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# An Index Theorem for General Equilibrium Models with Production

Timothy J. Kehoe

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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## 1. Introduction

It is well known that some variant of Brouwer's fixed point theorem can be used to prove the existence of equilibrium prices for a general model of economic competition. However, simple existence proofs leave many interesting questions unanswered: For example, is the equilibrium price vector unique? If not, is it locally unique? Does it vary continuously with the underlying parameters of the model? Answers to these questions are intimately linked with the applicability of such models to problems of comparative statics. Consequently, it is not surprising to find that a large amount of effort has been devoted to analyzing these questions. To provide complete answers, however, requires a more detailed set of assumptions than those required for simple existence proofs. For this reason we adopt a differentiable framework.

There have been many approaches to answering the question of when an equilibrium is unique (see Arrow and Hahn (1971) for a survey). Recent

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ones have included the use of a tool borrowed from the field of algebraic topology, the fixed point index (Dierker (1972, 1974, forthcoming), Mas-Colell (1976, forthcoming), and Varian (1975)). Researchers have recognized the close connection that this tool, when cast in a differentiable framework, has with the concept of regularity introduced by Debreu (1970) in response to the questions of local uniqueness and continuity. With the notable exceptions of Mas-Colell (1975, 1976, forthcoming) and Smale (1974), attention has been focused on pure exchange models that allow no production. One reason for this has been that, in contrast to the case of the pure exchange model where the excess demand function has been the natural subject of study, the model with production has had no obvious, single-valued, differentiable, function to investigate.

Using a variant of a device due to Todd (1977), we define a single-valued function whose fixed points are equivalent to equilibria of a model with an activity analysis production technology. We associate each fixed point of this function with an index that is an integer determined by the local properties of the function at that point. The global index theorem makes a statement about the sum of all the indices of equilibria allowing us to establish conditions sufficient for uniqueness.

## 2. The Model

Let us begin by describing a simple version of the Walrasian model of economic equilibrium. We assume that there is a finite number, $n$ , of perfectly divisible commodities. On the consumption side of the model responses of consumers to a vector of non-negative prices $\pi = (\pi_1, \ldots, \pi_n)'$ are aggregated into a vector of market excess demand functions

$\xi(\pi) = (\xi_1(\pi), \ldots, \xi_n(\pi))'$ . We take these functions to be completely arbitrary except for the following assumptions:

A.1 (Differentiability) Each $\xi_i$ is a continuously differentiable function defined, for the sake of simplicity, over the domain of all non-negative prices except the origin, $R^n_+\setminus\{0\}$ .

A.2 (Homogeneity) Each $\xi_i$ is homogeneous of degree zero; that is, $\xi_i(t\pi) = \xi_i(\pi)$ for all $t > 0$ .

A.3 (Walras's law) The vector function $\xi$ obeys Walras's law, $\pi'\xi(\pi) = 0$ .

The production technology is specified by an activity analysis matrix $A$ with $n$ rows and $m$ columns. Aggregate production is denoted $Ay$ where $y$ is an $m \times 1$ vector of non-negative activity levels. We assume that $A$ satisfies the following assumptions:

A.4 (Free disposal) $A$ includes $n$ free disposal activities, one for each commodity.

A.5 (Boundedness) There is no production without any inputs, $\{y | y \geq 0,\ Ay \geq 0\} = \{0\}$ .

Alternative forms of these assumptions are useful: The free disposal assumption implies that the $n \times n$ matrix $-I$ is a submatrix of $A$ . The boundedness assumption implies that there is some vector $\pi$ , strictly positive, such that $\pi'A < 0$ .

For our present purposes an economy is completely described by an excess demand function representing the consumption side and an activity analysis matrix representing the production side.

Definition. An equilibrium of an economy $(\xi, A)$ is a price vector $\hat{\pi}$ that satisfies the following conditions:

a. $\hat{\pi}'A \leq 0$ .

b. There exists $\hat{y} \geq 0$ such that $\xi(\hat{\pi}) = A\hat{y}$ .

c. $\Sigma_{i=1}^{n}\hat{\pi}_i = 1$ .

The condition $\hat{\pi}'A \leq 0$ implies that at prices $\hat{\pi}$ no excess profits can be made. $\xi(\hat{\pi}) = A\hat{y}$ requires that demand equal supply at equilibrium. This condition together with Walras's law implies that $\hat{\pi}'A\hat{y} = 0$ . Thus profits are maximized by the aggregate production plan $A\hat{y}$ . Since A.1 rules the vector $\pi = 0$ out of consideration we use the homogeneity assumption to restrict our attention to the unit simplex $S = \{\pi \in R^n | \pi_i \geq 0, \Sigma_{i=1}^{n}\pi_i = 1\}$ when examining equilibrium positions. At this stage it might seem more appropriate to include the vector of activity levels $\hat{y}$ directly in the definition of equilibrium. However, we shall soon impose conditions on $(\xi, A)$ that would make this addition superfluous.

We shall find it useful to consider whole spaces of economies. To do so, we must specify some topological structure on the space of economies satisfying A.1-A.3 and on the set of activity analysis matrices satisfying A.4-A.5. A topology on a space is specified by defining a system of open sets. Recall that in a metric space this is done by employing the concept of distance between two points in that space. We now give the space of economies the structure of a metric space: Let $\mathcal{A} \subset \{-I\} \times R^{n\times(m-n)}$ be the space of activity analysis matrices that satisfy

A.4 and A.5; here $-I$ is the submatrix of free disposal activities. We endow $\mathcal{A}$ with the standard topology on $R^{n\times(m-n)}$ by defining the metric $d(A^1, A^2) = (\sum_{i=1}^{n}\sum_{j=n+1}^{m}(a_{ij}^1 - a_{ij}^2)^2)^{1/2}$ for any $A^1, A^2 \in \mathcal{A}$. Let $\mathcal{D}$ be the space of $C^1$ functions satisfying A.1-A.3. We endow $\mathcal{D}$ with the topology of uniform $C^1$ convergence on compacta. Letting $M$ be some compact subset of $R_+^n\setminus\{0\}$, we define the metric

$$d_M(\xi^1, \xi^2) = \sup_{i,\pi\in M}|\xi_i^1(\pi) - \xi_i^2(\pi)| + \sup_{i,j,\pi\in M}\left|\frac{\partial\xi_i^1}{\partial\pi_j}(\pi) - \frac{\partial\xi_i^2}{\partial\pi_j}(\pi)\right| \quad \text{for any}$$

$\xi^1, \xi^2 \in \mathcal{D}$. Since we assume A.1 and A.2 we shall use $S$ as the compact domain on which maps $\xi^1$, $\xi^2$ are compared. (This topology is the same as the Whitney $C^1$ topology if we restrict the domain of our maps to the compact set $S$.) The space of economies $E = \mathcal{D} \times \mathcal{A}$ has the induced product topology: For any $(\xi^1, A^1), (\xi^2, A^2) \in E$ we define the metric $d[(\xi^1, A^1), (\xi^2, A^2)] = d_S(\xi^1, \xi^2) + d(A^1, A^2)$.

Definition. An economy $(\xi,A)$ is an element of the metric space $E = \mathcal{D} \times \mathcal{A}$.

## 3. The Mapping

Let us now define a mapping of $S$ into itself which whose fixed points are equivalent to equilibria of an economy $(\xi,A)$. Letting $N$ be any non-empty, closed, subset of $R^n$, we define the projection map $p^N : R^n \to N$ by the rule that associates any point $q \in R^n$ with the point $p^N(q)$ that is closest to $q$ in terms of Euclidean distance. It is well known that $p^N$ is continuous if $N$ is convex. Observe that our definition of equilibrium implies that any equilibrium is an element of the convex set $S_A = \{\pi \in R^n | \pi' A \leq 0, \sum_{i=1}^{n}\pi_i = 1\} \subset S$.

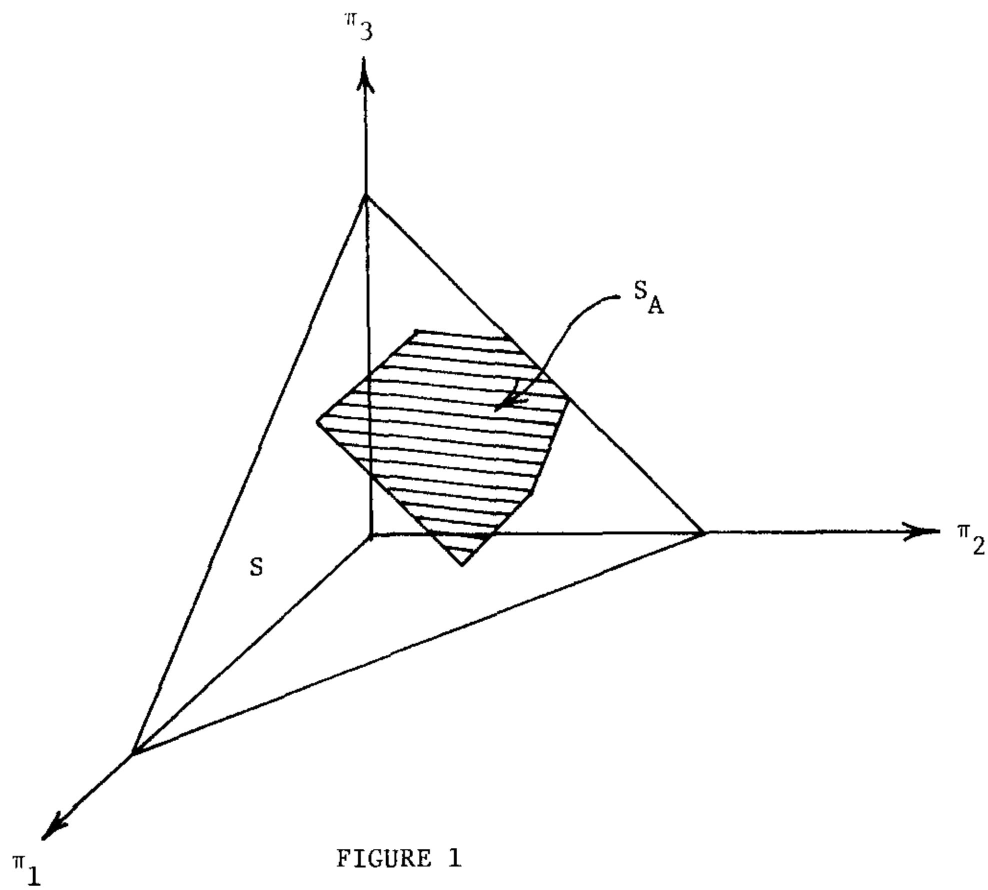


FIGURE 1

Definition. For any economy $(\xi,\Lambda) \in E$, define the map $g : S \to S$ by the rule

$$g(\pi) = p^S \Lambda(\pi + \xi(\pi)) .$$

It will sometimes be convenient to refer to $g$ as $g^{(\xi,\Lambda)}$ .

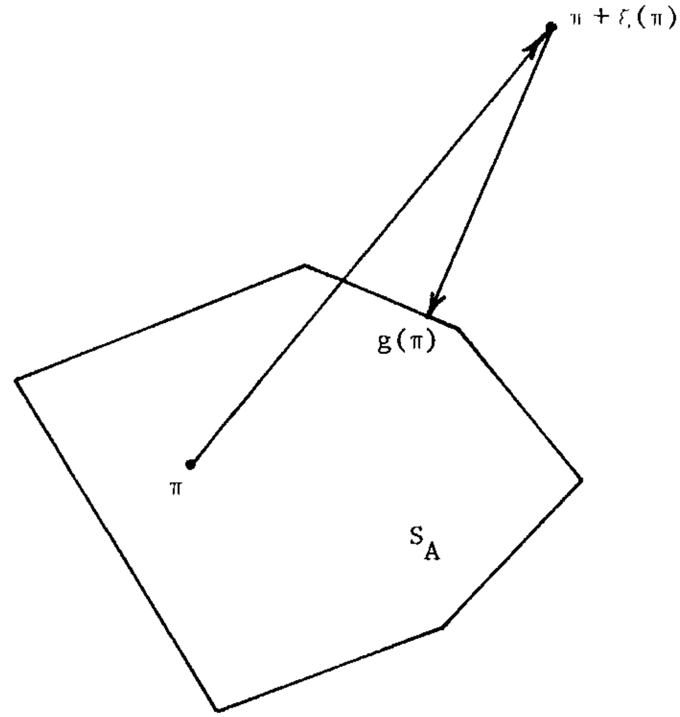


FIGURE 2

Note that, since $g$ is the composition of two continuous maps, it too is continuous. We shall apply Brouwer's fixed point theorem to this map $g$ to prove the existence of an equilibrium price vector for $g$.

Brouwer's Fixed Point Theorem. If $f : M \to M$ is a continuous map of some non-empty, compact, convex set $M \subset R^n$ into itself then $f$ leaves some point fixed; that is, there exists $\hat{x} \in M$ such that $\hat{x} = f(\hat{x})$.

The motivation for our definition of $g$ is clearly seen in the following theorem.

Theorem 1. Fixed points $\hat{\pi} = g(\hat{\pi})$ of the map $g$ and equilibria of the economy $(\xi, A) \in E$ are equivalent.

Proof. $\hat{\pi} = g(\hat{\pi})$ if and only if $\hat{\pi}$ is the unique solution to the quadratic programming problem

$$\min \ \frac{1}{2}\Sigma_{i=1}^{n}(p_i - \hat{\pi}_i - \xi_i(\hat{\pi}))^2$$

$$\text{s.t. } p'A \leq 0$$

$$p'e = 1 \ , \quad \text{where} \quad e = (1, \ldots, 1)' \ .$$

By the Kuhn-Tucker theorem there exists $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_m)'$ non-negative and $\hat{\lambda}$ such that

$$p - \hat{\pi} - \xi(\hat{\pi}) + A\hat{y} + \hat{\lambda}e = 0$$

and $p'A\hat{y} = 0$ . But $p = \hat{\pi}$ so that $\xi(\hat{\pi}) = A\hat{y} + \hat{\lambda}e$ . Applying Walras's law, A.3,

$$\begin{aligned} 0 &= \hat{\pi}'\xi(\hat{\pi}) \\ &= \hat{\pi}'A\hat{y} + \hat{\pi}\hat{\lambda}'e \\ &= \hat{\lambda} \ . \end{aligned}$$

Therefore since $\hat{\pi} \in S_A$ , the equilibrium conditions are satisfied. $\square$

Consequently, Brouwer's theorem implies the existence of equilibrium and hence the logical consistency of our model. The restriction of the domain of $g$ to $S$ , while entirely natural for proving existence of equilibrium, would make our discussion of regularity and index theory awkward. We want to study the derivatives of $g$ . To simplify matters we define $X$ to be a smooth (that is, $C^1$ ) $n$ dimensional manifold with boundary, embedded in $R^n$ so that it contains $S$ in its interior and does not contain the origin. We want to extend the domain of the map $g$ to $X$ .

Lemma 1. Let $\xi$ be any function satisfying A.1 and let $X \subset R^n$ be defined as above. $\xi$ can be extended to a $C^1$ map with domain $X$ .

Proof. We demonstrate our contention by constructing a $C^1$ map $\xi^* : R^n \to R^n$ that satisfies $\xi^*(\pi) = \xi(\pi)$ for all $\pi \in X \cap R^n_+$ . There are two steps in this construction. First we deal with the relatively minor technical problem of $\xi$ being undefined at the origin. Second we follow Saigal and Simon (1973) in extending $\xi$ to a $C^1$ map on all $R^n$ .

Since $X$ is compact and does not contain $0 \in R^n$ , the continuous function $\|\pi\|$ achieves a minimum $\alpha > 0$ on $X$ . Let $\beta$ be such that $\|\pi\| \geq \alpha > \beta > 0$ for all $\pi \in X$ . Take a smooth function $\theta : R^n \to R$ that satisfies

$$\theta(\pi) = 1 \quad \text{if} \quad \|\pi\| \leq \beta$$

$$\theta(\pi) = 0 \quad \text{if} \quad \|\pi\| \geq \alpha$$

$$0 < \theta(\pi) < 1 \quad \text{if} \quad \beta < \|\pi\| < \alpha \ .$$

The construction of such a function, known as a bump function, is a standard exercise (see Hirsch (1976), pp. 41-42). We define $\xi^0(\pi) = (1-\theta(\pi))\xi(\pi)$ for all $\pi \in R^n_+\backslash\{0\}$ and $\xi^0(\pi) = 0$ for $\pi = 0$ . Note that $\xi^0$ is $C^1$ , maps $R^n_+$ into $R^n$ , and agrees with $\xi$ on $X \cap R^n_+$ .

Now extend $\xi^0$ to a $C^1$ map on all $R^n$ employing the following recursive procedure. Given $\xi^{i-1}$ defined on $\{\pi \in R^n | \pi_i \geq 0, \ldots, \pi_n \geq 0\}$ define $\xi^i$ on $\{\pi \in R^n | \pi_{i+1} \geq 0, \ldots, \pi_n \geq 0\}$ by the rule

$$\xi^i(\pi_1, \ldots, \pi_n) = \begin{cases} \xi^i(\pi_1, \ldots, \pi_n) & \text{if } \pi_i \geq 0 \\ -\xi^{i-1}(\pi_1, \ldots, \pi_{i-1}, -\pi_i, \pi_{i+1}, \ldots, \pi_n) & \\ \quad + 2\xi^{i-1}(\pi_1, \ldots, \pi_{i-1}, 0, \pi_{i+1}, \ldots, \pi_n) & \text{if } \pi_i < 0 . \end{cases}$$

Note that $\xi^{i+1}$ extends $\xi^i$ and is $C^1$. Therefore $\xi^* = \xi^n$ extends $\xi^0$ to all $R^n$. □

We shall use this lemma to assume that $g$ is defined on $X$ (although there is no reason to expect that $\xi$ satisfies A.2 and A.3 at points not in $X \cap R^n_+$ ).

## 4. Regular Economies

When studying fixed points of $g$ we want to rule out certain degenerate situations. For this purpose we employ the notion of regular economy introduced by Debreu. References for the technical concepts employed in this and subsequent sections are the books on differential topology by Milnor (1972), Guillemin and Pollack (1974), and Hirsch (1976).

Definition. Consider a $C^1$ map $f : M \to N$ from a smooth manifold of dimension $m$ to a smooth manifold of dimension $n$. A point $x \in M$ is a regular point if $Df_x : T(M)_x \to T(N)_{f(x)}$ has rank $n$; in other words, is onto. A point $y \in N$ is a regular value if every point $x$ for which $f(x) = y$ is a regular point. Points in $M$ that are not regular points are critical points; points in $N$ that are not regular values are critical values.

By convention, any point $y$ for which the set $f^{-1}(y)$ is empty is a regular value. Also, if $m < n$, then clearly every point $x \in M$ is a critical point. We extend these concepts to maps such as $g$ that are not everywhere differentiable by requiring that the map $Df_x$ exist at a point $x$ for $x$ to be a regular point.

In the following analysis we focus our attention on economies $(\xi,A)$ for which 0 is a regular value of the map $(g-I) : X \to R^n$ where $I$ is the identity map on $R^n$. Unfortunately, the map $g$ defined in the previous section is not everywhere differentiable. The projection map $p^{S_A}$ has kinks because of the boundary of $S_A$. Consequently, $g$ is only piecewise differentiable. As we shall see, however, $g$ is a smooth map on any open set $U \subset X$ such that all points in the image $g(U)$ are contained in the same face of $S_A$. In order to ensure that $g$ is differentiable at every equilibrium, let us make the following non-degeneracy assumptions on $(\xi,A)$ :

<u>A.6</u>. No column of $A$ can be expressed as a linear combination of fewer than $n$ other columns.

<u>A.7</u>. Let $B(\hat{\pi})$ denote the submatrix of $A$ whose columns are all the activities earning zero profit at $\pi$. At every equilibrium $\hat{\pi}$ all $\hat{y}_b$ are strictly positive in the equation $\xi(\hat{\pi}) = \sum_{b \in B(\hat{\pi})} \hat{y}_b b$.

We now turn our attention to finding an expression for $Dg_{\hat{\pi}} - I$ for an equilibrium $\hat{\pi}$ of some economy $(\xi,A)$ satisfying A.1-A.7.

<u>Lemma 2</u>. Let $p$ be the map that projects any point $q \in R^n$ into the non-empty set $\{x | x'C = c'\}$. If the columns of $C$ are linearly independent then $p$ is a smooth function with Jacobian matrix $Dp_q = I - C(C'C)^{-1}C'$ for all $q \in R^n$.

The proof of this result is a straightforward application of the Kuhn-Tucker theorem. Of course $I - C(C'C)^{-1}C'$ is simply the orthogonal projection into the null space of $C$ as anyone familiar with least-squares regression techniques would expect.

We want to apply this lemma to $g - I$. Note that at any fixed point $\hat{\pi}$, $\hat{\pi}'B(\hat{\pi}) = 0$ and A.6 implies that $B(\hat{\pi})$ has fewer than $n$ columns and these are linearly independent. Let $C = [e \; B(\hat{\pi})]$; the columns of $C$ are linearly independent since $\hat{\pi}'e = 1$ while $\hat{\pi}'B(\hat{\pi}) = 0$.

<u>Theorem 2</u>. Let $(\xi, \Lambda)$ be any economy satisfying A.1-A.7. The map $g$ is differentiable in some open neighborhood of every fixed point $\hat{\pi}$ and the Jacobian matrix $Dg_{\hat{\pi}} - I$ equals $(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) - I$.

<u>Proof</u>. By assumption A.7 any activity $a^j$ not in $B(\hat{\pi})$ at equilibrium $\hat{\pi}$ is such that $\hat{\pi}'a^j = g(\hat{\pi})'a^j < 0$. Therefore $\hat{\pi}$ has an open neighborhood $U$ such that $g(\pi)'a^j < 0$ for all $\pi \in U$ by the continuity of $g$. Lemma 2 implies that if all the constraints $g(\pi)'B(\hat{\pi}) \leq 0$ are satisfied with equality then the vector $y$ varies continuously with $\pi$. But $\hat{y}_b > 0$ for $b \in B(\hat{\pi})$ implies that there is an open neighborhood $V \subset U$ of $\hat{\pi}$ where $g(\pi)'B(\hat{\pi}) = 0$. Using the chain rule we can differentiate $g$ to obtain

$$Dg_{\pi} - I = (I - C(C'C)^{-1}C')(I + D\xi_{\pi}) - I$$

for all $\pi \in V$. $\square$

<u>Definition</u>. An economy $(\xi, \Lambda) \in E$ that satisfies A.6 and A.7 and has $Dg_{\hat{\pi}} - I$ non-singular at every equilibrium is a regular economy. The set of regular economies is denoted $R \subset E$. Economies that are not regular are <u>critical economies</u>.

To justify this terminology we note that Theorem 2 implies that $0$ is a regular value of $g - I$ if $(\xi, A)$ is a regular economy. As we shall see, the conditions that define a regular economy are satisfied by almost all economies in $\mathcal{E}$. Of course, the sense of the phrase "almost all" has been made more precise. We shall also demonstrate that these conditions for regularity are equivalent to those given by Debreu (1970) for the special case of a pure exchange economy with all equilibria strictly positive.

Another useful concept is that of the equilibrium price correspondence.

<u>Definition</u>. The <u>equilibrium price correspondence</u> $\Pi : \mathcal{E} \to S$ associates with any economy the set of its equilibrium price vectors.

<u>Theorem 3</u>. $\Pi$ is an upper-semi-continuous, point-to-set correspondence.

<u>Proof</u>. Let $(\xi^i, A^i) \to (\xi, A)$ and $\pi^i \to \pi$ where $\pi^i \in \Pi(\xi^i, A^i)$. We want to show that $\pi \in \Pi(\xi, A)$. Let us define the production set $Y(A) = \{Ay \mid y \geq 0\}$. We can rephrase our equilibrium conditions as $\Pi(\xi, A) = \{\pi \in S \mid \pi' A \leq 0,\ \xi(\pi) \in Y(A)\}$. Now if all $A^i$ satisfy A.4–A.5, $A^i \to A$ implies $Y(A^i) \to Y(A)$. (We can use the Hausdorff distance for non-empty, closed subsets of $R^n$ to make precise the notion of convergence of sets.) $\xi^i(\pi^i)$ is jointly continuous in $\xi^i$ and $\pi^i$, where $\xi^i$ varies in the topology of uniform $C^1$ convergence on compacta; $\pi^{i\prime} A^i$ is likewise jointly continuous in $\pi^i$ and $A^i$. Therefore we have $\pi^{i\prime} A^i \to \pi' A \leq 0$ and $\xi^i(\pi^i) \to \xi(\pi)$. Since $Y(A^i) \to Y(A)$ and $\xi^i(\pi^i) \in Y(A^i)$, $\xi(\pi) \in Y(A)$. Thus $\pi \in \Pi(\xi, A)$. $\square$

One immediate consequence of this theorem is that for any economy $(\xi, A)$, the set of equilibria $\Pi(\xi, A)$ is a closed subset of the compact set $S$; therefore $\Pi(\xi, A)$ is compact. In the case of a regular economy, the inverse function theorem applied to $g - I$ at $\hat{\pi}$ implies that the equilibria are isolated. Thus any economy $(\xi, A) \in \mathcal{R}$ has only a finite number of equilibria.

<u>Theorem 4</u>. The equilibrium price correspondence $\Pi$ is continuous on $\mathcal{R}$ and the number of equilibria is locally constant.

<u>Proof</u>. We model our proof after that of Dierker (1974, forthcoming) for the pure exchange model. Let $(\xi, A)$ be a regular economy. The set $\Pi(\xi, A) = (g-I)^{-1}(0)$ consists of a finite number of equilibria $\pi^1, \ldots, \pi^k$. Each $\pi^i$ has a neighborhood that is mapped diffeomorphically by $g - I$ onto a neighborhood of $0$. We make use of the following result from Dierker (1974):

<u>Lemma</u>. Let $z$ map a neighborhood of $\pi \in X$ diffeomorphically onto a neighborhood of $0 \in R^n$ and $z(\pi) = 0$. There exist a neighborhood $V$ of $\pi$ and $\varepsilon > 0$ such that every $C^1$ function $\tilde{z}$ maps $V$ diffeomorphically onto some neighborhood of $0$ provided the $C^1$ distance of $\tilde{z}|V$ and $z|V$ is less than $\varepsilon$.

It follows that each $\pi^i$ has a neighborhood $V_i$ such that $(\tilde{g}-I) : V_i \to R^n$ is a diffeomorphism onto a neighborhood of the origin if $\tilde{g}$ is $C^1$ close to $g$ on $V_i$. Let $g = g^{(\xi,A)}$ and $\tilde{g} = g^{(\tilde{\xi},\tilde{A})}$ for some $(\tilde{\xi},\tilde{A}) \in \mathcal{E}$. We have no trouble with the differentiability of $g$ since we can choose $V_i$ small enough for $g|V_i$ to be $C^1$ by Theorem 2. The topology on $\mathcal{E}$ is fine enough for us to find a neighborhood $\mathcal{U} \subset \mathcal{E}$ of $(\xi, A)$ such that

$(\tilde{\xi},\tilde{A}) \in \mathcal{U}$ implies that $\tilde{g}|V_i$ is smooth and had $C^1$ distance less than $\varepsilon$ from $g|V_i$ .

Let each $V_i$ be small enough so that $V_i \cap V_{i'} = \emptyset$ for $i \neq i$ . Recall that $\Pi$ is upper-semi-continuous. The set $\bigcup_{i=1}^{k} cl(V_i)$ , where $cl(V_i)$ is the closure of $V_i$ , is a compact subset of $X$ , implying that $\Pi(\tilde{\xi},\tilde{A})$ $\bigcup_{i=1}^{k} V_i$ for all $(\tilde{\xi},\tilde{A}) \in \mathcal{U}$ . Thus the number of equilibria is locally constant. Furthermore, the correspondence $\Pi$ can be considered

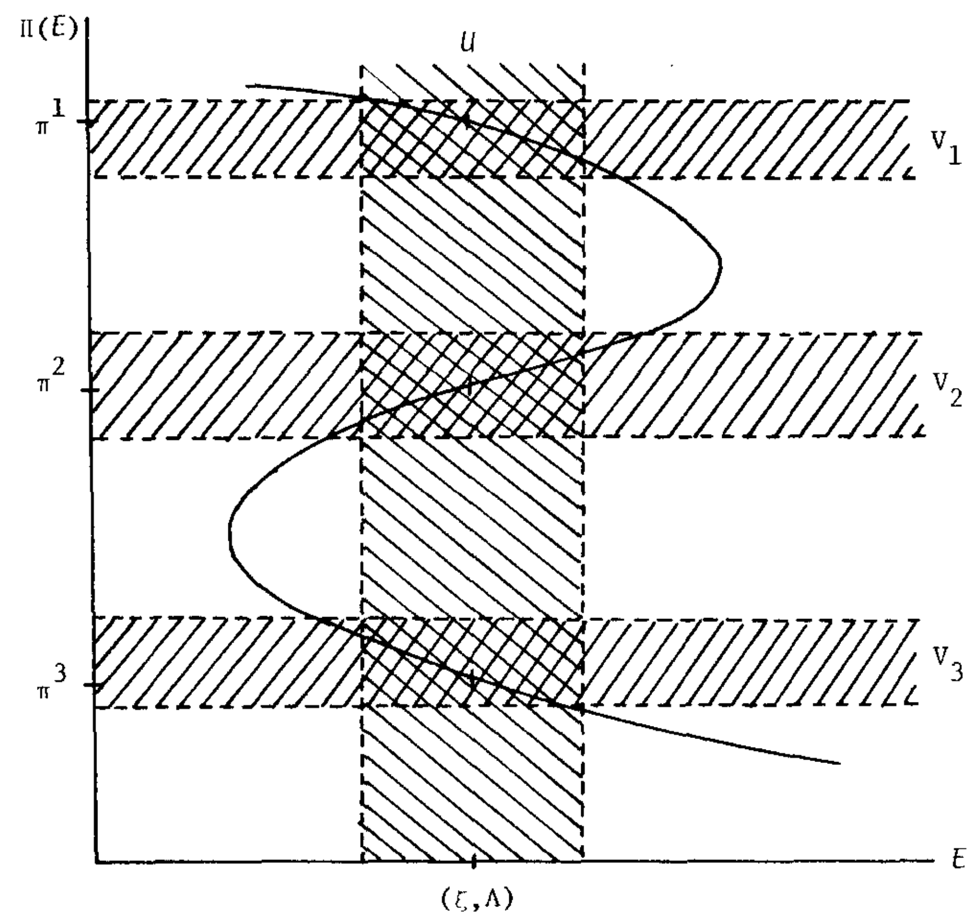


FIGURE 3

as the union of $k$ single-valued functions on $U$. They are continuous since $\Pi$ is upper-semi-continuous. □

When the economy under investigation is regular this theorem provides us with answers to the questions of local uniqueness and continuity posed earlier.

## 5. The Global Index Theorem

We are concerned with fixed points of the map $g$ defined on the manifold $X$. Although $g$ is not everywhere differentiable, if the economy $(\xi,A)$ satisfies A.6 and A.7 and has locally unique equilibria, we can smooth $g$ without disturbing its fixed points. Although this operation is not strictly necessary to derive the results that follow, carrying it out simplifies our exposition.

Lemma 3. Let $g : X \to X$ be defined as previously. If $(\xi,A) \in E$ satisfies A.6 and A.7 and if the fixed points of $g$ are isolated then for any $\varepsilon > 0$ there exist a $C^1$ map $g^* : X \to X$ and an open set $V \subset X$ containing $\{\pi \in X | g(\pi) = \pi\}$ which satisfy

a. $\|g^*(\pi) - g(\pi)\| \leq \varepsilon$ for all $\pi \in X$;

b. $g^*(\pi) = g(\pi)$ for all $\pi \in V$;

c. $g^*(\hat{\pi}) = \hat{\pi}$ implies $\hat{\pi} \in V$.

Proof. We make use of the following well known theorem (see, for example, Dieudonne, p. 133.)

<u>Weierstrass Approximation Theorem.</u> Let $X \subset R^n$ be a compact set and $g : X \to R^n$ be a continuous map. For any $\varepsilon > 0$ there exists a polynomial map $h : X \to R^n$ such that $||h(\pi) - g(\pi)|| \leq \varepsilon$ for all $\pi \varepsilon X$ .

The set $\{\pi \varepsilon X | g(\pi) = \pi\}$ consists of a finite number of points $\pi^1, \ldots, \pi^k$ . By Theorem 2, for each $\pi^i$ we can find some $\alpha_i > 0$ such that $g(\pi)$ is differentiable on some open ball $U_i \subset X$ centered at $\pi^i$ with radius $\alpha_i$ . Choose $\alpha_1, \ldots, \alpha_k$ small enough so that $U_i \cap U_{i'} = \emptyset$ for $i \neq i'$ . For each $\pi^i$ choose some $\beta_i$ such that $\alpha_i > \beta_i > 0$ . Let $V_i$ be the open ball centered at $\pi^i$ with radius $\beta_i$ . Note that the set $V = \bigcup_{i=1}^{k} V_i$ contains $\{\pi \varepsilon X | g(\pi) = \pi\}$ and that $g$ is differentiable on the set $U = \bigcup_{i=1}^{k} U_i$ . Let $\theta^i : R^n \to R$ be a smooth function such that $\theta^i(\pi) = 1$ if $||\pi^i - \pi|| \leq \beta_i$ , $\theta^i(\pi) = 0$ if $||\pi^i - \pi|| \geq \alpha_i$ , and $0 < \theta^i(\pi) < 1$ if $\beta_i < ||\pi^i - \pi|| < \alpha_i$ . Let $\theta : R^n \to R$ be the smooth function $\theta(\pi) = \sum_{i=1}^{k} \theta^i(\pi)$ . Note that $\theta(\pi) = 1$ for $\pi \varepsilon cl(V)$ . $\theta(\pi) = 0$ for $\pi \varepsilon X \backslash U$ , and $0 < \theta(\pi) < 1$ for $\pi \varepsilon U \backslash cl(V)$ .

We are now ready to construct a map $g^*$ satisfying the conditions of the lemma. The continuous function $||g(\pi) - \pi||$ achieves a minimum $\delta > 0$ on the compact set $X \backslash V$ . By the above theorem there exists a smooth map $h : X \to R^n$ such that $||h(\pi) - g(\pi)|| \leq \varepsilon < \delta$ for all $\pi \varepsilon X$ . Since $g$ maps $X$ into $S_\Lambda$ , that is, maps $X$ into its interior, we can choose $\varepsilon$ small enough so that $h : X \to X$ . Define the map $g^* : X \to X$ as $g^*(\pi) = \theta(\pi) g(\pi) + (1 - \theta(\pi)) h(\pi)$ . Note that $g^*$ is $C^1$ and agrees with $g$ on $V$ . Also note that for any $\pi \varepsilon X \backslash V$ , $||g^*(\pi) - g(\pi)|| \leq \varepsilon$ and $||g(\hat{\pi}) - \hat{\pi}|| \geq \delta > \varepsilon$ implies that $g(\pi) \neq \pi$ . Hence $g^*$ is the map that we desire. $\square$

We now present the formula for computing the local Lefschetz number of an isolated fixed point $\hat{\pi}$ of the map $g$. Actually, the local Lefschetz number of an isolated fixed point is a purely topological concept which requires only the continuity of $g$ in order to be defined. However, in keeping with our differentiable approach to the study of equilibria, we shall concern ourselves only with cases where $g$ is differentiable at its fixed points. The above formula indicates that if $(\xi,A) \in \mathcal{E}$ satisfies A.6 and A.7 and has locally unique equilibria, then, for our purposes, we can assume that $g^{(\xi,A)}$ is $C^1$.

For a $C^1$ map $g : X \to X$ we compute the local Lefschetz number of an isolated fixed point $\hat{\pi}$ as follows. Take a small closed ball $B$ containing $\hat{\pi}$ and no other fixed point. The rule $v(\pi) = [g(\pi) - \pi]/||g(\pi) - \pi||$ defines a smooth map from the boundary of $B$, $\partial B$, to the unit sphere, $\sum$. The local Lefschetz number is defined to be the degree of this map $v$ which can be any integer. Intuitively, the degree of $v$ measures the number of times $v$ wraps around $\sum$, taking orientation into account. To be more specific, we choose some regular value $y$ of $v$ and count the number of points in its preimage $v^{-1}(y)$, adding $+1$ for every point $\pi \in v^{-1}(y)$ such that the non-singular linear map $Dv_\pi : T(\partial B)_\pi \to T(\sum)_y$ preserves orientation and adding $-1$ for every point $\pi \in v^{-1}(y)$ such that $D\pi_v$ reverses orientation. It is well known that this claculation is independent of the regular value $y$ and the ball $B$ and that a regular value always exists (see Hirsch (1976), p. 124).

In the special case where $\hat{\pi}$ is a regular point of $g - I$ this reduces to the following rule (see Hirsch (1976), p. 122). $L_{\hat{\pi}}(g) = +1$ if $Dg_{\hat{\pi}} - I$ preserves orientation. $L_{\hat{\pi}}(g) = -1$ if $Dg_{\hat{\pi}} - I$ reverses orientation. Thus, $L_{\hat{\pi}}(g) = \mathrm{sgn}(\det[Dg_{\hat{\pi}} - I])$ when $Dg_{\hat{\pi}} - I$ is non-singular.

Theorem 5. Let $(\xi,A) \in \mathcal{E}$ be any economy that satisfies A.6 and A.7 and has locally unique equilibria. $\sum_{\pi=g(\pi)} L_\pi(g) = (-1)^n$ .

Proof. This theorem is a special case of a major theorem in algebraic topology, the Lefschetz fixed point theorem (see Dold (1965) and Guillemin and Pollack (1974), pp. 119-130). Saigal and Simon (1973) prove a version particularly suitable to our purposes. The basic idea is that if $g$ is a $C^1$ map $X \to X$ with isolated fixed points and if we calculate the local Lefschetz number $L_{\hat{\pi}}(g)$ of a fixed point $\hat{\pi}$ as above there is a global Lefschetz number $L(g) = \sum_{\pi=g(\pi)} L_\pi(g)$ which is a homotopy invariant. In fact, $L(g)$ is independent of $g$ ; it depends only on the manifold $X$ .

Let $c$ be the map $c : X \to X$ that maps every point into the constant $\bar{x} = (1/n, \ldots, 1/n)'$ . Since $X$ is convex we are able to construct the homotopy $G : X \times [0,1] \to X$ by defining $G(\pi,t) = (1-t)\bar{x} + tg(\pi)$ . Note that $c$ is a smooth map and $\bar{x}$ is the unique fixed point of $c$ . Therefore $L_c = L_{\bar{x}}(c)$ . Since $L(g)$ is a homotopy invariant, $L(g) = L(c) = L_{\bar{x}}(c) = \text{sgn}(\det[-I]) = (-1)^n$ . □

Figure 4 illustrates this theorem for the case where $g$ maps the unit interval into itself. Here $\sum_{\pi=g(\pi)} L_\pi(g) = L(g) = -1$ .

Although we can compute the local Lefschetz number of a fixed point whenever it is isolated, we shall have little use for it except when $(\xi,A)$ is a regular economy.

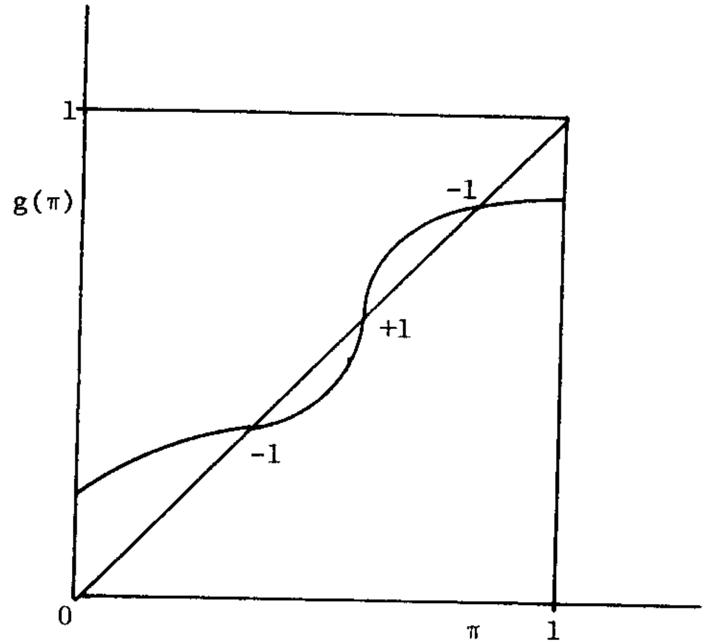


FIGURE 4

Definition. For any economy $(\xi,A) \in E$ satisfying A.6 and A.7, if an equilibrium $\hat{\pi}$ is such that $[(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}} - I]$ is non-singular then index $(\hat{\pi})$ is defined as $(-1)^n \text{sgn}(\det[(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}} - I])$ .

The advantage of this concept of index is that it can be calculated without reference to the map $g$ . Note that if $(\xi,A)$ is regular then index $(\hat{\pi}) = (-1)^n L_{\hat{\pi}}(g)$ . However, we leave index $(\hat{\pi})$ undefined in cases where $\hat{\pi}$ might not be a regular value of $g - I$ .

Let us now state our central result which is an immediate consequence of Theorem 6 and the definitions of index $(\hat{\pi})$ and regular economy.

Theorem 6. If $(\xi,A)$ satisfies A.1-A.7 and is a regular economy then index $(\hat{\pi})$ is defined and equal to +1 or -1 at every equilibrium $\hat{\pi}$ and

$$\sum_{\pi \in \Pi(\xi,A)} \text{index } (\pi) = +1 .$$

## 6. Calculation of the Index

In order to apply our results to specific economic models, we want to develop alternative expressions for the index $(\hat{\pi})$. One way to do this is to manipulate the matrix $[D\xi_{\hat{\pi}} - C(C'C)^{-1}C'D\xi_{\hat{\pi}} - C(C'C)^{-1}C']$ without changing the sign of the determinant. Elementary operations with this property can be found in any standard text on linear algebra; our reference is Gantmacher (1959). However, to the writer's knowledge, the following useful result in linear algebra is not found elsewhere.

Lemma 4. Let $C$ be an $n\times k$ matrix of full column rank $k \leq n$ and let $J$ be an $n\times n$ matrix. The determinant of $(I - C(C'C)^{-1}C')(I+J) - I$ has the same sign as

$$\det \begin{bmatrix} J & C \\ C' & 0 \end{bmatrix} .$$

Proof. $(I - C(C'C)^{-1}C')(I+J) - I = J - C(C'C)^{-1}C'J - C(C'C)^{-1}C'$. The determinant of this matrix equals that of

$$\begin{bmatrix} J - C(C'C)^{-1}C'J - C(C'C)^{-1}C' & 0 \\ C'J + C' & I \end{bmatrix} .$$

Adding the last row pre-multiplied by $C(C'C)^{-1}$ to the first, we do not change the determinant of this matrix. We now have

$$\det \begin{bmatrix} J & C(C'C)^{-1} \\ C'J + C' & I \end{bmatrix} .$$

Subtracting the first row pre-multiplied by $C'$ from the last we are left with

$$\det\begin{bmatrix} J & C(C'C)^{-1} \\ C' & 0 \end{bmatrix}.$$

Note that $C'C$ is positive definite and hence has a positive determinant. The sign of the above expression therefore remains unchanged when the matrix is post-multiplied by $\begin{bmatrix} I & 0 \\ 0 & C'C \end{bmatrix}$ where $I$ is $n\times n$.

$$\det\left(\begin{bmatrix} J & C(C'C)^{-1} \\ C' & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & C'C \end{bmatrix}\right) = \det\begin{bmatrix} J & C \\ C' & 0 \end{bmatrix}. \quad \square$$

We can use this lemma to calculate $\text{index}(\hat{\pi})$ of any regular equilibrium $\hat{\pi}$ of an economy satisfying A.1-A.7.

$$\text{index}(\hat{\pi}) = (-1)^n \text{sgn}\left(\det\begin{bmatrix} D\xi_{\hat{\pi}} & e' & B(\hat{\pi}) \\ e & 0 & 0 \\ B'(\hat{\pi}) & 0 & 0 \end{bmatrix}\right) = (-1)^n \text{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

An alternative expression is

$$\text{index}(\hat{\pi}) = \text{sgn}\left(\det\begin{bmatrix} -D\xi_{\hat{\pi}} + E & -B(\hat{\pi}) \\ B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

where $E$ is an $n\times n$ matrix whose every element is unity. To see why this is so, note that

$$\det \begin{bmatrix} -D\xi_{\hat{\pi}} + E & -B(\hat{\pi}) \\ B'(\hat{\pi}) & 0 \end{bmatrix} = (-1)^n \det \begin{bmatrix} 1 & e' & 0 \\ 0 & D\xi_{\hat{\pi}} - E & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} .$$

If we post-multiply the second column of the final matrix by $\hat{\pi}$ and subtract from the first column we do not change the determinant. The homogeneity assumption, A.2, when differentiated, implies that $D\xi_{\pi}\pi = 0$ for any $\pi \in R^n_+\backslash\{0\}$ . Also $e'\hat{\pi} = 1$ and $B'(\hat{\pi})\hat{\pi} = 0$ . Thus we are left with

$$(-1)^n \det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - E & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} = (-1)^n \det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} .$$

Let the matrix $\bar{J}$ be formed by deleting from $D\xi_{\hat{\pi}}$ all rows and columns $i$ for which $\hat{\pi}_i = 0$ and then deleting any one more row and column. Let $\bar{B}$ be formed by deleting from $B(\hat{\pi})$ columns corresponding to disposal activities and all rows $i$ for which $\hat{\pi}_i = 0$ . Note that Walras's law, A.3, when differentiated implies that $(D\xi_{\hat{\pi}})'\hat{\pi} = -\xi(\hat{\pi}) = -B(\hat{\pi})y_B$ . Using this observation and matrix manipulations similar to those above it is easy to demonstrate that

$$\text{index } (\hat{\pi}) = \text{sgn}\left(\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right)$$

if more than one $\hat{\pi}_i$ is strictly positive; index $(\hat{\pi}) = +1$ otherwise.

Now if $A = -I$ (the case of the pure exchange model) and if $\xi_i(\pi) > 0$ for any $\pi$ such that $\pi_i = 0$ then any equilibrium $\hat{\pi}$ is strictly positive and index $(\hat{\pi}) = \text{sgn}(\det[-\bar{J}])$ . Such is the situation analyzed by Dierker (1972, 1974, forthcoming) and Varian (1975) who calculate the same expression for the index. Moreover, since A.6 and A.7 are satisfied by any economy $(\xi, -I)$ , if $\det[-\bar{J}] \neq 0$ at any equilibrium then $(\xi, -I)$ is regular. This concept of regularity is therefore equivalent to that of Debreu (1970). Thus, this formulation of index $(\hat{\pi})$ allows us to compare our results with those previously known for pure exchange economies. Even for these economies our results are more general than those given elsewhere; we allow some prices to be zero at equilibrium.

One other point is worth mentioning. For the sake of clarity, we have defined the map $g$ on the n-manifold $X$ . However, since $g(X) \subset S_A$ , we could have well worked with the n-1-manifold $X \cap \{\pi \in R^n | \sum_{i=1}^n \pi_i = 1\}$ . Many writers find it more elegant to work with an n-1 manifold in this type of model, identifying $S$ with its natural projection into the last n-1 coordinates. Although this is less trivial than it might seem, we can use the homogeneity assumption A.2 and Walras's law A.3 to drop the first coordinate from consideration. In this setting the natural definition for index $(\hat{\pi})$ , at least when $\hat{\pi}$ is strictly positive, becomes $(-1)^{n-1}\text{sgn}(\det[\bar{J} - \bar{B}(\bar{B}'\bar{B})^{-1}\bar{B}'\bar{J} - \bar{B}(\bar{B}'\bar{B})^{-1}\bar{B}'])$ . Rather than going through the derivation of this, which requires some work, we simply note that by Lemma 4, this is equivalent to the expression for the index that we have already derived.

## 7. Genericity of Regular Economies

The value of the global index theorem depends on how common a situation regularity is in the space of economies. Intuitively, we view economies that are not regular, critical economies, as somehow degenerate. Thus, we might hope that such economies form a very small subset of $\mathcal{E}$ . In fact this is the case.

Definition. A property that holds for some subset $\mathcal{V}$ of a topological space $\mathcal{U}$ is a generic property of $\mathcal{U}$ if $\mathcal{V}$ is open and dense in $\mathcal{U}$ .

We shall argue that the property of regularity is generic, in other words, that $\mathcal{R}$ is open and dense in $\mathcal{E}$ .

It should be noted that genericity as defined above is stronger than the usual concept. Mathematicians often speak of some subset $\mathcal{V}$ of a space $\mathcal{U}$ as being residual if it contains the intersection of a countable number of open dense sets. For example, the set of irrational numbers is a residual subset of the reals. A property that holds for a residual subset of $\mathcal{U}$ is then called a generic property. We shall insist on the stronger definition, however. This is to ensure that the set of critical economies is not dense in $\mathcal{E}$ . The set of irrational numbers is not open; its complement, the set of rational numbers, is dense on the real line.

The difficult half of proving that regularity is a generic property of $\mathcal{E}$ is proving that $\mathcal{R}$ is dense in $\mathcal{E}$ ; Theorem 4 already implies that it is open. Our approach can be motivated by an argument reminiscent of the counting of equations and variables by Walras. Our equilibrium conditions can be expressed as a system of $n+k$ variables in $n+k+1$ equations

$$B'\hat{\pi} = 0$$

$$\xi(\hat{\pi}) = B\hat{y}$$

$$e'\hat{\pi} = 1$$

where $B$ is the $n \times k$ matrix of activities $B(\hat{\pi})$. Walras's law implies that one of the $n$ equations in $\xi(\hat{\pi}) = B\hat{y}$ can be eliminated. The conditions that ensure regularity also ensure that the equilibrium pair $(\hat{\pi}, \hat{y})$ is a locally unique solution to this system that is stable under small perturbations. If A.6 or the determinant condition does not hold we may not have enough independent equations to expect the equilibrium to be unique. On the other hand, if A.7 does not hold then we may have too many independent equations to expect a solution to exist. If we do not have as many independent equations as variables then it is intuitively plausible that some very slight perturbation in the underlying parameters of the economy could make the equations independent. Similarly, if there are not as many independent variables as equations then some slight perturbation of the system could make a solution impossible. What we need is freedom to make perturbations in a sufficient number of directions.

The proof that the set of matrices satisfying A.6 is open and dense in $\mathcal{A}$ is trivial. Therefore, we can demonstrate that $\mathcal{R}$ is dense in $\mathcal{E}$ by proving that for any economy $(\xi^0, A)$ that satisfies A.6 but not A.7 or not the non-zero determinant condition, there is another economy $(\xi^1, A)$ that is regular with $d_S(\xi^0, \xi^1) < \varepsilon$ for any $\varepsilon > 0$. We reduce the problem from one in the infinite dimensional vector space $\mathcal{D}$ to the finite dimensional vector space $R^n$ by parameterizing any excess demand function $\xi \in \mathcal{D}$ with a perturbation vector $v \in R^n$. Define the function $\delta(\pi, v) = (\delta_1(\pi, v), \ldots, \delta_n(\pi, v))'$,

$$\delta_i(\pi,v) = \frac{\sum_{\ell=1}^{n} \pi_\ell v_\ell}{\sum_{\ell=1}^{n}} - v_i \ , \quad i = 1, \ldots, n \ .$$

$\delta$ satisfies A.1-A.3 and therefore so does $\xi+\delta$ if $\xi$ does. For any economy $(\xi,A)$ we consider the family of economies $\{(\xi_v, A) \in E \mid \xi_v(\pi) = \xi(\pi) + \delta(\pi,v),\ v \in R^n\}$. Our goal is to demonstrate that the set of regular economies is dense in this $n$ parameter family. If it is, the topology on $E$ is such that for any $\varepsilon > 0$ we can find a $\varepsilon' > 0$ such that $\|v\| < \varepsilon'$ implies that $d[(\xi,A), (\xi_v, A)] < \varepsilon$. Consequently if regular economies are dense in this $n$ parameter family they are dense in $E$.

In $R^n$ the concept of density is related to that of Lebesque measure. Recall that a set $U \subset R^n$ has Lebesque measure zero if it is possible to cover $U$ with a countable number of rectangular solids with arbitrarily small volume. The complement in $R^n$ of a set with Lebesque measure zero has full Lebesque measure. This concept of Lebesque measure is easily extended to manifolds using local parameterizations (see Guillemin and Pollack (1974), pp. 204-205). It is well known that a set of full Lebesque measure is dense (see Milnor (1972), pp. 10-11). A theorem in differential topology that is a direct consequence of Sard's theorem is particularly suited to our purpose (Guillemin and Pollack (1974, pp. 67-69).

Transversality Density Theorem. Let $M$, $V$, and $N$ be smooth manifolds without boundary where $\dim M = m$, $\dim N = n$, and $m \leq n$ and let $y \in N$. Suppose that $F : M \times V \to N$ is a $C^1$ map such that for every $(x,v) \in M \times V$, rank $DF_v(x,v) = n$. The set of $v \in V$ for which $F(x,v) = y$ implies that rank $DF_x(x,v) = n$ has full Lebesque measure.

Before applying this theorem to our problem, a few preliminary definitions are necessary. For any n×k matrix B , $0 \leq k \leq n-1$ let

$$K_B = \{x \in R^n | B'x = 0,\ e'x = 1\} ,$$

$$O_B = \{x \in R^n | B'x = 0,\ e'x = 0\} .$$

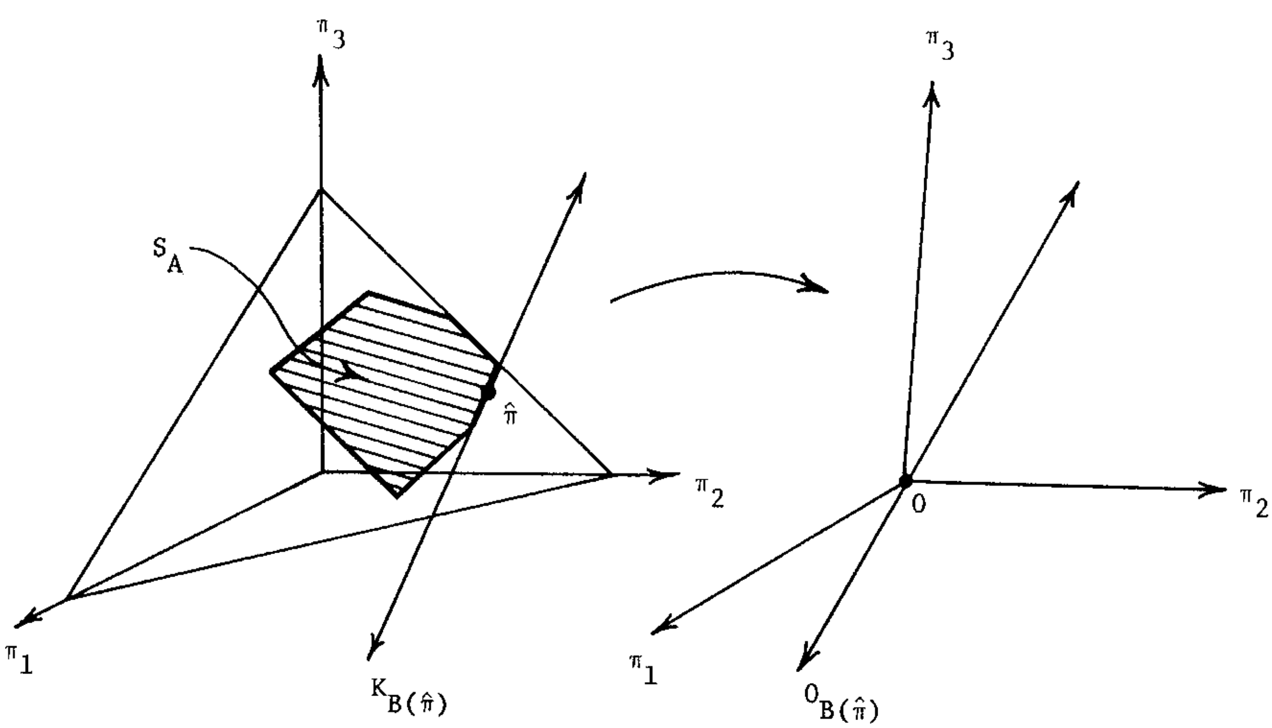


FIGURE 5

Note that at any equilibrium $\hat{\pi}$ of an economy $(\xi,A) \in E$ , both $K_{B(\hat{\pi})}$ and $K_{B(\hat{\pi})} \cap \text{int } X$ are smooth and without boundary. Consider the $C^1$ map $f^{B(\hat{\pi})} : (K_{B(\hat{\pi})} \cap \text{int } X) \to O_{B(\hat{\pi})}$ defined by the rule $f^{B(\hat{\pi})}(\pi) = p^{K_{B(\hat{\pi})}}(\pi + \xi(\pi)) - \pi = p^{O_{B(\hat{\pi})}}(\xi(\pi))$ . If $(\xi,A)$ satisfies A.6 and A.7 then $f^{B(\hat{\pi})}$ agrees with $g - I$ in some neighborhood of $\hat{\pi}$ on $K_{B(\hat{\pi})}$ . At an equilibrium $\hat{\pi}$ , $f^{B(\hat{\pi})}(\hat{\pi}) = 0$ . The derivative map $Df^{B(\hat{\pi})}_{\hat{\pi}} : T(K_{B(\hat{\pi})} \cap \text{int } X)_{\hat{\pi}} \to T(O_{B(\hat{\pi})})_0$ maps $O_{B(\hat{\pi})}$ into itself; on

$0_{B(\hat{\pi})}$ , $Df_{\hat{\pi}}^{B(\hat{\pi})} = (I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ .

For any economy $(\xi,A)$ we focus our attention on all possible maps $f^B$ where $B$ is some $n \times k$ submatrix of $A$ , $0 \leq k \leq n-1$ , such that $K_B \cap S_B \neq \emptyset$ . Obviously, every equilibrium $\hat{\pi}$ is such that $\hat{\pi} \in K_B$ and $f^B(\hat{\pi}) = 0$ for some such $B$ , namely $B(\hat{\pi})$ . The converse, however, does not necessarily hold; in addition to $\hat{\pi} \in K_B$ and $f^B(\hat{\pi}) = 0$ the condition $\hat{\pi} \in S_A$ is needed for $\hat{\pi}$ to be an equilibrium. To apply the transversality density theorem we use the function $\delta$ to parameterize our maps $f^B$ with the perturbation vector $v \in R^n$ . Define the $C^1$ map $F^B : (K_B \cap \text{int } X) \times R^n \to 0_B$ by the rule $F^B(\pi,v) = p^{0_B}(\xi(\pi) + \delta(\pi,v))$ .

In order to apply the transversality density theorem we demonstrate that for all $(\pi,v) \in (K_B \cap \text{int } X) \times R^n$ the derivative map $DF_v^B : R^n \to 0_B$ has rank $n-k-1$ , in other words is onto. $DF_v^B(\pi,v) = (I - C(C'C)^{-1}C')D\delta_v$ where $C = [B\ e]$ . Now

$$D\delta_v(\pi,v) = \begin{bmatrix} \pi_1 - 1 & \cdots & \pi_n \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_n - 1 \end{bmatrix} .$$

has rank $n-1$ since $D\delta_v(\pi,v)e = 0$ but the $(n-1) \times (n-1)$ matrix formed by deleting any row and column $j$ for which $\pi_j > 0$ is non-singular. Letting $\rho = \text{rank}(I - C(C'C)^{-1}C')D\delta_v$ , we note that

$$\rho+k+1 = \text{rank}\begin{bmatrix} (I - C(C'C)^{-1}C')D\delta_v & C \\ 0 & I \end{bmatrix} = \rho+k+1$$

$$= \text{rank}\begin{bmatrix} D\delta_v & C \\ (C'C)^{-1}C'D\delta_v & I \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} D\delta_v & C \\ 0 & 0 \end{bmatrix}$$

$$= \text{rank}[D\delta_v \quad e \quad B] .$$

Clearly $\text{rank}[D\delta_v \quad e \quad B] \geq n$ since $D\delta_v$ has rank $n-1$ , $D\delta_v e = 0$ , and $e'e = n$ . However, $(I - C(C'C)^{-1}C')D\delta_v$ maps into $0_B$ implying that $\rho = n-k-1$ .

We first demonstrate that for almost all $v \in R^n$ , the economy $(\mathcal{E}_v, A)$ satisfies A.7. Let $B^*$ be any $n \times k^*$ , $k^* > k$, submatrix of $A$ that has $B$ as a submatrix. $K_{B^*} \cap \text{int } X$ is an $n - k^* - 1$ submanifold of the $n-k-1$ manifold $K_B \cap \text{int } X$ . We restrict $F^B(\pi,v)$ to domain $(K_{B^*} \cap \text{int } X)$ . As a consequence of the transversality density theorem, for almost all $v \in R^n$ , $F^B(\pi,v) = 0$ implies that rank $DF^B_\pi = n-k-1$ . However, $k^* > k$ implies rank $DF^B_\pi \leq n - k^* - 1 < n-k-1$ . Thus for almost all $v \in R^n$ there is no $\pi \in K_B \cap \text{int } X$ such that $F^B(\pi,v) = 0$ . Intuitively, although $F^B$ takes $(K_{B^*} \cap \text{int } X) \times R^n$ into $0_B$ , the image $F^B(K_{B^*} \cap \text{int } X, v)$ is a very small subset of $0_B$ for any fixed $v \in R^n$ . Indeed for a set $U$ of full Lebesque measure in $R^n$ the image $F^B(K_{B^*} \cap \text{int } X, U)$ does not contain $\mathbf{0}$ . We can repeat this same argument for all possible combinations $B$ and $B^*$ . The intersection of a finite number of sets of full Lebesque measure also has full Lebesque measure.

Now if we apply the transversality density theorem to $F^B : (K_B \cap \text{int } X) \times R^n \to 0_B$ we establish that for almost all $v \in R^n$, $F^B(\pi,v) = 0$ implies that rank $DF^B_\pi = n-k-1$. Thus for almost all $v \in R^n$ if $\hat{\pi}$ is an equilibrium of $(\xi_v, A)$ then $DF^{B(\hat{\pi})}_{\hat{\pi}} = (I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ has rank $n-k-1$. Let $x \in R^n$ be such that

$$(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}x - C(C'C)^{-1}C = 0 .$$

Since the columns of $(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ and those of $C(C'C)^{-1}C$ are orthogonal and rank $C(C'C)^{-1}C = k+1$,

$$\{x \in R^n | (I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}x = 0\} \cap \{x \in R^n | C(C'C)^{-1}C'x = 0\} = \{0\} .$$

Pre-multiplying the above equation by $(I - C(C'C)^{-1}C')$, we have

$$(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}}x = 0 .$$

On the other hand, pre-multiplying by $C(C'C)^{-1}C'$ produces

$$C(C'C)^{-1}C'x = 0 .$$

Together these imply that $[(I - C(C'C)^{-1}C')D\xi_{\hat{\pi}} - C(C'C)^{-1}C']$ is non-singular. Again using the fact that the intersection of a finite number of sets of full Lebesque measure also has full Lebesque measure, we have proven

<u>Theorem 6</u>. The set of regular economies $R$ is open and dense in $E$.

## 8. Extensions and Conclusions

The above treatment of genericity of regular economies is rough in the sense that the perturbation of excess demand functions, while entirely natural for economies where all commodities enter into consumers' final demands, is not always appropriate in economies where production plays an important role. In such economies there are likely to be primary commodities which are inelastically supplied as inputs to the production process and intermediate commodities which are only produced in order to produce other commodities. It is possible to extend our argument to such situations but there are several minor technical problems. First we must deal with the possibility of excess demand being unbounded at some points on the boundary of $R^n_+$ in order to deal properly with primary commodities. Second, we must slightly alter our definition of regularity to deal with the possibility of prices of intermediate commodities being undefined at equilibria where no production takes place.

Another direction in which our discussion can be extended is production technologies with smooth production functions. It is relatively easy to exploit the properties of the profit functions in such a technology to show that our mapping and index theorem carry over when we substitute the input-output vectors for production functions used at equilibrium for activities.

These matters are discussed in more detail in the writer's doctoral dissertation which is currently in progress. Also discussed there are conditions that may be placed on the structure of an economic model such that the index theorem implies uniqueness of equilibrium. We remark on two sets of such conditions, one dealing with the production side of the economy alone, the other dealing with the consumption side alone. First, if we can impose conditions on $(\xi, A)$ that ensure that there are always $n-1$ activities in use at equilibrium then

$$\text{index}(\hat{\pi}) = \text{sgn}\left(\det\begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right) = \text{sgn}(\det[\bar{B}'\bar{B}]) = +1$$

at every equilibrium. The global index theorem then implies that there is a unique equilibrium. Recall that the conditions of the non-substitution theorem of input-output analysis imply that there are always $n-1$ activities in use at equilibrium. Second, if the excess demand function $\xi$ is such that $\bar{J}$ is always negative definite then there is a unique equilibrium if the production technology is such that $(\xi, A)$ satisfies A.4-A.7. This can be seen by noting that

$$\text{index}(\hat{\pi}) = \text{sgn}\left(\det\begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right) = \text{sgn}(\det[-\bar{J}]\det[-\bar{B}'\bar{J}^{-1}\bar{B}]) = +1$$

at every equilibrium. It is interesting to note that it has been shown by Kihlstrom, Mas-Colell, and Sonnenschein (1976) that negative definiteness of $\bar{J}$ implies that the weak axiom of revealed preference holds for $\xi$, a condition previously known to imply uniqueness.

A final remark should also be made with regard to the necessity of the conditions we have established for uniqueness of equilibrium. Conditions such as gross substitutability in $\xi$ are neither necessary nor sufficient for $\text{index}(\hat{\pi})$ to be positive. Constructing an economy with gross substitutes and with an equilibrium with $\text{index}(\hat{\pi}) = -1$ is fairly easy. Such an equilibrium cannot be unique. Such an observation might imply that non-uniqueness of equilibrium is a less pathological situation than is sometimes thought.

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