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The Instability of a Thermal Ocean Circulation

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ABSTRACT

A thermally driven meridional vortex in a diffusive ocean of finite depth is shown to be unstable. The dispersion relation at marginal stability is obtained approximately.

1. Introduction. As an explanation of the thermohaline features of the general ocean circulation and of the structure of the mean oceanic thermocline, the model of a rotating spherical shell of fluid forced to convect by a prescribed distribution of surface density (temperature) has been proposed and explored (Robinson and Stommel, 1959; Robinson, 1960). The purpose of this note is to initiate a study of the stability of the resultant flow and temperature distributions. An exact stability analysis of the thermal boundary layer and deep ocean flows obtained in the above-mentioned studies would involve complex profiles in a mixed shear-flow and thermal instability problem (in a system of variable rotation) and is obviously precluded. Instead, our approach will be to extract three aspects of the thermocline circulation which have been deduced to be of primary importance, to pose the simplest problem of steady convection in which these three effects dominate, and to explore the stability of this simpler system. We shall focus attention upon the importance of 1) the turbulent diffusion of heat in the vertical (although momentum diffusion be negligible), 2) the variation of the Coriolis parameter with latitude (β-plane geostrophy at very small Rossby number), 3) the longitudinal density gradient for the support of a thermocline (even though the insolation serves to produce a surface gradient which is primarily latitudinal). The basic flow consists of a meridional and vertical circulation. Thus our problem differs from the well-known baroclinic instability of a narrow zonal current of moderate Rossby number in a nondiffusing atmosphere.

2. Formulation. The equations of conservation of momentum, mass and heat relevant to the model proposed in the preceding section are, under the Boussinesq and β-plane approximation,
The notation is standard; primes indicate dimensionful quantities, the subscripts partial differentiation. The spatial variables \((x,y,z)\) are positive eastward, northward and upward; \(\alpha, \alpha'\) are respectively the (turbulent) thermometric conductivity and the thermal expansion coefficient. Geostrophy has been assumed; a validity criterion is obtained in § 5. The vorticity equations derivable from (1) – (4) are

\[
\begin{align*}
\beta v' - (f_0 + \beta y) w'z &= 0, \\
v'_z - \frac{\alpha g}{(f_0 + \beta y)} T'_x &= 0, \\
u'_z + \frac{\alpha g}{(f_0 + \beta y)} T'_y &= 0.
\end{align*}
\] (6a) (6b) (6c)

Insertion of \(v'\) from (6a) into (6b) yields

\[
\omega'_{yz} = \frac{\alpha \beta g}{(f_0 + \beta y)^2} T'_x,
\] (7)

which exhibits directly the rôle of the \(\beta\)-effect and of the longitudinal temperature gradient in the support of the convective circulation.

The ocean is taken to be of finite depth, and at the surface and bottom the temperature field is to be prescribed with the same constant longitudinal gradient, although the mean temperature of the surface will be higher than that of the bottom. Thus the boundary conditions are

\[
\begin{align*}
T'(x,y,z) &= (\Delta T)_H \frac{x}{L}, \\
T'(x,y,h) &= (\Delta T)_H \frac{x}{L} + (\Delta T)_T \nu, \\
\omega'(x,y,z) &= \omega'(x,y,h) = 0,
\end{align*}
\] (8a) (8b)
where $h$ is the height of the surface (assumed undistorted), $L$ the longitudinal scale length, and $(\Delta T)_H$, $(\Delta T)_V$ the characteristic horizontal and vertical temperature differences.

Note that we introduce independently a longitudinal gradient of density (as the driving mechanism for the fluid motion) and an over-all stable stratification. No boundary conditions are prescribed on vertical surfaces since we anticipate that higher order dynamics will in general be necessary to satisfy such conditions (boundary layers or coastal currents). In this sense, the open ocean we are considering is the "interior region" of a boundary layer problem and, as such, net transports of physical properties in and out of the region considered are allowed. The finite horizontal extent of the ocean, however, is to be felt by the introduction of characteristic scale lengths $L$, $l$ in the longitudinal and latitudinal direction respectively. Furthermore, the presence of meridional barriers (continents) enters tacitly through the imposed longitudinal temperature gradients (vide Robinson and Stommel, op. cit. § 3).

The linear vorticity eq. (6b), which is to be maintained for both the basic state and perturbation fields, provides an appropriate basis for nondimensionalization of the meridional component of the velocity field in terms of the thermal driving mechanism. The remaining two components are nondimensionalized after being weighted by the independent scale length ratios so as to provide a continuity equation free of constants; the vertical conduction time is chosen as a characteristic interval. We write each nondimensional field as the sum of basic and perturbation contributions, i.e.,

$$u' = \frac{V_o L}{l} (U + u), \quad v' = V_o (V + v),$$

$$w' = \frac{V_o h}{l} (W + w), \quad T' = (\Delta T)_H (T + T),$$

with $V_o = \frac{\alpha g (\Delta T)_H h}{f_o L}$. The independent variables are expressed as

$$x = L \xi, \quad y = l \eta, \quad z = h \zeta, \quad t = \frac{h^2}{\alpha} \tau,$$

and a nondimensional Coriolis parameter is defined by $f_o = \beta y = f_o (f \eta)$, whence

$$f(\eta) = 1 + \beta^* \eta, \quad \beta^* = \frac{\beta l}{f_o}.$$  

Within the restrictions imposed, the problem contains three parameters; for the remaining two we choose
The equations and boundary conditions for the basic state are

\[ \begin{align*}
\beta^* v - f(\eta) W_z &= 0, \quad (13) \\
f V_z - T_z &= 0, \quad (14) \\
f U_z + T_\eta &= 0, \quad (15) \\
U_z + V_\eta + W_z &= 0, \quad (16) \\
-T_{\xi\xi} + \Lambda (UT_\xi + VT_\eta + WT_z) &= 0, \quad (17)
\end{align*} \]

and

\[ \begin{align*}
T(\xi, \eta, 0) &= \xi, \\
T(\xi, \eta, 1) &= \xi + \theta, \\
W(\xi, \eta, 0) &= W(\xi, \eta, 1) = 0.
\end{align*} \]

The linear equations are of course unaltered for the perturbations, e. g.,

\[ \begin{align*}
\beta^* v - f w_z &= 0, \quad (19) \\
f v_z - T_z &= 0, \quad (20) \\
u_z + v_\eta + w_z &= 0, \quad (21)
\end{align*} \]

but the heat equation becomes

\[ T_\tau - T_{\xi\xi} + \Lambda (UT_\xi + T_\xi u + VT_\eta + T_\eta v + WT_z + T_z w) = 0, \quad (22) \]

and the boundary conditions

\[ \begin{align*}
T(\xi, \eta, 0) &= T(\xi, \eta, 1) = w(\xi, \eta, 0) = w(\xi, \eta, 1) = 0.
\end{align*} \]

Note that of the three parameters, two \((\Lambda, \beta^*)\) enter in the equations and one \((\theta)\) enters through a boundary condition.

From (17) and (22) \(\Lambda\) is seen to be a measure of the relative importance of convective to conductive processes in the fluid. Therefore, to be consistent with the discussion of the previous section (that these processes are both of importance in an oceanic thermocline), \(\Lambda\) for our model must not take on extreme values away from unity. In the stability analysis we shall seek eigenvalues of \(\Lambda\) for the onset of cellular motion, in the presence of the constraints of stratification and variable rotation, \(\theta, \beta^*\). From its definition, (12), \(\Lambda\) may be seen to have the usual interpretation of the ratio of driving force to dissipative mechanism; but in this case the parameter is weighted by rotational factors.
3. The Basic State. An exact solution to the full nonlinear problem posed by eq. (13)–(18) may be simply formulated by writing the temperature field in the form

$$T(\xi, \eta, \zeta) = \xi + \theta S(\eta, \zeta)$$  \hspace{1cm} (24)$$

and by assuming the velocity field to be independent of longitude, $\xi$. Then it is only the first term on the right-hand side which contributes to eq. (14) and thereby serves to determine the $V, W$ velocity distribution. The meridional cell

$$V = \frac{\zeta - 1/2}{f}, \hspace{1cm} W = \frac{\beta* \zeta}{2f^2} (\zeta - 1)$$  \hspace{1cm} (25)$$

consistently satisfies (13), (14) and (16). The stratification function $S(\eta, \zeta)$ and the longitudinal flow $U(\eta, \zeta)$ may now be determined from (15) and (17). The nonlinear problem degenerates into the solution of linear equations with known nonconstant coefficients given by (25), viz:

$$fU_\zeta + \theta S_\eta = 0,$$  \hspace{1cm} (26)$$

$$-S_\zeta + \Lambda \left[ \frac{U}{\theta} + \frac{\zeta - 1/2}{f} S_\eta + \frac{\beta* \zeta (\zeta - 1)}{2f^2} S_\zeta \right] = 0,$$  \hspace{1cm} (27)$$

with $S(\eta, o) = 0$, $S(\eta, 1) = 1$.

At this point we choose to consider two special examples rather than to attempt an exact solution to (24). We shall assume either that $\theta = 0$ or that $\Lambda$ be sufficiently small so that a power series solution is appropriate. In the former case, $S = U = 0$ and $\Lambda$ is unrestricted in magnitude. In the latter case we write

$$S = S_0(\zeta) + \Lambda S_1(\eta, \zeta) + \ldots, \hspace{1cm} U = \Lambda U_1(\eta, \zeta) + \ldots,$$

and find the temperature field through $S_1$ to be given by

$$T = \xi + \theta \zeta + \frac{\theta \Lambda \beta*}{12(1 + \beta* \eta)^2} \left[ \frac{\zeta^4}{2} - \zeta^3 + \frac{\zeta}{2} \right] + \ldots$$  \hspace{1cm} (28)$$

In the stability analysis we shall in fact use only the first two terms of (28). A formal estimate of the range of validity of this approach is to limit it within $O(\theta \Lambda \beta*/12)$. Physically two effects are to be neglected—a distortion of the vertical temperature gradient, and the existence of a latitudinal temperature gradient with a corresponding longitudinal flow. The first effect should alter our results very little because of the method of approximation employed in the next section, in which only the integral properties of the gradient are re-
levant. The second effect, that of the latitudinal gradient of temperature created by the variable Coriolis parameter, is minimized by our choice of boundary condition at the bottom and surface, i.e. by imposing $T_\eta = 0$. Other $\beta$-effects are seen to be of greater importance in the range of parameters we consider. Recall that in any case the meridional cell $(V, W)$ given by eq. (25) is an exact solution.

4. The Stability Problem. A single partial differential equation of fourth order may be obtained for the perturbation vertical velocity. From (19), $v$ is given directly in terms of $w$. Since the coefficients in (22) are independent of $\xi$, upon partial differentiation of the equation with respect to $\xi$, the functions $T, u$ do not appear in undifferentiated form. Thus (20) and (21) may be applied to yield

$$w_{\xi \xi} - w_{\xi \xi \xi} + \beta^* \frac{A}{f} \left[ \frac{U}{\beta^*} w_{\xi \xi} - T_\xi (2 w_{\xi} + \frac{f}{\beta^*} w_{\eta \xi}) + f V (2 w_{\xi \xi} + \frac{f}{\beta^*} w_{\xi \xi \eta}) + \frac{1}{\beta^*} T_\eta w_{\xi \xi} + T_\xi w_{\xi} + \frac{f}{\beta^*} W w_{\xi \xi} \right] = 0.$$  \hspace{1cm} (29)

Since by (17) and (18) $w_{\xi \xi} = \frac{\beta^*}{f^2} T_\xi$, the vanishing of the perturbation temperature on the bottom and surface implies a vanishing of the second derivative of the vertical velocity; condition (23) transforms to

$$w(\xi, \eta, 0) = w_{\xi \xi}(\xi, \eta, 0) = w(\xi, \eta, 1) = w_{\xi \xi}(\xi, \eta, 1) = 0.$$  \hspace{1cm} (30)

The coefficients of (29) will be evaluated from the basic state, (24), (25), with $T_\xi = 0, U = 0$ as discussed in the preceding section. Explicitly, the $w$-equation becomes

$$w_{\xi \xi} - w_{\xi \xi \xi} + \frac{A}{f} \left\{ - (w_{\eta \xi} + 2 \frac{\beta^*}{f} w_{\xi}) + (\zeta - \frac{1}{2}) (w_{\eta \xi \xi} + 2 \frac{\beta^*}{f} w_{\xi \xi}) + \frac{\beta^*}{2f} (\zeta - 1) w_{\xi \xi \xi} + \frac{\beta^*}{f} \theta w_{\xi} \right\}.  \hspace{1cm} (31)

The coefficients of this equation are functions of the two variables $\eta, \zeta$. We shall, however, employ an assumption which has found application in many oceanic problems previously; viz, that the Coriolis parameter may now be treated as a constant. Thus the variation of the Coriolis parameter which, e.g., supports the meridional velocity component, is allowed for by the exact derivation of (31) from the dynamical equations; but at this point we assume
Thus we set \( f = 1 \) but let \( \beta^* \) remain in the coefficients where it has appeared from previous differentiations. Since the coefficients are now considered to be dependent solely upon \( \zeta \), let

\[
\omega = \omega(\zeta) e^{i(\sigma \tau + \eta \xi + m \eta)},
\]

whence \( \omega \) satisfies

\[
i\sigma \omega_{\zeta} - \omega_{\zeta\zeta} + A^* \left[ - (2 + i\mu) \omega_{\zeta} + (2 + i\mu) \left( \zeta - \frac{1}{2} \right) \omega_{\zeta\zeta} + \frac{\zeta}{2} (\zeta - 1) \omega_{\zeta\zeta} \right] + iv \omega = 0,
\]

where

\[
A^* = A^* \beta^*, \quad \mu = \frac{m}{\beta^*}, \quad v = A \theta \beta^* n.
\]

The longitudinal and latitudinal wave-numbers \( n, m \) (or effective wave-numbers \( v, \mu \)) are to be taken as real, but the nondimensional frequency, \( \sigma \), is in general complex. Note that the longitudinal degree of freedom of the perturbation wave appears in the eigenvalue problem only because of the basic stability, i.e., \( n \) appears coupled to \( \theta \). We shall investigate (33) only for the case of marginal stability in the presence of possible overstability, i.e., for real \( \sigma \). Full generality is maintained by assuming \( \mu > 0 \), with \( v, \sigma < 0 \).

Although (33) is an ordinary differential equation, exact solution is not possible and we shall content ourselves with approximate results appropriate to the case that \( \omega(\zeta) \) is a smooth function without much distortion (the lowest vertical eigenmode). The nature of the boundary conditions (30) facilitates such an approximation, since, if \( \omega \) be expanded in a Fourier sine-series, the boundary conditions will be satisfied term by term. Upon multiplication of (33) successively by \( \sin n\pi\zeta \) (\( n \) an integer) and integration between \( \zeta = 0, 1 \), there results an infinite set of algebraic equations in the Fourier coefficients and the parameters, the determinant of which yields the eigenvalues. The approximation consists in truncating the infinite determinant to a finite determinant of any order; experience indicates that the associated expansion for the lowest eigenvalue usually converges rapidly. Since our interest lies primarily in qualitative results for the lowest vertical eigenmode, we shall take only the lowest consistent approximation, that of a two-by-two determinant which allows for a coupling of two Fourier modes in the vertical.

Formally, we let \( \omega = \sum_{p=1}^{\infty} a_p \sin p\pi \zeta \), which by (30) may be differentiated term-by-term and substituted into (33). Multiplying successively by \( \sin p\pi \zeta \),
sin2πζ, and integrating over the depth, we obtain under the approximation

\[ a_p = 0, \ p \geq 3 \]

the two equations

\[
[ - (i\sigma + 10) + 0.1 iv ] a_1 + A^* [i\mu + 0.4] a_2 = 0, \tag{34a}
\]

\[
A^* [0.45 i\mu - 0.15] a_1 + [20 (i\sigma + 40) - 0.5 iv] a_2 = 0. \tag{34b}
\]

The complex determinental equation is

\[
20 \sigma^2 - 10^3 i\sigma - 2.5 v \sigma + 85 iv - 8 \times 10^3 + 5 \times 10^{-2} v^2 + 
+ 10^{-2} A^{*2} (45 \mu^2 - 31 \mu + 6) = 0. \tag{35}
\]

Upon taking real and imaginary parts (for the case of marginal stability, \( \sigma \) real) the resulting equations can be combined into the forms

\[
\sigma^2 - \left[ 0.13 v + 7 \times 10^2 \left( \mu + \frac{0.13}{\mu} \right) \right] \sigma + 2.5 \times 10^{-3} v^2 
+ 60 \left( \mu + \frac{0.13}{\mu} \right) v - 4 \times 10^2 = 0. \tag{36a}
\]

\[
A^{*2} = \frac{3 \times 10^4}{\mu} [8.5 \times 10^{-2} v - \sigma]. \tag{36b}
\]

Equation (36a) may be regarded as the dispersion relationship for the marginally stable waves and (36b) as the eigenvalue of \( A^* \) at which waves of such characteristics become unstable. It should be borne in mind that the dispersion relation will be altered as a wave grows to finite amplitude. Since interest will lie mainly in a relatively high wave number range, it will be sufficiently accurate to take \( \mu + \frac{0.13}{\mu} \approx \mu \) in (36a). Recall that we choose to explore positive and negative values of \( v, \sigma \) for positive values of \( \mu \).

5. Upper Bounds on the Horizontal Wave-Numbers. The assumption that the perturbation wave fields remain geostrophic on the \( \beta \)-plane will of course become invalid for sufficiently small horizontal scales or sufficiently short periods. As a measure of the maximum wave-lengths and frequencies allowed, we shall investigate the equation for the vertical component of the perturbation vorticity. The necessary conditions obtained will be more stringent than conditions obtained directly from the momentum equations and will be sufficient conditions for the momentum equations to remain geostrophic. To this end
we obtain a more accurate form of eq. (19) by including the local and linearized accelerations of the horizontal velocities. The result is (for \( U = 0 \))

\[
\delta (\nu_T - \gamma^2 u_\eta) + \varepsilon [V\nu_\eta \xi + V_\eta u_\xi + W\nu_\xi + \gamma^2 (V_\eta u_\eta + V u_{\eta\eta} + W_\eta u_\xi + W u_{\eta\xi})] + \beta^* \nu - f w_\xi = 0.
\]

Three new parameters appear:

\[
\varepsilon = \frac{V_0}{f_0 L} = \frac{\alpha g (\Delta T) H h}{f_0^2 L^2}, \quad \text{the Rossby number of the basic flow,}
\]

\[
\delta = \frac{\nu l}{h^2 f_0 L}, \quad \text{a frequency ratio, and}
\]

\[
\gamma = \frac{L}{l}, \quad \text{the over-all horizontal scale ratio.}
\]

We shall assume \( \gamma \) to be \( o(1) \) and exclude its effect from further study. The frequency ratio \( \delta \) measures the importance of Coriolis to local accelerations of time scale \( h^2/\nu \). With \( \nu = 10 \text{ cm/sec}, h^2 = 10^{11} \text{ cm}^2, f_0 = 10^{-4} \text{ (sec)}^{-1}, L/l = 1, \delta = 10^{-6}.1 \) For an estimate on the upper bound of the Rossby number, let \( V_0 = 1 \text{ cm/sec} \), and \( L = 10^8 \) or \( 10^9 \text{ cm} \); then \( \varepsilon = 10^{-4} \) or \( 10^{-5} \).

For the range of interest in which the last two terms of (37) balance one another and are individually much larger than other terms, the amplitude of \( w \) relative to \( v \) will be proportional to \( \beta^* \) as is the case in the basic state (25). We divide (37) by \( \beta^* \), replace \( \partial/\partial \xi, \partial/\partial \eta, \partial/\partial \tau \) acting upon a perturbation field by \( in, im, is, \) and \( \partial/\partial \eta \) acting upon a basic field by \( \beta^* \) [or substitute from (25)]. Then, relative to the last two terms, contributions to the inertial terms arise of orders \( enm/\beta^*, en, em, em^2/\beta^*, e\beta^*; \) and to the time terms of orders \( \delta n\sigma, \delta m\sigma \). The strongest restrictions from the inertial terms may be written \( enu < 1, e\beta^* \mu^2 < 1 \); a safe limitation to impose is \( \mu, n \leq 10^2 \). Evaluating the time terms with the maximum wave-numbers, we find that they will remain negligible so long as \( \sigma < 10^4 \). This maximum frequency corresponds roughly to a monthly oscillation.

6. Interpretation of the Results. We shall first examine the case of an un-stratified ocean where our basic state is exact and \( \Lambda^* \) is unlimited. Setting \( \nu = 0 \) in (36a,b), we find from the eigenvalue relationship the condition that

\[\text{At this point, in attempting to evaluate parameters with characteristic oceanic numbers, the difficulty arises that our basic state is an undistorted thermal circulation and not a boundary-layered (thermocline) one. Thus we choose } \nu \text{ as an order of magnitude greater than deduced in the thermocline theory and } h^2 \text{ as intermediate between the values appropriate for the thermocline depth and the total depth of the ocean.}\]
σ < 0, i.e., only overstable perturbations are allowed. These solutions correspond to travelling plane waves with a northward component of phase velocity. It should be remarked that v = 0 cannot be achieved by n = 0 because of the necessity of a longitudinal temperature gradient to support the second derivative of the perturbation vertical velocity; but no information about the zonal phase velocity is contained in the eigenvalue problem. The quadratic dispersion relation has only one physically realizable root. The results for the range $\mu^2 \gg 0.3 \times 10^{-2}$ may be written

$$\sigma = -\frac{1}{2\mu}, \quad \Lambda^* = \frac{1.5 \times 10^4}{\mu^2}$$  \hspace{1cm} (38)$$

and are plotted in Fig. 1. Note that as $\Lambda^*$ increases, disturbances of larger scale become unstable. The frequency of the waves decreases with increasing wave number but is always very low; the period is always longer than a year. Such waves would appear stationary, although the actual unsteadiness of their temperature distribution is essential for their existence.

In the more general case $v \neq 0$, the condition of physical realizability (real $\Lambda^*$) becomes $8.5 \times 10^{-2} v - \sigma \geq 0$. This is interesting because the possibility exists for waves of many $\mu$-values to remain unstable even as $\Lambda^*$ vanishes (within the order of our approximation). This is indeed the case. Eigenvalue and dispersion diagrams are shown in Fig. 2. Again there is only one root of
the dispersion relation which can satisfy the necessary inequality. However, this root does not always correspond to an unstable wave; this corresponds to the break in the curves $\Lambda(\sigma)$ (for constant $\mu$) that occur on the positive $\sigma$-axis. The algebraic expressions

$$\sigma = 8.8 \times 10^{-2} \nu - \frac{.61}{\mu}, \quad \Lambda^* = \frac{3 \times 10^2}{\mu} \left[ \frac{.61}{\mu} - .3 \nu \right] \quad (39)$$

are valid when $(.3 \times 10^{-2} - 1.5 \times 10^{-4} \mu \nu) \ll \mu^2$, which hold for values of $\Lambda^*$ small enough to be consistent with our treatment of the basic state. The graphs, however, have been extended beyond this range.

Near the origin of Fig. 2 our approximations are most rigorously satisfied. For low wave numbers, $\sigma > 0$ so that the phase velocity now has a southward component, but for higher wave numbers there are northward travelling waves
unstable at eigenvalues of $A^* < 1$. To discuss the longitudinal velocities of the waves, we note that in order to have the eastward temperature gradient of the basic state negative, as is appropriate to a subtropical region of an ocean, $\Delta T_H < 0$ and thus $\Lambda < 0$. For stable stratification we then have $\theta < 0$, and thus from (33), $v$ has the opposite sign from $n$. Since the sign of $\sigma$ is essentially the sign of $u$ (except very near the origin), the phase velocity is always eastward. The group velocity may be derived from the first of (39). The longitudinal component does not differ significantly from the corresponding component of the phase velocity, but the latitudinal component $-\partial \sigma/\partial m = -\beta^*/m^2$ is always southward.

The most significant result of our analysis lies in the demonstration of how highly unstable is our basic meridional convective cell. In fact, waves of scales ranging from tens to hundreds of kilometers are unstable upon a basic longitudinal temperature gradient (and the associated flow) of any value. The value of $(\Delta T)_H$ may be regarded as merely fixing the frequency of the unstable mode at the point of marginal instability. The dispersion relationship will of course evolve with amplitude if one considers a single mode as growing upon the basic fields. More important, however, may be the nonlinear interactions between a multitude of unstable modes at finite amplitude. For although we have explored the growth of plane-wave perturbations only on the mean meridional cell, it is conceivable that merely the eigenvalue and characteristics of a particular wave will change because of presence of other unstable modes. Then little can be said of the final state of the fluid except that it will be turbulent. It may be, however, that one or a few of the unstable modes will dominate (contain a significant fraction of the total kinetic energy). We may note in passing that if the diagrams of Fig. 2 be examined in the ranges of monthly frequencies [for $|\sigma|$ values of $10^4$ and $|\Lambda|$ values of $10^3$] which are definitely outside the formal range of validity of our arguments], waves with horizontal scales of tens of kilometers are, in particular, present. The possibility thus exists that the considerations initiated here may bear some relevance to an ultimate understanding of the large amplitude transient motions observed by Swallow in the deep North Atlantic.

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