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Competitive Valuation of Cooperative Games

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1. *Introduction*

In many applications, it is important to be able to assess the relative "value" of playing various roles in a game. Previous approaches to valuation in cooperative game theory have involved axiomatic structures, n-person bargaining models, and a variety of probabilistic considerations. In this paper we take a new approach, similar to that taken by Shubik and Young in [1]. We consider the n participants of a given cooperative game as independent units, control of which is sought by two opposing forces (one supporting change, the other supporting the status quo). The optimal allocations of resources by the opposing forces indicate the relative values of the players in the original game.

The competitive game for control of the players has a natural interpretation in terms of lobbying for votes, and in a different context may be viewed as an extension of the "Colonel Blotto" games of the theory of warfare [2]. There are several different ways in which the "purchase"

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or "capture" of players can be modelled. An earlier study [1] was concerned primarily with voting games, and with a simple mechanism whereby fractions of the players were "bought" in proportion to the amounts bid for them. We more broadly concern ourselves with general games, and with a variety of mechanisms which determine the probability with which the support of a player is obtained. Our results demonstrate a relationship between the "competitive" values, and a family of values, characterized axiomatically by Dubey and Weber [3], which include the classical Shapley [4] and Banzhaf [5] values.

2. *The Competitive Supergame*

Consider a fixed n-person cooperative game with characteristic function v , played by the members of the player set $N = \{1, 2, \ldots, n\}$. We shall construct a competitive two-person game based on v . There are two related interpretations which can be given to the competitive game, each of which involves the idea of two "superplayers" playing a "supergame."*

In the first interpretation, we view the players in N as political actors. For any coalition $S \subset N$, $v(S)$ measures the extent to which the players in S can affect the current state of public affairs. We hypothesize two opposing superplayers, named I and II . I has the goal of bringing about change, while II seeks to protect the status quo. Hence, each will allocate the resources available to him among the players in N (for example, will make campaign contributions, or exert pressure through lobbying), in such a way as to gain the support of a coalition of influential politicians.

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*We deviate from the tradition of defining a supergame as the sequentially-repeated play of a single game.

The other interpretation is related to a classical military model. Each of the elements of N corresponds to a military objective (such as the winning of a particular battle, or the destruction of a missile silo), and $v(S)$ represents the worth of achieving the objectives comprising S. Superplayer I is the attacker, attempting to attain these objectives. II has the role of defender, and the goal of minimizing I's achievements. Each superplayer has a limited supply of strategic resources, and seeks an optimal allocation of these resources. Assume, for example, that an objective is won (with certainty) by the superplayer who commits the greater amount of resources to it. Then this interpretation of the competitive game closely corresponds to the class of "Colonel Blotto" games.

We continue our discussion within the framework of the first interpretation. The supergame takes place in two stages. First, the superplayers independently (and simultaneously) allocate their respective initial resource holdings among the n players of the underlying game. Assume that I allocates the amounts $(a_1, \ldots, a_n)$, and II the amounts $(b_1, \ldots, b_n)$. Each player in N then (independently) decides whether to support I or not. The probability that player k gives his support to superplayer I is $p_k(a_k, b_k)$, which depends only on the amounts allocated to k; the collection $\{p_k\}_{k=1}^{n}$ of as-yet-unspecified "support functions" determines the particular form of the supergame being played. Player I's payoff is the characteristic function value $v(S)$ of the coalition S of players who actually support him. The corresponding payoff to superplayer II is $v(N) - v(S)$. (We could, of course, make the supergame zero-sum by defining the payoff to II to be $-v(S)$; this would not affect any strategic considerations. However, the given definition, which yields a constant-sum supergame, will clarify later discussions.)

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Then the expected payoff to superplayer I , according to the preceding description of the supergame, is

$$P_I(a,b) = \sum_{S \subset N} \{ \prod_{i \in S} p_i(a_i, b_i) \prod_{j \notin S} (1 - p_j(a_j, b_j)) \} v(S) .$$

(One might note the similarity of this payoff function and the multilinear extension [6] of v .) The expected payoff to superplayer II is

$$\begin{aligned} P_{II}(a,b) &= v(N) - P_I(a,b) \\ &= \sum_{S \subset N} \{ \prod_{i \in S} p_i(a_i, b_i) \prod_{j \notin S} (1 - p_j(a_j, b_j)) \} [v(N) - v(S)] \\ &= \sum_{S \subset N} \{ \prod_{i \in S} p_i(a_i, b_i) \prod_{j \notin S} (1 - p_j(a_j, b_j)) \} v^*(S^c) . \end{aligned}$$

In the last expression, v^* is the "dual" of v , defined for all $T \subset N$ by $v^*(T) = v(N) - v(T^c)$. The dual of v can be thought of as a "blocking" game. If the players in N have agreed to cooperate in order to attain $v(N)$, but are debating the precise division of this amount among themselves, then a coalition T might be concerned not only with the amount $v(T)$ as a justification of its members' claims, but also with the amount $v^*(T) = v(N) - v(T^c)$, which is the amount which it cannot be unquestionably denied by the opposing coalition. Given this "blocking" interpretation of the dual, the form of the last expression for $P_{II}(a,b)$ is not surprising.

The competitive game has a hidden symmetry, regardless of what the underlying game v may be. If we are informed only of the payoff functions of a pair of superplayers, we cannot deduce whether the first is playing a role of superplayer I in a supergame based on a cooperative game v, or whether the second superplayer has the role of I in the supergame based on v^*. If v is constant-sum (that is, if $v(T) + v(T^c) = v(N)$ for all $T \subset N$) and if $p_i(a_i, b_i) + p_i(b_i, a_i) = 1$ (that is, the outcome functions are symmetric), then the supergame is overtly symmetric. This follows from the easily-verified fact that a constant-sum game is self-dual.

3. *Values and Equilibria*

Much game-theoretic work has centered on the problem of comparing the prospects of playing various positions in a variety of games. Given a family $\mathcal{G}$ of cooperative games, a *value* ϕ on $\mathcal{G}$ is a mapping which assigns a real number $\phi_k(v)$ to each player k of any game $v \in \mathcal{G}$. A classical approach to the study of values is from an axiomatic viewpoint. A number of "reasonable" criteria are proposed, and the values satisfying these criteria are characterized.

A game v, with player set N, is *monotonic* if $v(S) \geq v(T)$ whenever $S \supset T$; v is *strictly monotonic* if $v(S) > v(T)$ for all $S \supsetneq T$. A player $k \in N$ is a *dummy* in v if $v(S \cup k) = v(S) + v(k)$ for all $S \subset N/k$; k is a *zero-dummy* if, in addition, $v(k) = 0$. (Note that we frequently omit the braces when writing one-element sets.)

It is common to require that a value be a linear function on its domain $\mathcal{G}$. Other criteria include the following: if k is a dummy in v, then $\phi_k(v) = v(k)$; if v is monotonic, then all components of $\phi(v)$ are nonnegative; the value of a game to any of its players is unaffected by introducing a dummy player into the game. Of a slightly different nature is the criterion that the value be symmetric; that is, if π is a permutation of the player set of $v \in \mathcal{G}$, and if πv (defined in the usual manner) is in $\mathcal{G}$, then for every player k of the game, $\phi_k(v) = \phi_{\pi k}(\pi v)$.

Take any $0 \leq t \leq 1$. The *t-value* for any player k of an n-player game v is defined by

$$\phi_k^{(t)}(v) = \sum_{S \subset N \setminus k} t^s (1-t)^{n-s-1} [v(S \cup k) - v(S)] ,$$

where N is the player set of v , and where s generically denotes the cardinality of S . The values which satisfy the preceding criteria (over the family of all finite-player games) are precisely those which are weighted averages of t-values [3]. When all t-values are equally-weighted in the average, we obtain the well-known Shapley value; the 1/2-value of a game is its Banzhaf value.

A primary purpose of this paper is to indicate how the t-values arise in a competitive setting. Since the setting involves only the single n-person game under consideration (rather than a family of games, many of them not relevant to the situation being studied, over which the value is constrained by various criteria), our results provide strong support to the importance of such values.

In order to study competitive games, we require several equilibrium concepts. Consider any two-person game. A pair of strategies (one for each player) is an *equilibrium point* of the game if each player's strategy is a best response to his opponent's strategy. Assume that the initial resources of the two superplayers of a competitive supergame (as previously discussed) are in the amounts A and B , respectively. Thus the strategy space for I is the set $\mathcal{A} = \{a \in R_+^n : \sum a_k = A\}$, and the strategy space for II is $\mathcal{B} = \{b \in R_+^n : \sum b_k = B\}$. The strategy pair (a,b) is in equilibrium if $P_I(a,b) \geq P_I(a',b)$ for all $a' \in \mathcal{A}$, and $P_{II}(a,b) \geq P_{II}(a,b')$ for all $b' \in \mathcal{B}$. The equilibrium concept can be localized. A *local equilibrium point* is a strategy pair (a,b) which satisfies the preceding conditions as long as a' and b' are restricted to sufficiently small neighborhoods of a and b , respectively. The local equilibrium concept is a natural one for situations which evolve over time, and in which major strategy shifts quickly become known to one's opponent. In particular, pure strategy local equilibria tend to be self-reinforcing, and therefore stable.

4. *Competition for the Support of a Single Player*

Assume that the two superplayers respectively commit the resource amounts a and b in an attempt to win the support of player k in the underlying game. (Since we are only concerned with a single player in this section, we shall omit the subscripts from all expressions.) How might these commitments affect k's behavior? A reasonable assumption is that $p(a,b)$, the probability that k supports I , depends on the relative sizes of a and b , rather than on the absolute magnitudes of these amounts. Hence, we assume that p is homogeneous of degree zero; that is, for all $\ell > 0$, $p(\ell a, \ell b) = p(a,b)$. Certainly, it is reasonable to assume that $p(a,b)$ is monotone increasing in a , and monotone decreasing in b . This merely asserts that k is not averse to having additional resources allocated to him. We further assume that $p(0,0) = p(1,1)$, $p(1,0) = 1$, and $p(0,1) = 0$. The first of these is merely a convenient convention, while the latter two indicate that

k's loyalty can be ensured with high probability by a sufficiently disproportionate commitment by one of the superplayers.

Besides these structural considerations, there are at least two additional important characteristics of p . When the two superplayers commit equal amounts of resources to k , the resulting probability $p(1,1) = \gamma$ indicates the degree of "natural" support k provides to I . And when the commitments are unequal, the difference between $p(a,b)$ and $p(1,1)$ indicates the sensitivity of k's behavior to the relative difference between a and b .

For any $0 < \gamma < 1$ and $m > 0$, let $p(a,b) = \gamma a^m/[\gamma a^m + (1-\gamma)b^m]$. The family of such "support functions" serves to illustrate the preceding discussion. Clearly, $p(1,1) = \gamma$. For values of m near zero, $p(a,b)$ is near γ even when a and b differ substantially. However, if m is very large, slight differences between a and b can make $p(a,b)$ quite different from γ . For the sake of future discussions, we define $B(a,b)$ as the limit of $p(a,b)$ when $\gamma = 1/2$ and $m \to \infty$. This is the so-called "Colonel Blotto" outcome function, which satisfies $B(a,b) = 0,\ 1/2$, or 1 accordingly as $a < b$, $a = b$, or $a > b$.*

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*Another Blotto interpretation might give "ties" to the defender, whence for $a = b$ the defender scores 1.

5. *The Location of Equilibria*

Our first result is that pure strategy equilibria (local or global) of the competitive supergame must correspond to the t-values of the underlying cooperative game.

Theorem. Let v be an n-person monotonic game, in which no players are zero-dummies. Consider the two-person competitive game based upon v, and let the initial resources of the superplayers I and II be L and 1, respectively. Assume that the same support function p governs the behavior of all n players of the underlying game, and that p is continuously differentiable, and is strictly monotonic in each variable. Then, if the allocations $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ constitute a local equilibrium pair of pure strategies in the competitive game, it must be that both a and b are proportional to the $p(L,1)$-value of v.

Proof. We first demonstrate that $a > 0$ and $b > 0$ at any local equilibrium point. Let $\overline{p} = (p_1, \ldots, p_n)$ be the vector of probabilities determined by a and b; hence $p_i = p(a_i, b_i)$ for all i in the underlying player set N. The payoff function $P_I(a,b)$ can be written as a continuous function $P_I(\overline{p})$. Furthermore, the assumption that v is monotonic and without zero-dummies implies that P_I is strictly monotonic in p; that is, $\overline{p}' \gneq \overline{p}$ implies $P_I(\overline{p}') > P_I(\overline{p})$. Now, suppose that some $a_i = 0$. If b_i is also zero, then I can modify his strategy by reducing some positive a_j by a slight amount, and increasing a_i accordingly. This will decrease p_j slightly, while increasing p_i from $p(0,0)$ to 1. But $P_I(p_1, \ldots, p_i = 1, \ldots, p_j - \epsilon, \ldots, p_n) \approx P_I(p_1, \ldots, 1, \ldots, p_j, \ldots, p_n) > P_I(p_1, \ldots, p(1,1), \ldots, p_j, \ldots, p_n)$, where the approximation follows from continuity and the inequality from monotonicity. Hence, a local strategy change is advantageous for I , so (a,b) could not have been in equilibrium. On the other hand, if $a_i = 0$ and $b_i > 0$, then II can decrease b_i while keeping $p_i = 0$, and can correspondingly increase some b_j for which $a_j > 0$, thus decreasing p_j . It follows from the monotonicity of P_I that this local strategy change works to II's advantage, again contradicting the assumption that (a,b) is in equilibrium. Similar arguments show that no $b_i = 0$.

Since (a,b) is an equilibrium pair such that a and b are in the relative interiors of the superplayers' strategy spaces, it is necessary that all of the derivatives $\partial P_I/\partial a_k$ are equal and nonnegative at (a,b) , and that all of the derivatives $\partial P_I/\partial b_k = -\partial P_{II}/\partial b_k$ are equal and nonpositive.

Now,

$$\frac{\partial P_I}{\partial a_k}(a,b) = \frac{\partial}{\partial a_k}p(a_k, b_k)\cdot \sum_{S\subset N\setminus k} \{ \prod_{i\in S} p(a_i, b_i) \prod_{\substack{j\notin S \\ j\neq k}} (1-p(a_j, b_j))\}[v(S\cup k)-v(S)] ,$$

and

$$\frac{\partial P_I}{\partial b_k}(a,b) = \frac{\partial}{\partial b_k}p(a_k, b_k)\cdot \sum_{S\subset N\setminus k} \{ \prod_{i\in S} p(a_i, b_i) \prod_{\substack{j\notin S \\ j\neq k}} (1-p(a_j, b_j))\}[v(S\cup k)-v(S)] .$$

Recall that p is homogeneous of degree zero. Hence, by differentiation (this is an instance of Euler's equality),

$$a_k\cdot\frac{\partial}{\partial a_k}p(a_k, b_k) + b_k\cdot\frac{\partial}{\partial b_k}p(a_k, b_k) = 0 ,$$

and therefore

$$\frac{\partial}{\partial b_k}p(a_k, b_k) = -\frac{a_k}{b_k}\cdot\frac{\partial}{\partial a_k}p(a_k, b_k) .$$

The monotonicity of v, combined with the assumption that p is strictly monotonic in each variable, allows us to conclude that the derivatives $\partial P_I/\partial a_k$ and $\partial P_I/\partial b_k$ are nonzero at (a,b). The equality of all derivatives of the first type implies that $(\partial P_I/\partial a_k)(a,b)$ is constant for all k. Furthermore, the equality of all $(\partial P_I/\partial b_k)(a,b)$ implies that

$$\frac{\partial P_I}{\partial b_k}(a,b) = -\frac{a_k}{b_k}\cdot\frac{\partial P_I}{\partial a_k}(a,b)$$

is constant for all k. Hence, the ratio a_k/b_k is unvarying in k, and therefore $a = Lb$.

For any nonzero b_k, $p(Lb_k, b_k) = p(L,1)$. It therefore follows from the homogeneity of p that

$$\frac{\partial}{\partial a_k}p(a_k,\ b_k)\bigg|_{a_k=Lb_k} = \frac{\partial}{\partial a_k}p\left(\frac{a_k}{b_k},\ 1\right)\bigg|_{a_k=Lb_k} = \frac{1}{b_k}p_1(L,1)\ ,$$

where p_1 is the partial derivative of p with respect to its first variable. Let $\phi = (\phi_1, \ldots, \phi_n)$ be the $p(L,1)$-value of v. Then

$$\frac{\partial P_I}{\partial a_k}(Lb,b) = \frac{\partial}{\partial a_k}p(a_k,\ b_k)\bigg|_{a_k=Lb_k}\cdot\phi_k = \frac{1}{b_k}p_1(L,1)\phi_k = K\ ,$$

where $p_1(L,1)$ and K are positive constants, independent of k. Consequently, b is proportional to ϕ. This completes the proof. □

The assumption that v has no zero-dummies was included in the theorem for mathematical convenience. Clearly it is not optimal for either superplayer to allocate any of his resources to a zero-dummy. Since the

$p(L,1)$-value of a zero-dummy is zero, the proportionality asserted in the theorem will also hold for monotonic games involving such players.

Perhaps the proportionality of the optimal pure strategies (to one another) seems surprising. However, observe that the t-value of any game is equal to the $(1-t)$-value of its dual (this can be verified in a straightforward manner). Because of this, the hidden symmetry of the competitive supergame provides an intuitively-satisfying explanaton of the preceding theorem.

6. *The Existence of Equilibria*

The preceding result allows us to restrict our search for pure strategy equilibria to a single point. However, we have no guarantee that this point will indeed be an equilibrium point.

Let us limit our consideration to the class $\{p^m\}_{m>0}$ of support functions defined by

$$p^m(a,b) = \frac{\gamma a^m}{\gamma a^m + (1-\gamma)b^m} ,$$

where γ is a fixed constant strictly between 0 and 1. We indicated earlier that, for large values of m, p^m is similar to the Colonel Blotto support function B. It is well-known that many Colonel Blotto games (at least, most of those in which the superplayers begin with comparable initial resources) do not possess pure strategy equilibria. On the other hand, for small values of m, p^m is relatively stable with respect to small variations in its arguments. This suggests that equilibria may then be easier to come by. This is made precise in the following result.

Theorem. Let v be an n-person game which is strictly monotonic. Let the two superplayers have initial resources L and 1 in the competitive game based on v, and let the support function p^m govern the behavior of all n underlying players. Then, for all sufficiently small m, the allocations a^* and b^*, both proportional to the $p^m(L,1)$-value of v, constitute the unique global equilibrium point of the game.

Proof. Let $P_I^m(a,b)$ be the payoff function for superplayer I. We shall show that $P_I^m(a,b^*)$ is, for all sufficiently small m, a strictly concave function of a on the strategy space $\mathcal{A}$, and that the function attains its unique maximum at $a = a^*$. In a similar manner, it will follow that $P_I^m(a^*,b)$ is uniquely minimized (on $\mathcal{B}$) at $b = b^*$. Therefore, since the competitive game is constant-sum and all equilibria are equivalent and interchangeable, we will have shown that (a^*,b^*) is the unique equilibrium point of the game, as claimed.

For any players k and ℓ in the player set N of v, define

$$\beta_k^m(a,b) = \sum_{S\subset N\setminus k} \{ \prod_{i\in S} p^m(a_i,\ b_i) \prod_{\substack{j\notin S\\ j\neq k}} (1-p^m(a_j,\ b_j))\}[v(S\cup k) - v(S)] ,$$

and

$$\beta_{k,\ell}^m(a,b) = \sum_{S\subset N\setminus(k\cup\ell)} \{ \prod_{i\in S} p^m(a_i,\ b_i) \prod_{\substack{j\notin S\\ j\neq k,\ell}} (1-p^m(a_j,\ b_j))\}$$

$$\cdot [v(S\cup k\cup \ell) - v(S\cup k) - v(S\cup \ell) + v(S)] .$$

Then

$$\frac{\partial P_I^m}{\partial a_k}(a,b) = p_1^m(a_k,\ b_k)\beta_k^m(a,b) ,$$

$$\frac{\partial^2 P_I^m}{\partial a_k^2}(a,b) = p_{11}^m(a_k,\ b_k)\beta_k^m(a,b) .$$

and

$$\frac{\partial^2 P_I^m}{\partial a_k \partial a_\ell}(a,b) = p_1^m(a_k,\ b_k)p_1^m(a_\ell,\ b_\ell)\beta_{k,\ell}^m(a,b) .$$

The functions p_1^m and p_{11}^m are the first and second partial derivatives of p^m with respect to its first variable; that is,

$$p_1^m(a_k,\ b_k) = \gamma(1-\gamma)\cdot\frac{m}{a_k}\cdot\left(\frac{a_k}{b_k}\right)^m \Bigg/ \left\{\gamma\left(\frac{a_k}{b_k}\right)^m + (1-\gamma)\right\}^2$$

and

$$p_{11}^m(a_k,\ b_k) = \gamma(1-\gamma)\cdot\frac{m}{a_k^2}\cdot\left(\frac{a_k}{b_k}\right)^m\cdot\left[(1-\gamma)(m-1) - \gamma(m+1)\left(\frac{a_k}{b_k}\right)^m\right] \Bigg/ \left\{\gamma\left(\frac{a_k}{b_k}\right)^m + (1-\gamma)\right\}^3 .$$

Let b be any fixed nonnegative vector. The function P_I^m is strictly concave throughout the nonnegative orthant if the Hessian of P_I^m is negative definite for all nonnegative a . After deleting common positive factors from the rows and columns of the Hessian, we are left with the matrix $H^m(a,b) = (h_{k\ell})$, in which the k^{th} diagonal entry is

$$h_{kk} = \left[\gamma\left(\frac{a_k}{b_k}\right)^m + (1-\gamma)\right]\cdot\left[(1-\gamma)(m-1) - \gamma(m+1)\left(\frac{a_k}{b_k}\right)^m\right]\cdot\left(\frac{b_k}{a_k}\right)^m\cdot\beta_k^m(a,b)$$

and the $(k,\ell)^{\text{th}}$ off-diagonal entry is

$$h_{k\ell} = \gamma(1-\gamma)m\cdot\beta^m_{k,\ell}(a,b) \ .$$

(The factor $\gamma(1-\gamma)\cdot(m/a_k)\cdot(a_k/b_k)^m/(\gamma(a_k/b_k)^m + (1-\gamma))^2$ has been deleted from the k^{th} row, and the factor $(1/a_\ell)\cdot(a_\ell/b_\ell)^m/(\gamma(a_\ell/b_\ell)^m + (1-\gamma))^2$ from the ℓ^{th} column; these deletions don't affect the potential negative-definiteness of $H^m(a,b)$.)

For sufficiently small m (for example, $m < 1$), the k^{th} diagonal entry is negative, and is less than $\gamma(1-\gamma)(m-1)\beta^m_k(a,b)$. Furthermore, since v is strictly monotonic, each $\beta^m_k(a,b) \geq \min_{S\subset N\setminus k} (v(S\cup k) - v(S)) > 0$. And, since each $\beta^m_{k,\ell}(a,b) \leq v(N)$, each off-diagonal entry is positive and is no greater than $\gamma(1-\gamma)m\cdot v(N)$. We can apply Gershgorin's Theorem, which asserts that the eigenvalues of an n-by-n matrix $A = (a_{k\ell})$ lie in the union of the n circles (in the complex plane) with centers a_{kk} and radii $\sum_{\ell\neq k} |a_{k\ell}|$. $H^m(a,b)$ is symmetric, and therefore its eigenvalues are all real. Let $\mu = \min_{k,S\subset N\setminus k} (v(S\cup k) - v(S)) > 0$. Then each $|h_{kk}| > \gamma(1-\gamma)(1-m)\mu$. For each k , $\sum_{\ell\neq k} h_{k\ell} \leq \gamma(1-\gamma)m(n-1)v(N)$. Hence if $(1-m)\mu > m(n-1)v(N)$, all eigenvalues of $H^m(a,b)$ will be strictly negative. Therefore, $H^m(a,b)$ will be negative definite for all $m < \mu/nv(N)$. (Of course, this doesn't preclude negative-definiteness for larger values of m .)

It remains to be shown that, when a is restricted to $\mathfrak{a}$, $F^m_I(a,b^*)$ is maximized at $a = a^*$. But both a^* and b^* are proportional to $\beta^m(L,1)$, and consequently all derivatives $\partial F^m_I/\partial a_k$ are equal and positive at (a^*,b^*) . Hence, a local (and therefore, due to concavity, global) maximum occurs at a^* . □

What if v is monotonic, but not strictly so? This is the case, for example, with most political games. We can still show that, for sufficiently small m , (a^*,b^*) is a local equilibrium point of the competitive game.

Assume that v is monotonic, and (for notational convenience) without zero-dummies. The Hessian of P_I^m at (a^*,b^*) will be negative definite if, for all $k \in N$, we have $(1-m)\beta_k^m(L,1) > m(n-1)v(N)$. As m approaches 0, $p^m(a_k^*, b_k^*)$ approaches γ for every $k \in N$. Hence, $\beta_k^m(L,1)$ approaches $\beta_k^{(\gamma)}(v)$. Since v is monotonic, the γ-value of v is positive in all components. Let $\eta = \min_k \phi_k^{(\gamma)}(v) > 0$. Then for all sufficiently small m, and for all $k \in N$, $\beta_k^m(L,1) \geq \frac{1}{2}\eta$. If, in addition, $m < \frac{1}{2}\eta/nv(N)$, then $H^m(a^*,b^*)$ will certainly be negative definite. Since the appropriate partial derivatives are equal, as before, we conclude that (a^*,b^*) is a local equilibrium point.

7. *Symmetric Games*

Typically, if m is near zero, the competitive game has a global pure strategy equilibrium. Then, as m is increased, the equilibrium point loses its global character, but remains in equilibrium with respect to neighborhoods of decreasing size. Finally, when m is sufficiently large, the game may have no pure strategy equilibrium point at all. This behavior is most clear when the underlying game is symmetric.

Let v be symmetric (that is, $v(S)$ depends only on the cardinality of S) and monotonic (as usual, we rule out zero-dummies), with player set N. We consider support functions of the form $p^m(a_k, b_k) = a_k^m/(a_k^m + b_k^m)$ (that is, $\gamma = 1/2$), and we restrict our attention to the equal-resource case, $L = 1$. In this case, the matrix $H^m(a^*,b^*)$ has diagonal terms all equal to $-\beta_1$, and off-diagonal terms equal to $\frac{1}{4}m\beta_2$. The quantity β_1 is the Banzhaf value (the $\frac{1}{2}$-value) of the underlying game, which is (due to symmetry) the same for all players. (Note that $a^* = b^* = (1/n, \ldots, 1/n)$, and that both are proportional to the Banzhaf value.) Also, for any $k,\ell \in N$,

$$\beta_2 = \frac{1}{2^{n-2}} \sum_{S \subset N \setminus (k \cup \ell)} [v(S \cup k \cup \ell) - v(S \cup k) - v(S \cup \ell) + v(S)]$$

$$= \frac{1}{2^{n-2}} \cdot \frac{1}{n(n-1)} \cdot \sum_{s=1}^{n} [(2s-n)^2 - n] \binom{n}{s} v_s \ ,$$

where $v_{|S|} = v(S)$ for all $S \subset N$.

This matrix has an eigenvalue $\lambda = -\beta_1 - \frac{1}{4}m\beta_2$ of multiplicity $(n-1)$. The corresponding eigenspace is $\{x \in R^n : \Sigma x_i = 0\}$. This eigenspace is parallel to the hyperplane containing the strategy space $\mathcal{A}$, and the remaining eigenvector of $H^m(a^*,b^*)$ is orthogonal to this hyperplane. Therefore, $H^m(a,b^*)$ is negative definite throughout a neighborhood of a^* in $\mathcal{A}$ if and only if $\lambda < 0$.

The quantity β_1 is clearly positive. However, β_2 can either be positive or negative. Consider the three-player game defined by $v_0 = v_1 = 0$, $v_2 = 1$, $v_3 = 2$. For this game, $\beta_1 = 3/4$ and $\beta_2 = 1/2$. Hence, (a^*,b^*) is a local equilibrium point for every value of m . On the other hand, consider the six-player game defined by $v_0 = v_1 = v_2 = 0$, $v_3 = v_4 = v_5 = 1$, $v_6 = 2$. For this game, $\beta_1 = 11/32$ and $\beta_2 = -1/16$. Therefore, for $m < 22$, $\lambda < 0$ and (a^*,b^*) is a local equilibrium point. (Of course, for sufficiently small m , the equilibrium is global.) For larger values of m , the competitive game lacks an equilibrium in pure strategies.

Notice that the preceding six-person example was well-behaved, even for moderately large values of m . It would be of interest to learn whether there are games which lack pure strategy equilibria for relatively small values of m .

8. *Summary*

The preceding sections concern two related results: that supergames based on outcome functions of the form p^m have pure strategy equilibria (that is, optimal pure strategies) when m is small, and that these equilibria correspond to previously-studied "values" of n-person cooperative games. We discuss these results in turn.

Consider the qualitative difference between pure strategy equilibria, and equilibria which involve mixed strategies. In the former case, at least for two-person constant-sum games, secrecy in strategic choice is unnecessary. One may publicly announce his choice of a strategy, without fear of that announcement damaging his prospects. However, in the latter case secrecy is essential; one cannot afford to let his opponent learn of the particular randomly-chosen strategy which will be followed.

When can we expect secrecy to be important? One instance is when small strategy shifts can have a dramatic effect on the outcome of the game. Such is the case, for example, in traditonal Colonel Blotto games in which it is assumed that the slightest numerical superiority on the battlefield will assure one of victory. This is the limiting case of our outcome function p^m, when m becomes large. However, if this model of battlefield outcomes seems extreme, then we should turn our attention to smaller values of the parameter m.

If small strategy shifts have relatively small effect on the outcome, it seems more reasonable that a stable situation exists. This is the case, for example, in economic and political games which are played over time, in which local strategy shifts have only small effect on the state of the entire economic or political system. Our use of an outcome function p^m, with m small, places us within this case. Under the legislator/lobbyist interpretation of our work, we view a legislator as having developed (over time) a natural proclivity to support the agents of change a fraction γ of the time. The current pressures brought to bear upon the legislator (and contributions made to his campaign chest) by opposing forces will affect this degree of support, but the effect will not be drastic if the integrity of the legislator is high. This corresponds to the case of small values of m, and to the existence of stable patterns of legislative behavior. That mixed strategies are unnecessary means simply that "under-the-table" payments are not part of an optimal lobbying strategy.

When m is small, the optimal strategies of the superplayers of the competitive game indicate the relative "values" of the players in the underlying cooperative game. When the initial resources of the opposing sides are equal, then as we vary the parameter γ we obtain the full family of "t-values," which spans the class of values studied axiomatically in [3]. That these values arise from markedly-different approaches indicates that they measure some universal attribute of the components of a cooperative game. For the strategic planner, the implication of our work is that these values provide natural guidelines for the allocation of strategic resources.

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