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Notes on the Theory of the Thermocline

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ABSTRACT

This paper presents an exact analytical solution in closed form for a steady, laminar model of the oceanic thermocline. The model includes vertical diffusion and horizontal and vertical convection of heat. The solution satisfies boundary conditions that are in fair agreement with observation over the range of latitude $10^\circ$ to $45^\circ$. The behavior of the solution is given as a function of the imposed temperature and vertical velocity component at the bottom of the Ekman layer. A typical result of the analysis is that, as the surface temperature increases, the thermocline depth decreases, and the deep vertical velocity component increases. Subpolar gyres are shown to be excluded for some boundary conditions. The equations solved numerically by Stommel and Webster are shown to be exact and a derivation of Robinson's equations for an equatorial thermal regime is given.

1. Introduction. Observations show that the interior of the ocean is unsteady and turbulent. The present work, which presents an analysis of steady laminar, global flow, is therefore only a model of the real flow. In § 2 an exact analytical solution for the model is found by using Robinson and Welander's (1963) similarity transformation. The response of the model to different surface temperatures and vertical velocity components at the bottom of the Ekman layer (hereafter referred to as the Ekman velocity) is determined analytically. These results are in agreement with those of Stommel and Webster (1962) and can be summarized as follows for subtropical gyres. For increasing surface temperatures, the depth of the thermocline decreases and the asymptotic deep vertical velocity component increases. For increasingly negative Ekman velocities, the thermocline depth deepens and the asymptotic deep vertical velocity decreases. With an increasing mixing coefficient, this asymptotic velocity and the depth of the thermocline increase. All of these relations are monotonic in behavior. An intriguing result is the exclusion of subpolar gyres for certain ranges of these parameters.

A large class of transformations is shown to exist for which the horizontal convections of heat exactly cancel. In § 3 it is shown that the numerical
calculations of Stommel and Webster, which they thought were for approximate equations, give the exact solution for one member of this class of transformations—the one with constant surface temperature.

In § 4 the divergence near the equator of the theoretically derived velocity and temperature fields is used to derive the equations of motion appropriate to the equatorial region.

2. The Similarity Transformation and Exact Solution. Robinson (1960) has discussed the nondimensionalization and scaling of the equations appropriate to the present thermocline model. His results are, in a slightly different notation and in the spherical coordinates of Robinson and Welander (1963),

\[-v \sin \theta = -\frac{1}{\cos \theta} p_{\varphi}, \quad (2.1)\]

\[u \sin \theta = -p_{\theta}, \quad (2.2)\]

\[-p_z + T = 0, \quad (2.3)\]

\[\frac{1}{\cos \theta} u_{\varphi} + \frac{1}{\cos \theta} (v \cos \theta)_\theta + w_z = 0, \quad (2.4)\]

\[-\bar{\nu} T_{zz} + \frac{u}{\cos \theta} T_{\varphi} + v T_{\theta} + w T_z = 0. \quad (2.5)\]

All the variables have been nondimensionalized and scaled; subscripts indicate differentiation with respect to the subscripted variables; \((u, v, w)\) are the horizontal and vertical velocity components; \((\varphi, \theta, z)\) are longitude, latitude, and vertical distance, respectively; \(T\) is the temperature, \(p\) the pressure, and \(\bar{\nu}\) is the nondimensional (turbulent) thermometric conductivity. The following list gives the expressions by which the corresponding fields have been nondimensionalized and scaled: \((u, v), [\alpha^2 T_s^2 g^2/4 \bar{\nu}^2 \bar{R}^3]; w, [\alpha T_s g/2 \bar{\nu} \bar{R}^3]; z, [2 \bar{\nu} \bar{R}^2 \xi_0/\alpha T_s g]; \rho, p_0[2 \bar{\nu} \bar{R}^2 \alpha^2 \bar{R}^3 T_s g^2]; T, T_s; \alpha, \alpha_0.\) Here \(\bar{\nu} = 0.7 \times 10^{-4} \text{sec}^{-1}, g = 10.0 \text{m sec}^{-2}, \alpha = 2.5 \times 10^{-4} \text{°C}^{-1}, T_s = 20 \text{°C}, \bar{R} = 6.3 \times 10^3 \text{km}, \xi_0 = 1.0 \text{cm}^2 \text{sec}^{-1}, p_0 = 1.0 \text{g cm}^{-3}.\) The characteristic sizes of the nondimensionalized variables are: \((u, v), 1.26 \text{ cm sec}^{-1}; w, 4.46 \times 10^{-5} \text{cm sec}^{-1}; z, 220 \text{ m}; T, 20 \text{°C}; \alpha, 1.0 \text{ cm}^2 \text{sec}^{-1}.\)

To reduce the system to one equation, we follow Robinson and Welander and define the potential function,

\[M = -\int_{-\infty}^{z_1} \int_{-\infty}^{z_2} T dz_1 dz_2 + C(\theta, \varphi), \quad (2.6)\]

where \((z_1, z_2)\) are dummy variables and \(C(\theta, \varphi)\) is an arbitrary function of \(\theta, \varphi.\)
Expressions for $T$, $u$, and $v$ can be obtained by differentiating and integrating $M$ and by using (2.1, 2.2) and by requiring the condition that $u$ and $v$ vanish at depth. Substituting these expressions in (2.4) and integrating, we obtain

$$w = \frac{1}{\sin^2 \theta} M_{\phi} + F(\theta, \phi),$$

(2.7)

where $F$ is an arbitrary function of $\theta$ and $\phi$. For $z = -\infty$, then, $w = (-1/\sin^2 \theta) C_{\phi} + F$. All other physical fields in the model are not affected by $C$ or $F$, since their expression in terms of $M$ involves $z$ derivatives. Therefore, since $F$ and $C$ occur together in the expression for $w$, $F$ may be absorbed in $C$, and for convenience we set $F = 0$.

Substituting the expressions for $u$, $v$, $w$, and $T$ in (2.5), we have

$$- \frac{\kappa}{2} M_{zzz} \sin 2 \theta + M_{\theta z} M_{zz} \phi - M_{z \phi} M_{zz0} - \cot \theta M_{\phi} M_{zzz} = 0.$$  (2.8)

The similarity variable for this problem is

$$\eta = (\sin \theta)^m (\phi + E[\theta])^n z.$$  (2.9)

Here $E(\theta)$ is an arbitrary function of $\theta$. This expression for $\eta$ departs from that of Robinson and Welander in that it does not include $\sigma = \pm 1$ on the right-hand side. Robinson and Welander used $\sigma$ to maintain a convention on the sign of $\eta$, $\eta < 0$, but it is not necessary to use $\sigma$ to do this. Since $z < 0$, the product $(\sin \theta)^n (\phi + E)^m$ must be $> 0$. Since $\sin \theta > 0$, $(\phi + E)$ must be $> 0$. Suppose $(\phi + E) < 0$ and $n = 1/3$. This is a contingency for which $\sigma$ was used by Robinson and Welander. However, $1/3$ may be approximated to any accuracy by $2q/(2p + 1)$, $p$ and $q$ integers, and $(\phi + E)$ to this power is even. Therefore there is a physically continuous spectrum of solutions. There is no "discontinuity" at fractions like $1/3$. Hereafter the convention is $(\phi + E)^n > 0$, but note that, if $(\phi + E) < 0$, then $(\phi + E)^{3n + 1} < 0$.

The similarity function is defined as

$$M = \frac{1}{2} (\sin \theta)^m + 2 (\phi + E)^n + 1 G(\eta).$$  (2.10)

By differentiation we can obtain

$$T = -M_{zz} = -\frac{1}{2} (\sin \theta)^{3m} + 2 (\phi + E)^{3n + 1} G''',$$

(2.11)

$$w = -(M_{\phi} / \sin^2 \theta) = -\frac{1}{2} (\sin \theta)^m (\phi + E)^n [(n + 1) G + n \eta G''],$$  (2.12)

where the primes indicate differentiation with respect to $\eta$. From these equations may be deduced the useful relation.
If (2.11) and (2.12), together with similar expressions for $u$ and $v$, are substituted into (2.8), the terms in $(\theta, \varphi)$ cancel and there is left
\[ \varkappa G^{iv} - (n-m)\eta G''G''' + (n+1)G'''G + (2n-m)\eta G''G' - (2n-m)G''G' = 0. \] (2.14)

The contribution of the horizontal convection terms is $M_{\alpha\kappa}M_{\varphi\varphi} - M_{z\varphi}M_{z\varphi}$. Transformed, this expression becomes $(2n-m)[G''G'' + \eta G''G'' - \eta G''G'']$. Thus, the effect of the horizontal heat convections vanishes for $(2n-m) = 0$. It is shown in § 3 that the equations solved by Stommel and Webster (1962) are exactly equivalent to the case $n = -1/3, m = -2/3$. Equation (2.11) shows that these values for $n$ and $m$ correspond to constant surface temperature.

An exact solution to (2.14) is given by
\[ G = -a + b e^{(a/\alpha)} z \sin^{-1} \theta \] (2.15)

for the case $n = 0, m = -1$. For these values of $n$ and $m$, $(2n-m) \neq 0$, and the horizontal convections of heat do not sum to zero. In no obvious sense is this simple solution trivial. It was not discovered by Robinson and Welander because they included an $n$ in the denominator in their definition of the similarity form for $M$, similar to (2.10). They placed the $n$ in this position because they anticipated a physical discontinuity for $n = 0$, for then, by (2.9), the longitudinal gradient of the thermocline scale-depth vanishes. If this meant that $T\varphi$ vanished, then by (2.13), $w_{zz}$ would also vanish, and $w$ would have to be a constant or diverge at depth, both unrealistic thermoclines. However, $n = 0$ does not imply $T\varphi = 0$, because the longitudinal surface-temperature gradient does not vanish. Thus there is no physical discontinuity for $n = 0$, and the discontinuity found in the paper of Robinson and Welander is due to their choice of the similarity form.

The exact solution represents the principal advance of the present work. It makes possible the analytical computation of the thermocline response to varying boundary conditions and is a nontrivial example to which approximate solutions may be compared.

The expressions for the fields of the exact solution are
\[ T = -\frac{1}{2} \sin^{-1} \theta (\varphi + E) b (a^2/\alpha^2) e^{(a/\alpha)} \eta, \] (2.16)
\[ w = -\frac{1}{2} \sin^{-1} \theta (-a + b e^{(a/\alpha)} \eta), \] (2.17)
\[ \eta = z \sin^{-1} \theta, \] (2.18)
\[
  u = \frac{1}{2} \sin^{-2} \theta \left[ -\cos \theta (\varphi + E) \left(ba^2/x^2\right) \eta e^{(a/x) \eta} + 
  + (\sin \theta) (E_0 b a/x) e^{(a/x) \eta} \right],
\]
(2.19)

\[
  v = -\frac{1}{2} \sin^{-1} \theta (\cos^{-1} \theta) \left(ba/x\right) e^{(a/x) \eta}.
\]
(2.20)

This solution has constant potential vorticity for constant \( T \); that is, computations give \( \sin \theta/(\partial z/\partial T) = (a/x) T \). This is in approximate agreement with observations of the thermocline to the east of the Gulf Stream, and it is of further interest because theories of the Gulf Stream as an inertial boundary layer are simplified by the assumption of constant potential vorticity.

The exact solution must satisfy the boundary conditions. If we let \( E = 0 \), and if \( \varphi = -1 \) corresponds to the western North Atlantic, we may then assume \( T = T_0, w = w_0 \) at \( z = 0, \theta = 30^\circ N \); then

\[
  T_0 = ba^2/x^2, \quad w_0 = a - b.
\]
(2.21)

Eliminating \( b \), the resulting cubic is

\[
  \left(\frac{a}{w_0}\right)^3 - \left(\frac{a}{w_0}\right)^2 - \left(\frac{T_0 x^2}{w_0}\right)\frac{w_0^3}{w_0^3} = 0.
\]
(2.22)

It is necessary to solve for \( a > 0 \), because the deep vertical velocity component must be positive.

The above boundary conditions represent subtropical gyres in that the surface temperature decreases on the surface from west to east. A subpolar gyre in which the temperature increases on the surface from west to east may be modeled by taking the origin of \( \varphi \) on the west coast and matching boundary conditions at \( \varphi = +1 \). In this case the temperature boundary condition and the third term of the cubic change sign.

Fig. 1 shows the graph of \( a/w_0 \) as a function of \( (T_0 x^2)/w_0^3 \) for those solutions for which \( a > 0 \). Note the two regions, one for subtropical and one for subpolar gyres. Note also that only for \( 0 < (T_0 x^2)/w_0^3 < 4/27 \) are there subpolar solutions. So, for small \( w_0 \), when the graph becomes inappropriate, there are only subtropical gyres, and an expansion gives

\[
  a \approx (T_0 x^2)^{\frac{1}{6}} \left\{ 1 + \frac{1}{3} \left( \frac{w_0^3}{T_0 x^2} \right)^{\frac{1}{3}} + \ldots \right\} = (T_0 x^2)^{\frac{1}{6}} + \frac{1}{3} w_0 + \ldots.
\]

There are several unusual features in Fig. 1. For subpolar gyres there are two solutions for each value of \( T_0 x^2/w_0^3 \). Furthermore, if \( w_0 \) is fixed and \( T_0 x^2 \) is increased, then \( a \) will either increase or decrease, depending on the "branch" of the solution. But if \( T_0 x^2 \) is fixed and \( w_0 \) is increased, then \( a \) goes either to
o or to \( w_0 \). This indeterminateness in the solutions for the subpolar regions needs to be more fully investigated. Finally, note that, if \( w_0 < 0 \), there must be subtropical gyres. This suggests that the transformation that obtains for the Atlantic, hence the slope of the thermocline, is forced by the Ekman layer.

A detailed investigation of the behavior of the cubic equation leads to the following conclusions for subtropical gyres.

1. As \( w_0 \) and \( T_0 \) increase, \( a \), and therefore \( a/\kappa \), increase monotonically.
2. As \( \kappa \) increases, \( a \) increases monotonically, and \( a/\kappa \) decreases monotonically.

Figs. 2 and 3 show the behavior of \( a \) for fixed \( w_0 \) as a function of \( T_0 \kappa^2 \), and as a function of \( w_0 \) for fixed \( T_0 \kappa^2 \). Both graphs are for subtropical gyres.

A simple calculation at a value of \( z_0 \), for which \( \exp[(a/\kappa)\eta] = 1/2 \), gives

\[
\frac{[(u/cos \theta) T_\theta + vT_\theta]}{w T_z} = \frac{(a - w_0)}{(a + w_0)}. \tag{2.23}
\]

The absolute value of this quantity is \( > 1 \) if \( w_0 < 0 \). Therefore, in contrast to the findings of Robinson and Welander, it appears that, in at least some similarity solutions, the horizontal convections are not numerically small compared with the vertical convections. Of course, if \( 2n - m = 0 \), the right-hand side of (2.23) would be zero.

Fig. 4 is a cross section of the exact thermocline solution. For purposes of the drawing, I have taken \( a/\kappa = \frac{1}{2}, \varphi = -1 \), and \( T_0 = 1 \) at \( 30^\circ \text{N} \).

It does not seem to be possible, for reasonable values of \( w_0, T_0 \), and \( \kappa \), for this exact solution to have a vertical scale comparable to that of the Atlantic or Pacific. The exponential decay is governed by \( (z a/\kappa) \sin \theta \), and the char-
characteristic scale of $z$ is 220 m. From fig. 3 in Stommel and Robinson we may select, as the value of $z$ for which $T = e^{-1}T_{surface}$, 1000 m and 600 m for the Atlantic and Pacific, respectively. Thus, for $\theta = 30^\circ$, $a/\kappa$ must be between $1/5$ and $1/10$ for a reasonable theory. This calls for an expansion for large $\kappa$, for which we make an error of order $\frac{1}{3} w_0/\kappa$. ($w_0$, known to be of order unity, cannot give the large variations needed.) In this limit, $(a/\kappa) = (T_0/\kappa)^{1/3}$. So, for $a/\kappa = 1/5$, $T_0/\kappa = 1/125$. Since we believe that $0.1 < \kappa < 10$, the remaining discrepancy can only be traced to the boundary conditions. Since the constant surface-temperature boundary conditions, discussed in § 3, can achieve a realistic depth for reasonable $\kappa$, this is a possible explanation. Robinson and Welander's assertions (i) that the flow at depth is strongly influenced by the surface boundary conditions, and (ii) that the similarity
solutions do not match the observed boundary conditions with sufficient accuracy are therefore corroborated.

In Fig. 4 the isothermal line from the surface at 60°N is deepest at 19°N. Fig. 2 in Robinson and Stommel shows that this point lies at 40°N. This discrepancy between the model and observation is independent of \( w_0, T_0, \) and \( \kappa \). The heavy line in my Fig. 4 gives the locus of maximum depths. From a simple calculation it is \( z_{\text{max}} = -\left(\sin \theta\right) \kappa / a \).

The disagreements with observation outlined above probably are traceable to the inaccurate boundary conditions. In Robinson and Stommel’s work, \( uT_x \) was neglected and the equations resulting from the similarity transformation were linearized. These approximations made it possible to use the observed variations with latitude as boundary conditions, and good agreement with observation was then obtained for the vertical scale and for the maximum depths of the isotherms. To what extent this good agreement is invalidated by the approximations is unknown.
Figure 4. Plot of eq. (2.16) showing isotherms and the locus of maximum isotherm depths for $a/\kappa = \frac{1}{2}$, $\varphi = -1$, $E = 0$.

3. Relationship of this Study to the Work of Stommel and Webster. Stommel and Webster thought they were solving approximate equations. However, in this section it is shown that their equations were exact for a particular similarity transformation—the one for constant surface temperature.

If $2n - m = 0$, (2.14) becomes

$$\kappa G^{iv} + (n + 1) G G'''' + n \eta G' G''' = 0. \quad (3.1)$$

Considering (2.11) and (2.12), let

$$\omega = - [(n + 1) G + n \eta G'], \quad (3.2)$$

$$\Theta = - G''. \quad (3.3)$$

Then (3.1) becomes

$$\kappa \Theta'' - \omega \Theta'. \quad (3.4)$$

Differentiating (3.2), we find

$$\omega'' - (3n + 1) \Theta - n \eta \Theta' = 0. \quad (3.5)$$
If \( n = -1/3 \), (3.5) becomes

\[
\omega'' + \frac{1}{3} \eta \Theta' = 0.
\]  

(3.6)

Since \( \omega \) and \( \eta \) in Stommel and Webster have signs that are opposite to those in this paper, and since Stommel and Webster absorbed the \( 1/3 \) in their nondimensionalization, (3.4) and (3.6) here are the same as (1.1) and (1.2) in Stommel and Webster. For other \( n \), (3.5) must be used instead of (3.6).

We may now explain why, in the case \( 2n - m = 0 \) of Stommel and Webster, small \( \kappa \) and large \( -\omega_0 \) resulted in a thin, deep thermocline, while \( n = 0 \), \( m = -1 \) has an exponential as the only possible solution. If \( 2n - m = 0 \), an increasing downward convection of heat can only be balanced by the upward convection of cold deep water. Where the vertical velocity vanishes, the heat must be transferred by conduction alone, in an internal boundary layer that must grow increasingly sharp as the downward convection of heat increases. However, if horizontal convections can carry away some of the heat from this level of no vertical motion, there need be no internal boundary layer. Thus there are important qualitative differences between the solutions of different similarity transformations.

This discussion may help to explain the very sharp thermocline in equatorial regions, where the \( n = -1/3 \), \( m = -2/3 \) transformation is appropriate because of the small surface temperature gradients. There also, the Ekman velocities, if they exist, would be large.

If we take into account in Stommel and Webster the facts that their nondimensionalization absorbed the \( 1/3 \) in (3.6) and that they nondimensionalized their temperature scale with \( 1^\circ \text{C} \), we find that their calculations may be transferred to our spherical geometry and nondimensionalization if the characteristic sizes of the fields are taken as: \( T, 1^\circ \text{C} \); \( (u, v) \), \( 0.36 \text{ cm sec}^{-1} \); \( w, 2.37 \times 10^{-5} \text{ cm sec}^{-1} \); \( z, 777.0 \text{ m} \); \( \kappa, 1.0 \text{ cm}^2 \text{ sec}^{-1} \).

From figs. 1–4 in Stommel and Webster it is seen that parameters can be adjusted within plausible limits to obtain a good approximation to the thermocline in a small range of latitude. However, in this transformation, the surface temperature is constant, which is certainly unrealistic on the global scale.

4. The Equatorial Equations. Here it is verified that Robinson’s equations for the equatorial region are the correct ones to connect to the thermocline solutions in which \( m = -2/3 \), i.e. those in which the temperature does not vanish or diverge near the equator. These solutions can match only those Ekman velocities that are proportional to \( \theta^{-2/3} \). The equations are derived by the simple (but perhaps debatable) assertion that, at some latitude, \( \theta \), characteristic of the equatorial regime, the orders of the fields of \( (u, v, w, p) \) and the vertical scale are equal to their orders in the equatorial regime itself. This is the same assumption that Robinson used for the \( w \) field alone. Some difficulties
with this description are then discussed, and it is shown how a remark by Charney leads to a possible resolution.

In Cartesian coordinates, which are sufficient for our present purposes,

\[-n\sigma \varepsilon u_{zz} - fu + p_x + n(uu_x + vu_y + wu_z) = 0,\]  
\[-n\sigma \varepsilon v_{zz} + fu + p_y + n(uv_x + vv_y + wv_z) = 0,\]  
\[-n\sigma \varepsilon w_{zz} - T + p_z + n(uw_x + vw_y + ww_z) = 0,\]  
\[u_x + v_y + w_z = 0,\]  
\[-\varepsilon T_{zz} + uT_x + vT_y + wT_z = 0.\]  

These have been nondimensionalized as in Robinson's paper. Therefore, \[\varepsilon = \frac{(2Qx)}{\alpha T_{8g}R}, \ n = \frac{(\alpha T_{8g})}{4 \Omega^2 R}, \ \sigma = \frac{v}{\kappa},\] where \(v\) is the (turbulent) eddy viscosity. However, instead of using Robinson's values for these parameters, we choose the values from § 2, and in addition we take \(v = 1.0 \text{ cm}^2 \text{ sec}^{-1}\). Then \(\varepsilon = 4.4 \times 10^{-14}, \ n = 0.41, \ \sigma = 1.0.\)

Assume that the amplitude in \(\varepsilon\) of these fields in the equatorial system are as follows: \(u, \ v^A; \ v, \ v^B; \ w, \ v^C; \ z, \ v^D; \ p, \ v^E; \ y, \ v^F.\)

The last relation recognizes that the distance at which the matching is to be made is a small fraction of the radius of the earth. From (2.12) and analogous equations for other fields, it is seen that, for \(m = -2/3\), near the equator, the fields vary with \(\theta\) as follows:

- \(u, \ \theta^{-4/3}\);  
- \(v, \ \theta^{-1/3}\);  
- \(w, \ \theta^{-2/3}\);  
- \(z, \ \theta^{2/3}\);  
- \(p, \ \theta^{2/3}\).  

Near the equator, \(\theta\) may be replaced by \(y\). Now \(y\) at the point of matching has the size \(e^F\). So \(w\), for example, has been increased by \((e^F)^{-2/3}\). This is the increase of \(w\) over its amplitude at midlatitudes, which, from Robinson's work, is known to be \(e^{2/3}\). So the amplitude of \(w\) at this point is \(e^{2/3} \cdot (e^F)^{-2/3}\). This is to be set equal to the amplitude of \(w\) in the equatorial regime, \(e^C\).

Equating exponents, \(C = 2/3 - 2/3 F.\) In this way we find the following five equations in six unknowns.

\[A = 1/3 - 4/3 F \quad D = 1/3 + 2/3 F\]  
\[B = 1/3 - 1/3 F \quad E = 1/3 + 2/3 F\]  
\[C = 2/3 - 2/3 F\]  

In deriving a sixth equation, we recall that, in the thermocline solution, \(p_x = f\dot{v}, \ \alpha T_{zz} = uT_x + vT_y + wT_z,\) and all the convective terms are considered to be of the same order. Now, since \(u\) is increasing with \(y\) faster than any other field, the first departures from the midlatitudinal thermocline regime are expected in the first momentum equation. By virtue of the above considerations, this must first occur when any one of the viscous or nonlinear terms equals either the pressure gradient or the coriolis force. Because of the above
considerations, any such equation is equivalent to any other. Similarly, all equations expressing equality between members of the same group, e.g. the viscous term and any inertial term, are redundant with (5.6), as may be verified. Arbitrarily, a balance is selected between $p_x$ and $wuz$, giving $E = A + C - D$. Solution of these six equations yields $A = 1/5$, $B = 3/10$, $C = 3/5$, $D = 2/5$, $E = 2/5$, $F = 1/10$. The sizes of the fields are then: $u$, 75.6 cm sec$^{-1}$; $v$, 3.58 cm sec$^{-1}$; $w$, $3.38 \times 10^{-4}$ cm sec$^{-1}$; $z$, 28.3 m; $y$, 2.63° latitude; $T$, 20°C.

Although the amplitude of $u$ seems to be in fair agreement with observation, recent direct measurements by Knauss (personal communication) in the Cromwell Current show that $v$ is too small by a factor of 5. Knauss found $v/u = 1/5$, but here $v/u = 1/20$. Furthermore, observations show that the thermocline between 3°N and 6°N is relatively flat, whereas the transformation used here insists, by (2.9), that the thermocline deepens by a factor of 1.6 between 3°N and 6°N.

To determine the scales at the equator, Robinson made six assertions. In one assertion, all three terms in the continuity equation are of equal size, and in another, $w$ matches in amplitude with the interior. Because of the continuity equation, the six assertions result in seven equations. Charney has noticed that $u_x \sim v_y$ is equivalent to matching the amplitude of $w$. The scaling may be derived by using either assumption. If we make $w$ diverge faster than $\theta^{-2/3}$ and if we impose all of Robinson’s assumptions except $u_x \sim v_y$, we find $u_x \ll v_y$. This also brings $v/u$ closer to observation. Since the Ekman layer diverges like $y^{-1}$ and $y^{-2}$, we do expect $w$ to diverge faster than $y^{-2/3}$. Of course it is not possible to separate the Ekman layer from the thermocline in equatorial regions, but we might still expect the wind-driven velocities to diverge more rapidly than $y^{-2/3}$. Thus Robinson’s equations, as he asserted, are for a purely thermal regime.

Acknowledgment. I am happy to acknowledge much help and encouragement from A. R. Robinson and Henry Stommel.

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