five diagrams occur in the energy transfer shown in Fig. 4. The numbers indicate mode numbers; a number with a bar denotes an antiwave (Hasselmann 1966).

In Fig. 5, (a) shows that (i) if only the first mode contains energy initially, the only possible direction of energy flow is toward low frequencies and (ii) the second mode gains energy from the first mode, in agreement with Fig. 3. Diagram (b) shows that, if the first two modes contain energy initially, then there is also an energy flow toward high frequencies. All five diagrams taken together show that there will be (i) an energy loss from the first two modes at frequencies near that of the initial energy peak, (ii) an energy gain to the first mode at frequencies that are higher than that of the initial peak, and (iii) an energy gain to the first four modes at frequencies lower than that of the initial peak. This agrees with Fig. 4. Figs. 3 and 4 show also that the energy gain balances the energy loss, as it should for conservative interactions.

One feature of the energy transfer, which is not predictable from Fig. 5, is the relative strengths of different interactions. Figs. 3 and 4 show that the largest transfer rates occur for the modes that contain energy initially and that the rates of energy transfer to modes that initially contained no energy are relatively small.

The Coriolis frequency $f/2\pi$ ($f = 2 \Omega \sin \varphi$, with $\Omega$ the angular frequency of the earth and $\varphi$ the latitude) for the latitude $\varphi = 30^\circ 44' \text{N}$ (Cox et al. 1965) has been indicated on the frequency axis in Figs. 3 and 4 in order to point out the limitations of the computations due to the neglect of rotation. In the case of the constant Väisälä-frequency model, no energy can be transferred to fre-
Figure 6. Frequency spectrum of vertical isotherm fluctuations, $H(n)$, based on observations at the N.E.L. tower during August 5–7, 1959, in water 18 m deep off Mission Beach, California, and an idealized spectrum, $G(n)$, used in the computations shown in Figs. 7 and 8. $N/2 \pi$ is plotted on the abscissa, where the mean Väisälä frequency, $N$, for the water column is $3.98 \times 10^{-2}$ rad/sec. The spectrum, $H(n)$, is taken from Cox (1962).

frequencies higher than the high-frequency cut-off, $N/2 \pi$; if the rotation were included, no energy could be transferred to frequencies that are lower than the low-frequency cut-off, $f/2 \pi$. The computations would apply better at the equator if similar energy densities existed there. In computations for the shallow-ocean case, presented next, the no-rotation assumption is not as critical.

4.2. Mutual Internal-wave Interactions in the Shallow Ocean. The computations here are based on measurements of the vertical fluctuations of isotherms taken at the U.S. Navy Electronics Laboratory oceanographic tower, which is located in water 18 m deep off Mission Beach, California. Cox (1962) has analyzed two days of data (August 5–7, 1959) taken at this tower; he has given the average frequency spectrum for three separate spectra of the vertical fluctuations of a particular isotherm. Fig. 6 shows Cox’s spectrum, $H(n)$, together with the idealized spectrum, $G(n)$, chosen here for the computations. In order to make the constant Väisälä-frequency assumption a more reasonable approximation, the computations have been restricted to frequencies that are much lower than the mean Väisälä frequency.

It has been assumed that (i) only the first mode is present, i.e. $I(v) = 1$ for $v = 1$, and $I(v) = 0$ for $v > 1$ [see (21)], (ii) the mean depth of the fluctuations in Fig. 6 was $z = h/2 = 9$ m, and (iii) the maximum fluctuations occurred at that depth. In other words, the amplitude spectrum, $A(n, z)$, was assumed to be
Figure 7. Initial energy spectrum and energy-transfer rates for modes 1 and 2 as functions of frequency, \( n \). The initial spectrum based on Fig. 6 is isotropic in direction, with energy only in mode 1. The Väisälä frequency, \( N \), is constant at \( 3.98 \times 10^{-4} \) rad/sec; the depth is constant at 18 m. \( N/6 \pi \) and the Coriolis frequency, \( f/2 \pi \), for 32°46'N (at which the data used for Fig. 6 were taken) are plotted on the abscissa.

\[
A(n, z) = G(n) \sin^2 \left[ \pi(z + h)/h \right],
\]

where \( G(n) \) (shown in Fig. 6) has the form (24), with \( \epsilon_1 = 2.63 \times 10^3 \text{m}^2/\text{c/s} \), \( \epsilon_2 = 7.15 \times 10^{-4} \text{c/s} \), and \( \epsilon_3 = 0.49 c_2 \). Cox (1962) indicated that an inspection of the depth variations in the isotherms shows that the first mode is clearly dominant. The spectrum \( A(n, z) \) was then related to the energy spectrum, \( f(n, \alpha, \nu) \), with (20)-(23), as before. The depth was taken to be constant at 18 m, and the Väisälä frequency was taken to be constant at the mean value \( N(z) = 3.98 \times 10^{-2} \) rad/sec (2.6-min. period), where \( N(z) \) has been given by Cox (1962).

\( Q_{iii} \) was computed for two different spreading factors, \( S(\alpha) \); the results are shown in Table II. In example 1, the spectrum was assumed to be isotropic in direction; and in example 2, the spreading factor used was \( (8/3\pi) \cos^4 \alpha \) for \( 1 \alpha 1 < \pi/2 \), and zero for \( 1 \alpha 1 > \pi/2 \). The initial spectra, \( E_2(n, \alpha, \nu) \), and the transfer rates, \( \partial E_2(n, \alpha, \nu)/\partial t \), for examples 1 and 2 are shown in Figs. 7 and 8, respectively. The \( Q_{iii} \) were calculated from (8) by using \( Ti = 2 \times 10^3 \) sec and the transfer rates in Figs. 7 and 8. Table II shows that the \( Q_{iii} \) for the isotropic spectrum in example 1 is slightly lower than that for the narrow beam in example 2, as might be expected. The appropriate transfer diagram for examples 1 and 2 is (a) in Fig. 5.

Comparison of example 1 in Tables I and II shows that \( Q_{iii} \) for the deep-ocean case is about a factor of 50 larger than that for the shallow-ocean case. A factor of about 40 would be contributed by the smaller amplitude-to-depth ratio in the deep-ocean case, according to (9). Also, the deep-ocean \( Q_{iii} \) is
expected to be larger due to the fact that the initial spectrum in this case was a factor of $10^2$ narrower in frequency than that in the shallow-ocean case; however, this effect is difficult to estimate.

4.3. Interactions of Surfaces Waves and Internal Waves in the Shallow Ocean. Here are given the computations of the interaction of surface waves and internal waves for two examples: a swell spectrum that is isotropic in direction (example 1), and a swell spectrum with a narrow directional beam (example 2). The computations are based on the energy-transfer equation, (1), on the constant Väisälä-frequency model, and on observations in water about 20 m deep. The main properties of this interaction process are summarized in § 1 and are illustrated by Hasselmann’s (1966) transfer diagram in Fig. 9; the ingoing wave (o) and antiwave (ô) components represent surface waves, the outgoing wave component (v) an internal wave.

Reference is made again to the internal-wave measurements taken in water 18 m deep on August 5-7, 1959, off Mission Beach, California (Cox 1962); the surface waves were not measured. However, a few miles to the south, off Coronado, a

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$S(x)$</th>
<th>$10^{-2} Q_{iii}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>$1/2 \pi$</td>
<td>$</td>
</tr>
<tr>
<td>2</td>
<td>$(8/3\pi) \cos^4 x$</td>
<td>$</td>
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</tr>
</tbody>
</table>
wave recorder located in 11 m of water was operating during this period. A 26-minute wave-height record, which is typical of the 12-hour period beginning at 0800 h on August 5, 1959, was analyzed; the result is shown in Fig. 10 together with the idealized spectrum, $E_1(n)$, which was used as initial data for the computations.

The spectral transformations were as follows: $f(n, \alpha, v)$ and $E_2(n, \alpha, v)$ were still assumed to be given by (20) and (21). [The group velocity, $dn/dk$, in (20) for $v = 0$ was calculated from $n = \omega/2 \pi$, where $\omega$ is given by (7).] $E_1(n)$ is given in Fig. 10, and the computations were made for the same two spreading factors, $S'(\alpha)$, referred to in § 4.2 (Table III).

Fig. 11 shows the energy-transfer rate, $\partial E_2(n, \alpha, 1)/\partial t$, to the first internal mode for example 1. The initial swell spectrum is shown in the upper right-hand corner. Also included in Fig. 11, for comparison, is a part of the observed internal-wave spectrum, $E_2(n, \alpha, 1)$ (also assumed isotropic and containing only the first mode); the computations of mutual internal-wave interaction in Fig. 7 were based on this part. $Q_{ssi}$ and $Q_{ssi}$ were calculated from (11) and (16), respectively, by assuming $T_s = 20$ sec, $T_i = 2 \times 10^3$ sec, and by using the transfer rates in Fig. 11; contributions from the second and higher modes have been neglected (see below). The values of $Q_{ssi}$ and $Q_{ssi}$ are given in Table III.

The energy-transfer rates for example 2 are shown in Figs. 12 and 13. Fig. 12 shows the transfer rate, $\partial E_2(n, \alpha, 1)/\partial t$, to the first mode for the three directions, $\alpha = 60^\circ, 90^\circ, 120^\circ$, and Fig. 13 shows the energy-transfer rates for shallow-ocean surface-internal-wave interactions.

### Table III. Interaction $Q$'s for shallow-ocean surface-internal-wave interactions.

<table>
<thead>
<tr>
<th>Ex.</th>
<th>$S(\alpha)$</th>
<th>$10^{-6} \bar{Q}_{ssi}$</th>
<th>$10^{-10} Q_{ssi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/2 \pi$</td>
<td>$</td>
<td>\alpha</td>
</tr>
<tr>
<td>2</td>
<td>$(8/3 \pi) \cos^4 \alpha$</td>
<td>$</td>
<td>\alpha</td>
</tr>
<tr>
<td></td>
<td>$0$</td>
<td>$</td>
<td>\alpha</td>
</tr>
</tbody>
</table>
Figure 10. Frequency spectrum of swell amplitudes measured by a wave recorder in 11 m of water off Coronado, California, on August 5, 1959, and idealized spectrum, $E_1(n)$, used in the computations shown in Figs. 11-13.

Figure 11. Energy-transfer rate to internal mode \( \nu = 1 \) due to interactions of swell (\( \nu = 0 \)). The swell spectrum, from Fig. 10, is shown in the upper right-hand corner; its directional distribution is isotropic. Also shown, for comparison, is the internal-wave energy spectrum (which is isotropic with energy in mode 1 only) used for the initial spectrum in Fig. 7. The Väisälä frequency, \( N \), is constant at \( 3.98 \times 10^{-2} \text{ rad/sec} \), and the depth is constant at 18 m. \( N/12\pi \) is plotted on the abscissa.
Figure 12. Energy-transfer rate to internal mode \( \nu = 1 \) for three directions, \( \alpha = 60^\circ, 90^\circ, 120^\circ \), due to interactions of swell (\( \nu = 0 \)). The swell spectrum, from Fig. 10, is shown in the upper right-hand corner for two directions, \( \alpha = 0^\circ, 30^\circ \). The spreading factor is taken to be proportional to \( \cos^4 \alpha \) for \( |\alpha| < \pi/2 \) (and zero for \( |\alpha| > \pi/2 \)). The Väisälä frequency, \( N \), is constant at \( 3.98 \times 10^2 \) rad/sec, and the depth is constant at 18 m. \( N/12 \pi \) is plotted on the abscissa.

The energy-transfer rate to the internal mode \( \nu = 1 \) at \( \alpha = 90^\circ \) to modes \( \nu = 1, 2 \). The initial swell spectrum, \( E_2(n, \alpha, 0) \) for \( \alpha = 0^\circ, 30^\circ \), is shown in the upper right-hand corner of Figs. 12 and 13. Fig. 12 indicates that the energy-transfer rate is a maximum for \( \alpha = 90^\circ \), which is at right angles to the mean swell direction, as expected. Fig. 13 shows that the average energy-transfer rate to the first mode is more than 10 times the average energy-transfer rate to the second mode. According to (17) and (9), the transfer rates continue to decrease rapidly with increasing mode number, which justifies neglect of the contributions to \( Q_{ssi} \) and \( \tilde{Q}_{ssi} \) from the second and higher modes.

The values of \( Q_{ssi} \) and \( \tilde{Q}_{ssi} \) in Table III were calculated for example 2 by assuming

\[
\frac{\partial E_2}{\partial t}(n, \alpha, \nu) = \frac{\partial E_2}{\partial t}(n, 90^\circ, \nu) \cos^8 \alpha.
\]

(26)

Fig. 12 indicates that this may be a good approximation for the first mode. \( T_s \) and \( T_i \) are assumed to be 20 sec and \( 2 \times 10^3 \) sec, as in example 1. The contributions from the second and higher modes have been neglected. [For
Figure 13. Energy-transfer rate to internal modes $\nu = 1, 2$ for the direction $\alpha = 90^\circ$ due to interactions of swell ($\nu = 0$). The swell spectrum, from Fig. 10, is shown in the upper right-hand corner for two directions, $\alpha = 0^\circ, 30^\circ$. The spreading factor is taken to be proportional to $\cos^4 \alpha$ for $|\alpha| \leq \pi/2$ (and zero for $|\alpha| > \pi/2$). The Väisälä frequency, $N$, is constant at $3.98 \times 10^{-3}$ rad/sec, and the depth is constant at 18 m. $N/12 \pi$ is plotted on the abscissa.

example, if the second mode is neglected in example 2, then $\bar{Q}_{ssi} = 0.87 \times 10^6$; if the second mode is included, then $\bar{Q}_{ssi} = 0.86 \times 10^6$.]

Comparison of example 1 in Tables II and III shows that $\bar{Q}_{ssi}$ is an order of $10^4$ larger than $Q_{iii}$. In other words, if it is assumed that the observed surface-wave spectrum and internal-wave spectrum are isotropic in direction and that the constant energy-transfer rates obtain, then, from zero, it would take $10^4$ times longer to generate the internal-wave energy by swell interactions than to completely redistribute the internal-wave energy by mutual internal-wave interactions. Example 2 in Tables II and III indicates that this result would be nearly the same if the surface-wave spectrum and internal-wave spectrum had narrow directional beams. However, eq. (17) shows that, if the r.m.s. surface-wave amplitude had been larger by a factor of 10 ($2.1$ m instead of $0.21$ m) for the frequency interval $0.05 \leq \omega \leq 0.07$ c/s, then $\bar{Q}_{ssi}$ would have been of the same order as $Q_{iii}$. The internal waves moved toward the shore (Cox 1962) in a direction nearly normal to it. The direction of the swell was not measured, but it was probably closer to being perpendicular to the shore rather than perpendicular to the internal-wave direction as predicted by the interaction theory.
The generation of internal waves by surface-wave interactions is therefore negligible in this particular case for the theory and for the ocean model used. However, this interaction process may be more important in the case of a large swell that interacts in highly stratified shallow water.

The Qssi in Table III are one to two orders of magnitude larger than the Qv for molecular viscous damping of the surface waves in the deep-water limit; this indicates that viscous damping is stronger. According to the classical formula

\[ Q_v = \frac{\omega}{4 \mu k^2} = g^2 T^3/32 \pi^3 \mu = 4.8 \times 10^8 \]

for (the acceleration of gravity) \( g = 980 \text{ cm/sec}^2 \), (the dynamic viscosity) \( \mu = 10^{-2} \text{ cm}^2/\text{sec} \), and (the wave period) \( T = 17 \text{ sec} \), which corresponds to the observed spectral peak in Fig. 10. The interaction of surface waves and internal waves would have been stronger than the viscous damping if the r.m.s. surface-wave amplitude had been larger by a factor of 10 or more (i.e., greater than 2.1 m) as indicated by (12), or if the characteristic wave periods had been longer than 27 sec.

Conclusions.

5.1. Mutual Internal-wave Interactions. The largest energy-transfer rates for low frequencies occur for the highest mode-number internal waves having the largest ratio of vertical displacement to water depth. The largest energy-transfer rates for high frequencies occur for internal waves with the largest wave slopes. Since the low frequencies in the ocean generally have the largest energy density, the strongest mutual internal-wave interactions probably take place in the relatively shallow water near the coast, where the ratio of vertical displacement to depth may be the largest (assuming that the same mode numbers are dominant everywhere). Over the continental shelf off southern California, Summers and Emery (1963) have observed large internal waves of nearly semidiurnal period, including possible evidence of internal surf in waves approaching the coast. The characteristic interaction, Q’s, for internal waves of low mode number and low frequency is found to be of the order of \( 10^3 \) for 10-m amplitude and 12-h period waves in water 4 km deep and of the order of \( 10^2 \) for 0.5-m amplitude and 0.5-h period waves in water 20 m deep; this indicates stronger interactions in shallow depths.

5.2. Interactions of Surface Waves and Internal Waves. The largest energy-transfer rates occur for the largest amplitude surface waves, for the largest ratio of internal-wave frequency to surface-wave frequency, and for the best overlapping of surface-wave and internal-wave eigenfunctions over the depth. Since the highest surface-wave energy density in the ocean occurs for waves in the sea-swell-frequency range (periods of the order of 10 sec), the greatest rate of energy transfer from surface wave to internal waves takes place for
large amplitude swell that interacts to generate internal waves of low mode number in water that is highly stratified and has a depth comparable to a surface-wave length. In other words, the greatest transfer rates would occur in the sun-warmed water over the continental shelf.

5.3. **Comparison of the Interactions.** Comparison of the mutual internal-wave interaction with the interaction of surface waves and internal waves, based on a dimensional analysis of the energy equation and on computations of energy-transfer rates derived from ocean measurements, shows that, in general, the transfer rates for the mutual internal-wave interaction are much larger. In other words, the time scale for the energy input to internal waves due to interactions with surface waves is generally much larger than the time scale for the energy redistribution among the internal modes due to the wave-wave interactions of the internal waves. The two time scales could be equal in the rather extreme case of large-amplitude surface waves that interact in highly stratified shallow water.

Comparison of the interaction of surface waves and internal waves with the mutual surface-wave interaction, based on a dimensional analysis of the energy equation, shows that, except for the unimportant case of very weak interactions, the transfer rates for the interaction of surface waves and internal waves are much smaller in the ocean. In other words, the time scale for the energy loss to internal waves is generally much larger than the time scale for the energy redistribution within the surface mode.

5.4. **The Ocean Model.** The above conclusions are based on an attempt to apply the resonant energy-transfer theory to the ocean by using a specific model and certain ocean measurements. The conclusions are regarded as tentative because of a lack of knowledge about the partition of internal-wave energy with respect to horizontal direction and vertical mode number and because of the particular model used.

The constant Väisälä-frequency assumption is the one most open to criticism. This assumption, when coupled with the rigid surface assumption, may strongly affect the mutual internal-wave interactions, because the additional condition on the mode numbers greatly reduces the total number of interaction possibilities that satisfy the usual resonance conditions. In order to justify this assumption to some extent, the computations were restricted to frequencies that were low when compared with the mean Väisälä frequency. However, the interaction of surface waves and internal waves probably does not depend strongly on the constant Väisälä-frequency assumption. The time scale for this interaction depends on the frequency ratios of the surface waves and internal waves and on the eigenfunction matchings; these may not be expected to vary appreciably—for example, for different density distributions in which the bottom-to-top density difference remains constant.
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