12-1-1980

Housing Market with Indivisibility

Mamoru Kaneko

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 571

HOUSING MARKET WITH INDIVISIBILITY

MAMORU KANEKO

December 1980

HOUSING MARKET WITH INDIVISIBILITY*

by

Mamoru Kaneko**

November 1980

Abstract: In this paper we present a model of rental housing market in which houses are treated as indivisible commodities. We provide a recursive equation which determines a competitive equilibrium and argue that we can regard the competitive equilibrium as a representative of the set of all competitive equilibria. Using this representative equilibrium, we provide several propositions on comparative statics, that is, we consider how the competitive rents change when certain parameters of the model change.

*The author wishes to express his thanks to Professor K. Iwai for valuable discussions and comments.

**Institute of Socio-Economic Planning, University of Tsukuba, Sakura-mura, Niihari-gun, Ibaraki-ken 300-31 Japan, Current Address: Cowles Foundation for Research in Economics at Yale University, Box 2125, Yale Station, New Haven, Connecticut 06520 USA.

The research in this paper was partially sponsored by the Office of Naval Research Contract Number N00014-77-C-0518.

1. Introduction

Many authors have considered housing markets from theoretical points of view, formulating several types of mathematical models.[1] For example, Alonso studied a housing market on a linear or circular city in his classical work [1], and Backmann [2], Montesano [11], MacKinnon [6], Romanos [12] and others considered varieties of it more rigorously. One common feature among them is to formulate housing markets as models in which households or house-owners can choose sizes of their houses freely. That is, they assume that houses were perfectly divisible like the standard commodities in the usual equilibrium analysis. Of course, this can be observed in existent housing markets to a certain extent. We know, however, another feature of existent housing markets that households or house-owners can select any houses freely under their budgets but can not select sizes freely. For example, consider rental housing markets. That is, if house-owners have already built houses, then households can only select one or more from them. Then they encounter indivisibility. Existent housing markets have these two features, which are entangled with each other. We should also study the second extreme case, which would help our understanding of existent housing markets. The purpose of this paper is to provide a model of housing market in which houses are treated as indivisible commodities.

A general theory for this feature has been developed in a related field to game theory, i.e., Böhm-Bawerk [3], von Neumann and Morgenstern [16], Gale [4], Shapley and Shubik [13], Telser [15], Kaneko [7, 9].

---

[1] Of course, there are many more papers which studied housing markets from empirical points of view. But we do not refer to them in this paper.

It has concerned the relations among competitive equilibrium, the core of market game, or the dual prices of assignment problem and the existence of them.[2] The theory seems to be enough matured to be applied to the problem of housing markets, which motivated originally the theory. The attempt of this paper is on lines of this. Exactly speaking, this paper is an application of Kaneko [9] to a housing market.

In the model of this paper, houses are indivisible and every household wants to rent at most one unit of house. We treat other consumption commodities as one composite commodity, which we call "money." We consider competitive equilibria in this market model. We provide a certain recursive equation which determines a competitive equilibrium and argue that it can be regarded as a representative of the set of all competitive equilibria. Using these results, we present several propositions on comparative statics, that is, we consider how the competitive rents change when certain parameters of the model change.

This paper is written as follows. In Section 2, we describe a model of housing market. In Section 3, we define competitive equilibrium and provide several theorems which describe fully the structure of the set of all competitive equilibria in the market. In Section 4, we provide several propositions on comparative statics. Throughout this paper, we provide several numerical examples which would help our intuitive understanding of the model and the results.

---

[2] Koopmans and Beckmann [3], Heffley [5] and others considered location problems of industries in terms of assignment problems. The purpose of these studies are a little different from that of the above works. The work of Sweeney [14] is closer to the subject of this paper. The basic idea of this paper is the same as Sweeney's.

2. A Housing Market (M,N)

We consider a mathematical model of rental housing market (M,N) in which s-kinds of houses appear. We call a house an apartment in this paper. Let $M = \{1, 2, \dots, m\}$ and $N = \{1', 2', \dots, n'\}$. We call a member $i \in M$ a landlord and a member $j \in N$ a household. Each landlord $i \in M$ owns $w^i = (w^i_1, \dots, w^i_s)$ units of apartments which he can lease to households. We assume that each apartment is indivisible. Hence each w^i_k $(k = 1, \dots, s, \ i \in M)$ is a nonnegative integer. We assume:

$$\text{(A)} \quad \sum_{i \in M} w^i_k > 0 \quad \text{for all} \quad k = 1, \dots, s \quad \text{and} \quad \sum_{k=1}^{s} \sum_{i \in M} w^i_k \geqq n .$$

That is, the potential supply of each apartment is positive and the potential total supply is not smaller than the number of households. No household in N rents any apartment in the beginning. Each household $j \in N$ owns $I_i > 0$ amount of available money for renting an apartment. Of course, money should be interpreted as a "composite commodity." We also may call I_i household i's income. We assume that money is perfectly divisible. Hence I_i can be represented as a real number. We reorder the households as follows:

$$(2.1) \qquad I_1 \geqq I_2 \geqq \dots \geqq I_n .$$

We assume a very simple objective function for the landlords. Each landlord $i \in M$ has the evaluation function $u^i(x)$ on the set $X^i \equiv \{x = (x_1, \dots, x_s) : 0 \leqq x_k \leqq w^i_k$ and x_k is an integer for all $k = 1, \dots, s\}$ such that for all $x \in X^i$,

(B) $u^i(x) = \sum_{k=1}^{s} u^i(x_k e^k)$ and $u^i(x_k e^k) = a_k x_k$ for all $k = 1, \ldots, s$,

where e^k is the s-dimensional vector with $e^k_j = 1$ if $j = k$ and $e^k_j = 0$ if $j \neq k$ and a_k is a positive real number. Note that a_k is independent of landlords in M .

Here we should explain assumption (B). First we note that x in (B) means the units of remaining apartments, i.e., $w^i - x$ is the units of apartments which landlord i leases to households. a_k is the least value at which a landlord can lease one unit of the k^{th} apartment without decreasing his utility less than that of the state of not leasing it when the price of money is equal to 1. Hence when landlord i leases $w^i - x$ units of apartments at rents $r_1, r_2, \ldots, r_s$, his utility (profit) is

$$(2.2) \qquad u^i(x) + \sum_{k=1}^{s} r_k(w^i_k - x_k) = \sum_{k=1}^{s} a_k x_k + \sum_{k=1}^{s} r_k(w^i_k - x_k) .$$

It should be noted that a monotone transformation of (2.2) has the same meaning as (2.2). For example, subtracting $c = \sum_{k=1}^{s} a_k w^i_k$ from (2.2), we get

$$u^i(x) + \sum_{k=1}^{s} r_k(w^i_k - x_k) - \sum_{k=1}^{s} a_k w^i_k = \sum_{k=1}^{s} (r_k - a_k)(w^i_k - x_k) .$$

This formula means really landlord i's profit when he leases $w^i - x$ units of apartments. Kaneko [9] explained the conditions for a preference ordering to be represented as (2.2). See also [8]. We call a_k the evaluation value of the k^{th} apartment.[3]

In this paper we always assume that the price of money is equal to 1.

We assume that each household $j \in N$ has the same consumption set $Y \equiv \{0, e^1, \ldots, e^s\} \times R_+$, where R_+ is the set of all nonnegative real numbers. For convenience sake, we may use the notation $e^{s+1} = 0$ and $a_{s+1} = 0$ in the following. $(x,m) \in Y$ means that if $x = e^k$ $(k \leq s)$, then household j rents one unit of the k^{th} apartment and his available money is m after paying the rent and if $x = e^{s+1} = 0$, he does not rent any apartment and of course, $m = I_j$. This formulation already assumes that any households never rent more than one unit of apartments.[4]

---

[3]The evaluation function means the substitutional relation between apartments and money but not any relation between apartments and nominal profits. One possible interpretation of this evaluation function is as follows. When landlord i builds w_k^i-units of the k^{th} apartment and the cost is M measured in terms of the composite commodity, a_k is determined by

$$M/w_k^i = \sum_{t=1}^{T} a_k(1+\gamma)^t ,$$

where γ is the interest rate and T is the rentable term. If we employ this interpretation, we should reinterpret the initial endowment w_k^i as a potential supply which is not built before contract. This is the simplest interpretation, and we could provide a more complicated one. As we work in statics and do not concern this much more, we do not specify it. Anyway, a_k is measured in terms of money (the composite commodity) but not nominal value. If the price of money p is not equal to 1, the landlords' objective function is written as

$$u^i(x) + \sum_{k=1}^{s} r_k(w_k^i - x_k)/p \quad \text{or} \quad pu^i(x) + \sum_{k=1}^{s} r_k(w_k^i - x_k) .$$

[4]In Kaneko [9] a more general consumption set is employed and this property is assumed on preference relations.

Each household $j \in N$ has the same <u>preference relation</u> R on Y. $(x, m_1)R(y, m_2)$ means that each household prefers (x, m_1) to (y, m_2) or is indifferent between them. We assume that R is a weak ordering.[5] We define the <u>strict preference</u> P and the <u>indifferent relation</u> Q by

$$(x, m_1)P(y, m_2) \text{ iff not } (y, m_2)R(x, m_1) ,$$

$$(x, m_1)Q(y, m_2) \text{ iff } (x, m_1)R(y, m_2) \ \& \ (y, m_2)R(x, m_1) .$$

We assume:

(C) for all $(x,m) \in Y$, if $\delta > 0$, then $(x, m+\delta)P(x,m)$

(D) if $(x, m_1)P(y, m_2)$, then there is an $m_3 \geq 0$ such that $(x, m_1)Q(y, m_3)$,

(E) if $(x, m_1)Q(y, m_2)$, $m_1 < m_2$ and $\delta > 0$, then $(x, m_1 + \delta)P(y, m_2 + \delta)$.

Assumptions (C) and (D) are called "Monotonicity with respect to money" and "Archimedean property," respectively. These are quite mild and need no particular comment. Assumption (E) deserves a small comment. It means that the marginal utility of money is diminishing, in other words, the housing is a normal commodity, i.e., as income rises, the demand of a better apartment increases. If the conclusion of it is replaced by that $(x, m_1 + \delta)Q(y, m_2 + \delta)$, this condition together with

---

[5] R is said to be a <u>weak ordering</u> iff $(x, m_1)R(y, m_2)$ or $(y, m_2)R(x, m_1)$ for all $(x, m_1), (y, m_2) \in Y$ and $(x, m_1)R(y, m_2) \ \& \ (y, m_2)R(z, m_2)$ imply $(x, m_1)R(z, m_3)$.

(C) and (D) becomes a condition for transferable utility, in other words, a condition for consumer's surplus to be well-defined.[6] As well-known, this case cannot permit any income effect. When we consider a market such as a housing market, we cannot neglect the income effect because rents are not negligible relatively to income. This consideration makes it easier to understand assumption (E). That is, as the income level becomes lower, additional income δ becomes more important. This is the income effect and the implication of (E).

The following example provides a preference relation satisfying the above assumptions. In the succeeding example, we will use the same terminologies and so note it.

Example 1. Let $h_1, h_2, \ldots, h_s$ be positive real numbers but $h_{s+1} = 0$, and let $g(m)$ be a strictly concave, continuous and increasing function on R_+ with $\lim_{m\to\infty} g(m) = +\infty$. The utility function $U(e^k, m) = h_k + g(m)$ for $k = 1, \ldots, s, s+1$ gives a preference relation R on Y, i.e., $U(e^k, m_1) \geqq U(e^t, m_2)$ iff $(e^k, m_1)R(e^t, m_2)$. This preference relation satisfies the above assumptions. Of course, any monotone transformation of this $U(x,m)$ is also a utility function representing R.[7]

---

[6]See Kaneko [8].

[7]Sweeney [14] introduced the concept of convexity of R in the usual sense of equilibrium analysis as follows: If $(x, m_1), (y, m_2) \in Y$, $0 \leqq \gamma \leqq 1$, and $(\gamma(x, m_1) + (1-\gamma)(y, m_2)) \in Y$, then $(\gamma(x, m_1) + (1-\gamma)(y, m_2))R(x, m_1)$ or $(\gamma(x, m_1) + (1-\gamma)(y, m_2))R(y, m_2)$. But we can easily find an example which satisfies the convexity but not assumption (E). For example, the preference relation generated by $U(e^k, m) = h_k + m^2$ satisfies the convexity but not (E). Let $h_1 = 21$ and $h_2 = 10$. Then $U(e^1, 5) = 21 + 25 = 10 + 36 = U(e^2, 6)$, but $U(e^1, 6) = 57 < 59 = U(e^2, 7)$.

<u>Lemma 1(i)</u>. If $(x, m_1)Q(y, m_2)$ and $0 < \delta \leqq m_2 < m_1$, then $(x, m_1 - \delta)P(y, m_2 - \delta)$.

(ii). If $(e^k, 0)P(e^t, 0)$ and $(e^k, m_1)Q(e^t, m_2)$, then $m_1 < m_2$.

<u>Proof</u>. See Appendix.

We assume for simplicity that when households consume no amount of money, there is no indifference relation between any pair of apartments, i.e., for any k and t with $k \neq t$, $(e^k, 0)P(e^t, 0)$ or $(e^t, 0)P(e^k, 0)$.[8] Further we reorder $1, \ldots, s$ such that

(F) $(e^1, 0)P(e^2, 0)P \ldots P(e^s, 0)$.[9]

Assumption (F) is equivalent to the following condition (2.3), which gives us a clearer meaning of assumption (F):

$$(2.3) \qquad (e^1, m)P(e^2, m)P \ldots P(e^s, m) \quad \text{for all} \quad m \geqq 0 .$$

That is, if the amounts of consumable money are the same, the households have the same preference relation on the apartments as that given by (F). The proof is easy: if $(e^{k+1}, m_1)Q(e^k, m)$, then $m_1 > m$ by Lemma 1(ii), which implies $(e^k, m)P(e^{k+1}, m)$.

We define a function $G(k)$ $(k = 1, \ldots, s)$ by

---

[8] It is easily verified under our assumptions that $(e^k, 0)Q(e^t, 0)$ if and only if $(e^k, m)Q(e^t, m)$ for all $m \geqq 0$. So, further, if $a_k = a_t$, then we can regard these apartments as exactly the same one. Hence this assumption would be innocuous.

[9] Note that we do not necessarily assume that the apartments are ordered according to the distances from the center of the city. We just order them according to the qualities. See Example 2.

$$(2.4) \qquad G(k) = \sum_{t=1}^{k} \sum_{i \in M} w_t^i .$$

If $G(k) \leqq n$, then it can correspond to the household $G(k)$. In this case we call $G(k)$ the k^{th} marginal household. Assumption (A) implies

$$(2.5) \qquad G(f-1) < n \leqq G(f) \quad \text{for some} \quad f \leqq s .$$

We call f the marginal apartment. These concepts, the marginal households and the marginal apartment, will play essential roles in this paper.[10]

We define a vector $(p_1, \ldots, p_{f-1})$ backward recursively by

$$(2.6) \qquad \begin{aligned} &(e^f, I_{G(f-1)} - a_f) Q (e^{f-1}, I_{G(f-1)} - p_{f-1}) \\ &(e^{f-1}, I_{G(f-2)} - p_{f-1}) Q (e^{f-2}, I_{G(f-2)} - p_{f-2}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &(e^2, I_{G(1)} - p_2) Q (e^1, I_{G(1)} - p_1) . \end{aligned}$$

Note that $I_{G(k)}$ is the income of the marginal household $G(k)$ $(k = 1, \ldots, f-1)$.

In fact, we will construct a competitive equilibrium using this rent vector $(p_1, \ldots, p_{f-1})$, i.e., will show that this rent vector forms a competitive rent vector under appropriate assumptions. The following lemma provides a condition for (2.6) to own a unique solution.

---

[10] We order the households according to the income levels and attach households with larger incomes to better apartments. But these are just notations but not an *a priori* assumption. In Theorem 3, we derive this fact as a result. This is much different from the studies of Beckmann [2] or Montesano [11].

<u>Lemma 2</u>. If $I_n \geqq a_f$ and $(e^f, I_n - a_f)P(e^1, 0)$, then there is a unique vector $(p_1, \ldots, p_{f-1})$ which satisfies (2.6), and it has the following property:

$$(2.7) \qquad p_1 > p_2 > \ldots > p_{f-1} > a_f .$$

<u>Proof</u>. See Appendix.

We assume the supposition of Lemma 2 and the following condition:

(G) $I_n \geqq a_f$, $(e^f, I_n - a_f)P(e^1, 0)$ and $(e^f, I_n - a_f)P(e^k, I_n - a_k)$ for all k $(f < k \leqq s+1)$

(H) $p_k > a_k$ for all $k = 1, \ldots, f-1$.

Assumption (G) means that the household with the least income can rent the marginal apartment if the rent is the least, i.e., the landlords' evaluation value a_f and that he prefers it to the best apartment with zero consumption, and that he chooses the f^{th} apartment among ones which are worse than it even when the rents are the landlords' evaluation values. p_{f-1} means the maximal amount which household $G(f-1)$ pays for the $f-1^{th}$ apartment if he can also rent the f^{th} one at a_f . p_{f-2} is the maximal amount which household $G(f-2)$ pays the $f-2^{th}$ apartment if he can also rent the $f-1^{th}$ one at p_{f-1} . Further we define $p_{f-3}, \ldots, p_1$ backward recursively. Assumption (H) means that these values are greater than the landlords' evaluation values. Under these conditions, it will be shown in the next section that these values $p_1, \ldots, p_{f-1}$ form a competitive equilibrium.

Now we should consider what we defined. Let j be a household

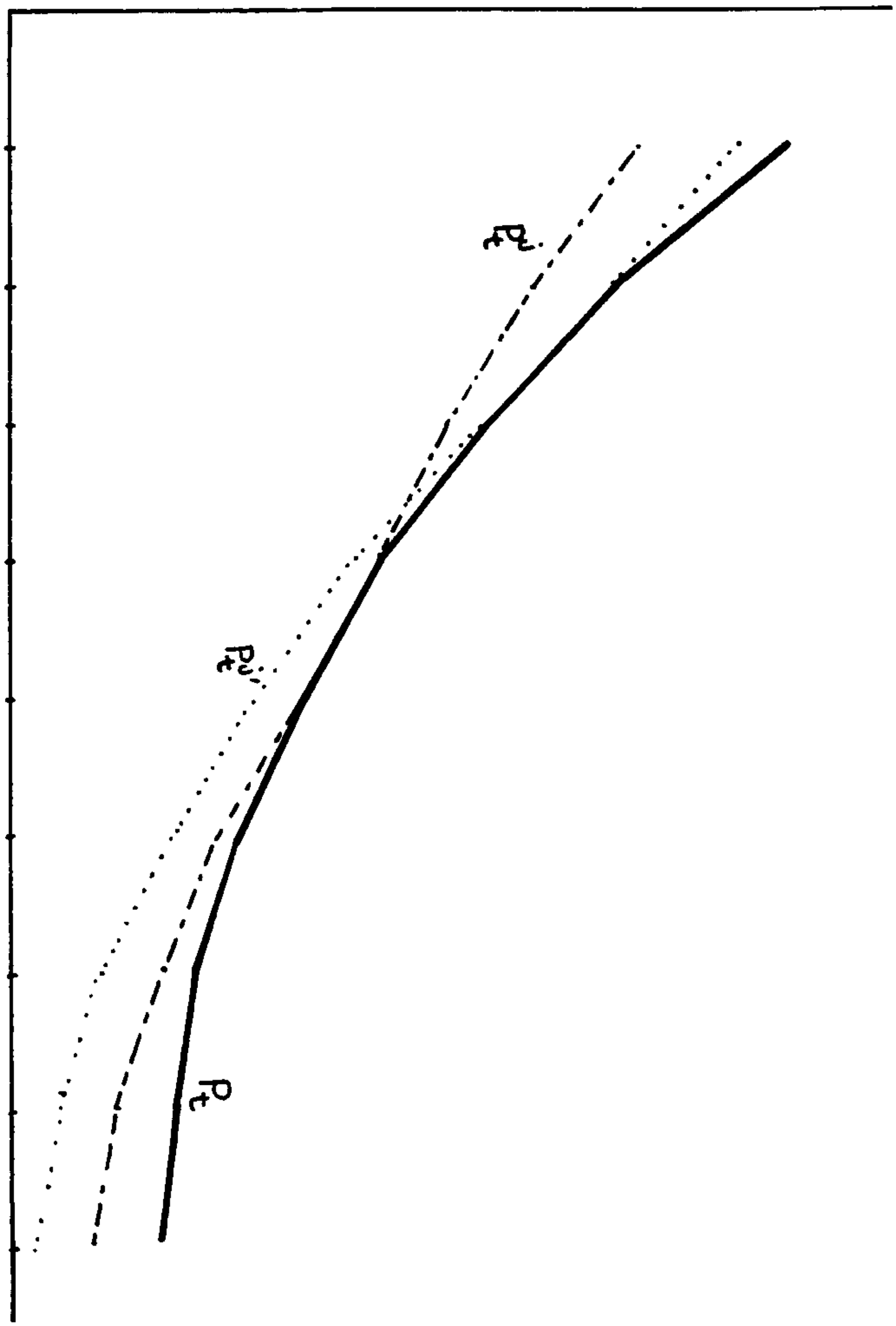


Figure 1

with $I_{G(k-1)+1} \leqq I_j \leqq I_{G(k)}$ and let us define $p^j = (p_1^j, \ldots, p_s^j)$ by

$$(e^t, I_j - p_t^j) Q (e^k, I_j - p_k) \quad \text{for all} \quad t = 1, 2, \ldots, s ,$$

where p is the vector given by (2.6). p^j is the rent vector such that if the rent of the k^{th} apartment is p_k, then it makes the household indifferent between the k^{th} one and every other. The following lemma implies that p is the envelope curve of $p^1, p^2, \ldots, p^n$.[11] See Figure 1. From this fact, we know that p has a shape like a convex curve, though, of course, it is not necessarily exactly convex.

The following lemma becomes necessary in the next section.

<u>Lemma 3</u>. Let I be an income level such that $I_{G(k-1)+1} \leqq I \leqq I_{G(k)}$ $(1 \leqq k \leqq f-1)$. Then the following propositions hold:

(i) $(e^k, I - p_k) R (e^{k+1}, I - p_{k+1}) R \ldots R (e^{f-1}, I - p_{f-1}) R (e^f, I - a_f)$ and $(e^f, I - a_f) P (e^t, I - a_t)$ for all $t > f$ with $I \geqq a_t$.

(ii) $(e^k, I - p_k) R (e^{k-1}, I - p_{k-1}) R \ldots R (e^{t+1}, I - p_{t+1}) R (e^t, I - p_t)$ if $p_t \leqq I$.

<u>Proof</u>. See Appendix.

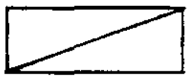
Before discussing the main subjects of this paper, we check the consistency of our model by the following example, and it also helps our understanding of the mathematical model.

<u>Example 2</u>. Let us consider a model of a city with one center. Every household is employed by factories, banks or universities, etc., which are located in the center of the city. Every day he goes to his office

---

[11] Lemma 3 implies $p_t^j \leqq p_t$ for all $t \leqq f$ and, by definition, it holds that $p_k^{G(k)} = p_k$ & $p_{k+1}^{G(k)} = p_{k+1}$ for all $k \leqq f-1$.

in the center. Everyone uses the same transportation system. Let the potential supply of apartments, i.e., $\sum_{i \in M} w^i$, be represented by Table 1.

 means that there is no supply of apartment of the kind. Note that it is not necessary to specify exact numbers of supply. Let the households' utility function be

$$V(s,t,m) = 6\sqrt{10s} + 5\sqrt{80-t} + 2\sqrt{m} ,$$

where s = number of rooms, e.g., $s = 3$ means a 3-room apartment,

t = distance from the center, measured in terms of minutes, e.g., $t = 30$ means that it takes 30 minutes to go to the center,

m = amount of available money after paying rent, e.g., $m = 750$ means that a household can spend 750\$ on consumption for a month.

h_k's in Example 1 are reached by calculating $6\sqrt{10s} + 5\sqrt{80-t}$, and we get Table 1 by reordering them. Let the landlords' evaluation values of apartments be given by Table 2. Further we assume that the marginal apartment is the 13^{th}, and that the incomes of the marginal households are given by Table 3 and I_n = 700\$/month . Finally, we get Table 4 solving the following backward equation:

$$h_{13} + 2\sqrt{800-160} = h_{12} + 2\sqrt{800-p_{12}}$$

$$h_{12} + 2\sqrt{900-p_{12}} = h_{11} + 2\sqrt{900-p_{11}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_2 + 2\sqrt{1900-p_2} = h_1 + 2\sqrt{1900-p_1} .$$

It is not difficult to verify that this example satisfies all assumptions (A) through (H). Note that as will be shown, $(p_1, \ldots, p_{f-1})$ forms

t \ s	1		2		3		4	
10	10	60.80	4	68.66				
20	13	57.70	6	65.56	2	71.59		
30	14	54.33	9	62.19	5	68.22	1	73.31
40	16	50.59	12	58.45	8	64.48	3	69.57
50	17	46.36	15	54.22	11	60.25	7	65.34

Table 1. | k | h_k |

t : distance (minutes)

s : number of rooms

k : apartment number

t \ s	1		2		3		4	
10	10	180	4	220				
20	13	160	6	200	2	240		
30	14	140	9	180	5	220	1	260
40	16	120	12	160	8	200	3	240
50	17	100	15	140	11	180	7	220

Table 2. | k | a_k |

a_k : the evaluation value ($/month)

t \ s	1		2		3		4	
10	10	1000	4	1600				
20	13	700	6	1400	2	1800		
30	14	ϕ	9	1100	5	1500	1	1900
40	16	ϕ	12	800	8	1200	3	1700
50	17	ϕ	15	ϕ	11	900	7	1300

Table 3. | k | $I_{G(k)}$ | $I_{G(k)}$: the income of household G(k) and I_n = 700

t \ s	1		2		3		4	
10	10	241.6	4	485.1				
20	13	160	6	383.2	2	588.0		
30	14		9	281.8	5	470.4	1	649.6
40	16		12	178.8	8	349.9	3	516.6
50	17		15		11	226.3	7	376.2

Table 4. | k | p_k | (\$/month) and p_{13} = 160

a competitive equilibrium of this market.[12]

From this example we know that our model is applicable to complicated housing markets to a certain extent, but that it cannot be applicable to housing markets in multi-center cities. For example, we cannot analyze market models like that of Romanos [12]. The reason for this is to assume that every household has the same preference relation.

3. The Competitive Equilibria in the Market (M,N)

We are now in a position to discuss the main subjects of this paper, that is, we define competitive equilibrium and provide theorems which describe the structure of the set of all competitive equilibria.

Definition. $(r,x) = (r_1, \ldots, r_s, x^1, \ldots, x^m, x^{1'}, \ldots, x^{n'})$ is said to be a competitive equilibrium iff

(3.1) $$x^i \in X^i \text{ for all } i \in M, \quad x^j \in \{e^1, \ldots, e^s, e^{s+1}\}$$
$$\text{for all } j \in N \text{ and } r_k \geqq 0 \text{ for all } k = 1, \ldots, s,$$

(3.2) $$\sum_{i \in M} x^i + \sum_{j \in N} x^j = \sum_{i \in M} w^i,$$

(3.3) $$\text{for all } i \in M, \quad u^i(x^i) + r(w^i - x^i) = \max_{x \in X^i} (u^i(x) + r(w^i - x)),$$

(3.4) $$\text{for all } j \in N, \quad rx^j \leqq I_j \text{ and } (x^j, I_j - rx^j)R(x,m)$$
$$\text{for all } (x,m) \in Y \text{ such that } rx + m = I_j.$$

[12] In Beckmann [2] and Montesano [11], the transportation costs are taken into account explicitly. If it is assumed ([2], [11]) that all households go the same number of times to the center, it is easy to introduce the transportation costs into our example.

We call $r = (r_1, \ldots, r_s)$ a <u>competitive rent vector</u> and each r_k $(k = 1, \ldots, s)$ a <u>competitive rent</u> of the k^{th} apartment.

Condition (3.1) means that each commodity bundle belongs to the appropriate set. (3.2) means the equivalence of the total supply and the total demand of the apartments. (3.3) means the utility maximization of the landlords. (3.4) means the budget constraint and the utility maximization of the households under the budget constraint. Note that the equivalence of the total demand and the total supply of money is automatically satisfied.

First we show the existence of a competitive equilibrium.

<u>Theorem 1</u>. The rent vector $r = (r_1, \ldots, r_s)$ which is given in the following (i) or (ii) is a competitive rent vector in each case:

(i) If $G(f-1) < n < G(f)$, then

$$(3.5) \qquad r_k = \begin{cases} p_k & \text{if } k < f \\ a_k & \text{otherwise,} \end{cases}$$

where $(p_1, \ldots, p_{f-1})$ is the vector defined by (2.6).

(ii) If $G(f) = n$, then r is defined by

$$(3.6) \qquad r_k = a_k \text{ for all } k \geqq f+1, \quad a_f \leqq r_f \leqq p_f^* \text{ and}$$
$$(e^k, I_{G(k)} - r_k)Q(e^{k+1}, I_{G(k)} - r_{k+1}) \text{ for all } k = 1, 2, \ldots, f-1,$$

where p_f^* is defined by $(e^f, I_n - p_f^*)Q(e^t, I_n - a_t)$ for some t such that $t \geqq f+1$ and $(e^t, I_n - a_t)R(e^k, I_n - a_k)$ for all k with $k \geqq f+1$ and $I_n \geqq a_k$.[13]

[13] Note that we permit $k = s+1$ or $t = s+1$.

This theorem says not only the existence of a competitive equilibrium but also the procedure of computing a competitive rent vector. In each case the backward recursive equation (2.6) provides a competitive equilibrium. So, Table 4 provides a competitive rent vector of the market of Example 2. In the case (ii), there exists multi-competitive rent vectors. We will consider further this case in the following.

In Kaneko [9], the existence of a competitive equilibrium is proved under a much more general assumption but with additional one.[14] For example, a variety of landlords' and households' preference relations is permitted. Hence the existence theorem can be applied to a multi-center model with multi-transportation systems. The existence theorem, however, provides no explicit shape of rent curve.

Proof. We prove in the case (i) that the rent vector is competitive. As we can show similarly it in the case (ii), we omit the proof of (ii).

Since Lemma 2 ensures the existence and the uniqueness of $(p_1, \ldots, p_{f-1})$, it is sufficient to construct a vector $x = (x^1, \ldots, x^n, x^{1'}, \ldots, x^{n'})$ and to show that (r,x) is a competitive equilibrium. We define $(x^1, \ldots, x\ , x^{1'}, \ldots, x^{n'})$ as follows:

$$(3.7) \qquad x_k^i = \begin{cases} 0 & \text{if } k < f \\ w_k^i & \text{if } k > f \end{cases},$$

$$\sum_{i \in M} x_f^i + \sum_{j \in N} x_f^j = \sum_{i \in M} w_f^i \quad \text{and} \quad 0 \leq x_f^i \leq w_f^i \quad \text{for all } i \in M ,$$

$$(3.8) \qquad x^j = e^k \quad \text{if } G(k-1) < j \leq G(k) \quad \text{and } k = 1, \ldots, f .$$

---

[14] That is, assumption (D) of [9] is not assumed in this paper and Example 2 does not satisfy it. But this assumption concerns the exclusion of corner solutions and is not so strong. It is easy to modify Example 2 to satisfy assumption (D).

It is easily verified that $(x^1, \ldots, x^m, x^{1'}, \ldots, x^{n'})$ satisfies (3.1) and (3.2). We should check (3.3) and (3.4).

Since $r_k > a_k$ for all $k = 1, \ldots, f-1$ by assumption (H) and $r_k = a_k$ for all $k = f, \ldots, s$ by (3.5), it holds that for all $i \in M$,

$$u^i(x^i) + r(w^i - x^i) = \max_{x \in X^i} (u^i(x) + r(w^i - x)) .$$

Let j be a household such that $G(k-1) + 1 \leqq j \leqq G(k)$. Lemma 3 says that if $r_t \leqq I_j$, then $(e^k, I_j - r_k)R(e^t, I_j - r_t)$.

Q.E.D.

Theorem 2. Let us consider the case (ii) of Theorem 1. Each r_f with $a_f \leqq r_k \leqq p_f^*$ determines exactly one competitive rent vector satisfying (3.6). If $r_f > r'_f$ and r, r' are the competitive rent vectors determined by r_f and r'_f, then it holds that

(3.9) $$0 < r_1 - r'_1 < r_2 - r'_2 < \ldots < r_{f-1} - r'_{f-1} < r_f - r'_f ,$$

(3.10) $$1 < r_1/r'_1 < r_2/r'_2 < \ldots < r_{f-1}/r'_{f-1} < r_f/r'_f .$$

Proof. It can be proved in the same way as Lemma 1 that each r_f determines exactly one vector. Since it can be shown similarly by Lemma 2, that $r'_1 > r'_2 > \ldots > r'_f$. (3.10) follows (3.9). But we omit the proof of (3.9) because it is almost the same as that of Theorem 5, which is much more important.

Q.E.D.

<u>Example 3</u>. Let us consider the housing market of Example 2, and let $n = G(13)$. Then the competitive rent vector r^* determined by p^*_{13} is provided by Table 5. Table 5 provides also the vector determined by $r_{13} = a_{13}$, the differences and the proportions between all pairs of their rents.

t \ s	1			2			3			4		
10	10	295.2	53.6	4	532.2	47.1						
		241.6	1.222		485.1	1.097						
20	13	216.9	56.9	6	432.7	49.5	2	633.1	45.1			
		160	1.356		383.2	1.129		588.0	1.077			
30	14	140	0	9	334.1	52.3	5	517.8	47.4	1	693.6	44.0
		140	1		281.8	1.186		470.4	1.101		649.6	1.068
40	16	120	0	12	234.9	56.1	8	400.2	50.3	3	563.1	46.5
		120	1		178.8	1.314		349.9	1.144		516.6	1.090
50	17	100	0	15	140	0	11	280.5	54.2	7	425.8	49.6
		100	1		140	1		226.3	1.240		376.2	1.132

Table 5.

k	r^*_k	$r^*_k - r_k$
	r_k	r^*_k/r_k

This result could be explained verbally as follows. The competitive rent vectors we think of are determined by the marginal households $G(k)$, i.e., (3.6). A household $G(k)$ is indifferent between $(e^k, I_{G(k)} - r'_k)$ and $(e^{k+1}, I_{G(k)} - r'_{k+1})$. In this case he should pay a higher rent for the k^{th} apartment than for the $k+1^{th}$ one because the k^{th} one is better than the other. So the remaining income is lower in the case of renting the k^{th} one than that in the other. If both rents rise a little, but if he also is indifferent between them, then the increment of the better one should be smaller than that of another, because he should pay the additional payment from his smaller budget. This is the implication of assumption (E).

We considered the competitive rent vector of the special type, i.e., that defined by (3.5) or (3.6). But this can be a representative of all competitive rent vectors. That is, an arbitrary one has similar properties and is not so different from that defined by (3.5) or (3.6). To show this, we need two theorems.

Theorem 3. Let (r,x) be an arbitrary competitive equilibrium. Then the following propositions hold:

(i) For every $j \in N$, $x^j = e^k$ for some $k \leqq f$, and for every $i \in M$, $x_k^i = 0$ for all $k \leqq f-1$.

(ii) If $I_{j_1} < I_{j_2}$ $(j_1, j_2 \in N)$, $x^{j_1} = e^{k_1}$ and $x^{j_2} = e^{k_2}$, then $k_1 \geqq k_2$.

(iii) $r_k \geqq a_k$ for all $k = 1, \ldots, f$ and $r_k \leqq a_k$ for all $k = f+1, \ldots, s$.

(iv) $r_1 > r_2 > \ldots > r_{f-1} > r_f$.

Proof. Note that if $r_k < a_k$, then the landlords never lease any unit of the k^{th} apartment and that if $r_k > a_k$, then the landlords lease all units of the k^{th} one.

First we show (ii). Suppose $k_1 < k_2$. By assumption (F), $(e^{k_1}, 0)P(e^{k_2}, 0)$. Since (r,x) is a competitive equilibrium, we have $(e^{k_1}, I_{j_1} - r_{k_1})R(e^{k_2}, I_{j_1} - r_{k_2})$. By assumptions (C) and (D) there is a $\delta \geqq 0$ such that $(e^{k_1}, I_{j_1} - r_{k_1})Q(e^{k_2}, I_{j_1} - r_{k_2} + \delta)$. By Lemma 1(ii), we have $I_{j_1} - r_{k_1} < I_{j_1} - r_{k_2} + \delta$. Then we have $(e^{k_1}, I_{j_2} - r_{k_1})P(e^{k_2}, I_{j_2} - r_{k_2} + \delta)R(e^{k_2}, I_{j_2} - r_{k_2})$ by assumptions (E) and (C). This is a contradiction to that (r,x) is a competitive equilibrium.

Next we show (i). Suppose that there is a household j with $x^j = 0$. If $x^{j'} = e^k$ for some $j' \in N$ with $I_{j'} < I_j$ and some $k \leqq s$, then $(e^k, I_{j'} - r_k)R(0, I_{j'})$. Hence we get $(e^k, I_j - r_k)P(0, I_j)$ by assumption (E), which is a contradiction. So we have shown that if $x^j = 0$, then $x^{j'} = 0$ for all j' with $I_{j'} < I_j$. Hence we can assume $x^n = 0$ without loss of generality. If $r_f \leqq a_f$, then $(e^f, I_n - r_f)P(0, I_n)$ by assumption (G), which is a contradiction. So $r_f > a_f$. The total supply of the f^{th} apartment is $\sum_{i \in M} w_f^i$. Hence there are $\sum_{i \in M} w_f^i$ number of households who rent the f^{th} apartments. This and the supposition $x^n = 0$ imply that some household j with $I_j \geqq I_{G(f-1)}$ rents the f^{th} apartment. Hence

$$(e^f, I_j - r_f)R(e^{f-1}, I_j - r_{f-1}) .$$

Since $(e^{f-1}, I_{G(f-1)} - p_{f-1})Q(e^f, I_{G(f-1)} - a_f)$ by (2.6) and $p_{f-1} > a_f$ by Lemma 2, we have $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)$ by assumption (E). Since $r_f > a_f$, $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)P(e^f, I_j - r_f)R(e^{f-1}, I_j - r_{f-1})$, which implies $r_{f-1} > p_{f-1} > a_{f-1}$ by assumptions (C) and (G). Repeating this argument, we get

$$r_t > a_t \quad \text{for all} \quad t = 1, \ldots, f .$$

Then the total supply $\sum_{k=1}^{f} \sum_{i \varepsilon M} w_k^i$ exceeds the number of households who rent one unit of apartment, because n does not rent any apartment. This is a contradiction. Hence every household rents one unit of apartment.

If some household j rents the k^{th} apartment with $k > f$, then $(e^k, I_j - r_k)R(e^f, I_j - r_f)$. But since $(e^f, I_j - a_f)P(e^k, I_j - a_k)$ by Lemma 3 and $r_k \geqq a_k$, $(e^f, I_j - a_f)P(e^k, I_j - a_k)R(e^k, I_j - r_k)R(e^f, I_j - r_f)$ which implies $a_f < r_f$. So the argument of the above paragraph is applicable to this case and a contradiction is derived. Hence $x^j = e^k$ for some $k \leqq f$. If $x_k^i > 0$ for some $i \varepsilon M$ and $k \leqq f-1$, then there is a $j \varepsilon N$ by (ii)such that $I_j \geqq I_{G(f-1)}$ and $x^j = e^f$. In this case we can prove analogously to the above paragraph that $r_k > p_k > a_k$ for all $k \leqq f-1$. Hence we have $x_k^i = 0$ for all $k \leqq f-1$ by the first remark of the proof. This is a contradiction.

Proposition (iii) follows proposition (i) and the note at the beginning of this proof.

Finally we show (iv). Suppose $r_k \leqq r_{k'}$ for some k and $k' \leqq f$ with $k < k'$. Then no household rents the k'^{th} apartment because $(e^k, I_j - r_k)P(e^{k'}, I_j - r_k)R(e^{k'}, I_j - r_{k'})$ by (2.3) and assumption (C). This is a contradiction to (i). Q.E.D.

In order to investigate furthermore the structure of the set of all competitive rent vectors, we define two concepts. We call $r = (r_1, \ldots, r_s)$ the <u>maximal</u> (<u>minimal</u>) <u>competitive rent vector</u> iff

(3.10) $\quad$ r is a competitive rent vector,

(3.11) $\quad$ for any competitive rent vector r', $r_k \geqq (\leqq)\ r'_k$ for all $k = 1, \ldots, s$.

It should be noted that if the maximal (minimal) competitive rent vector exists, then it is unique.

Theorem 4(i).[15] Let $G(f-1) < n < G(f)$. Then the maximal competitive rent vector is given by (3.5). Next let $r = (r_1, \ldots, r_s)$ be defined by

(3.12) $$(e^f, I_{G(f-1)+1} - a_f)Q(e^{f-1}, I_{G(f-1)+1} - r_{f-1})$$
$$(e^{f-1}, I_{G(f-2)+1} - r_{f-1})Q(e^{f-2}, I_{G(f-2)+1} - r_{f-2})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(e^2, I_{G(1)+1} - r_2)Q(e^1, I_{G(1)+1} - r_1)$$

$r_f = a_f$ and $r_k = \max(0, p_k)$ for all $f < k \leqq s$, where p_k's are the numbers such that $(e^f, I_n - a_f)Q(e^k, I_n - p_k)$.

If $r_k \geqq a_k$ for all $k = 1, \ldots, f-1$, then this $(r_1, \ldots, r_s)$ is the minimal competitive rent vector.

(ii). Let $G(f) = n$. The rent vector r determined by (3.6) and p_f^* is the maximal rent vector. The rent vector given by (3.12) is also the minimal competitive rent vector under the assumption that $r_k \geqq a_k$ for all $k = 1, \ldots, f-1$.

From Theorem 4 we get the following corollary.

Corollary 1. Let $G(f-1) < n < G(f)$. Suppose $I_{G(k)} = I_{G(k)+1}$ for all $k = 1, 2, \ldots, f-1$. Then it holds that for any arbitrary competitive rent vector r' ,

(3.13) $$r'_k = r_k \quad \text{for all } k = 1, \ldots, f ,$$

where r is the maximal competitive rent vector.

---

[15] In fact, the maximal and minimal competitive rent vectors correspond to, respectively, the imputations (u^*, v_*) and (u_*, v^*) in the core of the assignment game given in Shapley and Shubik [13, Theorem 3].

If n is large and if the potential supply $\sum_{i\varepsilon M} w_k^i$ of each apartment is large relative to the number of the kinds of apartments, i.e., f or s, then it seldom happens that $G(f) = n$. Further it often happens in this case that $I_{G(k)} = I_{G(k)+1}$ for all $k = 1, \ldots, f-1$. Although this does not hold exactly, we can say that this holds approximately. These assumptions are plausible when we consider housing markets in not too small towns. Further the rents of k^{th} apartments $(k > f)$ are not important because they are not rented actually at all. Therefore we can regard the maximal competitive rent vector as a representative. In the next section we assume:

(I) $G(f-1) < n < G(f)$.

Further we employ the maximal competitive rent vector as a representative of the set of all competitive rent vectors to consider comparative statics.

Proof of Theorem 4. We show that r given by (3.5) of Theorem 1 is the maximal competitive rent vector in the case (i). We can prove almost similarly the other propositions, and so we omit the proofs of them.

Let r' be an arbitrary competitive rent vector. Theorem 3(iii) says that $r_k' \leqq a_k$ for all $k \geqq f+1$. Suppose $r_f' > a_f$. Then the total supply of the f^{th} apartment is $\sum_{i\varepsilon M} w_f^i$. Since r' is a competitive rent vector and $G(f-1) < n < G(f)$, there is a household j with $I_j \geqq I_{G(f-1)}$ who rents one unit of the f^{th} apartment. This implies $(e^f, I_j - r_f')R(e^{f-1}, I_j - r_{f-1}')$. By Lemma 3 we have $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)$, where $p = (p_1, \ldots, p_{f-1})$ is the vector defined by (2.5). Hence we get $(e^{f-1}, I_j - p_{f-1})R(e^f, I_j - a_f)P$ $(e^f, I_j - r_f')R(e^{f-1}, I_j - r_{f-1}')$ by assumption (C), which implies

$p_{f-1} < r'_{f-1}$. Repeating this argument we can get $r'_k > p_k > a_k$ for all $k \leq f$. This means that the total supply is at least $\sum_{k=1} \sum_{i \in M} w^i_k$, which exceeds n . This is a contradiction. Hence we have shown that $r'_f = a_f$.

By Theorem 3 we can assume without loss of generality that in an arbitrary competitive equilibrium (r', x) each k^{th} marginal household $G(k)$ $(k \leq f-1)$ rents one unit of the k^{th} apartment, i.e., $x^{G(k)} = e^k$. Since r' is a competitive rent vector, we have $(e^{f-1}, I_{G(f-1)} - r'_{f-1}) R (e^f, I_{G(f-1)} - a_f)$. But since $(e^{f-1}, I_{G(f-1)} - p_{f-1}) Q (e^f, I_{G(f-1)} - a_f)$ by (2.5), we have $I_{G(f-1)} - r'_{f-1} \geq I_{G(f-1)} - p_{f-1}$ by assumption (C), i.e., $r'_{f-1} \leq p_{f-1} = r_{f-1}$. Hence we have $(e^{f-2}, I_{G(f-2)} - r'_{f-2}) R (e^{f-1}, I_{G(f-2)} - r'_{f-1}) R (e^{f-1}, I_{G(f-2)} - r_{f-1}) Q$ $(e^{f-2}, I_{G(f-2)} - r_{f-2})$ by assumption (C), (3.5) and the supposition that r' is a competitive rent vector. This implies $r'_{f-2} \leq r_{f-2}$ by assumption (C). Similarly we get $r'_k \leq r_k$ for all $k \leq f-1$.

Q.E.D.

4. Comparative Statics

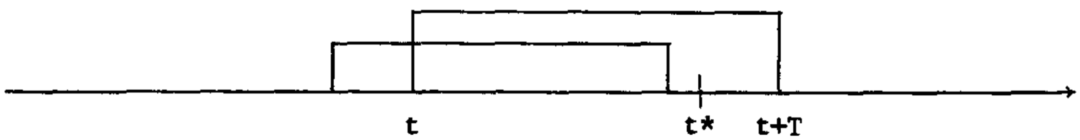
In this section we consider effects of some changes of certain parameters of the housing market (M,N) upon the competitive rent vectors. Let (M^*, N^*) be a new housing market which is yielded by changes of certain parameters from the housing market (M,N) , and which is also assumed to satisfy all the assumptions (A) through (I) of the previous sections. $M^* = \{1^*, \ldots, m^*\}$ is the set of all landlords, and $N^* = \{1^{*'}, \ldots, n^{*'}\}$ is the set of all households. Of course, members in M^* or N^* may be different from those in M or N respectively. Because we think that economic agents are also parameters in our economy.

But the kinds of apartments do not change, i.e., there are s kinds of apartments. The landlords in M^* have the evaluation functions in the form given by assumption (B) with the evaluation values $a_1^*, \ldots, a_s^*$. The households in M^* have the same preference relation R as that of the households in N. Similar to Section 2 we denote the k^{th} marginal household in (M^*, N^*) by $G^*(k)$, i.e.,

$$(4.1) \qquad G^*(k) = (\sum_{t=1}^{k} \sum_{i \in M^*} w_t^i)^* .$$

Let f and f^* be the marginal apartments in (M,N) and (M^*, N^*) respectively, i.e., $G(f-1) < n < G(f)$ and $G^*(f^*-1) < n^* < G^*(f^*)$.

Here we should give a brief interpretation to our comparative statics. The two markets (M,N) and (M^*, N^*) may be considered to be ones held at two different times. Let (M,N) and (M^*, N^*) be markets held at times t and t^* respectively $(t < t^*)$. Every apartment appearing in (M,N) and (M^*, N^*) has a term of contract, which may be different or may be determined endogenously. Now suppose that an apartment owned by a landlord in M is engaged in T year contract with a household at time t. If $t^* < t+T$, then the apartment does not appear in the market (M^*, N^*) unless the contract has been cancelled by time t^*. Hence if the apartment appears in (M^*, N^*), then $t^* \geq t+T$ or the contract is cancelled before time t^* and it has not been engaged between $t+T$ (or time when the contract is cancelled) and time t^*. Of course, apartments appearing in (M^*, N^*) also may not appear in (M,N), that is, they are newly-built ones or ones the contracts of which were made before t and have expired by time t^* or have been cancelled. Thus we provided one interpretation of our comparative statics. But it is much easier and does not yield any conceptual difficulty to interpret (M,N) and (M^*, N^*) as two different cities.[16]



The main result of this section is the following theorem.

<u>Theorem 5</u>. Let $f^* \geqq f$, and assume

$$(4.2) \qquad I_{G^*(1)} - I_{G(1)} \geqq I_{G^*(2)} - I_{G(2)} \geqq \cdots \geqq I_{G^*(f-1)} - I_{G(f-1)} .$$ [17]

Let r and r^* be the maximal competitive rent vectors in (M,N) and (M^*, N^*) respectively. Let k be a number with $1 \leqq k \leqq f-1$. Then the following propositions hold:

(i) $I_{G^*(k)} - I_{G(k)} \gtreqless r^*_k - r_k$ if and only if $r^*_k - r_k \gtreqless r^*_{k+1} - r_{k+1}$.

(ii) If $r^*_k - r_k > r^*_{k+1} - r_{k+1}$, then $r^*_1 - r_1 > r^*_2 - r_2 > \ldots > r^*_k - r_k$.

(iii) If $r^*_k - r_k < r^*_{k+1} - r_{k+1}$, then $r^*_{k+1} - r_{k+1} < \ldots < r^*_f - r_f$.

---

[16] It is conceptually difficult to construct an argument which makes our interpretation complete or consistent in a dynamic situation. Still, I think that the first interpretation is very valuable and helps our understanding of existent housing markets.

[17] In the case of $f^* < f$, this theorem remains true, replacing f by f^* .

Proof. (i) Let $I_{G*(k)} - I_{G(k)} < r_k^* - r_k$. Let $b_k = (r_k^* - r_k) - (I_{G*(k)} - I_{G(k)})$. Since $(e^k, I_{G*(k)} - r_k^*)Q(e^{k+1}, I_{G*(k)} - r_{k+1}^*)$ by (2.6) and $r_k^* > r_{k+1}^*$, we get $(e^{k+1}, I_{G(k)} - r_{k+1})Q(e^k, I_{G(k)} - r_k) = (e^k, I_{G*(k)} - r_k^* + b_k)P(e^{k+1}, I_{G*(k)} - r_{k+1}^* + b_k)$ by assumption (E) and (2.6). This implies $I_{G(k)} - r_{k+1} > I_{G*(k)} - r_{k+1}^* + b_k$, i.e., $r_{k+1}^* - r_{k+1} > r_k^* - r_k$. Let $I_{G*(k)} - I_{G(k)} > r_k^* - r_k$. Then we put $b_k = (I_{G*(k)} - I_{G(k)}) - (r_k^* - r_k)$. Since $(e^k, I_{G(k)} - r_k)Q(e^{k+1}, I_{G(k)} - r_{k+1})$ by (2.6) and $r_k > r_{k+1}$, we get $(e^{k+1}, I_{G*(k)} - r_{k+1}^*)Q(e^k, I_{G*(k)} - r_k^*) = (e^k, I_{G(k)} - r_k + b_k)P(e^{k+1}, I_{G(k)} - r_{k+1} + b_k)$ by assumption (E) and (2.6). This implies $I_{G*(k)} - r_{k+1}^* > I_{G(k)} - r_{k+1} + b_k$, i.e., $r_k^* - r_k > r_{k+1}^* - r_{k+1}$. Similarly we can prove that $I_{G*(k)} - I_{G(k)} = r_k^* - r_k$ implies $r_k^* - r_k = r_{k+1}^* - r_{k+1}$.

(ii) Let $r_k^* - r_k > r_{k+1}^* - r_{k+1}$. Then we get, by (i) and (4.2), $I_{G*(k-1)} - I_{G(k-1)} \geqq I_{G*(k)} - I_{G(k)} > r_k^* - r_k$. Let $b_k = (I_{G*(k-1)} - I_{G(k-1)}) - (r_k^* - r_k)$. Since $(e^{k-1}, I_{G(k-1)} - r_{k-1})Q(e^k, I_{G(k-1)} - r_k)$ by (2.6) and $r_{k-1} > r_k$, we get $(e^{k-1}, I_{G(k-1)} - r_{k-1} + b_k)P(e^k, I_{G(k-1)} - r_k) = (e^k, I_{G*(k-1)} - r_k^*)Q(e^{k-1}, I_{G*(k-1)} - r_{k-1}^*)$ by assumption (E) and (2.6). This implies $I_{G(k-1)} - r_{k-1} + b_k > I_{G*(k-1)} - r_{k-1}^*$, i.e., $r_{k-1}^* - r_{k-1} > r_k^* - r_k$. We can repeat the above argument. So we get the result of (ii).

(iii) Let $r_k^* - r_k < r_{k+1}^* - r_{k+1}$. Then we get, by (i) and (4.2), $I_{G*(k+1)} - I_{G(k+1)} \leqq I_{G*(k)} - I_{G(k)} < r_k^* - r_k < r_{k+1}^* - r_{k+1}$. Hence it follows from (i) that $r_{k+1}^* - r_{k+1} < r_{k+2}^* - r_{k+2}$. We repeat this argument, and so we get the result of (iii).

Q.E.D.

Although this theorem provides a general criterion for changes in rents, the causal relations between changes in parameters and those in rents are not clear. So, we should consider the causal relations.

Corollary 2. Assume that the marginal apartments in (M,N) and (M^*, N^*) are the same, i.e., $f = f^*$ and that $a_f = a_f^*$. Assume that

$$(4.3) \qquad I_{G^*(1)} - I_{G(1)} \geqq \cdots \geqq I_{G^*(f-1)} - I_{G(f-1)} > 0 .$$

Let r and r^* be the maximal competitive rent vectors in (M,N) and (M^*, N^*) respectively. Then it holds that

$$(4.4) \qquad I_{G^*(k)} - I_{G(k)} > r_k^* - r_k \quad \text{for all } k = 1, \ldots, f-1 ,$$

$$(4.5) \qquad r_1^* - r_1 > \cdots > r_{f-1}^* - r_{f-1} > 0 .$$ [18]

[18] Let us consider the case where the price of money (the composite commodity) changes into $p > 1$. Let $a_k^* = a_k$ for all $k = 1, \ldots, s$ and $f = f^*$. Note that the evaluation values a_k or a_k^* reflect the costs of building the apartments measured in terms of money. See footnote 2. Further we assume that the nominal incomes of marginal households do not change, i.e., $I_{G(k)} = I_{G^*(k)}$ for all $k = 1, \ldots, f-1$. Corollary 2 is applicable to these markets. That is, transforming the price p into 1, we get a housing market (M^{**}, N^{**}) such that $a_k^{**} = a_k$ for all $k = 1, \ldots, s$ and $I_{G^{**}(k)} = I_{G(k)}/p$ for all $k = 1, \ldots, f-1$. Of course, (M^{**}, N^{**}) is the same as (M^*, N^*) except the units of the prices. If $r_1^{**}, \ldots, r_s^{**}$ are the maximal competitive rents, then it holds that

$$r_1 - r_1^{**} > \cdots > r_{f-1} - r_{f-1}^{**} > 0 .$$

That is, all of the real rents decrease and further the real rents of better apartments decrease more. But the nominal rents $pr_1^{**}, pr_2^{**}, \ldots, pr_s^{**}$ have the property that $r_k < pr_k^{**}$ for all $k \geqq f$. Therefore there may exist the possibility that the nominal rents of worse apartments rise but simultaneously those of better ones decrease. Regretfully the author has not found any example for this possibility. In all his examples, it holds that $r_k < pr_k^{**}$ for all $k = 1, \ldots, s$.

Proof. If is clear that $r_f = r_f^* = a_f = a_f^*$. Let $r_{f-1}^* \leq r_{f-1}$. Then since $(e^{f-1}, I_{G(f-1)} - r_{f-1}) Q (e^f, I_{G(f-1)} - a_f)$ and $r_{f-1} > a_f$, we get, by assumption (E), $(e^{f-1}, I_{G*(f-1)} - r_{f-1}) P (e^f, I_{G*(f-1)} - a_f)$. Further we have $(e^{f-1}, I_{G*(f-1)} - r_{f-1}^*) R (e^{f-1}, I_{G*(f-1)} - r_{f-1})$ by assumption (C). Hence $(e^{f-1}, I_{G*(f-1)} - r_{f-1}^*) P (e^f, I_{G*(f-1)} - a_f)$, which contradicts (2.6). Hence we get $r_{f-1}^* > r_{f-1}$. Hence $r_{f-1}^* - r_{f-1} > r_f^* - r_f = 0$. Therefore we get (4.4) and (4.5) by Theorem 5(i) and (ii).

Q.E.D.

This corollary says as follows: Assume that the marginal apartment and the landlords' evaluation value for it are the same, and that for each $k < f$, the income of the k^{th} marginal household rises more than (exactly speaking, not smaller than) that of the k+1-marginal household. Then the increments in the maximal competitive rents of more preferred apartments are larger than those of less preferred ones. (4.4) says that the increments in the rents do not exceed those in the incomes. These results depend greatly upon assumption (E). The dependence could be explained intuitively as follows. Let us consider the situation where household j is indifferent between $(e^k, I_j - r_k)$ and $(e^{k+1}, I_j - r_{k+1})$. Since the k^{th} apartment is better than the $k+1^{th}$ one, r_k is higher than r_{k+1} . Hence the consumption level is lower in $(e^k, I_j - r_k)$ than that in $(e^{k+1}, I_j - r_{k+1})$. If his income rises, but if both the rents also rise in the same magnitude which is less than the increment in income, then he chooses the better one, the k^{th} apartment, because the increments in consumption are the same but $I_j - r_k < I_j - r_{k+1}$ and so the increment in $(e^k, I_j - r_k)$ makes him more comfortable than that in $(e^{k+1}, I_j - r_{k+1})$. Hence the demand for the better one increases and violates the market clearing condition. So the rent of the better one rises more than that of the worse. See Figure 2.

This theorem is applicable to the case where the income levels increase proportionally, i.e., for some a, $I_{G*(k)} = (1+a)I_{G(k)}$ for all $k \leqq f-1$. This case is considered in the following example.

Example 4. Let us consider the housing market (M,N) given in Example 2. Let $I_{G*(k)} = (1+0.1)I_{G(k)}$ for all $k \leqq 12$, which is written as Table 6. Let the other parameters be fixed. Then the maximal competitive rent vectors are given as Table 7.

Although the income of each marginal household rises uniformly at 10%, the rates of the increments in the rents are not equal. The rate of the most preferred apartment is 4.6% but that of the 12th one is 0.7%. In this example we get the interesting result that the maximal competitive rent of a more preferred apartment rises more than that of a less in both sense of absolute magnitude and proposition. But the proposition with respect to proportion is not generally true.

Lemma 4. Let $f \leqq f^*$. Assume that $I_{G*(k)} = I_{G(k)}$ for all $k \leqq f-1$. If $r_k^* > r_k$ for some $k \leqq f$, then

$$r_t^* > r_t \quad \text{for all } t \leqq k. \tag{4.6}$$

Proof. Let $r_k^* > r_k$. Since $(e^{k-1}, I_{G(k-1)} - r_{k-1})Q(e^k, I_{G(k-1)} - r_k)P(e^k, I_{G(k-1)} - r_k^*)Q(e^{k-1}, I_{G(k-1)} - r_{k-1}^*)$ by (2.6) and assumption (C), we have $r_{k-1} < r_{k-1}^*$. Repeating this argument, we get (4.6).

Q.E.D.

t \ s	1		2		3		4	
10	10	1100	4	1760				
20	13	770	6	1540	2	1980		
30	14	ϕ	9	1210	5	1650	1	2090
40	16	ϕ	12	880	8	1320	3	1870
50	17	ϕ	15	ϕ	11	990	7	1430

Table 6.

k	$I_{G*(k)}$

t \ s	1			2			3			4		
10	10	246.5	4.9	4	504.8	19.7						
		241.6	1.020		485.1	1.041						
20	13	160	0	6	396.8	13.6	2	613.9	25.9			
		160	1		383.2	1.0355		588.0	1.044			
30	14	140	0	9	289.2	7.4	5	489.2	18.8	1	679.3	29.7
		140	1		281.8	1.026		470.4	1.040		649.6	1.046
40	16	120	0	12	180.0	1.2	8	361.4	11.5	3	538.3	21.6
		120	1		178.8	1.007		349.9	1.033		516.6	1.042
50	17	100	0	15	140	0	11	230.4	4.1	7	389.3	13.1
		100	1		140	1		226.3	1.018		376.2	1.035

Table 7.

k		
k	r_k^*	$r_k^* - r_k$
	r_k	r_k^*/r_k

Corollary 3. Assume that $f = f^*$, and that $I_{G^*(k)} = I_{G(k)}$ for all $k = 1, \ldots, f-1$. And assume that $a_f^* > a_f$. Let r and r^* be the maximal competitive rent vectors in (M,N) and (M^*, N^*) respectively. Then it holds that

$$(4.7) \qquad 0 < r_1^* - r_1 < \ldots < r_{f-1}^* - r_{f-1} < a_f^* - a_f ,$$

$$(4.8) \qquad 1 < r_1^*/r_1 < \ldots < r_{f-1}^*/r_{f-1} < a_f^*/a_f .$$

Proof. Since $r_f = a_f < a_f^* = r_f^*$, we get, by Lemma 4, $r_t^* > r_t$ for all $t \leqq f$. Hence $r_1^* - r_1 > I_{G^*(1)} - I_{G(1)} = 0$, which implies that Theorem 5(iii) is applicable to this case. Therefore, we get (4.7). Since $r_1 > r_2 > \ldots > r_f$ by Lemma 2, we get (4.8).

Q.E.D.

When the marginal apartment and the income levels of the marginal households do not change, but when the evaluation value a_f of the marginal apartment rises, the maximal competitive rents also rise. In this case, however, the shape of the increments in the rents has a different tendency from that of Corollary 2 but rather a converse one, that is, the increment of a less preferred apartment in the rent is larger than that of a more preferred one. This is the same as Theorem 2. As the evaluation value a_f reflects the cost of building one unit of the marginal apartment f, we may think that the increment of the evaluation value a_f is that of the cost of building the apartment.[19] An increment of the cost of building the marginal apartment is one reason for the rises in the rents given in Corollary 3. See

[19] See footnote 2.

Figure 3.

This result could be explained verbally in the same way as the explanation after Theorem 2. Example 3 can become an example for this corollary with a little change. So, we do not give any example here.

Corollary 4. Assume that $a_f^* \geq a_f$, and that $I_{G(k)} = I_{G*(k)}$ for all $k \leq f-1$. And assume that $f < f*$. Let r and $r*$ be the maximal competitive rent vectors in (M,N) and (M*, N*) respectively. Then it holds that

$$(4.9) \qquad 0 < r_1^* - r_1 < \dots < r_f^* - r_f ,$$

$$(4.10) \qquad 1 < r_1^*/r_1 < \dots < r_f^*/r_f .$$

Proof. Since $r_f^* > a_f^* \geq a_f = r_f$ by assumption (G), we have, by Lemma 4, $r_t^* > r_t$ for all $t \leq f$. Hence $r_1^* - r_1 > I_{G*(1)} - I_{G(1)} = 0$, which implies that Theorem 5(iii) is applicable to this case. Therefore, we get (4.9). Since $r_1 > r_2 > \dots > r_f$, by Lemma 2, we get (4.10).

Q.E.D.

When the incomes of the marginal households and the landlords' evaluation values do not change, but when the marginal apartment moves to a worse one, the shape of the increments in the rents has the same tendency as that of Corollary 3. That is, the rent of a less preferred apartment rises more than that of a more preferred one. This case may occur when the population of an urban area participating in the housing market increses, i.e., the demand of apartments increases more than the supply. See Figure 4. The crucial assumption of this corollary is that $I_{G*(k)} = I_{G(k)}$ for all $k < f$, i.e., though the population of the area increases, the increment of population is in the group of households with lower income levels. Hence we can say nothing exactly by this corollary in the case where the population increases uniformly on each group of households with an income level. If we want to consider such a case, then we should take into account the effect of changes in the marginal households' incomes like Corollary 2.

We investigated the effect of change of each parameter upon the maximal competitive rent vector. Theorem 5 can be applicable to the case when more than one parameter changes simultaneously. Corollary 5 says that there are three possibilities, (4.5), (4.7) and a compromise between these two. See Figure 5.

Corollary 5. Let $f \leqq f^*$. Assume (4.2). Let r and r^* be the maximal competitive rent vectors in (M,N) and (M^*, N^*) . Then there are k_1 and k_2 and that $1 \leqq k_1 \leqq k_2 \leqq f$ and

$$r_1^* - r_1 > r_2^* - r_2 > \ldots > r_{k_1}^* - r_{k_1} , \tag{4.11}$$

$$r_{k_1}^* - r_{k_1} = r_{k_1+1}^* - r_{k_1+1} = \ldots = r_{k_2}^* - r_{k_2} , \tag{4.12}$$

$$r_{k_2}^* - r_{k_2} < r_{k_2+1}^* - r_{k_2+1} < \ldots < r_f^* - r_f . \tag{4.13}$$

Proof. Obvious from Theorem 5.

Example 5. Let us consider the housing market (M,N) given in Example 2. Let $I_{G*(k)} = (1+0.1)I_{G(k)}$ for all $k \leqq 12$, which are given as Table 6. Let $f = f*$ and $a^*_{13} = 260$. Then we get Table 8. Approximately, it holds in this example that $k_1 = 10$ and $k_2 = 11$.

t \ s	1			2			3			4		
10	10	341.0	99.4	4	588.4	103.3						
		241.6	1.411		485.1	1.213						
20	13	260	100	6	484.2	101.0	2	694.2	106.2			
		160	1.625		383.2	1.264		588.0	1.181			
30	14	240	100	9	381.5	99.7	5	573.3	102.9	1	757.7	108.1
		140	1.714		281.8	1.353		470.4	1.219		649.6	1.166
40	16	220	100	12	278.5	99.7	8	450.3	100.4	3	620.8	104.2
		120	1.833		178.8	1.558		349.9	1.287		516.6	1.202
50	17	200	100	15	240	100	11	325.7	99.4	7	477.1	100.9
		100	2.00		140	1.714		226.3	1.439		376.2	1.268

Table 8.

k	r^*_k	$r^*_k - r_k$
	r_k	r^*_k / r_k

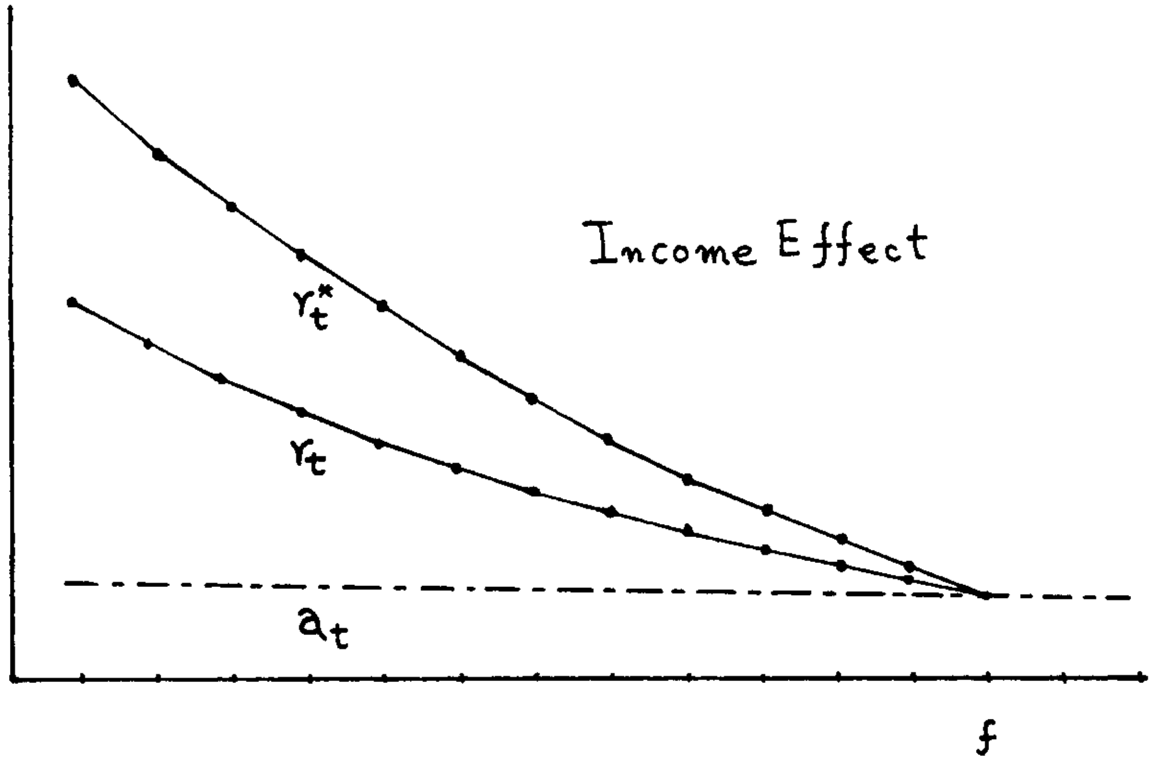


Figure 2

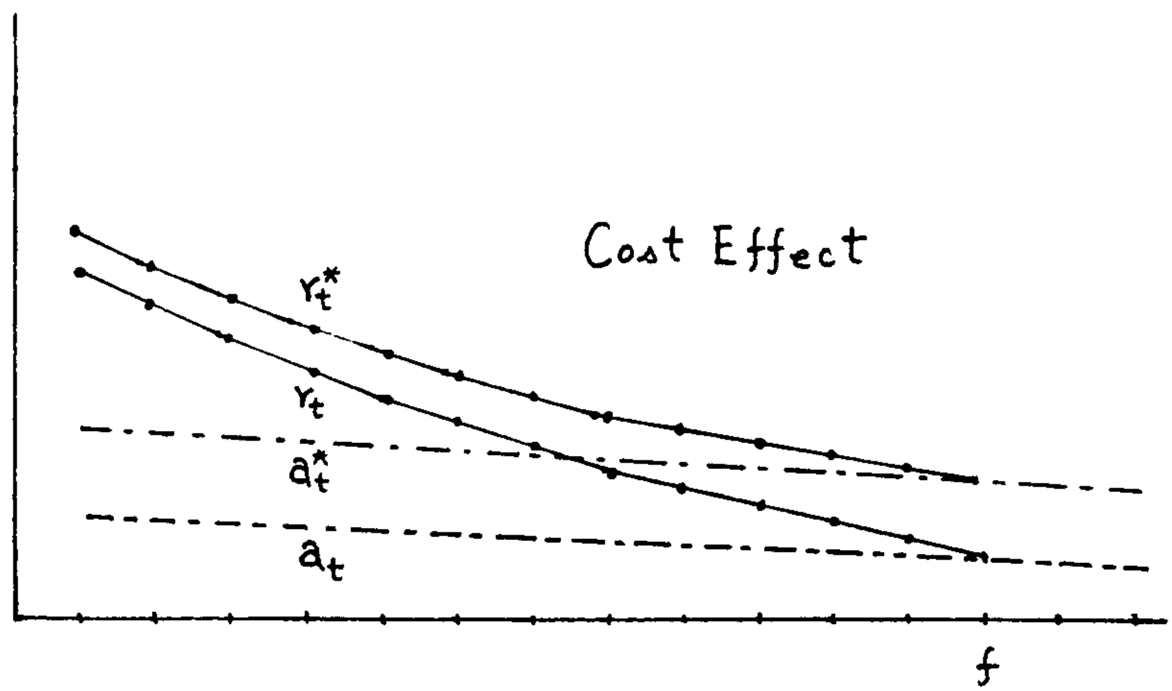


Figure 3

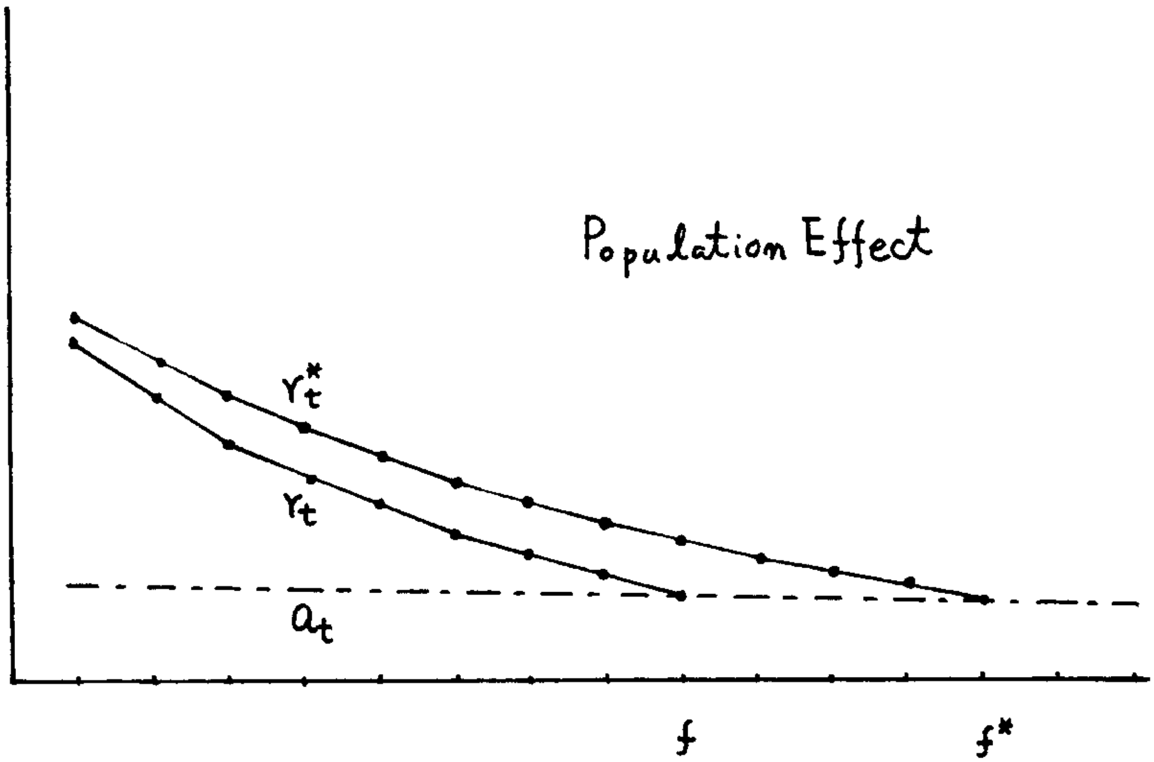


Figure 4

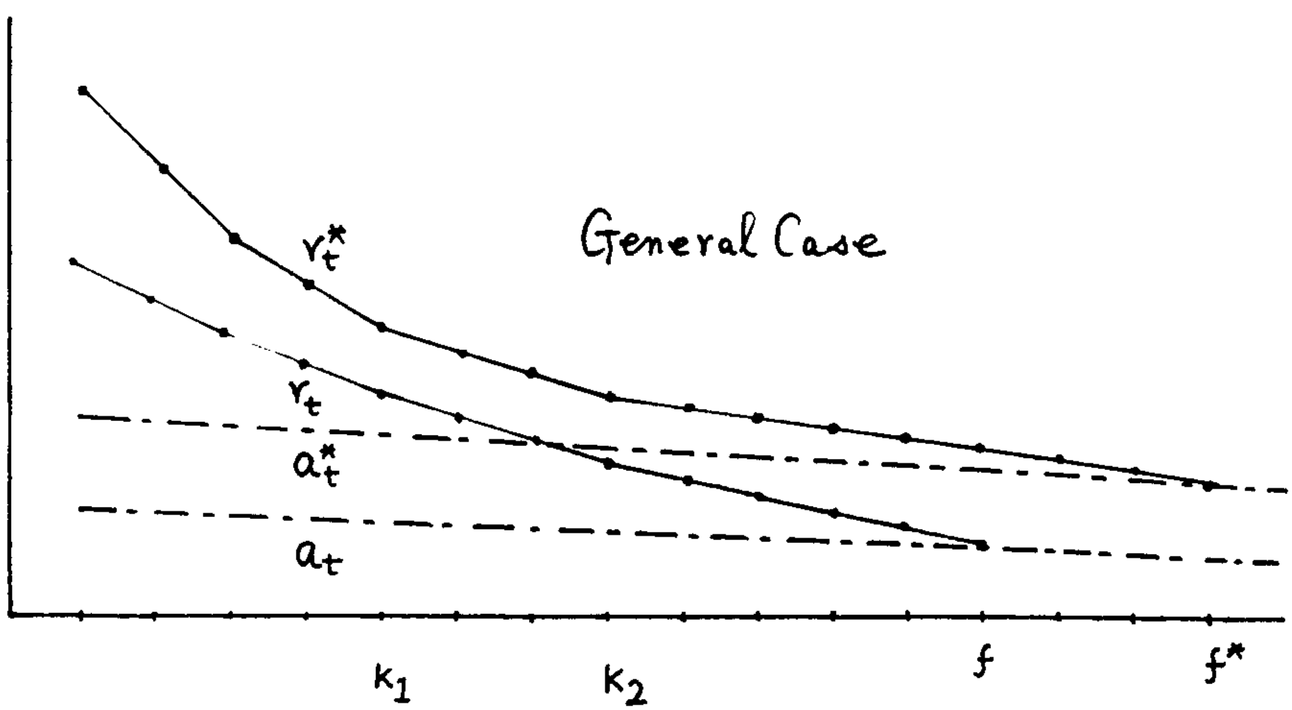


Figure 5

5. Conclusion

We constructed a simple mathematical model of housing market in which apartments are treated as indivisible commodities and all the other commodities are treated as a single composite commodity. We got the constructive proof, the recursive equation (2.6), of the existence of a competitive equilibrium and argued that the equilibrium given by the equation could be regarded as a representative of all competitive equilibria. Further we showed that there are several but limited tendencies of variations in rents when parameters change. We discussed that this is caused mainly by the assumption of diminishing marginal utility, assumption (E).

As pointed out, our analysis made many assumptions for simplification. For example, the theory of this paper is applicable to the case where there is a unique transportation system and all households go the same number of times to the center. Without these the analysis becomes much more difficult. Although Kaneko [9] provided several general properties, e.g., the existence of a competitive equilibrium, it cannot answer explicitly what happens in the market. For the purpose of analysis of a general housing market, we would need the algorithm for calculation of a competitive equilibrium because we cannot expect to get any explicit equation like (2.6) which determines a competitive rent vector. Now I plan to construct such an algorithm.

APPENDIX

Proof of Lemma 1(i). Suppose $(y, m_2 - \delta)R(x, m_1 - \delta)$. By assumptions (C) and (D), there is a $b \geq 0$ such that $(y, m_2 - \delta)Q(x, m_1 - \delta + b)$. Since $m_1 - \delta + b > m_2 - \delta$, we have $(y, m_2)P(x, m_1 + b)$ by assumption (E). But by assumption (C) we have $(x, m_1 + b)R(x, m_1)Q(y, m_2)$, i.e., $(x, m_1 + b)R(y, m_2)$, which is a contradiction.

(ii). By assumptions (C) and (D) there is a $b > 0$ such that $(e^k, 0)Q(e^t, b)$. By assumption (E), $(e^k, m_1)P(e^t, m_1 + b)$. Hence $(e^t, m_2)Q(e^k, m_1)P(e^t, m_1 + b)$. This implies $m_2 > m_1 + b > m_1$ by assumption (C).

Q.E.D.

Proof of Lemma 2. Since $(e^f, I_{G(f-1)} - a_f)R(e^f, I_n - a_f)P(e^1, 0)P \ldots P(e^{f-1}, 0)$ by assumptions (C), (F) and the supposition of this lemma, there is a b_{f-1} such that $(e^f, I_{G(f-1)} - a_f)Q(e^{f-1}, b_{f-1})$. This b_{f-1} is unique by assumption (C). Let $p_{f-1} = I_{G(f-1)} - b_{f-1}$. Since $(e^{f-1}, I_{G(f-2)} - p_{f-1})R(e^{f-1}, I_{G(f-1)} - p_{f-1})Q(e^f, I_{G(f-1)} - a_n)P(e^1, 0)P \ldots P(e^{f-2}, 0)$ by assumptions (C), (F) and the supposition of this lemma, there is a b_{f-2} such that $(e^{f-1}, I_{G(f-2)} - p_{f-1})Q(e^{f-2}, b_{f-2})$. Let $p_{f-2} = I_{G(f-2)} - b_{f-2}$. This b_{f-2} is also unique. Repeating this argument we get $(p_1, \ldots, p_{f-1})$. Lemma 1(ii) implies that if $(e^k, I_{G(k)} - p_k)Q(e^{k+1}, I_{G(k)} - p_{k+1})$, then $I_{G(k)} - p_k < I_{G(k)} - p_{k+1}$, i.e., $p_k > p_{k+1}$. This is (2.6).

Q.E.D.

<u>Proof of Lemma 3</u>. Let $k \leq t \leq f-1$. Since $(e^t, I_{G(t)} - p_k)Q(e^{t+1}, I_{G(t)} - p_{k+1})$ and $p_k > p_{k+1}$, we have, by assumption (E), $(e^t, I - p_k)R(e^{t+1}, I - p_{k+1})$. This is the first proposition. Let $t > f$ and let $I_n \geq a_t$. Since $(e^f, I_n - a_f)P(e^t, I_n - a_t)$ by assumption (G), there is a $b_t > 0$ by assumptions (C) and (D) such that $(e^f, I_n - a_f)Q(e^t, I_n - a_t + b_t)$. By assumption (H) and Lemma 2, $I_n - a_f < I_n - a_t + b_t$. Hence, we have, by assumption (E), $(e^f, I - a_f)R(e^t, I - a_t + b_t)P(e^t, I - a_t)$. Let $I \geq a_t > I_n$. Since $(e^f, 0)P(e^t, 0)$ by assumption (F), there is a $b_t > 0$ such that $(e^f, 0)Q(e^t, b_t)$. Hence $(e^f, I - a_t)R(e^t, I - a_t + b_t)$ $P(e^t, I - a_t)$. Since $I - a_f \geq I_n - a_f \geq 0$ but $a_t > I_n$, we have $I - a_f > I - a_t$, which implies $(e^f, I - a_f)P(e^f, I - a_t)$. So we have $(e^f, I - a_f)P(e^t, I - a_t)$.

Let $t < k$. Then since $(e^t, I_{G(t)} - p_t)Q(e^{t+1}, I_{G(t)} - p_{t+1})$ and $p_t > p_{t+1}$, we have, by Lemma 2(i), $(e^{t+1}, I - p_{t+1})R(e^t, I - p_t)$ if $p_t \leq I$.

Q.E.D.

REFERENCES

[1] W. Alonso. Location and Land Use. Cambridge: Harvard University Press, 1964.

[2] M. J. Beckmann. "On the Distribution of Urban Rent and Residential Density," J. Econ. Theory 1 (1969), pp. 60-67.

[3] E. von Böhm-Bawerk. Positive Theory of Capital, translated by W. Smart. New York, 1923 (original publication, 1891).

[4] D. Gale. Theory of Linear Economic Models. New York: McGraw-Hill, 1960.

[5] D. Heffley. "Efficient Spatial Allocation in the Quadratic Assignment Problem," J. Urban Economics 3 (1976), pp. 309-322.

[6] J. MacKinnon. "Urban General Equilibrium Models and Simplicial Search Algorithms," J. Urban Economics 1 (1974), pp. 161-183.

[7] M. Kaneko. "On the Core and Competitive Equilibria of a Market with Indivisible Goods," Naval Research Logistics Quarterly, 23 (1976), pp. 321-337.

[8] ________. "Note on Transferable Utility," International Journal of Game Theory 5 (1976), pp. 183-185.

[9] ________. "The Central Assignment Game and the Assignment Markets," Cowles Foundation Discussion Paper No. 563, 1980.

[10] T. C. Koopmans and M. J. Beckmann. "Assignment Problems and the Location of Economic Activities," Econometrica 25 (1957), pp. 53-76.

[11] A. Montesano. "A Restatement of Beckmann's Model on the Distribution of Urban Rent and Residential Density," J. Econ. Theory 4 (1972), pp. 329-354.

[12] M. Rommanos. "Household Location in a Linear Multi-Center Metropolitan Area," Regional Science and Urban Economics 7 (1977), pp. 233-250.

[13] L. S. Shapley and M. Shubik. "The Assignment Game I: The Core," International Journal of Game Theory 1 (1972), pp. 111-130.

[14] J. L. Sweeney. "Quality, Commodity Hierarchies, and Housing Markets," Econometrica 42 (1974), pp. 147-167.

[15] L. Telser. Competition, Collution and Game Theory. Aldine Atherton, 1972.

[16] J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. Princeton: Princeton University Press, 3rd ed., 1953.