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The Wave-drift Current

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ABSTRACT

The irrotational wave-drift current associated with the one-dimensional equilibrium wave spectrum is calculated, and by comparison with observation, it is shown that in many situations this current may make an important contribution to surface-drift velocities. One of the interesting features of the irrotational current profile is a remarkable affinity to the logarithmic profile of similarity theory, both formally and quantitatively.

1. Introduction. Consider in a general manner the conditions near the surface of a wind-driven sea. First, consider a spectrum of nearly irrotational surface waves and an associated irrotational wave-drift current that may be defined as the Stokes mass transport of each component:

\[ S = a^2 \sigma k e^{-2kz}, \]  

where \( a \) is the wave amplitude, \( \sigma \) is the wave frequency, \( k \) is the wave number, and \( z \) is the depth (measured downward from the surface) integrated over the spectrum.

Second, consider a spectrum of turbulent eddies and planetary motions varying from large-scale current systems and tidal motions to scales of the order of the smallest wave motions. At a given place, the total drift current is formally the sum of the above two components. It is the purpose of this paper to evaluate the wave-drift current, and, by comparison with observations, to show that in many situations it may make an important contribution to the total drift experienced on or near the sea surface. Because of the complexity of the problem, only a completely irrotational theory is given, but in the discussion an attempt is made to consider qualitatively the possible effects on the drift profile of the diffusion of vorticity through the surface boundary. Basically, of course, the principles of the analysis of Longuet-Higgins (1953) apply, but

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2. To terms \( O(a^2) \).
the difficulties of a complete treatment for a wind-driven sea are formidable because of the nature of the boundary conditions and the role of the convective as well as the conductive terms in the diffusion of vorticity.

2. The Equilibrium Wave-drift Profile. The wave-drift current is calculated for the one-dimensional energy spectrum

\[ F(\sigma) = \beta g^2 \sigma^{-5}, \]  

where \( g \) is the acceleration of gravity; \( \beta = 1.5 \times 10^{-2} \), which is the estimate given by Longuet-Higgins (1962). This spectrum, in the range of frequencies between the upper and lower frequency bounds, \( \sigma_U \) and \( \sigma_L \) (Fig. 1), approximates the conditions along the direction of the wind in a fully developed sea. The upper bound (\( \sigma_U \)) is estimated to be the frequency at which surface tension and gravity combine to give a minimum wave speed (i.e., \( \sigma_U = 60 \text{ sec}^{-1} \)) and the lower bound is the frequency of the energy maximum of the spectrum.

First, by definition the effective amplitude \( a \) of a small range (\( \Delta \sigma \)) of wave components in the spectrum about a mean frequency \( (\sigma) \) is given by

\[ a^2 = 2F(\sigma)\Delta \sigma. \]  

Therefore, by substituting eq. (2) into eq. (3),

\[ a^2 = 2\beta g^2 \sigma^{-5}\Delta \sigma, \]

and, substituting this equation into eq. (1) for the mass transport, using the wave equation \( gk = \sigma^2 \), \( S \) is obtained as a function of \( \sigma \):

\[ S = 2\beta g \sigma^{-2} e^{(-2\sigma^2)/g} \Delta \sigma. \]

The total wave-drift current is simply

\[ \sum_{\sigma_L, \sigma_U} S, \]

or, in the limit \( \Delta \sigma \to 0 \),

\[ U = \int_{\sigma_L}^{\sigma_U} 2\beta g \sigma^{-2} e^{(-2\sigma^2)/g} d\sigma. \]
Finally, integrating eq. (4), and assuming $\sigma_U \gg \sigma_L$,

$$U = 2\beta g \left( \frac{1}{\sigma_L} e^{-2\sigma_L^2/g} - (2\pi \sigma_L/g)^{1/2} P_{2,\sigma_L(g/\sigma_L)}^{\infty} \right),$$

where $P$ is the normal probability function.

Eq. (5) has several interesting features that will be discussed below and then compared with observations.

(i) **The Wave-surface Current.** By re-expressing eq. (4) in terms of $F(\sigma)$ at $z = 0$, the surface current,

$$U_s = \int_{\sigma_L}^{\infty} \frac{2\sigma^3 F(\sigma)}{g} d\sigma,$$

is the third moment of the energy spectrum.

Evaluating by means of eq. (5),

$$U_s = \frac{2\beta g}{\sigma_L} \frac{3\sigma}{\sigma_L} \text{cms/sec}.$$  

Hence, the magnitude of $U_s$ is inversely proportional to the frequency of the maximum in the energy spectrum.

(ii) **The Wave-current Profile.** In Fig. 2, $U$ is plotted against $z$ for $\sigma_L = 10 \text{ sec}^{-1}$ and $1 \text{ sec}^{-1}$. The notable feature is that the central sections of both profiles have an approximate logarithmic slope as indicated by the arrows in Fig. 2.

If eq. (5) is expressed in terms of the nondimensional depth, $\xi = (\sigma_L^2 z)/g$, and is then divided by $\sigma_L/2\beta g$,

$$\frac{U\sigma_L}{2\beta g} = e^{-2\xi} - (2\pi)^{1/2} \xi^{1/2} P_{2,\xi}^{\infty}.$$  

Eq. (7) is a universal profile for the nondimensional velocity, $U\sigma_L/2\beta g$, which is valid for all $\sigma_L$. Wave-drift current profiles for specific $\sigma_L$ are therefore similar and may be obtained on a semilogarithmic graph by a translation along the depth axis,

$$\ln z = \ln \xi + \ln (g/\sigma_L^2),$$

and by a change in scale along the velocity axis. This is readily observable in Fig. 2. There is indeed a close analogy between eq. (7) and the universal logarithmic profile over solid surfaces,
\[ U = \frac{U_* \ln \frac{z}{z_o}}{k} \]  

where \( k \) is VonKarman's constant and \( z_o \) is a roughness length; \( U_* = (\tau/\rho)^{1/2} \) is the frictional velocity of the fluid of density, \( \rho \), and \( \tau \) is the boundary-layer stress.

Eq. (8) represents a family of curves of the same general nature: (i) the velocity \( 2\beta g/\sigma_L \) corresponds to the velocity \( U_*/k \), and both expressions contain a universal constant; (ii) the nondimensional parameter, \( \xi \), corresponds to \( z/z_o \), or in other words, the length, \( g/\sigma_L^2 \), is analogous to the roughness length, \( z_o \). In § 3, when the magnitude of the logarithmic slopes of these profiles is considered, this analogy is shown to have even greater significance.

![Figure 2. Calculated wave-drift profiles.](image-url)
3. Comparison with Observations. Observations of phenomena near the surface of lakes have been discussed (Bye 1965). It is assumed below that these observations are applicable in a fully developed sea (in the absence of non-local currents, e.g., tidal currents or large-scale circulations) and may be compared with the analysis.

(i) The Surface Current. Under a wide variety of conditions, the surface current is given by the relations (Van Dorn 1953 and Bye 1965, respectively)

\[ U_s \sim 0.033 U_a \sim 22 U_* \]  

where \( U_a \) is the wind speed measured at 1 m; \( U_* = (\tau/\rho)^{1/2} \) is the frictional velocity of water in which \( \rho \) is the density of water and \( \tau \) is the surface stress. Now, substituting for \( U_a \) in eq. (9) from the empirical relationship

\[ \sigma_{\text{max}} = (3/2)^{1/2} g U_a^{-1}, \]

where \( \sigma_{\text{max}} \) is the frequency of the maximum in the energy spectrum, which is derived from the Neumann wave spectrum for a fully developed sea (Kinsman 1965: 395), and equating \( \sigma_L \) with \( \sigma_{\text{max}} \),

\[ U_s = \frac{40}{\sigma_L} \]  

and

\[ U_* = \frac{1.8}{\sigma_L}. \]

Eq. (10a) agrees in form with the present analysis, and the irrotational surface current, though somewhat smaller (0.75), is in approximate agreement with the observational estimate.

(ii) The Current Profile. The only observations of the drift-current profile near the surface appear to have been made in a steady wind of 7–8 m/sec that had been blowing for several hours with a fetch of 14.5 km (Bye 1965). The profile was observed to have the logarithmic equation

\[ U = U_s - \frac{U_*}{k} \ln \frac{z}{z_0}. \]

Now, substituting for \( U_* \) from the empirical eq. (10b),

\[ U = \frac{1.8}{\sigma_L} \left( 22 - \frac{1}{k} \ln \frac{z}{z_0} \right). \]
hence the velocity difference ($\Delta U$) between two levels, $z_1$ and $z_2$, is given by

$$\Delta U = U_2 - U_1 = 4.4 \ln \frac{z_1}{z_2}. \quad (11)$$

Compare $\Delta U$ in eq. (11) with the velocity difference ($\Delta U_m$) measured directly from the approximately logarithmic central regions of the wave-drift profiles in Fig. 2; either may be used since the profiles are similar;

$$\Delta U_m = \frac{5.5}{\sigma_L} \ln \frac{z_1}{z_2}. \quad (12)$$

Again there is order of magnitude agreement between observation and the irrotational theory, although the irrotational logarithmic slope is larger than the observed, in contrast to the estimates of the surface current. In view of the approximations involved in the comparison, however, this difference is probably not significant.

Therefore, provided the above observations are accepted as applicable to our analysis, it appears that the total drift associated with a fully developed sea may be explained essentially as an irrotational wave-drift current, the profile of which shows a remarkable affinity to the logarithmic profile of similarity theory, both formally and quantitatively.

The thickness ($\delta_w$) of this wave-boundary layer, defined as the depth at which the wave-drift current is reduced to $e^{-1}U_8$, is given by

$$\delta_w = \frac{100}{\sigma_L^2}; \quad (\text{cf. Fig. 2})$$

$\delta_w$ is about 0.25 of the average height ($H_{1/10}$) of 0.1 of the surface waves, which, from the Sverdrup-Munk empirical relation (Kinsman 1965: 390), may be written

$$H_{1/10} = 3.4 \times 10^{-4} U_a^2,$$

$$= 0.31 \frac{U_s^2}{\sigma_L^2}, \quad \text{by eq. (9)}$$

$$= \frac{500}{\sigma_L^2}. \quad \text{by eq. (10)}$$

Hence, the wave drift occurs mainly above the trough line. 3.

3. Drift in the context of this paper is essentially Lagrangian, such as might be observed with small neutrally buoyant particles to which surface floats approximate. From an Eulerian point of view, the entire Stokes mass transport occurs rigorously in motions above the trough line.
(iii) The roughness length \( z_0 \) over the sea. Note also that the present theory predicts that the governing length in the air-water interface has the dimensions \( g/\sigma_L^2 \). This may be the reason for some of the experimental scatter in determinations of the roughness length, \( z_0 \), over the sea. It is more appropriate to relate \( z_0 \) to the frequency of the energy maximum in the wave spectrum than to the wind speed or frictional velocity. Only in a fully developed sea, in which eq. (10b) is valid, is it expected that \( z_0 \) would have the dimensional form proposed by Ellison (1956),

\[
\frac{z_0}{g} = \frac{U^2}{U_*},
\]

where \( \alpha \) is a constant.

Eq. (13) may well represent the limiting curve for \( z \) under fully developed conditions. However, smaller values of \( z_0 \) obtain in fetch-limited situations or where the duration of the wind has not been sufficient to obtain the equilibrium wave spectrum.

4. Discussion. This analysis will be discussed two ways. First, consider the energetics of the wave-boundary layer. It may appear that such a discussion is not pertinent, since an irrotational theory has been proposed. However, if it is assumed that a turbulent stress \( \tau \) in the boundary layer is provided by some mechanism outside the present theory (for example, wave breaking) there must be an energy transfer from the wave-drift current. In particular, if \( \tau \) is approximately constant with depth, the rate of work on the surface by the tangential stress is exactly balanced by the energy flux from the wave-drift current to turbulence; thus,

\[
W = \tau U_s = \tau \int_{-\infty}^{\infty} \left( \frac{\partial U}{\partial z} \right) dz.
\]

In other words, energy from the tangential stress is "filtered" by the wave spectrum to become an input to turbulence at all levels in the boundary layer. The tangential energy flux effectively passes through rather than into the wave spectrum. Of course, this energy flux to turbulence is in addition to that derived from normal pressure fluctuations through wave-breaking mechanisms.

Second, consider qualitatively the effect on the drift profile of the vorticity that undoubtedly penetrates the sea surface through processes such as those described above. Consider now the approach to a fully developed sea. As the duration of the wind increases, longer and longer wavelengths attain their equilibrium-energy state until each component in the spectrum is finally in equilibrium. Now, equilibrium simply means that a component is, on an average, just maintained by the surface forces; Lamb (1932: 623–624) has shown that it may be completely irrotational, even when viscous effects are
included. Hence, as the sea develops and as longer and longer wavelengths attain equilibrium, it is expected that an essentially irrotational boundary layer develops downward from the surface. In the limiting case of the fully developed sea, this layer is approximated by the wave-boundary layer discussed in the present paper. However, at depths greater than the thickness of the layer [cf. § 3 (ii)], the irrotational component of the drift profile is almost certainly of minor importance, and the flow is dominated by rotational factors beyond the scope of the present analysis.

It is concluded, therefore, that (i) in a fully developed sea, the irrotational drift current may be the dominant component of the total drift current, and (ii) under developing or fetch-limited conditions, the importance of the irrotational drift is probably intimately related to the conditions themselves and to their rate of change.

Simultaneous observations on the growth of the wave spectrum and the drift profile under a variety of conditions are required to obtain the true picture.

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