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A Note on the Saddlepoint Approximation in the First Order Non-Circular Autoregression

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ABSTRACT

In approximating the small sample distribution of the least squares estimator of the coefficient in a non-circular autoregression the saddlepoint method is complicated by the presence of a branch point of the integrand within the natural contour of integration. Some new approximations are given based on a contour looping around the branch point; and a uniform approximation which is valid as the saddlepoint crosses the branch point is also developed.

KEYWORDS

Steepest descents and ascents, elliptic integral, uniform approximation, parabolic cylinder function.

1. INTRODUCTION

In an earlier paper (Phillips (1978)) it was found that the derivation of the saddlepoint approximation to the distribution of the least squares estimator of the autoregressive coefficient in a non-circular autoregression was complicated in certain cases by the existence of a branch point of the integrand within the natural contour of integration. This complication meant that the usual saddlepoint approximation was unavailable in a sizeable region of the tail of the distribution for values of the autoregressive coefficient greater

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I am grateful to Professor H. E. Daniels for helpful discussions on the problem of the saddlepoint method near a singularity and for a copy of his research note (Daniels (1977)) on this problem.

than 0.5. The present paper gives some new approximations to cover this case.

2. APPROXIMATIONS BASED ON A LOOP AROUND THE BRANCH POINT

We work with the model $y_t = \alpha y_{t-1} + u_t$ $(t = \ldots -1,0,1,\ldots)$ where $|\alpha| < 1$ and the u_t are i.i.d. $N(0,\sigma^2)$. The least squares estimator of α is given by $\hat{\alpha} = \sum_{t=1}^{T} y_t y_{t-1} / \sum_{t=1}^{T} y_{t-1}^2 = y'C_1y/y'C_2y$ where $y' = (y_0,\ldots,y_T)$, and the density of $\hat{\alpha}$ is given by the expression

$$(1) \qquad p_T(x) = \frac{1}{2\pi i} \int \left. \frac{\partial M(z_1, u - xz_1)}{\partial u} \right|_{u=0} dz_1$$

where the integration is taken along the imaginary axis in the z_1 plane or an allowable deformation of this path and $M(z_1,z_2)$ is the joint moment generating function of the quadratic forms $y'C_1y$ and $y'C_2y$. In fact, we find that $M(z_1,z_2) = (1-\alpha^2)^{\frac{1}{2}} h_{T+1}^{-\frac{1}{2}}$ where

$$(2) \qquad h_{T+1} = \frac{(\alpha+z_1)^T}{w^T(1-w^2)} \left\{ 1 - \left(\frac{(\alpha+z_1)^2+\alpha^2}{\alpha+z_1} \right) w + \alpha^2 w^2 - \left[w^{2T+2} - \left(\frac{(\alpha+z_1)^2+\alpha^2}{\alpha+z_1} \right) w^{2T+1} + \alpha^2 w^{2T} \right] \right\}$$

and w satisfies $w + w^{-1} = (1+\alpha^2-2z_2)/(\alpha+z_1)$. After changing the variable of integration in (1) from z_1 to w and neglecting the term in square brackets on the right side of (2) (which is exponentially small as $T \to \infty$) we obtain (details are available on request):

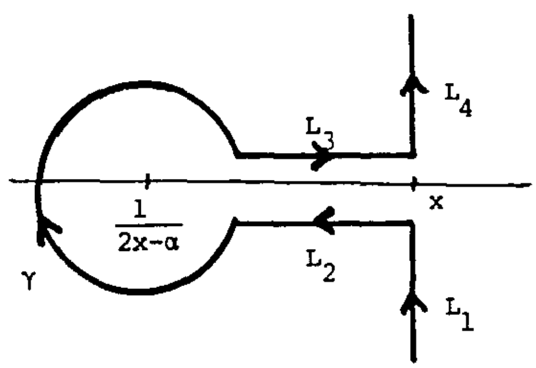
$$(3) \qquad p_T(x) \sim \frac{(T-3)(1-\alpha^2)^{\frac{1}{2}}}{2\pi i (1+\alpha^2-2x\alpha)^{(T-1)/2}} \int_C (w^2-2xw+1)^{(T-3)/2} \phi(w)\, dw$$

where

$$\phi(w) = \frac{(1-w^2)^{3/2}}{a(w)^{\frac{1}{2}}} \left[1 + \left(\frac{1}{T-3} \right) \frac{(w^2-2xw+1)(1+\alpha^2 w^2) - 2w^2(1+\alpha^2-2x)}{a(w)^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{-1}} \right],$$

$a(w) = (1-\alpha w)^2 (1+\alpha w - 2wx)(1-2\alpha x+\alpha w)$ and the path of integration C is taken inside the unit disk $|w| \leq 1$ from $e^{-i\theta}$ to $e^{i\theta}$ where $x = \cos\theta$ (see Daniels (1956) p.172).

The factor $(w^2-2xw+1)$ has a saddlepoint at $w=x$ on the real axis and the natural contour of integration is then the straight line connecting $e^{-i\theta}$ to $e^{i\theta}$ which, at $w=x$, is the line of steepest descent through the saddlepoint. However, the integrand of (3) has a branch point on the real axis at $w = 1/(2x-\alpha)$. We cut the w plane along the real axis to the right of the branch point and, then, when the saddlepoint lies outside the branch point (i.e. when $1+\alpha x-2x^2<0$ or when $x>1/(2x-\alpha)$ in the right hand tail) we deform the path of integration in a loop around the branch point as in Figure 1. We have a



similar configuration when $x<0$ in the left tail and this will be of greater importance when $\alpha<0$ also. In what follows we will assume that $(2x-\alpha)^{-1}>0$.

In Figure 1 the circle γ has centre $(2x-\alpha)^{-1}$ and radius ε. Along L_1, $w=x-iy(1-x^2)^{\frac{1}{2}}$ with $0\leq y\leq 1$ and, along L_4, $w=x+iy(1-x^2)^{\frac{1}{2}}$. We select that branch of $(1+\alpha w-2wx)^{\frac{1}{2}}$ for which the integral is positive and, along L_2, we set $(1+\alpha w-2wx)^{\frac{1}{2}} = |1+\alpha y-2y^2|^{\frac{1}{2}}e^{\pi i/2}$ with $(2x-\alpha)^{-1}+\varepsilon \leq y \leq x$; and, along L_3, we set $(1+\alpha w-2wx)^{1/2} = |1+\alpha y-2y^2|^{1/2}e^{3\pi i/2}$ for the same values of y. Taking each of these integrals in turn we find first that the integral around γ vanishes as the circle shrinks to the point $(2x-\alpha)^{-1}$. Along L_1 and L_4, we expand $\phi(w)$ in a Taylor series about its appropriate value on the real axis where $w=x$ and find for the dominant term in each case

$$\frac{1}{2\pi i}\int_{L_1}(w^2-2xw+1)^{(T-3)/2}\phi(w)\,dw$$
$$\sim \frac{(1-x^2)^{(T-2)/2}}{2\pi}\left(\frac{(1-x^2)^{3/2}}{(1-\alpha x)^2|1+\alpha x-2x^2|^{\frac{1}{2}}e^{\pi i/2}}\right)\int_0^1(1-y^2)^{(T-3)/2}dy$$

and

$$\frac{1}{2\pi i}\int_{L_4}(w^2-2xw+1)^{(T-3)/2}\phi(w)\,dw$$
$$\sim \frac{(1-x^2)^{(T-2)/2}}{2\pi}\left(\frac{(1-x^2)^{3/2}}{(1-\alpha x)^2|1+\alpha x-2x^2|^{\frac{1}{2}}e^{3\pi i/2}}\right)\int_0^1(1-y^2)^{(T-3)/2}dy$$

so these integrals cancel on addition.

L_2 and L_3 are paths of steepest ascent for $(w^2-2xw+1)^{(T-3)/2}$ from the saddlepoint at $w=x$. Most of the contribution to the integrals along these paths comes from the vicinity of $w=(2x-\alpha)^{-1}$ on the real axis. To evaluate these integrals we first define

$$\psi(w) = \frac{(1-w^2)^{3/2}}{(1-\alpha w)(1-2\alpha x+\alpha w)^{\frac{1}{2}}}$$

so that

(4) $$\phi(w) = \frac{\psi(w)}{(1+w(\alpha-2x))^{\frac{1}{2}}}\,[1+O(T^{-1})]$$

We now consider the integral

(5) $$\frac{1}{2\pi i}\int_{L_2}\frac{(w^2-2xw+1)^{(T-3)/2}\psi(w)}{(2x-\alpha)^{\frac{1}{2}}((2x-\alpha)^{-1}-w)^{\frac{1}{2}}}\,dw$$

Along L_2, $w=x-y(1-x^2)^{\frac{1}{2}}$ for $0\le y\le\lambda(x)$ where $\lambda(x) = (2x-\alpha)^{-1}(1-x^2)^{-\frac{1}{2}}(2x^2-\alpha x-1)$; and $(w^2-2xw+1)=(1-x^2)(1+y^2)$. (5) becomes

$$\frac{1}{2\pi}\,\frac{(1-x^2)^{(T-2-\frac{1}{2})/2}}{(2x-\alpha)^{\frac{1}{2}}}\int_0^{\lambda(x)}\frac{(1+y^2)^{(T-3)/2}}{(\lambda(x)-y)^{\frac{1}{2}}}\,\psi(x-y(1-x^2)^{\frac{1}{2}})\,dy$$

$$(6) \quad \sim \frac{1}{2\pi} \frac{(1-x^2)^{(T-2-\frac{1}{2})/2}\psi(x-\lambda(x)(1-x^2)^{\frac{1}{2}})}{(2x-\alpha)^{\frac{1}{2}}} \int_0^{\lambda(x)} \frac{(1+y^2)^{(T-3)/2}}{(\lambda(x)-y)^{\frac{1}{2}}} dy$$

since $\psi(.)$ is analytic on L_2 and most of the contribution to the integral comes from the vicinity of $y=\lambda(x)$. The corresponding integral along L_3 takes on the same value (5).

Hence, adding the contributions from the separate components of the contour in Figure 1, we obtain

$$(7) \quad p_T(x) \sim \frac{(T-3)(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}\psi(x-\lambda(x)(1-x^2)^{\frac{1}{2}})}{(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}} \int_0^{\lambda(x)} \frac{(1+y^2)^{(T-3)/2}}{(\lambda(x)-y)^{\frac{1}{2}}} dy$$

The integral that occurs in (7) can be calculated exactly. We distinguish cases of odd and even T.

When T is odd we let $n = (T-3)/2$ and then

$$\int \frac{(1+y^2)^n}{(\lambda(x)-y)^{\frac{1}{2}}} dy = Q(y)(\lambda(x)-y)^{\frac{1}{2}}$$

where $Q(y)$ is a polynomial of degree $2n$ (c.f. Gradshteyn and Ryzhik (1965), p.80). We write $Q(y) = \sum_{k=0}^{2n} q_k y^k$ and the coefficients q_k can be recovered from the identity

$$(1+y^2)^n = Q'(y)(\lambda(x)-y) - \tfrac{1}{2}Q(y).$$

For the definite integral in (7) we need only the constant coefficient q_0, which we calculate to be

$$q_0 = -2-2\lambda(x)\sum_{k=1}^{n}\binom{n}{n-k}\frac{\lambda(x)^{2k-1}2.3.4...(2k)}{\frac{3}{2}.\frac{5}{2}...(2k+\frac{1}{2})}$$

Then

$$\int_0^{\lambda(x)} \frac{(1+y^2)^{(T-3)/2}}{(\lambda(x)-y)^{\frac{1}{2}}} dy = 2\lambda(x)^{\frac{1}{2}}\left\{1 + \lambda(x)\sum_{k=1}^{n}\binom{n}{n-k}\frac{\lambda(x)^{2k-1}2.3...(2k)}{\frac{3}{2}.\frac{5}{2}...(2k+\frac{1}{2})}\right\}$$

Noting that $x-\lambda(x)(1-x^2)^{\frac{1}{2}} = (2x-\alpha)^{-1}$ we see that (7) becomes

$$(8)\quad p_T(x) \sim \frac{(T-3)(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}\psi((2x-\alpha)^{-1})}{\pi(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}}$$

$$\cdot\ 2\lambda(x)^{\frac{1}{2}}\left\{1+\lambda(x)\sum_{k=1}^{n}\binom{n}{n-k}\frac{\lambda(x)^{2k-1}2.3\ldots(2k)}{\frac{3}{2}\cdot\frac{5}{2}\ldots(2k+\frac{1}{2})}\right\}$$

where $\lambda(x) = (2x-\alpha)^{-1}(1-x^2)^{-\frac{1}{2}}(2x^2-\alpha x-1)$.

When T is even, we set $n = (T-2)/2$ and write

$$\int_0^{\lambda(x)} \frac{(1+y^2)^{(T-3)/2}dy}{(\lambda(x)-y)^{\frac{1}{2}}} = \int_0^{\lambda(x)} \frac{(1+y^2)^{(T-2)/2}dy}{\{(\lambda(x)-y)(y+i)(y-i)\}^{\frac{1}{2}}}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\int_0^{\lambda(x)} \frac{y^{2k}dy}{\{(\lambda(x)-y)(y+i)(y-i)\}^{\frac{1}{2}}}$$

$$(9)\quad = \sum_{k=0}^{n}\binom{n}{k}\int_{-\lambda(x)}^{0} \frac{v^{2k}dv}{\{(v+\lambda(x))(v+i)(v-i)\}^{\frac{1}{2}}}$$

This is now an elliptic integral and can be evaluated from the recurrence relation

$$(4k-1)R_{2k} = 2(1-2k)\lambda(x)R_{2k-1} + (3-4k)R_{2k-2} + 2(1-k)\ (x)R_{2k-3}$$

where $R_{2k} = \int_{-\lambda(x)}^{0} \{(v+\lambda(x))(v+i)(v-i)\}^{-\frac{1}{2}}v^{2k}dv$

by which we can express each of the integrals in (9) in terms of the two basic integrals R_1 and R_o. The latter can be reduced in terms of elliptic integrals of the first and second kinds as follows (see Byrd and Friedman (1971), pp.86-87 and p.206):

$$(10)\quad R_o = \int_{-\lambda(x)}^{0} \frac{dv}{\{(v+\lambda(x))(v+i)(v-i)\}^{\frac{1}{2}}} = (1+\lambda(x)^2)^{-\frac{1}{2}} F(\phi,k),$$

$$R_1 = \int_{-\lambda(x)}^{0} \frac{vdv}{\{(v+\lambda(x))(v+i)(v-i)\}^{\frac{1}{2}}} = -\frac{\lambda(x) + (1+\lambda(x)^2)^{\frac{1}{2}}}{(1+\lambda(x)^2)^{\frac{1}{2}}} F(\phi,k)$$

$$+ 2(1+\lambda(x)^2)^{\frac{1}{2}}\left[F(\phi,k) - E(\phi,k) + \frac{sn(u_1)dn(u_1)}{1+cn(u_1)}\right]$$

$$(11)\quad = -\frac{\lambda(x) + (1+\lambda(x)^2)^{\frac{1}{2}}}{(1+\lambda(x)^2)^{\frac{1}{2}}} F(\phi,k)$$

$$+ 2(1+\lambda(x)^2)^{\frac{1}{2}}\left[F(\phi,k) - E(\phi,k) + \frac{\sin(\phi)(1-k^2\sin^2(\phi))^{\frac{1}{2}}}{1+\cos(\phi)}\right]$$

where $F(\phi,k) = \int_0^{\phi} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} = u_1$

$$E(\phi,k) = \int_0^{\phi} (1-k^2\sin^2\theta)^{\frac{1}{2}} d\theta,$$

$$\phi = \cos^{-1}\left(\frac{(1+\lambda(x)^2)^{\frac{1}{2}}-\lambda(x)}{(1+\lambda(x)^2)^{\frac{1}{2}}+\lambda(x)}\right),$$

$$k = \left\{\frac{(1+\lambda(x)^2)^{\frac{1}{2}}+\lambda(x)}{2(1+\lambda(x)^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$$

and sn( ), cn( ) and dn( ) are the Jacobian elliptic functions.

The approximation for T even is clearly more awkward to evaluate than when T is odd. But both cases are in turn more complicated than the usual saddlepoint approximation which is available when $x<(2x-\alpha)^{-1}$. An alternative approximation which is much simpler in form can be obtained by approximating the integral that occurs in (7) rather than evaluating it exactly. The following derivation was suggested to me by Professor H.E.Daniels (1977):

$$(12)\quad \int_0^{\lambda(x)} \frac{(1+y^2)^{(T-3)/2}}{(\lambda(x)-y)^{\frac{1}{2}}} dy \sim \int_0^{\lambda(x)} \frac{e^{\frac{1}{2}(T-3)y^2}}{(\lambda(x)-y)^{\frac{1}{2}}} dy = \int_0^{\lambda(x)} \frac{e^{\frac{1}{2}(T-3)(\lambda(x)-v)^2}}{v^{\frac{1}{2}}} dv$$

$$= (T-3)^{-\frac{1}{2}} e^{\frac{1}{2}(T-3)\lambda(x)^2} \int_0^{(T-3)^{\frac{1}{2}}\lambda(x)} w^{-\frac{1}{2}} e^{-(T-3)^{\frac{1}{2}}\lambda(x)w+\frac{1}{2}w^2} dw$$

$$\sim (T-3)^{\frac{1}{2}} e^{\frac{1}{2}(T-3)\lambda(x)^2} \int_0^{\infty} w^{-\frac{1}{2}} e^{-(T-3)^{\frac{1}{2}}\lambda(x)w} dw$$

$$(13) \qquad = \frac{\pi^{\frac{1}{2}} e^{\frac{1}{2}(T-3)\lambda(x)^2}}{(T-3)^{\frac{1}{2}}\lambda(x)^{\frac{1}{2}}}$$

Using the approximation (13) in (7) we obtain the approximate density

$$(14) \qquad p_T(x) \sim \frac{(T-3)^{\frac{1}{2}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}\psi((2x-\alpha)^{-1}e^{\frac{1}{2}(T-3)\lambda(x)^2}}{\pi^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}\lambda(x)^{\frac{1}{2}}}$$

In deriving (13) we replace a finite integral by an infinite integral, so that the approximation will be most satisfactory when $(T-3)^{\frac{1}{2}}\lambda(x)$ is large. Note that when $\lambda(x)$ is small the approximation (13) will behave in a different way from the true value of the integral (12). In particular, as $\lambda(x) \to 0$ (i.e. as x approaches $(2x-\alpha)^{-1}$ in Figure 1), (13) tends to infinity, while the integral itself tends to zero. In this respect the approximate density (14) compares with the usual saddlepoint approximation obtained when $x < (2x-\alpha)^{-1}$ and x approaches the branch point from the left side.

3. A UNIFORM APPROXIMATION IN THE VICINITY OF THE BRANCH POINT

As the saddlepoint $w=x$ moves past the branch point $(2x-\alpha)^{-1}$ on the real axis the usual saddlepoint approximation for $x < (2x-\alpha)^{-1}$ and the approximations given in the previous section for $x > (2x-\alpha)^{-1}$ all break down. An approximation which holds uniformly for values of x in the vicinity of $(2x-\alpha)^{-1}$ therefore seems desirable.

To find such a uniform approximation we return to (3) and (4) from which we have

(15) $$p_T(x) \sim \frac{(T-3)(1-\alpha^2)^{\frac{1}{2}}}{(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}} \frac{1}{2\pi i}\int_C \frac{(w^2-2xw+1)^{(T-3)/2}\psi(w)\,dw}{((2x-\alpha)^{-1}-w)^{\frac{1}{2}}}$$

Using a line of approach developed by Professor Daniels (1977) we can now write the integral that occurs in (15) as

$$\frac{1}{2\pi i}\int_C (w^2-2xw+1)^{(T-3)/2}\psi(w)\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty u^{-\frac{1}{2}}e^{-u((2x-\alpha)^{-1}-w)}\,du\,dw$$

$$= \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty u^{-\frac{1}{2}}du\, \frac{1}{2\pi i}\int_C (w^2-2xw+1)^{(T-3)/2}\psi(w)\,e^{-u(2x-\alpha)^{-1}-w)}\,dw$$

We now let $w = x+iz(1-x^2)^{\frac{1}{2}}$ and let C* be the contour in the z plane corresponding to C. We obtain

$$\frac{(1-x^2)^{(T-2)/2}}{\Gamma(\frac{1}{2})}\int_0^\infty u^{-\frac{1}{2}}e^{-u((2x-\alpha)^{-1}-x)}du\,\frac{1}{2\pi}\int_{C^*}(1-z^2)^{(T-3)/2}\psi(x+iz(1-x^2)^{\frac{1}{2}})\, e^{uiz(1-x^2)^{\frac{1}{2}}}\,dz$$

$$\sim \frac{(1-x^2)^{(T-2)/2}}{\Gamma(\frac{1}{2})}\int_0^\infty u^{-\frac{1}{2}}e^{-u((2x-\alpha)^{-1}-x)}du\,\frac{1}{2\pi}\int_{C^*}\psi(x+iz(1-x^2)^{\frac{1}{2}})\, e^{-\frac{1}{2}(T-3)z^2+uiz(1-x^2)^{\frac{1}{2}}}\,dz$$

$$= \frac{(1-x^2)^{(T-2)/2}}{\Gamma(\frac{1}{2})(2\pi)^{\frac{1}{2}}}\int_0^\infty u^{-\frac{1}{2}}\exp\{-u((2x-\alpha)^{-1}-x) - \tfrac{1}{2}(T-3)^{-1}u^2(1-x^2)\}du \cdot \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{C^*}\psi(x+iz(1-x^2)^{\frac{1}{2}}\exp\left\{-\tfrac{1}{2}(T-3)\left(z-\frac{iu(1-x^2)^{\frac{1}{2}}}{T-3}\right)^2\right\}dz$$

We now let C* be the contour crossing the imaginary axis orthogonally at $z=(T-3)^{-1}iu(1-x^2)^{\frac{1}{2}}$. Most of the contribution to the integral comes from the vicinity of this point and the usual saddlepoint approximation gives us

$$\frac{(1-x^2)^{(T-2)/2}(T-3)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})(2\pi)^{\frac{1}{2}}}\int_0^\infty u^{-\frac{1}{2}}\exp\{-u((2x-\alpha)^{-1}-x)-\tfrac{1}{2}(T-3)^{-1}u^2(1-x^2)\}\,\psi(x-(T-3)^{-1}u(1-x^2))\,du$$

(16) $$= \frac{(1-x^2)^{(T-3)/2+\frac{1}{2}}}{2^{\frac{1}{2}}\pi}\int_0^\infty w^{-\frac{1}{2}}\exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}\psi(x-w(1-x^2)^{\frac{1}{2}})\,dw$$

where $\mu(x) = (1-x^2)^{-\frac{1}{2}}(2x-\alpha)^{-1}(1+\alpha x-2x^2) = -\lambda(x)$. We could now expand $\psi(x-w(1-x^2)^{\frac{1}{2}})$ in a Taylor series about its value when $w=0$, and then integrate term by term. The integrals in the resulting series can be expressed quite simply in terms of parabolic cylinder functions. However, this reduction of (16) neglects the fact that when $\mu(x) < 0$ an important contribution to the integral comes from the vicinity of $w = -\mu(x)$. Indeed, when $\mu(x) < 0$ the analysis of the previous section is relevant and we recall that the approximations there are based on the value of the analytic part of the integrand at the branch point $(2x-\alpha)^{-1}$ rather than the saddlepoint at x.

To cope with this problem and to ensure that the uniform approximation to (16) reconciles with the usual approximations on either side of $(2x-\alpha)^{-1}$ we expand $\psi(.)$ as follows:

$$(17)\quad \psi(x-w(1-x^2)^{\frac{1}{2}}) = \psi(x) + \left(\frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)}\right)w + w(w+\mu(x))G(w)$$

We take $G(w)$ to be defined by (17) and note that the singularities of $G(w)$ at $w=0$ and $w=-\mu(x)$ can be removed by writing

$$G(w) = \left\{\frac{\mu(x)^{-1}}{w} - \frac{\mu(x)^{-1}}{w+\mu(x)}\right\}\left\{\psi(x-w(1-x^2)^{\frac{1}{2}}) - \psi(x) - \frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)} \cdot w\right\}$$

$$= \frac{\mu(x)^{-1}}{w}\left\{\psi(x-w(1-x^2)^{\frac{1}{2}}) - \psi(x)\right\} - \frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)^2}$$

$$+ \frac{1}{w+\mu(x)}\left\{w\,\frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)^2} + \mu(x)\,\frac{\psi(x)-\psi(x-w(1-x^2)^{\frac{1}{2}})}{\mu(x)^2}\right\}$$

$$= \mu(x)^{-1}\sum_{r=1}^{\infty}\frac{(-1)^r}{r!}\psi^{(r)}(x)w^{r-1}(1-x^2)^{r/2} - \frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)^2}$$

$$- \frac{1}{w+\mu(x)}\sum_{r=1}^{\infty}\frac{(-1)^r}{r!}\psi^{(r)}(x+\mu(x)(1-x^2)^{\frac{1}{2}})(w+\mu(x))^r.$$

(16) is now

$$(18)\quad \frac{(1-x^2)^{(T-3)/2+\frac{1}{2}}}{2^{\frac{1}{2}}\pi}\left\{\int_0^\infty w^{-\frac{1}{2}}\exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}dw\,\psi(x)\right.$$

$$+\int_0^\infty w^{\frac{1}{2}}\exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}dw\left[\frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)}\right]$$

$$\left.+\int_0^\infty w^{\frac{1}{2}}\exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}(w+\mu(x))G(w)dw\right\}$$

The last integral in (18) is, by integrating by parts,

$$(T-3)^{-1}\int_0^\infty \exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}(\tfrac{1}{2}w^{-\frac{1}{2}}G(w)\ +\ w^{\frac{1}{2}}G'(w))dw$$

which is of $O(T^{-1})$ relative to the first two integrals in (18). We note also that

$$\int_0^\infty w^{\alpha}\exp\{-(T-3)[w\mu(x)+\frac{w^2}{2}]\}dw$$

$$=\ (T-3)^{-(1+\alpha)/2}\int_0^\infty v^{\alpha}\exp\{-(T-3)^{\frac{1}{2}}\mu(x)v-\tfrac{1}{2}v^2\}dv$$

$$(19)\quad =\ (T-3)^{-(1+\alpha)/2}\Gamma(1+\alpha)e^{\frac{1}{4}(T-3)\mu(x)^2}D_{-1-\alpha}((T-3)^{\frac{1}{2}}\mu(x))$$

where $D_\nu(z)$ is the parabolic cylinder function with variable z and parameter ν (Erdeyli (1953) Ch.8). Using (19) in the first two integrals of (18) we have as the dominant term

$$\frac{(1-x^2)^{(T-2-\frac{1}{2})/2}}{2^{\frac{1}{2}}\pi}\left\{(T-3)^{-\frac{1}{4}}\Gamma(\tfrac{1}{2})e^{\frac{1}{4}(T-3)\mu(x)^2}D_{-\frac{1}{2}}((T-3)^{\frac{1}{2}}\mu(x))\psi(x)\right.$$

$$\left.+\ (T-3)^{-\frac{3}{4}}\Gamma(\tfrac{3}{2})e^{\frac{1}{4}(T-3)\mu(x)^2}D_{-3/2}((T-3)^{\frac{1}{2}}\mu(x))\left[\frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)}\right]\right\}$$

From (15) the resulting approximation for the density is given by

$$(20)\quad p_T(x) \sim \frac{(T-3)^{\frac{1}{2}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}e^{\frac{1}{2}(T-3)\mu(x)^2}}{(2\pi)^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}}$$

$$\cdot\left\{D_{-\frac{1}{2}}((T-3)^{\frac{1}{2}}\mu(x))\psi(x) + \tfrac{1}{2}(T-3)^{-\frac{1}{2}}D_{-3/2}((T-3)\mu(x))\left\{\frac{\psi(x)-\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})}{\mu(x)}\right\}\right\}$$

This approximation holds uniformly as x moves past the branch point $(2x-\alpha)^{-1}$ i.e. as $\mu(x)$ becomes negative, because $D_\nu(z)$ is an entire function of z. The approximation can be computed by using the following expression for $D_\nu(z)$ in terms of the confluent hypergeometric function

$$D_\nu(z) = 2^{\nu/2}e^{-z^2/4}\left\{\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)}\,{}_1F_1(-\tfrac{1}{2}\nu,\tfrac{1}{2};\tfrac{1}{2}z^2) + \frac{z}{2^{\frac{1}{2}}}\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})}\,{}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}\nu,\tfrac{3}{2};\tfrac{1}{2}z^2)\right\}$$

(Erdéyli (1953) p.117).

The uniform approximation (20) can be related to the formulae that apply on either side of the branch point by taking an asymptotic expansion of $D_\nu(z)$. We have two expansions which are relevant in the present case:

$$(21)\quad D_\nu(z) = z^\nu e^{-\frac{1}{4}z^2}[1 + O(z^{-2})] \qquad -\tfrac{3}{4}\pi < \arg(z) < \tfrac{3}{4}\pi$$

and

$$(22)\quad D_\nu(z) = z^\nu e^{-\frac{1}{4}z^2}[1 + O(z^{-2})] - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}e^{\nu\pi i}z^{-\nu-1}e^{\frac{1}{4}z^2}[1 + O(z^{-2})]$$

(Erdéyli (1953) p.123) $\qquad \pi/4 < \arg(z) < 5\pi/4$

When $x < (2x-\alpha)^{-1}$ or $\mu(x) > 0$ we use (21) in (20) and obtain for the dominant term

$$p_T(x) \sim \frac{(T-3)^{\frac{1}{2}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}\psi(x)}{(2\pi)^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}\mu(x)^{\frac{1}{2}}}$$

$$(23)\qquad = \frac{(T-3)^{\frac{1}{2}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T+1)/2}}{(2\pi)^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}(1+\alpha x-2x^2)^{\frac{1}{2}}(1-\alpha x)^{3/2}}$$

The saddlepoint approximation available in this case is given by

$$(24)\quad p_T(x) \sim \frac{(T-3)\Gamma(\frac{T-1}{2})(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T+1)/2}}{2\pi^{\frac{1}{2}}\Gamma(\frac{T}{2})(1+\alpha^2-2x\alpha)^{(T-1)/2}(1-\alpha x)^{3/2}(1+\alpha x-2x^2)^{\frac{1}{2}}}$$

and using the asymptotic relation $\Gamma(\frac{T-1}{2})/\Gamma(\frac{T}{2}) = (\frac{1}{2}T)^{-\frac{1}{2}}[1 + O(T^{-1})]$ it is clear that (24) and (23) are the same up to a relative error of $O(T^{-1})$.

When $x > (2x-\alpha)^{-1}$ or $\mu(x) < 0$ we need to use (22) in (20). We see from (22) that the dominant terms in the expansions in the present case are

$$D_{-\frac{1}{2}}((T-3)^{\frac{1}{2}}\mu(x)) \sim - \frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{2}\pi i}\Gamma(\frac{1}{2})^{-1}e^{\frac{1}{4}(T-3)\mu(x)^2}}{(T-3)^{\frac{1}{4}}\mu(x)^{\frac{1}{2}}}$$

$$= \frac{i2^{1/2}e^{\frac{1}{4}(T-3)\mu(x)^2}}{(T-3)^{\frac{1}{4}}\mu(x)^{\frac{1}{2}}}$$

and

$$D_{-3/2}((T-3)^{\frac{1}{2}}\mu(x)) \sim - (2\pi)^{\frac{1}{2}}\Gamma(\tfrac{3}{2})^{-1}e^{-3\pi i/2}(T-3)^{\frac{1}{4}}\mu(x)^{\frac{1}{2}}e^{\frac{1}{4}(T-3)\mu(x)^2}$$

$$= - 2^{3/2}(T-3)^{\frac{1}{4}}i\mu(x)^{\frac{1}{2}}e^{\frac{1}{4}(T-3)\mu(x)^2}$$

Substituting in (20) we obtain

$$p_T(x) \sim \frac{(T-3)^{\frac{1}{4}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2-\frac{1}{2})/2}e^{\frac{1}{4}(T-3)\mu(x)^2}}{(2\pi)^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}(2x-\alpha)^{\frac{1}{2}}}$$

$$\cdot \{2^{\frac{1}{2}}(T-3)^{-\frac{1}{4}}i\mu(x)^{-\frac{1}{2}}e^{\frac{1}{4}(T-3)\mu(x)^2}\psi(x+\mu(x)(1-x^2)^{\frac{1}{2}})\}$$

$$= \frac{(T-3)^{\frac{1}{2}}(1-\alpha^2)^{\frac{1}{2}}(1-x^2)^{(T-2)/2}e^{\frac{1}{2}(T-3)\mu(x)^2}\psi((2x-\alpha)^{-1})}{\pi^{\frac{1}{2}}(1+\alpha^2-2x\alpha)^{(T-1)/2}}$$

which corresponds exactly with (14).

REFERENCES

Byrd, P.F. and M.D.Friedman (1971), "Handbook of Elliptic Integrals", (second edition), Berlin: Springer Verlag.

Daniels, H.E. (1956), "The Approximate Distribution of Serial Correlation Coefficients," Biometrika, 43, 169-185.

Daniels, H.E. (1977), "The saddlepoint method near a singularity", Research Note, University of Birmingham.

Erdeyli, A. (1953), "Higher Transcendental Functions", Volume 11, New York: McGraw Hill.

Gradshteyn, I.S. and I.M.Ryzhik, "Table of Integrals Series and Products", (Fourth edition), New York: Academic Press.

Phillips, P.C.B. (1978), "Edgeworth and Saddlepoint Approximations in a First Order Non-Circular Autoregression", Biometrika (forthcoming).