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Ratio Equilibrium in an Economy with an Externality

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1. Introduction and Definitions

It is a familiar proposition that in an economy with an externality, market mechanism may fail to work well. Of course, this proposition is also true in a public goods economy. In order to solve the difficulty, Kaneko [7, 8] introduced the concept of ratio equilibrium, which is a modification of Lindahl equilibrium, and provided a majority negotiation model called "voting game." He proved the existence of a ratio equilibrium and the equivalence of the ratio equilibria and the core of the voting game. This means that a ratio equilibrium may be achieved as a result of the majority negotiation of the voting game. In a public goods economy beneficiaries' right to consumption of the public goods and liability to burden of their costs are clear. These are, however, not clear in an economy with an externality (external economies and external diseconomies). It makes us unable to treat such an economy as a variation of a public goods economy to introduce a certain liability rule, which was initially proposed by Coase [2]. The purpose of this paper is to apply the concept of ratio equilibrium to an economy with external diseconomies by introducing a concept of allowance level which is considered as a specification of Coase's liability rule.

We consider a community which consists of a damaging firm 1, and suffered households, $2, \ldots, n$. Firm 1's revenue function is $f(q)$ with $f(0) = 0$, where activity level q is measured in terms of private cost of production.[1] $f(q)$ is defined on the set of all nonnegative real numbers E_+ . If firm 1 has no liability nor needs to compensate households for anything, then 1's profit is $f(q) - q$. Household i's utility function $U^i(q,m)$ is defined on the nonnegative orthant of the 2-dimensional Euclidean

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[1] Ratio equilibrium is determined independently of the measurement of activity level. See Ito and Kaneko [5].

space E_+^2 . q denotes firm 1's activity level and m household i's consumption level. Household i has the initial endowment of money $I_i > 0$ $(i = 2, \ldots, n)$. We call the consumption good "money."

Since an increase of firm 1's activity level raises his revenue and lowers the households' utility levels, a conflict between firm 1 and households $2, \ldots, n$ necessarily occurs. For the resolution of such a conflict, it is necessary to specify a liability rule: Who should compensate whom? Liability rule is a device for conflict resolution. We introduce a concept of allowance level $\bar{q} \geqq 0$ as a specification of liability rule as follows. The firm can choose freely an activity level if it is not greater than $\bar{q}$. If the firm desires to raise its activity level beyond $\bar{q}$, it must obtain the households' consent. Without their consent, firm 1 can not operate at a higher activity level than the allowance level $\bar{q}$. In this case the firm may compensate them to obtain their consent. Or if households desire to lower firm 1's activity level smaller than $\bar{q}$, then they compensate firm 1 so that they obtain firm 1's consent, because firm 1 has the right to operate freely at any activity level not greater than the allowance level $\bar{q}$. The introduction of allowance level to the community enables us to treat the problem of compensation for external diseconomies in the same way with that of cost share in a public goods economy. Hence we can define ratio equilibrium in our community. In this paper, we consider the behavior of ratio equilibrium in our community when an allowance level is exogenously given. But we do not consider a decision problem of allowance level.

Let an allowance level $\bar{q} \geqq 0$ be exogenously given. We denote, by $E(I,\bar{q}) = E(I_2, \ldots, I_n, \bar{q})$, the community with income levels $I = (I_2, \ldots, I_n)$ and allowance level $\bar{q}$. We call $(r, q^*) = (r_1, \ldots, r_n, q^*)$ a <u>ratio equilibrium</u> in $E(I,\bar{q})$ if

(1.1) $r_1 = \sum_{i=2}^{n} r_i$ and $q^* \geqq 0$,

(1.2) $f(q^*) - q^* - r_1(q^* - \bar{q}) \geqq f(q) - q - r_1(q - \bar{q})$ for all $q \geqq 0$

(1.3) for all $i = 2, \ldots, n$, $I_i + r_i(q^* - \bar{q}) \geqq 0$ and

$U^i(q^*, I_i + r_i(q^* - \bar{q})) \geqq U^i(q, I_i + r_i(q - \bar{q}))$ for all $q \geqq 0$

satisfying $I_i + r_i(q - \bar{q}) \geqq 0$.[2]

Note that r_i $(i = 1, \ldots, n)$ may be negative.

This definition can be interpreted as follows. Under a given allowance level $\bar{q}$ and a vector $r = (r_1, r_2, \ldots, r_n)$, firm 1 pays (or receives) total compensation $r_1(q - \bar{q})$ and household i receives (pays) compensation $r_i(q - \bar{q})$ $(i = 2, \ldots, n)$ when firm 1 operates at activity level q . The total compensation that firm 1 pays (receives) should be equal to the sum of compensations received (paid) by households $2, \ldots, n$, i.e., $r_1 = \sum_{i=2}^{n} r_i$. Condition (1.2) means the profit maximization of firm 1 under the assumption that r_1 is fixed. Condition (1.3) means the utility maximization of the households under the assumption that $r_2, \ldots, r_n$ are fixed. The definition requires that the "demands" of all households and firm 1 for firm's activity level coincide.[3]

We can ensure the existence of a ratio equilibrium in the community under natural assumptions:

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[2]Although we define the ratio equilibrium in a different way from Kaneko [7, 8], this definition is equivalent to that of [7, 8].

[3]Davis and Winston [3] considered a price mechanism in an economy with externality introducing the concept of allowance level. Their essential idea is the same as ours.

Proposition 1. Assume that $f(q)$ is a continuous and concave function with $f(\hat{q}) - \hat{q} = \max_{q \geqq 0}\{f(q) - q\}$ for some $\hat{q} \geqq 0$ and that $U^i(q,m)$ $(i = 2, \ldots, n)$ is a continuous, quasi-concave function of (q,m) and is monotonically increasing with respect to m. Then there exists a ratio equilibrium (r, q^*) in $E(I,\bar{q})$ for any $\bar{q} \geqq 0$.

This proposition can be proved without difficulty, modifying Bergstrom's proof of the existence of a competitive equilibrium. See [1].

Now we can prove a proposition which orients us to our purpose. Let us consider the case where $U^i(q,m)$ can be represented as

$$(1.4) \qquad U^i(q,m) = g^i(u^i(q) + m) \quad \text{for all} \quad (q,m) \in E^2_+ ,$$

where g^i is a monotonic function.[4] This condition means that there exists no income effect in household's demand for firm 1's activity level. Then we have the following proposition:

Proposition 2. Let $E(I^1, \bar{q}^1)$ and $E(I^2, \bar{q}^2)$ be communities with income levels $I^1 = (I^1_2, \ldots, I^1_n)$, $I^2 = (I^2_2, \ldots, I^2_n)$ and allowance levels $\bar{q}^1$, $\bar{q}^2$, respectively.[5] We assume that $U^i(q,m)$ is quasi-concave and satisfies (1.4) for all $i = 2, \ldots, n$. If (r, q^*) satisfies

$$(1.5) \qquad I^j_i + r_i(q^* - \bar{q}^j) > 0 \quad \text{for all} \quad i = 2, \ldots, n \quad \text{and} \quad j = 1, 2 ,$$

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[4]Kaneko [6] provided a necessary and sufficient condition for a preference relation to be represented as (1.4).

[5]Note that $E(I^1, \bar{q}^1)$ and $E(I^2, \bar{q}^2)$ consist of the same households $2, \ldots, n$ and that household i in $E(I^1, \bar{q}^1)$ has the same utility function with that of i in $E(I^2, \bar{q}^2)$.

then (r, q^*) is a ratio equilibrium in $E(I^1, \bar{q}^1)$ if and only if (r, q^*) is a ratio equilibrium in $E(I^2, \bar{q}^2)$.

As it is not difficult to prove this proposition, we omit the proof.

Proposition 2 means that in the absence of income effect a ratio equilibrium (r, q^*) is independent of income levels I and allowance level $\bar{q}$. Of course, we should note that when the income levels or the allowance level in $E(I,\bar{q})$ change, the income distribution derived from a ratio equilibrium also changes. This kind of result is often called Coase's theorem.[6] Our main purpose, however, is to investigate the behavior of ratio equilibrium in $E(I,\bar{q})$ in the presence of income effect.

2. Limit Properties of Ratio Equilibrium

In this section we will consider the "global" behavior of ratio equilibrium in $E(I,\bar{q})$ when the income levels $I_2, \ldots, I_n$ change. That is, we will show two limit properties of ratio equilibrium when $I_2, \ldots, I_n$ become small or large.

First we assume that the firm's revenue function $f(q)$ is a strictly concave, continuous function of $q \geqq 0$ and is continuously differentiable on the open interval $(0, +\infty)$ with

$$(2.1) \qquad \lim_{q\to\infty} \frac{df(q)}{dq} \leqq 0 \quad \text{and} \quad \lim_{q\to+\infty} \frac{df(q)}{dq} = +\infty \ .$$

Second, we assume that the households' utility functions $U^i(q,m)$, $i = 2, \ldots, n$ are decreasing functions of q , increasing functions of m , strictly concave, continuous functions of (q,m) in E_+^2 and are

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[6] See Inada and Kuga [4].

continuously differentiable on $\{(q,m) \in E_+^2 : m > 0\}$ with

$$(2.2)\qquad \lim_{m \to +0} \frac{\partial U^i(q,m)}{\partial m} = +\infty \quad \text{for any} \quad q \geqq 0 \ .$$

Note that (2.2) reflects an income effect. As these assumptions are standard and familiar, we would need no explanation.

Let $\hat{q}$ denote the activity level such that

$$(2.3)\qquad f(\hat{q}) - \hat{q} = \max_{q \geqq 0}\{f(q) - q\} \ .$$

This $\hat{q}$ is positive and unique under our assumptions.

<u>Lemma 3</u>. Let (r, q^*) be a ratio equilibrium in $E(I,\bar{q})$. Then

(i) (r, q^*) is an inner solution;[7]

(ii) $r_i > 0$ for all $i = 1, 2, \ldots, n$.

<u>Proof</u>. (i) Since $I_2, \ldots, I_n > 0$, we have, from (2.2), $0 < I_i + r_i(q^* - \bar{q})$ for all $i = 2, \ldots, n$. Also we have $0 < q^*$ from (2.1).

(ii) Since (r, q^*) is an inner solution, necessary conditions for it to be a ratio equilibrium are

$$(2.4)\qquad \frac{\partial U^i}{\partial q} + r_i \frac{\partial U^i}{\partial m} = 0 \quad \text{for all} \quad i = 2, \ldots, n \ ,$$

$$(2.5)\qquad \frac{df}{dq} - (1 + r_1) = 0 \ .$$

If there is at least one i $(2 \leqq i \leqq n)$ such that $r_i \leqq 0$, then (2.4)

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[7] A ratio equilibrium (r, q^*) is called an <u>inner solution</u> if $q^* > 0$ and $I_i + r_i(q^* - \bar{q}) > 0$ for all $i = 2, \ldots, n$.

does not hold, because $\partial U^i/\partial q < 0$ and $\partial U^i/\partial m > 0$. Therefore we have $r_i > 0$ for all $i = 2, \ldots, n$ and $r_1 > 0$ because of $r_1 = \sum_{i=2}^{n} r_i$. Q.E.D.

The main result of this section is the following propositions.

Proposition 4. Let $(r(\lambda), q(\lambda))$ be a ratio equilibrium in $E(\lambda I_i, q) = E((\lambda I^2, \lambda I^3, \ldots, \lambda I^n), \bar{q})$ for all $\lambda > 0$. We assume $\bar{q} < \hat{q}$. Then we have (i) and (ii):

(i) There is a $\lambda_0 > 0$ such that $\bar{q} < q(\lambda) < \hat{q}$ for all $\lambda \leqq \lambda_0$.

(ii) Assume that for any $q \geqq 0$,

$$\lim_{m\to\infty} \left.\frac{\partial U^i/\partial q}{\partial U^i/\partial m}\right|_{(q,m)} = -\infty \quad \text{for all } i = 2, \ldots, n . \tag{2.6}$$

Then $\lim_{\lambda\to\infty} q(\lambda) = 0$.

Proof. We show only (ii).

From Lemma 3, necessary conditions for a ratio equilibrium are

$$\frac{\partial U^i}{\partial q} + r_i(\lambda)\frac{\partial U^i}{\partial m} = 0 \quad \text{for all } i = 2, \ldots, n ,$$

$$\frac{df}{dq} - (1 + r_1(\lambda)) = 0 .$$

Suppose that $q(\lambda) \to 0$ $(\lambda \to \infty)$ does not hold. Then we can take a subsequence $\{\lambda^\nu\}$ such that $q(\lambda^\nu) \to q_0 > 0$ and $\lambda^\nu \to \infty$ $(\nu \to \infty)$. Since $1 + r_1(\lambda^\nu) = \frac{df(q\ (\lambda^\nu))}{dq}$ for any ν , there is a number $M > 0$ such that $r_1(\lambda^\nu) \leqq M$ for any ν . Since $r_i(\lambda^\nu) > 0$ $(i = 2, \ldots, n)$ and $r_1(\lambda^\nu) = \sum_{i=2}^{n} r_i(\lambda^\nu)$ for all ν , there is a number $M' > 0$ such that

$M' \geqq r_i(\lambda^\nu)$ for all $i = 2, \ldots, n$ and all ν. Hence $\lambda^\nu I_i + r_i(\lambda^\nu)(q(\lambda^\nu) - \bar{q}) \to +\infty$ $(\nu \to \infty)$. Using $q(\lambda^\nu) \to q^0 > 0$ $(\nu \to \infty)$, we have

$$\lim_{\nu\to\infty} r_i(\lambda^\nu) = -\left.\frac{\partial U^i}{\partial q}\middle/\frac{\partial U^i}{\partial m}\right|_{(q(\lambda^\nu),\lambda^\nu I_i + r_i(\lambda^\nu)(q(\lambda^\nu)-\bar{q}))} = +\infty .$$

This is a contradiction. Hence $\lim_{\lambda\to\infty} q(\lambda) = 0$. Q.E.D.

Proposition 4(i) says that when the households' income levels are very low, the firm always compensates the households and increases the activity level beyond the allowance level $\bar{q}$ in equilibrium. When the income levels are very low, the households are eager to get income but hardly want to decrease the firm's activity level. By this reason the firm can obtain the households' consent by paying small compensations to them, and so the firm can raise the activity level beyond the allowance level. Proposition 4(ii) says, conversely, that when the income levels are sufficiently high, the households compensate the firm to decrease his activity level to the zero-level. The case of Proposition 2 is usually interpreted as a case with sufficient large income levels, i.e., in this case we can approximately regard it as a case without income effect. Hence we can say from Propositions 2 and 4(ii) that when the income levels are sufficiently high, a ratio equilibrium is approximately independent of the income levels and the allowance level, but the activity level in equilibrium is always small. Finally note that condition (2.6) is necessary for this proposition, but that it would be economically trivial and is satisfied by many plausible examples.

3. Local Property of Ratio Equilibrium

In this section we will investigate relationships between allowance level and ratio equilibrium, and effects of some changes of allowance level or the households' income levels upon ratio equilibrium.

We add the following assumption on the households' utility functions U^i, $i = 2, \ldots, n$ to the previous assumptions that U^i's are functions of C^2 and

$$(3.1) \qquad \left.\frac{\partial^2 U^i}{\partial q \partial m}\right|_{(q,m)} \leqq 0 \quad \text{for all } (q,m) \text{ with } q \geqq 0 \text{ and } m > 0 .$$

Condition (3.1) means that the marginal utility of money is a nonincreasing function of firm's activity level. In other words, when the activity level becomes lower, i.e., the environment for the households becomes better, each household can gain greater (not smaller) additional utility by an additional income.

Lemma 5. (i) $\dfrac{U_1^i}{U_2^i}(q,m) = \left.\dfrac{\partial U^i}{\partial q}\Big/\dfrac{\partial U^i}{\partial m}\right|_{(q,m)}$ is a nonincreasing function of q and m with $q \geqq 0$ and $m > 0$.[8]

(ii) For any positive r_i, $\dfrac{U_1^i}{U_2^i}(q, I_i + r_i(q - \overline{q}))$ is a nonincreasing function of q, where $q \geqq 0$ and $I_i + r_i(q - \overline{q}) > 0$.

(iii) Let $D_i(r_i, I_i, \overline{q})$ denote household i's demand for the firm's activity level in $E(I, \overline{q})$.[9] For any positive r_i, $D_i(r_i, I_i, \overline{q})$ is a

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[8] We can employ this proposition as an assumption in place of (3.1) for the following discussions. But since Condition (3.1) has a clearer meaning than (i), we employ (3.1) as an assumption.

[9] $D_i(r_i, I_i, \overline{q}) = q_i$ is defined by $U^i(q_i, I_i + r_i(q_i - \overline{q}))$ $= \max\{U^i(q, I_i + r_i(q - \overline{q})) : q \geqq 0 \text{ and } I_i + r_i(q - \overline{q}) \geqq 0\}$.

nonincreasing function of $I_i > 0$ and a nondecreasing function of $\bar{q} \geqq 0$.

Proof. We will use the following notations: $U_1^i = \frac{\partial U^i}{\partial q}$, $U_2^i = \frac{\partial U^i}{\partial m}$, $U_{11}^i = \frac{\partial^2 U^i}{\partial q^2}$, $U_{12}^i = \frac{\partial^2 U^i}{\partial m \partial q}$, and $U_{22}^i = \frac{\partial^2 U^i}{\partial m^2}$.

(i) Since $U_1^i < 0$, $U_2^i > 0$ and $U_{11}^i, U_{12}^i, U_{22}^i \leqq 0$, by the assumptions of the previous section and (3.1), we have

$$\frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}\right) = \frac{U_2^i U_{11}^i - U_1^i U_{12}^i}{(U_2^i)^2} \leqq 0 \quad \text{and} \quad \frac{\partial}{\partial m}\left(\frac{U_1^i}{U_2^i}\right) = \frac{U_1^i U_{12}^i - U_1^i U_{22}^i}{(U_2^i)^2} \leqq 0 .$$

(ii) It follows from (i) that $\frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}(q,\ I_i + r_i(q - \bar{q}))\right) = \frac{\partial}{\partial q}\left(\frac{U_1^i}{U_2^i}\right) + r_i \frac{\partial}{\partial m}\left(\frac{U_1^i}{U_2^i}\right) \leqq 0$.

(iii) Let $I_i^1 < I_i^2$, $q^1 = D_i(r_i,\ I_i^1,\ \bar{q})$ and $q^2 = D_i(r_i,\ I_i^2,\ \bar{q})$. Since $\frac{U_1^i}{U_2^i}$ is a nonincreasing function of m , we have

$$-r_i = \frac{U_1^i}{U_2^i}(q\ ,\ I_i^1 + r_i(q^1 - \bar{q})) \geqq \frac{U_1^i}{U_2^i}(q^1,\ I_i^2 + r_i(q^1 - \bar{q})) .$$

Hence we have $q^1 \geqq q^2$ because of (ii) and $-r_i = \frac{U_1^i}{U_2^i}(q^2,\ I_i^2 + r_i(q^2 - \bar{q}))$.

It is similarly verified that $D_i(r_i,\ I_i,\ \bar{q})$ is a nondecreasing function of $\bar{q}$. Q.E.D.

Now we analyze the relationships between an allowance level $\bar{q}$ and an equilibrium activity level q_R (which is given by a ratio equilibrium) using a concept of marginal social cost.[10]

[10] The marginal social cost can be always well-defined, but the concept of social cost itself can not be defined without Assumption (1.4).

<u>Proposition 6</u>. Let (r, q_R) be a ratio equilibrium in $E(I,\bar{q})$. Then $q_R \leqq \bar{q}$ if and only if

$$\text{(3.2)} \qquad \frac{df}{dq}(\bar{q}) \leqq 1 + \sum_{i=2}^{n} - \frac{U_1^i}{U_2^i}(\bar{q}, I_i) .$$

<u>Proof</u>. <u>Necessity</u>: From $q_R \leqq \bar{q}$, we have, by the concavity of f and Lemma 5(ii),

$$1 + r_1 = \frac{df}{dq}(q_R) \geqq \frac{df}{dq}(\bar{q})$$

and

$$-r_i = \frac{U_1^i}{U_2^i}(q_R, I_i + r_i(q_R - \bar{q})) \geqq \frac{U_1^i}{U_2^i}(\bar{q}, I_i) \quad \text{for all } i = 2, \ldots, n .$$

Hence we have

$$1 = 1 + r_1 - \sum_{i=2}^{n} r_i \geqq \frac{df}{dq}(\bar{q}) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}, I_i) .$$

<u>Sufficiency</u>: Suppose (3.2) and $q_R > \bar{q}$. Since (r, q_R) is a ratio equilibrium, it must hold that

$$r_1 = \frac{df}{dq}(q_R) \quad \text{and} \quad -r_i = \frac{U_1^i}{U_2^i}(q_R, I_i + r_i(q_R - \bar{q})) \quad \text{for all } i = 2, \ldots, n .$$

From $q_R > \bar{q}$, we have $1 + r_1 = \frac{df}{dq}(q_R) < \frac{df}{dq}(\bar{q})$. Further we have, by Lemma 5(ii),

$$-r_i = \frac{U_1^i}{U_2^i}(q_R,\ I_i + r_i(q_R - \bar{q})) \leqq \frac{U_1^i}{U_2^i}(\bar{q},\ I_i) \quad \text{for all} \quad i = 2, \ldots, n .$$

Therefore,

$$1 = 1 + r_1 - \sum_{i=2}^{n} r_i < \frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q,\ I_i) \leqq 1 .$$

This is a contradiction. Q.E.D.

Note that (3.2) is a condition only at an allocation of an allowance level without compensation and that Samuelson's condition for an allocation to be Pareto optimal is that (3.2) holds in equation at the allocation.

If (3.2) holds in equation, i.e., the marginal social cost is equal to the marginal revenue at an allowance level, then the equilibrium activity level is equal to the allowance level. This allowance level has a special property: In the ratio equilibrium $(r,\ q_R)$ in the community with the allowance level, not only the equilibrium activity level coincides with the allowance level but also any compensations are not made. We call such an allowance level a <u>neutral allowance level</u>, denoted by $\bar{q}_N$, i.e., if $(r,\ q_R)$ is a ratio equilibrium in $E(I,\ \bar{q}_N)$ then $q_R = \bar{q}_N$.

<u>Proposition 7</u>. Assume that $I = (I_2, \ldots, I_n)$ is fixed. Then there exists one and only one neutral allowance level $\bar{q}_N$, and the following (3.3) is a necessary and sufficient condition for it:

$$\text{(3.3)} \qquad \frac{df}{dq}(\bar{q}_N) = 1 + \sum_{i=2}^{n} - \frac{U_1^i}{U_2^i}(\bar{q}_N,\ I_i) .$$

Proof.[11] It follows from Proposition 6 that (3.3) is a necessary and sufficient condition for a neutral allowance level. Also if a neutral allowance level exists, it is unique because $\frac{df}{dq}(q)$ is a decreasing function and $\frac{U_1^i}{U_2^i}(q, I_i)$ is a nonincreasing function of q for all $i = 2, \ldots, n$.

So we show the existence of a neutral allowance level. Since U^i is a continuously differentiable on $\{(q,m) \in E_+^2 : m > 0\}$ for all $i = 2, \ldots, n$ and $\lim_{q\to+\infty} \frac{df}{dq}(q) = +\infty$, we have $\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i) > 1$ for any sufficiently small $q > 0$. Also, since $\frac{df}{dq}(\hat{q}) = 1$ and $\frac{U_1^i}{U_2^i}(\hat{q}, I_i) < 0$ for all $i = 2, \ldots, n$, we have

$$\frac{df}{dq}(\hat{q}) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\hat{q}, I_i) < 1 .$$

Further since $\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i)$ is a continuous function of q, there exists a $\overline{q}_N$ satisfying (3.3). Q.E.D.

Corollary 8. Let $\overline{q}_N$ be the neutral allowance level in the community with I and let (r, q_R) be a ratio equilibrium in $E(I,\overline{q})$. Then $q_R \geqq \overline{q}$ if and only if $\overline{q} \leqq \overline{q}_N$.

Proof. $\frac{df}{dq}(q) + \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(q, I_i)$ is a decreasing function of q by the strict concavity of f and Lemma 5(i). Hence $q_R \geqq \overline{q}$ if and only if $\overline{q} \leqq \overline{q}_N$ by Propositions 6 and 7. Q.E.D.

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[11] A neutral allowance level exists under the assumptions of Section 2, and the assumptions of this section are not used in the following existence proof.

By this corollary we can illustrate the relation of an allowance level, an equilibrium activity level and the neutral allowance level as Figure 1.

We have the following proposition on effects that a change of an allowance level or households' income has on an equilibrium activity level.

<u>Proposition 9</u>. Let $\bar{q}_N(I)$ be the neutral allowance level of the community with I and let $q_R(I, \bar{q})$ be an equilibrium activity level in $E(I,\bar{q})$. Assume $\bar{q} \geqq \bar{q}_N(I)$. Then

(i) If Δ is a vector such that $\Delta \geqq 0$ and $\Delta \neq 0$, then $q_R(I,\bar{q}) \geqq q_R(I+\Delta, \bar{q})$.

(ii) If $\bar{q} < \bar{q}' < q$, then $q_R(I, \bar{q}') \geqq q_R(I,\bar{q})$.

When (3.1) holds in strict inequality, the results of (i) and (ii) hold also in strict inequality.

<u>Proof</u>. (i) Let (r^0, q^0) and (r^1, q^1) be ratio equilibria in $E(I,\bar{q})$ and $E(I+\Delta, \bar{q})$ respectively. Suppose $q^1 > q^0$. Hence we have

$$1 + r_1^0 = \frac{df}{dq}(q^0) > \frac{df}{dq}(q^1) = 1 + r_1^1 ,$$

There is an i $(2 \leqq i \leqq n)$ such that $r_i^0 > r_i^1$. Since $\bar{q} \geqq q^0$ by $\bar{q} \geqq q_N(I)$ and Corollary 9, we have, by Lemma 5(i) and (ii),

$$-r_i^1 = \frac{U_1^i}{U_2^i}(q^1, I_i + \Delta_i + r_i^1(q^1 - \bar{q})) \leqq \frac{U_1^i}{U_2^i}(q_i, I_i + r_i^1(q^1 - \bar{q}))$$

$$\leqq \frac{U_1^i}{U_2^i}(q^0, I_i + r_i^1(q^0 - \bar{q})) \leqq \frac{U_1^i}{U_2^i}(q^0, I_i + r_i^0(q^0 - \bar{q})) = -r_i^0 .$$

This is a contradiction. Therefore $q^0 \geqq q^1$.

(ii) As we can prove (ii) analogously to (i), we omit its proof.

Q.E.D.

We have shown monotonical effects that changes of an allowance level and the households' income level have on an equilibrium activity level. Such monotonical properties, however, may not hold for any allowance level $\bar{q} < \bar{q}_N(I)$. When $\bar{q} < \bar{q}_N(I)$, the equilibrium activity level $q_R(I,\bar{q})$ is higher than $\bar{q}$ by Corollary 8. In this case, there may be the possibility that when the allowance level decreases or the households' income levels increase, the equilibrium activity level increases. It is illustrated as Figure 1. Roughly speaking, when the allowance level decreases, the households agree with a higher activity level so that they can get greater compensations. This possibility is intuitively paradoxical. But we have not succeeded in constructing any counter example for Proposition 9 with $\bar{q} < \bar{q}_N(I,\bar{q})$ nor proving it. We can provide only a sufficient condition for it.

<u>Proposition 10</u>. Assume that $D_i(r_i, I_i, \bar{q})$ is a nondecreasing function of $r_i > 0$ for all $i = 2, \ldots, n$. Then (i) and (ii) of Proposition 9 are true.

<u>Proof</u>. (i) Suppose $q^1 \equiv q_R(I+\Delta, \bar{q}) > q_R(I,\bar{q}) \equiv q^0$. By the strict concavity of f , we have

$$1 + r_1^0 = \frac{df}{dq}(q^0) > \frac{df}{dq}(q^1) = 1 + r_1^1 .$$

Hence there is an i $(2 \leqq i \leqq n)$ such that $r_i^0 > r_i^1$. By the assumption and Lemma 5(iii) we have

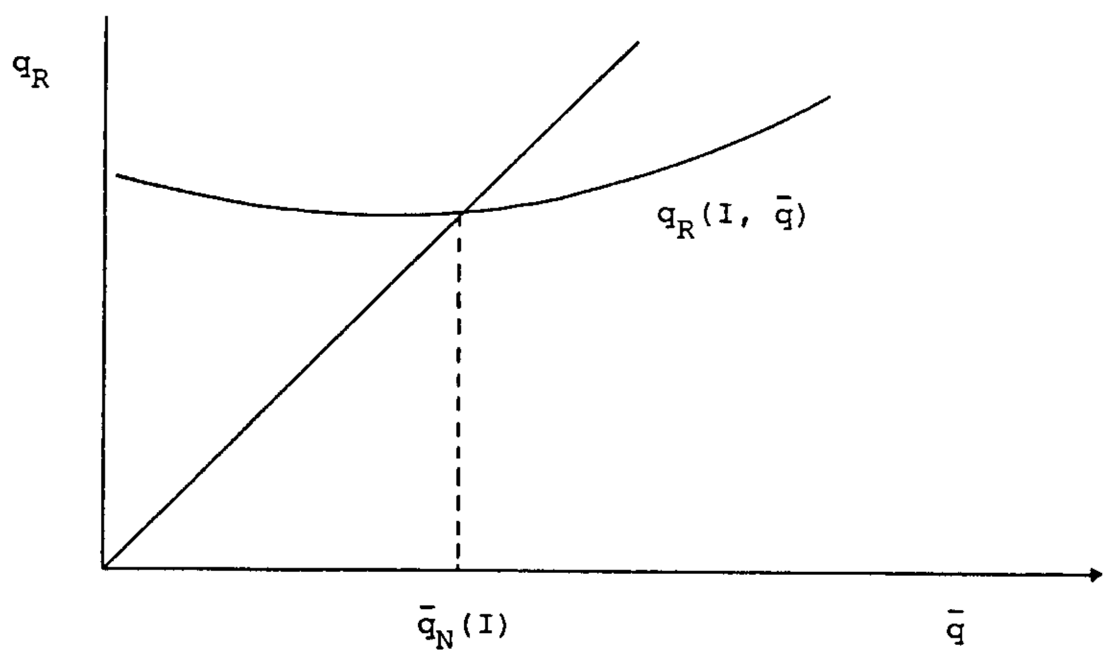

Figure 1

$$q^0 = D_i(r_i^0, I_i, \bar{q}) \geqq D_i(r_i^1, I_i, \bar{q})$$

$$\geqq D_i(r_i^1, I_i + \Delta_i, \bar{q}) = q^1 .$$

This is a contradiction.

(ii) As we can prove (ii) analogously to (i), we omit the proof.

Q.E.D.

The firm's activity may be thought of as a supply of an environmental good by measuring it in $-(q - \bar{q})$ and r_i may be considered as the price and so, we can discuss the demand for the environmental good in a standard way. Then the assumption of Proposition 10 is that the demand for the environmental good is a nonincreasing function of r_i . Hence it is just a Giffen's paradox that the assumption of Proposition 10 does not hold. But it may be reasonable to assume that any environmental good is not a Giffen's good when households' incomes are fairly high.

4. Property of the Neutral Allowance Level

In the previous section, we have characterized local properties of ratio equilibrium when the households' income levels or an allowance level change. As shown in Propositions 6, 9 and Corollary 8, the neutral allowance level plays an important role as a criterion for the local properties. In this section we will investigate the behavior of the neutral allowance level when the households' income levels change.

Proposition 11. Let Δ be a vector in E_+^{n-1} such that $\Delta \geqq 0$ and $\Delta \neq 0$. Then $\bar{q}_N(I) \geqq \bar{q}_N(I+\Delta)$.

Proof. Suppose $\bar{q}_N < \bar{q}_N(I+\Delta)$. Then $\frac{df}{dq}(\bar{q}_N(I)) > \frac{df}{dq}(\bar{q}_N(I+\Delta))$ by the strict concavity of f , and

$$\sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N(I),\ I_i) \geqq \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N(I+\Delta),\ I_i+\Delta_i)$$

by Lemma 5(i). But these can not satisfy (3.3) of Proposition 7. Q.E.D.

Hence the neutral allowance level does not behave in an irregular form such as the behavior of $q_R(I,\bar{q})$ pointed out in the previous section. Further we can provide limit properties of the neutral allowance level.

Proposition 12. Let $\{I^\nu\} = \{(I_2^\nu, \ldots, I_n^\nu)\}$ be a sequence of income vectors.

(i) If $(I_2^\nu, \ldots, I_n^\nu) \to (+0, \ldots, +0)$, then $\bar{q}_N(I^\nu) \to \hat{q}$ $(\nu \to \infty)$.

(ii) Assume that for any $q \geqq 0$, $\lim_{m\to\infty} \frac{U_1^i}{U_2^i}(q,m) = -\infty$ for all $i = 2, \ldots, n$. If $(I_2^\nu, \ldots, I_n^\nu) \to (\infty, \ldots, \infty)$, then $q_N(I^\nu) \to 0$ $(\nu \to \infty)$.

Proof.[12] (i) For any $\hat{q}$ $(0 \leqq q \leqq \hat{q})$, we have, by (2.2) $\lim_{\nu\to\infty} \frac{\partial U^i}{\partial m}(q,\ I_i^\nu) = +\infty$, and so, by the fact that $0 \leqq \bar{q}_N^\nu \leqq \hat{q}$,

$$\sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\bar{q}_N^\nu,\ I_i^\nu) \to 0 \quad (\nu \to \infty)\ .^{13}$$

Hence we have, by (3.3) of Proposition 7, $\lim_{\nu\to\infty} \frac{df}{dq}(\bar{q}_N^\nu) = 1$. Therefore we

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[12] Note that we do not use the assumption added in Section 3 in the proof of Proposition 12.

[13] $\bar{q}_N^\nu$ stands for $\bar{q}_N(I^\nu)$.

have $\overline{q}_N^{\nu} \to \hat{q} \quad (\nu \to \infty)$.

(ii) For any $q \geqq 0$, we have $\lim_{\nu\to\infty} \frac{U_1^i}{U_2^i}(q, I_i^{\nu}) = -\infty$. So we have

$$\lim_{\nu\to\infty} \sum_{i=2}^{n} \frac{U_1^i}{U_2^i}(\overline{q}_N^{\nu}, I_i^{\nu}) = -\infty .$$

Hence $\lim_{\nu\to\infty} \frac{df}{dq}(\overline{q}_N^{\nu}) = \infty$. Therefore we have $\overline{q}_N^{\nu} \to 0 \quad (\nu \to \infty)$. Q.E.D.

From Propositions 11 and 12, we can draw Figure 2 which indicates the full behavior of equilibrium activity level when an allowance leven and the households' income levels change. All curves do not intersect each other and are nondecreasing in the southeast triangle of the box-diagram of Figure 2, which is shown in Proposition 9, and they may intersect each other and may be decreasing in the north-west triangle, which is pointed out in the previous section. Further, Propositions 4(ii) and 12 say that curves become flat when they approach the bottom of the box-diagram.

5. Conclusion

We have shown that an economy with external diseconomies can be considered as a variation of a public good economy and can be treated in the same way by introducing the concept of allowance level. It is a device for the resolution of the conflict among the damaging firm and the suffered households to introduce an allowance level and to make negotiations. The resolution of the conflict means just that all the members' consent is reached. As we have shown, ratio equilibrium (which is the result of the negotiation)[14]

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[14] The argument in Kaneko [7, 8] can be applicable to the economy of this paper without essential change. That is, the core of the voting game coincides with the ratio equilibria in the economy with an exogenously given allowance level.

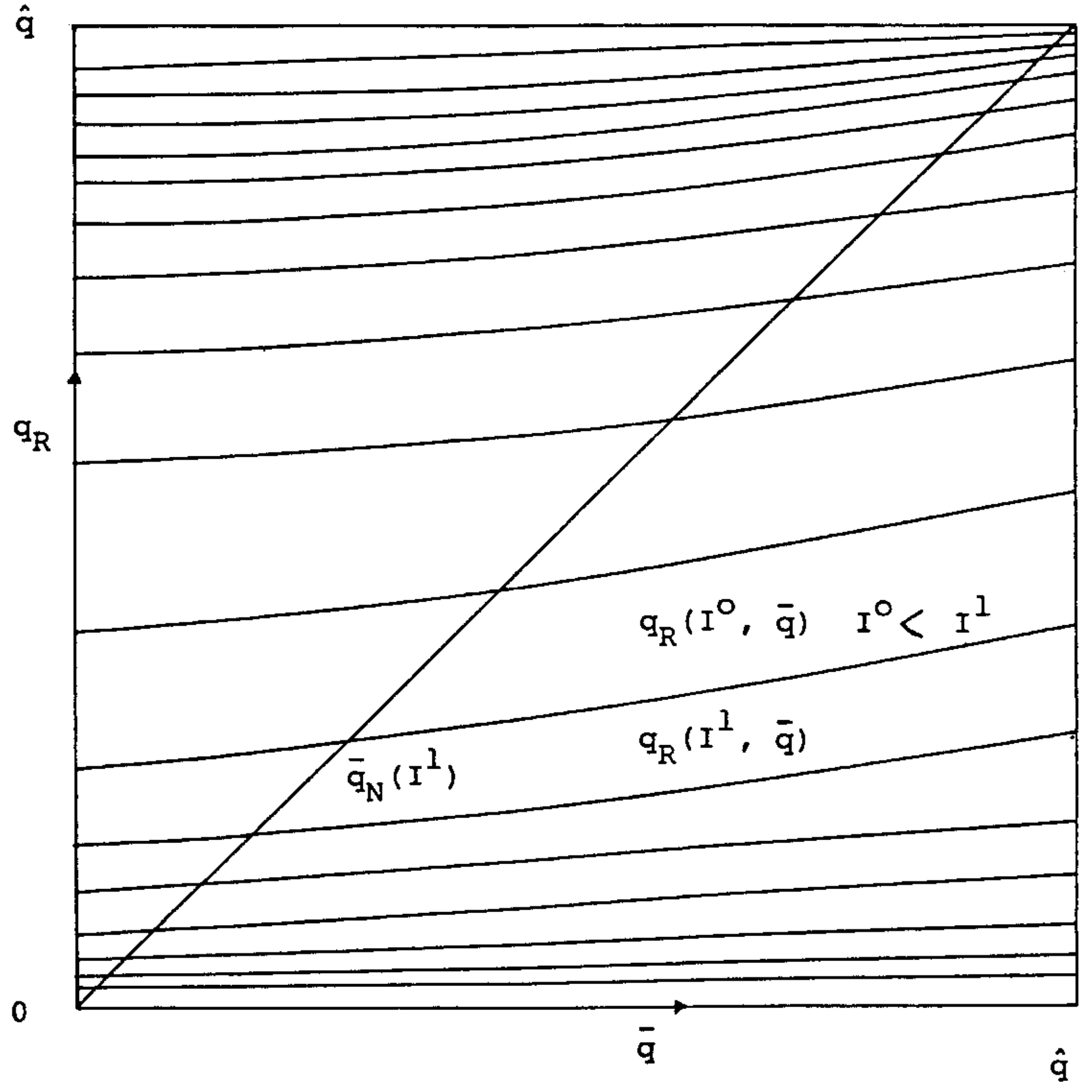

Figure 2

has almost regular behavior , though it may have the possibility that it behaves irregularly, which was pointed out in Section 3. There remain, however, two interrelated problems which we should consider. One is the problem of the determination of an allowance level. Another one is the problem of consideration of our study from view point of social welfare. The latter is rather conceptual but more important than the former. The latter would concern income distribution among not only the damaging firm and the suffered households but also the households who work in the firm and others. Probably, only the answer to the latter can answer the former completely. But, now these problems remain still open.

If the society which includes the community of the firm and the households is fairly rich, then redistribution of incomes is not an improtant problem from view point of social welfare. Further, if the external diseconomies are not accumulated, e.g., noises, smokes without heavy metals, etc., then it is sufficient to consider the community as a static model. Hence in this case, we can focus ourselves on the resolution of the conflict. Then we can propose the neutral allowance level as a convenient device for it. Because if the neutral allowance level is set, then the state of it without income negotiation and assures the consent of all the members in the community. But the neutral allowance level is known by an observer only if the environment of the community, i.e., the initial endowment of income, the utility functions and the firm's cost function, etc., are known by the observer. This is neither easy nor costless. For this problem, the game theoretical approach would be helpful. That is, we should consider the possibility of designing a game such that a neutral allowance level is determined as a result of the game.

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