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Explosion-generated Waves in Water of Variable Depth

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ABSTRACT

This study shows that the theory of geometric optics adequately predicts the behavior of a dispersive wave system radiating from a large chemical explosion over sloping bottom topography. The phase and amplitude of an equivalent wave train (reconstructed from the observed data for uniform depth equal to that at the explosion site) are shown to be consistent with the phase and amplitude predicted by theory for the wave system produced by an initial parabolic crater in the water. Since the potential energy of the linearized source represents only 1.7% of the available thermal energy of the explosive charge, it is surmised that wave-making by explosions is a relatively inefficient process.

Introduction. Although an approximate solution for the problem of water-wave generation by an axi-symmetric source in water of finite depth has been available for some years (Kranzer and Keller 1959), its practical application to explosion problems has been handicapped by lack of precise information on realistic input conditions. Although the initial-value problem can be entered with either the impulse applied to the water surface or with some initial deformation of the surface, the most appropriate values for either case remain to be elucidated for a specific explosion with an arbitrary initial geometry. One way out of this dilemma is to observe the waves produced by an explosion in water of uniform depth at several distances from the source, and to either compare the results of these observations with the computed wave systems from sources intuitively derived, or utilize the observations to compute an equivalent linear source by an inversion of the theory (Van Dorn 1963).

This paper represents an attempt at such an analysis, utilizing wave observations obtained during the HYDRA II A high-explosive test program at San Clemente Island, California, in the summer of 1961. The test series consisted of thirteen 10,000-lb high-explosive charges of HBX-1 detonated at various
charge submergences in water 300 feet\(^2\) deep. Although test participation included wave records from eight of these 13 explosions, this paper deals with only those results obtained from Shot 11, for which exemplary records were obtained at all three recording stations. Because the water depths were not uniform over the wave-travel paths in these experiments, it was necessary to reconstruct equivalent wave trains as they would have existed in water of uniform depth before conclusions could be drawn regarding the nature of the source.

Experimental Plan. The layout of the HYDRA experimental test facility at San Clemente Island is shown in Fig. 1. The charge position (point o) was 15.7 feet below the surface in 300 feet of water, about 2600 feet from shore. This area was particularly advantageous for wave observations because of the very low ambient swell and wind-wave background in the lee of the island. Wave measurements were conducted at three observation stations (A, B, and C) along lines extending both inshore and offshore from the shot site. The bottom topography, which has an important influence on the nature of the wave development, is characterized by a gentle and fairly continuous slope from the shore to the 300-foot contour, beyond which the descent is steeper. Although it would have been most advantageous to have had all of the wave stations located along a single line normal to the bottom contours, the station positions were to some extent influenced by the operational requirements of other measurement programs.

Two types of instrumentation were used for wave measurement; motion pictures of spars anchored in shallow water were taken from a camera station on the shore, and subsurface (15-foot) pressure measurements were made in deep water beyond the shot site.

Fig. 2 is a schematic illustration of the photo-spar installations at Sts. B and C. A 15-foot length of shelby tubing 2 inches in diameter was painted with alternating white and red bands 1 foot in length, and mounted rigidly on the top of a standard Navy 28-inch net float. This assembly, with about 150 pounds of excess buoyancy, was submerged until the spar projected about half its length above the surface. The spars exhibited relatively little motion.

2. In 1963, as a part of its publication policy, the Sears Foundation adopted the September 1962 recommendation of the Intergovernmental Oceanographic Commission and the January 1963 recommendation of the National Academy of Sciences Committee on Oceanography: "that member countries of the to take the necessary steps to encourage, as far as possible, the use of the Metric System and Celsius Scale in their oceanographic publications. Dr. Van Dorn's paper was prepared in American units prior to the adoption of this policy, and, rather than insert metric equivalents throughout the text, they are given here for the linear inches and feet used in this paper.

\[
\begin{align*}
1/8'' &= 0.32 \text{ cm}; \\
1/4'' &= 0.63 \text{ cm}; \\
1/4' &= 3.2 \text{ cm}; \\
3'' &= 5.1 \text{ cm}; \\
10'' &= 25.4 \text{ cm}; \\
12'' &\text{ or } 1' &= 30.5 \text{ cm}; \\
28' &= 71\text{ cm}; \\
o.1 &= 3 \text{ cm}; \\
o.25' &= 7.6 \text{ cm}; \\
5' &= 1.5 \text{ m}; \\
5.6' - 140' &= 1.7 - 42.7 \text{ m}; \\
7.5' &= 2.3 \text{ m}; \\
15' &= 4.6 \text{ m}; \\
15.7' &= 4.8 \text{ m}; \\
28' &= 8.5 \text{ m}; \\
29.5' &= 9 \text{ m}; \\
50' &= 15.2 \text{ m}; \\
108' &= 33 \text{ m}; \\
205' &= 62.5 \text{ m}; \\
300' &= 91 \text{ m}; \\
558' &= 170 \text{ m}; \\
950' &= 290 \text{ m}; \\
1000' &= 305 \text{ m}; \\
1100' &= 335 \text{ m}; \\
1550' &= 472 \text{ m}; \\
2250' &= 686 \text{ m}; \\
2600' &= 792.5 \text{ m}. \\
\end{align*}
\]
During these tests and were never more than 10° from the vertical at any time. No attempt was made to correct the data for spar inclination.

To record the migration of the water surface up and down the striped spars, a Mitchell 35-mm Photo-triangulation Camera with a 10-inch lens was set up on the beach and oriented to include both spars in its field of view. Tests showed that this camera and lens, when loaded with Anscochrome color film, could resolve water migration of less than 0.1 foot at the maximum range used

Figure 1. Geometry of the Hydra II A test site, San Clemente Island, California.
in this operation (1100 feet). Operation of the camera was automatic following an initiating electric pulse from a remote control station.

Fiducial zero-time was established by the blurred frames as the earth shock wave reached the camera location. Time control was furnished by a stop watch photographed in the margin of each data frame.

Film read-out was performed on a Kodak Microfilm Reader. Each tenth frame was read, but, where necessary, every frame was read to positively define the troughs and crests. The resolution of the read-out was again about 0.1 foot. A typical data frame is shown in Fig. 3.

As the waves radiated to deeper water, they passed out of range of the camera. Recording at St. A was therefore accomplished by monitoring subsurface pressure. Two Statham Absolute-pressure Transducers (0–30 and 0–40 psia) were encased and protected from shock pressure as shown in Fig. 4. These transducers were mounted on a taut-wire mooring as shown in Fig. 5. The transducers were excited by a 1.1-kc carrier, and the modulated AC signals were amplified, demodulated, and recorded on an oscillograph aboard a barge nearby. The barge was evacuated during the actual shots, so the system was also programmed to operate autonomously upon receipt of a starting signal. Fiducial zero was established by the arrival of the shock wave at the transducers; an Uysse-Nardin Chronometer provided time control.

Wave Measurements. The wave measurements obtained at Sts. A, B, and C are plotted in Fig. 6. For Sts. B and C, the plots consist of surface elevations read directly from camera films, with no corrections involved. The wave record for St. A (40 psi transducer) is a free-hand curve drawn through crests and troughs of the recorded oscillogram, since the original record was too long to be reproduced directly. These data, together with their respective arrival times, provide sufficient information for wave analysis. The amplitude data were multiplied by a calibration factor obtained before and after the shot from the amplitude of a known electrical step-signal. Since the transducers were
submerged beneath the surface, it was necessary—as shown later—to apply an orbital-pressure correction, which depends upon wave frequency and also upon the local tide height relative to the known tide height at the time of transducer installation.

Comparison of the wave records obtained for different depths of water reveals the profound effect of depth in governing the history of the wave train. At St. A the wave train was still quite compressed and the second or third wave was the highest. At St. C the train was greatly expanded in time, and the eleventh wave was the highest. The effect of shoaling water is also reflected in the amplitude of the highest wave, which was about 30% higher than it would have been in water of constant depth equal to that at the shot site.

**Procedure.** In order to compare these results with predictions for impulsively generated waves in water of uniform depth, it was first necessary to reconstruct from these data the wave trains that would have been produced from the same shots in water of uniform depth, $h = 300$ feet, the depth at
the charge position. The procedure consisted of two steps. (i) The wave-arrival times were calculated for the existing stations in order to determine whether or not the phase propagation speed was everywhere linear. The phase travel times are independent of the nature of the source and are affected only by the depth of burst or the shot-yield to the extent that the source is finite in time. (ii) The amplitude of the wave envelope as a function of frequency was calculated later to deduce the nature of the equivalent source. The amplitude spectrum of a wave train is independent of the phases and can be computed separately. Interest in the phases stems principally from the desirability of predicting which waves of the reconstructed train will be highest, and when they will have passed.

The procedure followed in making the above calculations is the application of geometric optics in a linear continuum; the procedure is set down here in some detail, since it seems not to have been previously attempted in problems of this nature. Probably the reason for this omission is that solutions for the

Figure 4. Pressure transducer in shock-proof case. Capillary tubing protects transducer from initial high-peak shock pressure.

Figure 5. Method of installation of pressure transducer in deep water (St. A).
The equations describing the field of motion for arbitrary topography cannot be obtained in closed form, but must be found by numerical approximation. In particular, these equations are functions of the local water depth, $h$, and wave number, $k$, both of which are related to the wave frequency, $\omega$, by the Hamiltonian equation,

$$\omega^2 = gk \tanh kh,$$

where $g$ is gravitational acceleration; this equation must be solved for $k$ wherever the motion is to be specified. In a two-dimensional Cartesian system, two orthogonal components of the wave number will ordinarily have to be considered. This difficulty will be avoided here by selecting a propagation path such that the normal coordinate everywhere vanishes; i.e. along a ray.

**Calculation of the Phase Arrivals.** In water of uniform depth, $h$, the (constant) phases, $\theta$, of a dispersive wave system will propagate along those curves in the space-time plane that are solutions of the differential
equation $dr/dt = c(k)$, where $c$ is the local phase velocity and $r$ and $t$ are the space and time coordinates, respectively. These solutions have the form $\theta(r, t) = \text{constant}$. In the case of a centered (radially symmetric) system resulting from an initial deformation of the water surface given by $y = y_0 f(r)$, $t = 0$, these solutions become (Van Dorn 1961)

$$\cos (\omega t - kr) = m = \text{constant}. \quad (2)$$

If attention is focused on, say, the positive maxima (crests) of the system observed to be passing a station at a distance $r = r'$ from the origin, then $m = 1$, and the times of arrival of successive crests in their order of occurrence, $n$, will be

$$t_n = \frac{2\pi (n-1) + kr'}{\omega_n}, \quad n = 1, 2, \ldots, \quad (3)$$

where $k_n$ and $\omega_n$ are now the local values of the wave number and frequency of the $n$th crest at time $t_n$. But $t_n = r'/V(k_n)$ is also the time of arrival of $k_n$ propagating at constant group velocity

$$V_n = V(k_n, h) = \frac{1}{2} \left[ \frac{g}{k_n} \tanh k_n h \right]^{\frac{1}{2}} \left[ 1 + \frac{2 k_n h}{\sinh 2 k_n h} \right] \quad (4)$$

over the distance $r'$. Thus the solution of (3) for a given $n$ is obtained by finding that value of $k_n$ which satisfies the identity

$$\frac{2\pi (n-1) + kr'}{\omega_n} = V_n. \quad (5)$$

The above reasoning can be generalized to compute phase-arrival times in water of variable depth along a ray path, $s$, by summing the increments, $\nabla t_n$, required for the phases to propagate in depths, $h(s)$—assumed constant—over small, equal path increments, $\nabla s$, which suitably approximate the bottom profile.

Before applying this procedure to the wave systems considered here, the conditions required for the applicability of linear theory and geometric optics were examined; namely, that, if $\eta[\omega(k, h)]$ is the wave amplitude,

$$\eta k/(kh)^3 \ll 1, \quad (6.a)$$

and

$$kh \gg dh/ds, \quad (6.b)$$

at all frequencies considered. The limiting situation arises in both cases when $\eta$ is large and $k$ and $h$ are small, i.e. near the front of the wave train at the shallowest station. From the record for St. C, the time interval between the first
two crests was about 13 seconds and $\eta = 0.25$ feet. From (1), $kh = 0.22$, and (6.1) is satisfied. But the bottom slope $dh/ds = 0.1$; thus the wave amplitudes computed in the next section may be somewhat in error at low frequencies. As will be shown, however, the higher waves at this station are associated with a value of $kh$ of order 1.0, and the method should apply without reservation.

The arrival times for the first ten crests at all three stations were determined first by dividing the bottom profiles into 50-foot increments of distance arranged in order of decreasing depth and then by programming eqs. (3) and (5) on the University of California’s CDC 1604 Computer at La Jolla. The path chosen, the direct route in all three cases, coincided very nearly with the slope gradient for Sts. A and B. The direct route to St. C deviated about 30° from the slope gradient, but the water depth was everywhere great enough so that the least-time path can be shown to differ insignificantly from this route for any frequency considered here. Fig. 6 also shows the computed crest arrivals (small arrows), which agree quite closely with those observed. It is concluded that the linear theory adequately predicts phase arrivals in water of variable depth.

**Wave-amplitude changes in water of variable depth.** Within the regime where the laws of geometric optics apply and where the phase speed is not a function of amplitude, energy is considered to be conserved between adjacent rays and also between adjacent frequencies. That is to say, if one forgets the phases altogether and considers the field of wave energy to be defined as a continuum of “patches” (any such patch being bounded in the direction of propagation by adjacent frequencies and by adjacent rays parallel to this direction), then, as this patch propagates through the continuum at its characteristic group velocity, the local wave amplitude will vary as the square root of the energy, i.e. inversely as the square root of the patch area. In the special case where the field of motion is centered and where the origin of the rays coincides with the center of the disturbance, conservation of energy in a wave patch characterized by frequency $\omega$ is given by the expression

$$E(\omega)/\pi \rho g = \eta_1^2 Z_1^2 =$$

$$\eta_1^2[k_0(\omega) \int_0^{S_1} ds/k(s,\omega)] [-V_1(\omega) \int_0^{S_1} ds \partial (1/2 V^2)/\partial k] = \text{constant}, \quad (7)$$

where, in addition to the symbols already defined, $E(\omega)$ is energy per unit frequency, $\rho$ is density, and the subscripts $\circ$ and $1$ refer to the origin and to any point of observation, respectively. The quantity, $Z_1$, defined by eq. (7), might be termed the Wave Intensity Factor, and is directly related to the

3. The sum is independent of the order of summation, and the trial and error solution converges rapidly if the first trial in each case was that value of $k$ determined for the preceding increment.
4. A derivation of this relation is given in the Appendix (p. 139).
wave-patch dimensions. In the case of an explosion, of course, the wave-amplitude spectrum is unknown at the origin, but it is known in principle at all observation stations; hence the applicability of theory to the present situation can be tested by first evaluating the intensity factor $Z$ at each station for a number of frequencies and then comparing the product $(\eta Z)$ at a given frequency between stations.

This somewhat elaborate computational procedure was accomplished in the following manner. First, as in the preceding section, the direct routes to all three stations were divided into $n$ small, equal (50-foot) intervals, $\Delta s$; and the mean depth, $h(s)$, was tabulated for each interval. Equation (7) was then rewritten as

$$Z^2(\omega) = k_0 V_1 \Delta s^2 \sum_{k=0}^{n-1} (1/k) \sum_{\omega} \beta,$$

where

$$\beta = -\partial (1/2 V^2)/\partial k =$$

$$(8/g \tanh kh) \left[ \frac{1}{(1 + 2 kh/\sinh 2 kh)^3} \right] ;$$

and again, $k(\omega)$ is computed by trial and error from eq. (1).

The envelope amplitudes, $\eta(\omega)$, were obtained from the observed wave trains, $\eta(t)$, at the three stations, as follows. Any frequency, $\omega$, propagates in water of depth $h$ at group velocity

$$V = \partial \omega / \partial k = \partial (gkh \tanh kh)^{1/2} / \partial k ;$$

and therefore, the time required for a given frequency to propagate to a point at a distance $s$, from the origin in water of variable depth, $h(s)$, will be

$$t = \int_0^s ds/V = 2\Delta s \sum_{k=0}^{n-1} \left[ \left( \frac{g}{k} \tanh kh \right)^{1/2} \frac{2 kh}{\sinh 2 kh} \right]^{-1},$$

where the $h = h(s)$ are the same depths previously tabulated. Smooth envelopes were drawn around the plotted wave records in Fig. 6; from these envelopes the amplitudes, $\eta(t)$, were scaled directly at times determined from eq. (11) corresponding to 12 selected frequencies spanning the appropriate time domain in frequency-wave-number space. These spectral amplitudes have no permanent relation to the actual waves within the envelope, but each represents the amplitude any wave would have if it arrived at the recording station at the same time as any of the 12 frequencies. This statement obtains because frequencies and individual waves propagate at different speeds and, in general, along different paths.
TABLE I. SPECTRAL AMPLITUDES AND INTENSITY CORRECTIONS.

<table>
<thead>
<tr>
<th>Frequency $\omega$ (sec)$^{-1}$</th>
<th>Station</th>
<th>Arrival Time</th>
<th>Double Spectral Amplitude $2 \eta(\omega)$ ft</th>
<th>Intensity Factor $Z(\omega) \times 10^{-3}$</th>
<th>Products $2 \eta Z \times 10^{-3}$</th>
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<td>.183</td>
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<td>2.21</td>
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* First minimum of wave envelope taken to be point of zero-wave amplitude.

The spectral amplitudes, $\eta(\omega)$, at St. A were then corrected for orbital-pressure attenuation by the factor $[\cosh k(z + h)/\cosh kh]$, where $z$ is the transducer depth at the local tide stage and $k$ is the wave number corresponding to each frequency. The corresponding values of $Z(\omega)$ were then computed from (8).

Table 1 lists the frequencies, arrival times, amplitudes, $Z$ values, and products, $\eta Z$, for all three stations. If the influence of variable water depth has
been correctly taken into account in this analysis, the \( \eta Z \) values should agree between stations at any particular frequency. Considering the marked fluctuations in the amplitudes of individual waves and the difficulty of drawing the most representative smooth curve through their extrema, the \( \eta Z \) values agree fairly well over most of the frequency range; there appears to be a general tendency, particularly at midfrequencies, for the values to be highest in shallow water and lowest in deep water. This effect is probably due to reflections from the shoreline, which are ignored in this analysis. Some recent unpublished qualitative observations by the author in connection with wave-tank experiments support this view. Monochromatic wave trains directed against beach slopes typical of the HYDRA environment indicate that reflections manifest themselves as an increase in the wave amplitude observed offshore, but without change of phase. The observed enhancement appeared to be somewhat frequency-sensitive and was of the order of 30°/o in some cases. In the present situation the wave trains are both centered and dispersive; thus any influence of reflection would be most important at the station nearest the shore and would die out rapidly in deeper water. Pending more precise results, then, the tentative conclusion is that the application of geometric optics to shoaling water waves is also confirmed.

**Reconstruction of the Amplitude Spectrum of the Wave Train in Water of Constant Depth.** The linear theory for waves produced by an explosion in water of finite depth (Kranzer and Keller 1959) gives the phase and amplitude spectrum of the wave train as a function of time and distance, starting with the Hankel transform of the initial impulse or surface elevation at time zero. To deduce the nature of the source, it is therefore necessary to reconstruct the amplitude spectrum of the observed wave train as it would have been recorded at (any) range, \( R \), in water of uniform depth. For uniform depth, eq. (8) becomes

\[
Z_0^2(\omega) = V_o R^3 \beta_0.
\]  

(12)

If one takes the RMS value, \( (\Sigma \eta^2 Z^2/3)^{1/2} \), of the products \( \eta Z \) listed in Table 1 at each frequency as the value that is least susceptible to random error, the spectral amplitudes will be given by

\[
\eta(\omega)_R = (\Sigma \eta^2 Z^2/3)^{1/2}/Z_0;
\]

(13)

and the corresponding arrival times are obtained from (11); for constant depth, eq. (11) reduces to

\[
t(\omega) = R/V_0(\omega).
\]

(14)
TABLE II. Spectral Amplitudes at Twelve Frequencies for Shot 11, Corrected to Uniform Water Depth $h = 300$ feet at Range $R = 1000$ feet.

<table>
<thead>
<tr>
<th>Frequency $\omega$ (sec$^{-1}$)</th>
<th>Arrival Time $t$ (sec)</th>
<th>$(\Sigma \eta^2 Z^2/3)^{1/2} \times 10^{-3}$</th>
<th>$Z_0 \times 10^{-3}$</th>
<th>Spectral Amplitude $\eta(\omega)$ feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>.183</td>
<td>11.9</td>
<td>0.35</td>
<td>1.35</td>
<td>0.13</td>
</tr>
<tr>
<td>.343</td>
<td>17.8</td>
<td>1.18</td>
<td>1.79</td>
<td>0.33</td>
</tr>
<tr>
<td>.433</td>
<td>23.9</td>
<td>1.51</td>
<td>1.93</td>
<td>0.39</td>
</tr>
<tr>
<td>.507</td>
<td>29.8</td>
<td>1.87</td>
<td>1.64</td>
<td>0.57</td>
</tr>
<tr>
<td>.585</td>
<td>35.6</td>
<td>2.52</td>
<td>1.42</td>
<td>0.89</td>
</tr>
<tr>
<td>.673</td>
<td>41.8</td>
<td>2.92</td>
<td>1.25</td>
<td>1.17</td>
</tr>
<tr>
<td>.765</td>
<td>47.6</td>
<td>3.25</td>
<td>1.15</td>
<td>1.41</td>
</tr>
<tr>
<td>.863</td>
<td>53.6</td>
<td>3.43</td>
<td>1.08</td>
<td>1.61</td>
</tr>
<tr>
<td>.956</td>
<td>59.6</td>
<td>3.03</td>
<td>1.03</td>
<td>1.47</td>
</tr>
<tr>
<td>1.050</td>
<td>65.3</td>
<td>2.25</td>
<td>0.98</td>
<td>1.15</td>
</tr>
<tr>
<td>1.143</td>
<td>71.3</td>
<td>1.42</td>
<td>0.94</td>
<td>0.75</td>
</tr>
<tr>
<td>1.240</td>
<td>77.2</td>
<td>0.00</td>
<td>0.90</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table II gives numerical results for the case $h = 300$ feet, $R = 1000$ feet; the reconstructed envelope of the wave train is shown in Fig. 7, where wave train circumscribed by the envelope was calculated from the theory, as described below.

Figure 7. Reconstructed wave envelope (spectrum) for an explosion in 300 feet of water, as observed at a range of 1000 feet. The solid line is the wave train predicted by linear theory for an initial parabolic crater in the water.
THE EQUIVALENT SOURCE. Although it is possible in principle to reconstruct a model for an equivalent source from the wave-amplitude spectrum observed at a distance in water of uniform depth (Van Dorn, 1963), the inherent nature of the solution requires an extremely well-dispersed spectrum, i.e. a very remote source, where the spectral amplitude changes only slightly over one wave length. Moreover, the distribution of source strength with radius is extremely sensitive to the higher derivatives of the spectrum, such that this method is proscribed by the approximate nature by which the spectrum for uniform depth was derived. It appears simpler and more accurate in the present case to make a guess from the general appearance of the reconstructed spectrum as to the nature of the source, compute the spectrum predicted by theory, and make small adjustments until satisfactory agreement is obtained with the reconstructed spectrum.

The node in the reconstructed spectrum at about $t = 77$ sec. suggests that such a spectrum would be predicted by an initial parabolic crater in the water, since it is known that such craters are produced by explosions on land over a considerable range of charge submergence. For the case of an initial parabolic depression of the form $y = y_0[1 - (r/r_0)^2]$, where $y_0$ and $r_0$ are the depth and radius of the depression, respectively, the linear theory (Kranzer and Keller 1959) predicts a wave train having an amplitude-time history at a distance $r$ in water of depth $h$ described by

$$\eta(t) = (2y_0/rk^2h) \left[ \frac{kV(k)}{-dV/dk} \right]^{\frac{1}{2}} J_2(r_0 k) \cos \left[ kr \left( \frac{2}{1 + 2kh/sinh 2kh} - 1 \right) \right]. \quad (15)$$

This solution was normalized to the reconstructed train as follows. Let $t_0 = \sqrt{gh}$ and $t'$ be the respective times of arrival of the front and first node of the envelope of the wave train. Then the wave number, $k'$, propagating with group-velocity, $V'$, which arrives at station $r$ at time $t'$, will be the root of the relation

$$V' = t_0/t' = \frac{1}{3} \left( \frac{g}{k'} \tanh k'h \right)^{\frac{1}{2}} \left[ 1 + \frac{2k'h}{\sinh 2k'h} \right]. \quad (16)$$

Since the first node corresponds to the first zero of the Bessel function, $J_2(r_0 k')$, which occurs when its argument has the value 5.14, the radius of the depression is determined as $r_0 = 5.14/k'$. In the present case the appropriate values are (for $t' = 77$ sec.) $k' = 0.048$, $r_0 = 108$ feet, which is in good agreement with the radius of the explosion plume determined independently from photographic data. With $r_0$ determined, eq. (15) was programmed on the computer and the value of $y_0 = 29.5$ feet was found (after a trial or two) to give the best fit to the reconstructed wave train. Both the reconstructed envelope and the computed wave systems are shown in Fig. 7. The amplitude is generally satisfactory, although the theory predicts a small negative bore at
the front of the disturbance due to a net removal of water by the explosion. By analogy with craters on land, however, it is known that the material thrown out of the crater rapidly descends to form a raised lip around the crater zone. A more accurate initial model taking this phenomenon into account would not show a bore.

Discussion

Limitations of the Linear Solution. The procedure followed in this analysis was to force a linear solution to a given set of experimental data. Evidently, the solution in no way describes the hydrodynamic phenomena leading to the wave train, nor does it start with a realistic in-put; the normalized crater depth, \( y_0 = 29.5 \) feet is manifestly ridiculous for an explosion of 10,000-lbs of HBX-1 at 15.7 feet, beneath the surface. The fact that the solution works so well is an expression of the fact that the important features of the early stages of the ensuing wave train are remarkably insensitive to the details of the source. That is to say, the train of waves produced by dropping a round ball or a square brick into water is virtually identical except for the energy at high frequencies that controls the way the train dies away. Since the coda is apt to be relatively unimportant from the standpoint of wave effects, it can usually be ignored.

Wave-making Efficiency. The linear solution for wave generation assumes that all the potential energy of the normalized crater is converted into wave motion. The potential energy of a crater given by \( y = y_0[1 - (r/r_0)^2] \) is easily shown to be \( E_p = -\rho g y_0^2 r_0^2 / 6 \). In the present case \( (y_0 = 29.5 \) feet; \( r_0 = 108 \) feet), \( E_p = 3.4 \times 10^7 \) ft-lbs = 1.1 \times 10^7 \) cal. Taking the available thermal energy of HBX-1 as 1445 cal/g, the wave-making efficiency of Shot 11 was about 1.7\%. Now, the explosions in this test series occurred over a range of charge submergences from 5.6 to 140 feet, with no wave systems differing in amplitude by more than 50\% from those discussed here. Thus, such events are markedly less efficient as wave producers than, say, a piston driven by expanding gas from an enclosed explosion, which can convert something like 50\% of the available thermal energy into waves.

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APPENDIX

Derivation of the Wave-Intensity Equation. In a Cartesian coordinate system where the local water depth, \( h(x,y) \), is a known function of position, consider a pair of adjacent rays, \( s_1, s_2 \), radiating from the origin of a disturbance at point \( o \) (Fig. 8). Let \( \omega \) and \( \omega + d\omega \) be in the instantaneous positions of two adjacent frequencies propagating from \( o \) at their respective group velocities, \( V(x,y,k) \). If \( E(\omega) \) is the energy per unit frequency of the source disturbance, and energy is conserved between rays, then the energy in the dark patch (Fig. 8) will always be \( E(\omega)d\omega \delta\theta/2\pi \). This energy is also equal to the local wave energy within the patch:

\[
E(\omega)d\omega \delta\theta/2\pi = \frac{1}{2} g \eta^2 d\sigma d\xi,
\]

(A1)

where \( \eta, \theta, \) and \( g \) are as previously defined.

Now, if \( K = K(x,y,k) = [g k \tanh kh(x,y)]^{\frac{1}{2}} \) is the Hamiltonian of the wave equation, then, by the usual concept of group velocity,

\[
V(x,y,k) = \partial K/\partial k.
\]

(A2)

The wave number \( k \) can be thought of as the magnitude of a vector propagating at group velocity \( V(k) \) in \( x, y \) space. Let \( l \) and \( m \) be the magnitudes of the components of this vector in the \( x \) and \( y \) directions such that \( k^2 = l^2 + m^2 \).

5. This development was kindly provided by Professor G. E. Backus, Institute of Geophysics and Planetary Physics, University of California.
It follows from (A2) and the properties of the Hamiltonian that

\[ \frac{dx}{dt} = V \frac{l}{k}, \quad \frac{dy}{dt} = V \frac{m}{k}, \quad \frac{dl}{dt} = -\frac{\partial K}{\partial x}, \quad \frac{dm}{dt} = -\frac{\partial K}{\partial y}. \]  

(A3)

Now, along a ray, \( \omega = [gk \tanh kh(x, y)]^{1/2} \) = constant, so that \( k(x, y, \omega) \) is everywhere known. Define a parameter, \( \varepsilon \), by \( dt = k d\varepsilon /V \). Hence \( d/dt = (V/k) d/d\varepsilon \), and thus eqs. (A3), can be expressed in terms of an 'elastic' time base linked to real time through \( V \) and \( k \):

\[ \frac{dx}{d\varepsilon} = l, \quad \frac{dy}{d\varepsilon} = m \]

(A4)

\[ \frac{d^2x}{d\varepsilon^2} = \frac{\partial}{\partial x} \left[ \frac{k^2}{2} \right], \quad \frac{d^2y}{d\varepsilon^2} = \frac{\partial}{\partial y} \left[ \frac{k^2}{2} \right]. \]

Eqs. (A4) are the analogue equations of motion of a unit particle in the field of a potential \( k \), where \( l \) and \( m \) are the analogue velocities in the \( x \) and \( y \) directions, and \( \varepsilon \) is analogue time. Similar equations have been derived by Eckart (1950) in discussing the ray-particle analogy.

We wish now to apply these results to the conservation of energy within the patch given by eq. (A1). In the special case where the path \( s \) can be chosen normal to the bottom contours, \( \partial K/\partial y = y = 0; K[x, k(x)] = \omega \); and \( dx/dt = V[x, k(x, \omega)] \). This situation is shown in Fig. 9. Let \( x_1 \) and \( x_1 + dx_1 \) be the positions of \( \omega \) and \( \omega + d\omega \) at some time following the disturbance at \( 0 \), and let \( x_2 \) and \( x_2 + dx_2 \) be their positions at a later time. The elapsed time, \( t_1 \), for \( \omega \) to propagate along \( s \) from \( x_1 \) to \( x_2 \) will be

\[ t_1 = \int_{x_1}^{x_2} \frac{dx}{V}, \]

(A5)

and the time, \( t_2 \), for \( \omega + d\omega \) to propagate from \( x_1 + dx_1 \) to \( x_2 + dx_2 \) will be, approximately,

\[ t_2 = t_1 + \frac{dx_2}{V_2} - \frac{dx_1}{V_1} + d\omega \frac{dt}{\partial \omega}, \]

(A6)

where \( V_1 \) and \( V_2 \) are the group velocities near \( x_1 \) and \( x_2 \). But the last term in (A6) can be written

\[ d\omega \frac{dt}{\partial \omega} = -d\omega \int_{x_1}^{x_2} \frac{\partial V}{V^2} \left[ \frac{\partial k}{\partial \omega} \right] dx = d\omega \int_{x_1}^{x_2} \frac{\partial}{\partial k} \left[ \frac{1}{2 V^2} \right] dx. \]

(A7)
Now, taking $x_1$ at the source ($dx_1 = 0$), setting $t_1 = t_2$, and combining (A6) and (A7), one obtains

$$\frac{dx_2}{d\omega} = V_2\int \frac{\partial}{\partial k} \left[ \frac{1}{2V^2} \right] dx,$$

(A8)

which gives the change in patch length with distance traveled in the $x$ direction.

The $y$ axis is parallel to the contours and therefore $m = \text{constant}$ in (A4). Hence, along the path $s_1$,

$$y_2 = y_1 + m \int_{\epsilon_1}^{\epsilon_2} d\epsilon \cdot \frac{dy_1 + m \int_{x_1}^{x_2} \frac{dx}{k}}{x_2}.$$

(A9)

But if $x_1$ is at the origin, $y_1 = 0$, and $m = k_0 \delta \theta$, where $k_0$ is the wave number at the origin. Eq. (A2) then becomes

$$y_2 = \delta \theta k_0 \int_{0}^{x_2} \frac{dx}{k}.$$

(A10)

Now, from Fig. 8, $d\xi = dx_2$, $d\sigma = y_2$, and therefore (A1) can, with rearrangement of terms, be written

$$\frac{E(\omega)}{\pi \rho g} = \eta^2 \left[ k_0 \int_{0}^{x_2} \frac{dx}{k} \right] \left[ -V_2 \int_{0}^{x_2} \frac{\partial}{\partial k} \left[ \frac{1}{2V^2} \right] dx \right],$$

(A11)

which is the same as eq. (7).